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Matrix reduction and the $\mathfrak{su}(2\,|\,2)$ superalgebra in AdS/CFT correspondence

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We study the supersymmetry generators Q, S on the 1-loop vectorless sector of $\mathcal{N}=4$ super Yang-Mills theory, by reduction to the plane-wave matrix model. Using a coherent basis in the $\mathfrak{su}(2\,|\,2)$ sector, a comparison with the algebra given by Beisert is presented, and some parameters (up to one loop) are determined. We make a final comparison of these supercharges with the results that can be obtained from the string action by working in the light-cone gauge and discretizing the string.

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I. INTRODUCTION

The gauge/string duality has been the object of study for more than a decade by means of the anti–de Sitter/conformal field theory (AdS/CFT) correspondence [1–3], between IIB superstrings on $\mathrm{AdS}_5\times S^5$ and $\mathcal{N}=4$ $U(N_c)$ super Yang-Mills (SYM) theory in four dimensions. But while the results calculated from the gauge theory are perturbative in 't Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N_c$, the calculations on the string side are valid for strong coupling λ.

This strong/weak property of the duality limited its study to operators/states in sectors protected by supersymmetry, as these would receive no quantum corrections. But a heuristic comparison of the algebraic structures in the weak/strong coupling limits was possible by taking the plane-wave limit, or BMN limit [4]. On the gauge theory side, the BMN limit is taken by considering single trace operators, i.e. N_c very large, with large R charge of $\mathfrak{so}(6)$ $J\sim\sqrt{N_c}$ and conformal dimension Δ, keeping $\Delta - J$ finite. These operators consist of a chiral primary (trace of a large number of a complex field) with some impurities (other complex fields, bosonic or fermionic). Even though the 't Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N_c$ is very large, one can use perturbation theory provided some effective coupling $\lambda' = g_{\mathrm{YM}}^2 N_c/J^2 \sim g_{\mathrm{YM}}^2$ is kept fixed and small.

On the string side we start from the Green-Schwarz action on the $\mathrm{AdS}_5\times S^5$ [5], with $J\sim\sqrt{N_c}$ now being the angular momentum in one of the directions of S^5. We also take the energy E (generator of time translations in AdS_5) to be large, obeying $E-J$ finite, thus originating pointlike closed strings with large angular momentum in S^5. In light-cone gauge, the quantity $E-J$ is just the light-cone Hamiltonian, and the light-cone momentum $P_+ = E+J$ is very large. In this case, there is an effective coupling just $\tilde\lambda = 4\lambda/P_+^2$, which is equivalent to λ' in the limit $J\to\infty$ $(P_+/2\to J)$. This limit allowed a direct comparison of the dilatation operator in SYM (anomalous dimensions of operators in the conformal field theory) to the energies $(E-J)$ of pointlike semiclassical string oscillations in the plane-wave geometry.

The algebra of symmetries $\mathfrak{psu}(2,2\,|\,4)$ is central in $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence, as both the gauge theory and its string theory dual have the same underlying supersymmetry algebra. The two-dimensional sigma model which gives us the perturbative string theory in $\mathrm{AdS}_5\times S^5$ [5] has a manifest global symmetry under $\mathrm{PSU}(2,2\,|\,4)$ [6,7], which is the same group of internal and space-time symmetries of the $\mathcal{N}=4$ SYM. (See [8] and references therein.)

In particular, one can use the algebra to compare the scattering of particles in the duality. For large 't Hooft coupling the scattering is best described by string theory, but for small 't Hooft coupling the spin-chain description is more adequate. It was shown by Beisert that the nonperturbative S matrix is almost completely determined by the centrally extended algebra $\mathfrak{su}(2\,|\,2)\oplus\mathfrak{su}(2\,|\,2)$ [9,10], up to an overall dressing phase (determined by a crossing symmetry restriction [11–14]). Each of these centrally extended algebras $\mathfrak{su}(2\,|\,2)$ has the following structure: bosonic (kinematical) generators $\mathbf{R}^a_b$, $\mathbf{L}^\alpha_\beta$, corresponding to the rotation generators of the bosonic subalgebra $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$; fermionic (dynamical) supersymmetry generators $\mathbf{Q}^\alpha_a$, $\mathbf{Q}^{\dagger b}_\beta$; and three central charges $\mathbf{H}$, $\mathbf{C}$, and $\mathbf{C}^\dagger$ (Hamiltonian, generator of space translations, and of boosts).[1] Their commutation relations are

