

# Expansion-free evolving spheres must have inhomogeneous energy density distributions

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In a recent paper a systematic study on shearing expansion-free spherically symmetric distributions was presented. As a particular case of such systems, the Skripkin model was mentioned, which corresponds to a nondissipative perfect fluid with a constant energy density. Here we show that such a model is inconsistent with junction conditions. It is shown that in general for any nondissipative fluid distribution, the expansion-free condition requires the energy density to be inhomogeneous. As an example we consider the case of dust, which allows for a complete integration.

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## I. INTRODUCTION

In a recent paper [1] a general study on shearing expansion-free (vanishing expansion scalar  $\Theta$ ) spherical fluid evolution is presented, which includes pressure anisotropy and dissipation. The interest of such models stems from the fact that the expansion-free condition necessarily implies the appearance of a cavity.

Indeed, it is intuitively clear that in the case of an overall expansion, the increase in volume due to the increasing area of the external boundary surface must be compensated with the increase of the area of the internal boundary surface (delimiting the cavity) in order to keep  $\Theta$  vanishing. The argument in the case of collapse is similar.

More rigorously, it was shown in [1] that for two concentric fluid shells in the process of expansion, in the neighborhood of the center, the  $\Theta = 0$  condition is violated, implying thereby that such a condition requires that the innermost shell of fluid should be away from the center, initiating therefrom the formation of the cavity (see [1] for details).

The natural appearance of a vacuum cavity within the fluid distribution in expansion-free solutions suggests that they might be relevant for the modeling of voids observed at cosmological scales.

In the particular case of a nondissipative isotropic fluid, with constant energy density, the Skripkin model [2] is recovered.

The purpose of this Brief Report is twofold. We apply the results of [1] to prove that the Skripkin model is ruled out by junction conditions. Second, we consider the case of dust, and provide a complete integration of this model.

## II. THE EXPANSION-FREE SPHERE

We consider a spherically symmetric distribution of collapsing fluid, bounded by a spherical surface  $\Sigma^{(e)}$ . The

fluid is assumed to be locally anisotropic (principal stresses unequal) but nondissipative.

Choosing comoving coordinates inside  $\Sigma^{(e)}$ , the general interior metric can be written

$$ds^2 = -A^2 dt^2 + B^2 dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where  $A$ ,  $B$ , and  $R$  are functions of  $t$  and  $r$  and are assumed positive. We number the coordinates  $x^0 = t$ ,  $x^1 = r$ ,  $x^2 = \theta$ , and  $x^3 = \phi$ .

Outside  $\Sigma^{(e)}$  we assume we have the Schwarzschild spacetime, described by

$$ds^2 = -\left[1 - \frac{2M}{r}\right]dv^2 - 2drdv + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (2)$$

where  $M = \text{const}$  denotes the total mass, and  $v$  is the retarded time.

The matching of (1) to the Schwarzschild spacetime (2) on  $\Sigma^{(e)}$  requires the continuity of the first and second differential forms, which implies (see [1] for details)

$$m(t, r) \stackrel{\Sigma^{(e)}}{=} M, \quad P_r \stackrel{\Sigma^{(e)}}{=} 0, \quad (3)$$

where  $\stackrel{\Sigma^{(e)}}{=}$  means that both sides of the equation are evaluated on  $\Sigma^{(e)}$ , and  $m$  is the mass function introduced by Misner and Sharp [3] (see also [4]) given by

$$m = \frac{R^3}{2} R_{23}{}^{23} = \frac{R}{2} \left[ \left(\frac{\dot{R}}{A}\right)^2 - \left(\frac{R'}{B}\right)^2 + 1 \right]. \quad (4)$$

As we mentioned in the introduction, the expansion-free models present an internal vacuum cavity. If we call  $\Sigma^{(i)}$  the boundary surface between the cavity and the fluid, then the matching of the Minkowski spacetime within the cavity to the fluid distribution, implies

$$m(t, r) \stackrel{\Sigma^{(i)}}{=} 0, \quad P_r \stackrel{\Sigma^{(i)}}{=} 0. \quad (5)$$