$$\begin{aligned}
[\mathbf{L}^\alpha_\beta, \mathbf{J}^\gamma] &= \delta^\gamma_\beta \mathbf{J}^\alpha - \tfrac{1}{2}\delta^\alpha_\beta \mathbf{J}^\gamma,\\
[\mathbf{R}^a_b, \mathbf{J}^c] &= \delta^c_b \mathbf{J}^a - \tfrac{1}{2}\delta^a_b \mathbf{J}^c,\\
[\mathbf{L}^\alpha_\beta, \mathbf{J}_\gamma] &= -\delta^\alpha_\gamma \mathbf{J}_\beta + \tfrac{1}{2}\delta^\alpha_\beta \mathbf{J}_\gamma,\\
[\mathbf{R}^a_b, \mathbf{J}_c] &= -\delta^a_c \mathbf{J}_b + \tfrac{1}{2}\delta^a_b \mathbf{J}_c,\\
\{\mathbf{Q}^\alpha_a, \mathbf{Q}^{\dagger b}_\beta\} &= \delta^b_a \mathbf{L}^\alpha_\beta + \delta^\alpha_\beta \mathbf{R}^b_a + \tfrac{1}{2}\delta^b_a\delta^\alpha_\beta \mathbf{H},\\
\{\mathbf{Q}^\alpha_a, \mathbf{Q}^\beta_b\} &= \epsilon^{\alpha\beta}\epsilon_{ab}\mathbf{C},\\
\{\mathbf{Q}^{\dagger a}_\alpha, \mathbf{Q}^{\dagger b}_\beta\} &= \epsilon^{ab}\epsilon_{\alpha\beta}\mathbf{C}^\dagger.
\end{aligned} \tag{1}$$

In the above expressions, $\mathbf{J}^M$ (where $M\in\{a,\alpha\}$, a being bosonic indices and α being the fermionic ones), is any

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[1]Note that $(\mathbf{Q}^\alpha_a)^\dagger = \mathbf{Q}^{\dagger a}_\alpha$ and the same relation holds to the central elements $\mathbf{C}$ and $\mathbf{C}^\dagger$.

element of the Lie algebra. From the elements of the algebra, the dilatation operator (or central charge Hamiltonian) has been studied in detail (see [8] and references therein).

Much has been done on the study of sectors of this superconformal algebra on the string side [15–19]. On the gauge side, Beisert perturbatively studied and determined the action of the generators of the superalgebra $\mathfrak{su}(2|2)$ up to two loops, by first restricting to the subalgebra $\mathfrak{su}(2|3)$ whose fundamental representation consists of three complex scalars and two complex fermions [20], and finally considering an infinite chain of one of the scalar operators [10]. Using Bethe ansatz techniques it was later conjectured an all loop result in this sector for the action of the algebra generators [9].

In this work, we present the supersymmetric (SUSY) algebra in terms of a matrix model reduction of Yang-Mills theory in the large N limit. The matrix model has played a very useful role in large N theories. In fact, the $\frac{1}{2}$ Bogomol'nyi-Prasad-Sommerfield (BPS) sector of $\mathcal{N}=4$ SYM is completely described in terms of a complex matrix model [21–25], and the $\frac{1}{4}$ BPS generalization is also of great interest (work in progress). Presently, the interest is in the detailed construction and comparison of supercharges and their commutation relations both on the Yang-Mills theory and on the string side. We will demonstrate that the algebra given by Beisert in [9] (at least at one loop) is correctly reproduced from the reduced matrix model point of view.

In [26] it was seen that the plane-wave matrix theory [4,27] arises when compactifying $\mathcal{N}=4$ SYM in $\mathbb{R}\times S^3$ followed by a consistent truncation in order to keep only the lowest Kaluza-Klein modes (see also [28,29]). These modes have masses proportional to a mass parameter, given by $(\frac{m}{3})^3=\frac{32\pi^2}{g_{\rm YM}^2}$. This theory was shown (in some sectors) to still be integrable up to four loops [30,31]. The study of this model is simpler than the full $\mathcal{N}=4$ SYM, and can be found in Sec. II. In this section we present a detailed study of the supercharges Q and S, following the approach of [26]. In Sec. III we restrict the action of the generators of the algebra to a subsector $\mathfrak{su}(2|2)$. The results presented in this paper are one loop, and we compare our results with the nonlocal generators presented in [9], evaluating some of the parameters defining these generators.

Some methods have been employed in the gauge theory side that allowed a comparison of the Hamiltonian to string theory equivalent algebra generator. Such methods include the use of coherent states [32–34], collective field theory, and string field theory [35,36]. In this framework one can compare a discrete (first quantized) version of the supercharges on the string side with the oscillator expansion of the charges in SYM, in the BMN limit.

In Sec. III we use a coherent state basis to write the supercharges, which are then shown, in Sec. IV, to have the same structure as the first quantized version of the algebra generators determined from the string action. In Sec. V we present a summary of the results and some future paths of investigation.

II. $\mathcal{N}=4$ SYM ON $\mathbb{R}\times S^3$: A REVIEW

In this first section, we summarize the method of finding the supercharges of $\mathfrak{su}(2,2|4)$ up to 1 loop, as can be found in [26,28].[2]

The action for $\mathcal{N}=4$ SYM in four dimensions can be obtained from dimensional reduction of the $\mathcal{N}=1$ 10-dimensional SYM on a 6-torus. Using the notation where the $D=10$ Dirac matrices split into SO(1, 3) × SO(6), the action becomes

$$S=\frac{2}{g_{\rm YM}^2}\int d^4x\sqrt{|g|}\,{\rm Tr}\Big\{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-\frac{1}{2}D^{\mu}\phi_iD_{\mu}\phi_i -\frac{\mathcal{R}}{12}\phi_i^2+\frac{1}{4}[\phi_i,\phi_j]^2-2i\lambda_A^{\dagger}\sigma^{\mu}D_{\mu}\lambda^A +(\rho_i)^{AB}\lambda_A^{\dagger}i\sigma^2[\phi_i,\lambda_B^*]-(\rho_i^{\dagger})_{AB}(\lambda^A)^Ti\sigma^2[\phi^i,\lambda^B]\Big\}.$$

We have a vector field A_μ, six real scalars ϕ_i, and four Weyl spinors $\lambda_{\alpha A}$ (all in the adjoint representation of the gauge group). The six scalars transform in a **6** of the R-symmetry group $SO(6)\equiv SU(4)_R$, while the spinors transform in a **4**. Coordinate indices are $x^\mu=(t,x^a)$, $\mu=0,\ldots,3$, with the spatial coordinates having (curved) indices $a=1$, 2, and 3. The metric is given by

$$ds^2=-dt^2+R^2(d\theta^2+\sin^2\theta d\psi^2+\sin^2\theta\sin^2\psi d\chi^2),$$

where R is the radius of S^3, and $\mathcal{R}=\frac{6}{R^2}$ is the Ricci scalar.

Some notation.—From this point on we will be considering $\sigma^\mu\equiv(\mathbf{1},\sigma^a)$ and $\bar{\sigma}^\mu=(-\mathbf{1},\sigma^a)$, where the σ^a are the usual Pauli matrices pulled back to S^3. Also, $\rho_i^{AB}\equiv\sigma_i^{AB}$ are the Clebsch-Gordan coefficients of SU(4) that relate two **4** irreducible representations (irreps) with one **6**. These coefficients have several properties, in particular $\rho_i^{AB}=\frac{1}{2}\varepsilon^{ABCD}(\rho_i^{\dagger})_{CD}$, and allow us to write

$$\phi_i=\tfrac{1}{2}\rho_i^{AB}\Phi_{AB}=\tfrac{1}{2}(\rho_i^{\dagger})_{AB}\Phi^{AB}.$$

Finally, one comment about the Weyl spinors. We know that in $D=10$ we start from a 32-component complex spinor, and by imposing a Majorana-Weyl condition, obtain a 16-component (after fixing the κ-symmetry) spinor L. This spinor can be written in terms of Weyl spinors as

$$L=\begin{pmatrix}\lambda^{\alpha A}\\ i(\sigma^2)^{\alpha\beta}\lambda_{\beta A}^*\end{pmatrix},$$

[2] We will be following the notation of [26], in which a different basis for the γ matrices is used. The same procedure could be done by following [28] choice of basis.

with $\alpha = 1, 2$ and $A = 1, 2, 3$, and 4. The $\lambda^{\alpha A}$ are four 2-component Weyl spinors.[3]