Now, it can be shown (see [1] for details) that as a consequence of the expansion-free condition and one of the Einstein field equations, we can write the line element

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as

$$ds^2 = -\left(\frac{R^2 \dot{R}}{\tau_1}\right)^2 dt^2 + \frac{1}{R^4} dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (6)$$

where  $\tau_1$  is a function of time and the dot stands for differentiation with respect to  $t$  (a unit constant with dimensions  $[r^4]$  is assumed to multiply  $dr^2$ ). This is the general metric for a spherically symmetric anisotropic perfect fluid undergoing shearing and expansion-free evolution (observe that it has the same form as for the isotropic fluid [5]).

For the line element (6), the Einstein equation  $G_{00}^- = 8\pi T_{00}^-$ , becomes [Eq. (76) in [1]]

$$8\pi\mu = -2R^3 R'' - 5R^2 R'^2 + \frac{1}{R^2} - 3\frac{\tau_1^2}{R^6}, \quad (7)$$

where  $\mu$  is the energy density and the prime stands for  $r$  differentiation. The first integral of (7) is

$$R'^2 = \frac{1}{R^4} + \frac{\tau_2 - 2m}{R^5} + \frac{\tau_1^2}{R^8}, \quad (8)$$

where  $\tau_2(t)$  is an arbitrary function of  $t$ .

Using the proper time derivative  $D_T$ ,

$$D_T = \frac{1}{A} \frac{\partial}{\partial t}, \quad (9)$$

and the proper radial derivative  $D_R$ ,

$$D_R = \frac{1}{R'} \frac{\partial}{\partial r}, \quad (10)$$

where  $R$  defines the areal radius of a spherical surface inside  $\Sigma^{(e)}$  (as measured from its area), the following equations are easily obtained. From (4),

$$D_T m = -4\pi P_r U R^2, \quad (11)$$

where  $P_r$  denotes the radial pressure and  $U$  is the velocity of the collapsing fluid,

$$U = D_T R < 0 \quad (\text{in the case of collapse}), \quad (12)$$

being the variation of the areal radius with respect to proper time. From (4), too, we can obtain

$$D_R m = 4\pi\mu R^2, \quad (13)$$

implying

$$m = 4\pi \int_0^R \mu R^2 dR, \quad (14)$$

with the assumption of a regular center to the distribution  $m(0) = 0$ .

From (6) with (12) it follows

$$U = \frac{\dot{R}}{A} = \frac{\tau_1}{R^2}, \quad B = \frac{1}{R^2}. \quad (15)$$

Substituting (15) into (4) and using (8) we obtain

$$\tau_2 = 0. \quad (16)$$

The Einstein equation  $G_{11}^- = 8\pi T_{11}^-$  reads [Eq. (78) in [1]]

$$8\pi P_r = \frac{\tau_1^2}{R^5 \dot{R}} \left(3\frac{\dot{R}}{R} - 2\frac{\dot{\tau}_1}{\tau_1}\right) + R^3 R' \left(2\frac{\dot{R}'}{R} + 5\frac{R'}{R}\right) - \frac{1}{R^2}, \quad (17)$$

where  $P_r$  denotes the radial pressure. Using (4), (8), and (15) in (17) we have

$$8\pi P_r = -\frac{2\dot{m}}{R^2 \dot{R}}. \quad (18)$$

Observe that (18) is fully consistent with the junction conditions (3) and (5). Now, if we assume as Skripkin [2], that  $\mu = \mu_0 = \text{const}$ , we obtain, using (14),

$$\dot{m} = 4\pi\mu_0 R^2 \dot{R}. \quad (19)$$

Feeding back (19) into (18) produces

$$P_r = -\mu_0 = \text{const}, \quad (20)$$

which by virtue of the junction condition

$$P_r \stackrel{\Sigma^{(e)}}{=} 0 \quad (21)$$

implies

$$P_r = \mu_0 = 0. \quad (22)$$

Thus, the Skripkin model is ruled out by junction conditions. Observe that the isotropy of pressure is not explicitly used. However this last condition follows from the constancy of the energy density and the expansion-free condition, as can be seen from the Bianchi identity  $T_{;\beta}^{-\alpha\beta} V_\alpha = 0$ , where  $V^\alpha$  denotes the four velocity of the fluid, reading [Eq. (80) in [1]]

$$\dot{\mu} + 2(P_\perp - P_r) \frac{\dot{R}}{R} = 0, \quad (23)$$

where  $P_\perp$  denotes the tangential pressure. From (23) we have that if the expansion-free fluid is isotropic,  $P_r = P_\perp$ , then the energy density  $\mu$  is only  $r$  dependent, and vice versa.