A. SUSY transformations and corresponding charges

The SUSY transformations are given by

$$\begin{aligned}\delta_\eta A_\mu &= 2i(\lambda_A^\dagger \sigma_\mu \eta^A - \eta_A^\dagger \sigma_\mu \lambda^A),\\ \delta_\eta \Phi^{AB} &= 2i(-\lambda_E^\dagger i\sigma^2 \varepsilon^{ABEF}\eta_F^* - (\lambda^A)^T i\sigma^2 \eta^B\\ &\quad - (\lambda^B)^T i\sigma^2 \eta^A),\\ \delta_\eta \lambda^A &= \tfrac{1}{2}F_{\mu\nu}\sigma^{\mu\nu}\eta^A + 2D_\mu \Phi^{AB}\bar\sigma^\mu i\sigma^2 \eta_B^*\\ &\quad + \Phi^{AB}\bar\sigma^\mu i\sigma^2 \nabla_\mu \eta_B^* - 2i[\Phi^{AC}, \Phi_{CB}]\eta^B.\end{aligned}$$

We want to build the Noether charge $Q\eta$. To do so we need to take into consideration the pairs of canonical variables. From the action, we have the following (anti-)commutation relations:

$$\begin{aligned}[F_{0\mu}, A^\nu] &= \delta^\nu_\mu,\\ [D_0\phi_i, \phi_j] &= \delta_{ij} \Rightarrow [D_0\Phi_{AB}, \Phi^{CD}]\\ &= \tfrac{1}{2}(\delta_A^D\delta_B^C - \delta_A^C\delta_B^D),\\ \{-i(\lambda_A^\dagger\sigma^0)_\alpha, \lambda^{B\beta}\} &= \delta_\alpha^\beta \delta_A^B.\end{aligned}$$

Also one has to take into consideration that $\eta^{\alpha A}$ are Killing spinors, which in $\mathbb{R}\times S^3$ obey the equation $\nabla_\mu \eta = \pm \frac{i}{2R}\sigma_\mu \eta$, and so will give us two solutions $\eta_\pm$. We will then obtain two charges $Q \equiv Q_L$ and $\bar Q \equiv Q_R$, corresponding to η_+ and η_-, respectively.

The fermionic Noether charges are thus[4]

$$\begin{aligned}Q\eta = \frac{2}{g_{\rm YM}^2}\int_{S^3} d\Omega\, \mathrm{Tr}\{&-2i\lambda_A^\dagger \sigma^0 \delta_\eta \lambda^A\\ &- 2i(\lambda^A)^T\sigma^0\delta_\eta \lambda_A^*\}.\end{aligned}$$

For the purposes of this paper, we will simplify the calculations by setting the vector field to zero (we will be looking only at the sector of scalars and spinors). This truncation is consistent with the one-loop calculation we will be performing.

[3]In the basis used in [28], the separation of L into $L = (L_+L_-)^T$ becomes a separation into $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, for which one uses dotted/undotted indices $\dot\alpha$ and α. In the basis used in [26] this separation is not obvious.

[4]For comparison purposes, one could also write this charge, in the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ formalism, as

$$Q_\epsilon = \frac{2}{g_{\rm YM}^2}\int_{s^3} \mathrm{Tr}\{i\bar\lambda_A^{\dot\alpha}\bar\sigma^0_{\dot\alpha\alpha}\delta_\epsilon\lambda^{\alpha A} + i\bar\lambda_\alpha^A(\sigma^0)^{\alpha\dot\alpha}\delta_\epsilon\lambda_{\dot\alpha A}\}.$$

The nonvector sector of the charges $Q\eta$ is given by

$$\begin{aligned}Q\eta = -\frac{2}{g_{\rm YM}^2}\int_{S^3} d\Omega\, \mathrm{Tr}\{&2i\lambda_A^\dagger(2\Pi^{AB}\sigma^0 i\sigma^2\eta_B^*\\ &+ 2\nabla_a\Phi^{AB}\bar\sigma^a i\sigma^2\eta_B^* + \Phi^{AB}\bar\sigma^\mu i\sigma^2\nabla_\mu\eta_B^*\\ &- 2i[\Phi^{AC}, \Phi_{CB}]\eta^B) + 2i(\lambda^A)^T\\ &\times(-2\Pi_{AB}(\bar\sigma^0)^T i\sigma^2\eta^B - 2\nabla_a\Phi_{AB}(\bar\sigma^a)^T i\sigma^2\eta^B\\ &- \Phi_{AB}(\bar\sigma^\mu)^T i\sigma^2\nabla_\mu\eta^B - 2i[\Phi_{AC}, \Phi^{CB}]\eta_B^*)\},\end{aligned} \tag{2}$$

where Π_{AB} is the momentum conjugate to the bosonic field Φ^{AB}.

We now have an expression for the supercharges. The next step is to evaluate it on $\mathbb{R}\times S^3$: we expand the four-dimensional fields in terms of the spherical harmonics of S^3 and then perform the integration of the sphere.