In the next section we consider the case of dust,  $P_\perp = P_r = 0$ , with  $\mu = \mu(r)$ .

### III. THE EXPANSION-FREE DUST

The Bianchi identity  $T_{;\beta}^{-\alpha\beta} \chi_\alpha = 0$ , where  $\chi_\alpha$  is a unit four vector along the radial direction, reads [Eq. (81) in [1]]

$$P_r' + (\mu + P_r) \frac{\dot{R}'}{R} + 2(\mu + 2P_r - P_\perp) \frac{R'}{R} = 0, \quad (24)$$

and for dust it becomes

$$\frac{\dot{R}'}{\dot{R}} + \frac{2R'}{R} = 0, \quad (25)$$

whose integration gives

$$\dot{R} = \frac{f(t)}{R^2}, \quad R' = \frac{g(r)}{R^2}, \quad (26)$$

with  $f(t)$  and  $g(r)$  denoting arbitrary functions of their arguments. Then from (26) we obtain

$$R^3 = \psi(t) + \chi(r), \quad (27)$$

with

$$\psi(t) = 3 \int f(t) dt \quad \chi(r) = 3 \int g(r) dr. \quad (28)$$

Without loss of generality we may choose  $\tau_1(t) = f(t)$ , implying, because of (6), (15), and (26),

$$A = 1, \quad U = \dot{R}. \quad (29)$$

Then, from the junction condition (5), using (4) and (6) we obtain

$$\dot{R}^2 \stackrel{\Sigma^{(i)}}{=} g - 1, \quad (30)$$

producing

$$R \stackrel{\Sigma^{(i)}}{=} (g - 1)^{1/2} t + R(0). \quad (31)$$

Evaluating (27) at  $\Sigma^{(i)}$  and considering (31) we obtain

$$\psi(t) \stackrel{\Sigma^{(i)}}{=} [(g - 1)^{1/2} t + R(0)]^3 - \chi, \quad (32)$$

thereby providing the explicit time dependence of the models.

In order to find the  $r$  dependence ( $\phi$  or  $\chi$ ) we proceed as follows. Because of (23) we know that  $\mu = \mu(r)$ , then evaluating (7) at  $t = 0$ , we obtain a differential equation for  $\phi(r)$  [or  $\chi(r)$ ], which may be integrated for any given function  $\mu = \mu(r)$ .

#### IV. CONCLUSIONS

We have seen so far that expansion-free condition together with junction conditions rule out the Skripkin model (constant energy density and isotropic pressure). In principle there could be constant energy density models, if we allow for the presence of dissipation, however we do not know at this point if such models may satisfy the whole set of junction and physical conditions.

Next, we consider the case of dust with  $\mu = \mu(r)$ . These models can be completely integrated for any given function  $\mu(r)$ . Of course such models are members of the Lemaître-Tolman-Bondi spacetimes [6–8].

Indeed, the general metric [9] for these spacetimes,

$$ds^2 = -dt^2 + \frac{R^2}{1-K} dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (33)$$

appears to be identical to (6) with (26) and (29) and the identification  $g^2 = 1 - K$ .

Before concluding we would like to emphasize once again that the main appeal of the expansion-free models resides in the unavoidable appearance of a vacuum cavity within the fluid distribution.

This fact suggests that such models might be used to describe the formation of voids observed at cosmological scales (see [10,11] and references therein) for very different kinds of fluid distributions (dust, anisotropic fluids, and dissipative fluids).

Thus, in the dust case discussed above, analytical solutions are available, which are relatively simple to analyze but still may contain some of the essential features of a realistic situation.

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