B. Harmonic expansion on S^3 and the plane-wave limit

Each field, defined by its spin, will have a decomposition in spherical harmonics on S^3. These spherical harmonics can be labeled by the irreducible representations (m_L, m_R) of the isometry group $\mathrm{SO}(4) \equiv \mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R$. As such, we have

(i) Spin 0: We have scalar spherical harmonics $Y_{(0)}^{kI}$, in the irrep $(k+1, k+1)$. Their mass will be $(k+1)/R$.

(ii) Spin $\frac{1}{2}$: In this case we will use spinor spherical harmonics: $Y_{(1/2)}^{kI+}$, in the irrep $(k+2, k+1)$; $Y_{(1/2)}^{kI-}$, in the irrep $(k+1, k+2)$. Both have mass $(k+3/2)/R$.

As usual, k labels different irreducible representations, and I enumerates elements of a particular irreducible representation ($I = 1\cdots d$, where d is the dimension of the representation).

The expansions of the fields in the corresponding harmonics are

$$\begin{aligned}\phi_i(x^\mu) &= \sum_{k=0}^{\infty}\sum_{I=1}^{(k+1)^2}\phi_i^{kI}(t)Y_{(0)}^{kI}(x^a),\\ \lambda_\alpha^A(x^\mu) &= \sum_{k=0}^{\infty}\sum_{I=1}^{(k+1)(k+2)}\sum_{\pm}\lambda^{A,kI\pm}(t)Y_{(1/2)\alpha}^{kI\pm}(x^a).\end{aligned}$$

Note that spinor spherical harmonics are 2-dimensional commuting Weyl spinors. The Killing spinor η^A (parameter of the superconformal transformations) will have the same expansion as λ^A, with coefficients $\eta^{A,kI\pm}(t)$.

Plane-wave limit [26,27].—We want to truncate the infinite tower of Kaluza-Klein modes to the lowest supermultiplet. One can then climb up the various states (with increasing masses) by acting with the two supercharges $Q_L = (\mathbf{2}, \mathbf{1}, \bar{\mathbf{4}})$ and $Q_R = (\mathbf{1}, \mathbf{2}, \mathbf{4})$, where the numbers correspond to representations of $SU(2)_L \otimes SU(2)_R \otimes SU(4)$. Focusing on the zero modes of the Kaluza-Klein tower we find 6 scalar spherical harmonics, constant on S^3, and 4 lowest spinor spherical harmonics $S_\alpha^{\hat\alpha\pm}$, in irrep $(2,1)\oplus(1,2)$ of $SU(2)_L \otimes SU(2)_R$ (the hatted index refers to the degeneracy of the solution), solutions to the killing spinor equation for a Weyl spinor.

The fields with only these zero modes become

$$\phi_i(x^\mu) = X_i(t),$$
$$\lambda_\alpha^A(x^\mu) = \sum_{\hat\alpha=1}^{2}(\theta_{\hat\alpha}^{A+}(t)S_\alpha^{\hat\alpha+}(x^a) + \theta_{\hat\alpha}^{A-}(t)S_\alpha^{\hat\alpha-}(x^a)).$$

If we restrict ourselves to half of the supercharges Q_L, then together with the bosonic symmetries will generate the subalgebra $\mathfrak{su}(2|4)$. The restriction to the Q_L charges leads us to consider only the zero modes that are $SU(2)_R$ singlets. Then we keep all the lowest scalar harmonics, and only two spinor harmonics $S_\alpha^{\hat\alpha+}$ (instead of the 4 if we included $S_\alpha^{\hat\alpha-}$). The conjugate momenta π_i will have the same expansion as its conjugate field ϕ_i, that is $\pi_i(x^\mu) = \Pi_i(t)$.

Now we can proceed to the actual integration on the supercharges. Going back to (2), we find that[5]:

$$\begin{aligned}Q_L = Q\eta^+ = \mathrm{Tr}\bigg\{&\left(\frac{1}{R}X^{AB} + 2i\Pi^{AB}\right)\theta_A^{+\dagger}i\sigma^2\eta_B^{+*}\\ &- \sqrt{2}[X_{AC}, X^{CB}]\theta_{\hat\alpha}^{+A}\varepsilon^{\hat\alpha\hat\beta}\eta_{B\hat\beta}^{+*}\\ &+ \left(\frac{1}{R}X_{AB} - 2i\Pi_{AB}\right)(\theta^{+A})^T i\sigma^2\eta^{+B}\\ &- \sqrt{2}[X^{AC}, X_{CB}]\theta_{A\hat\alpha}^{+\dagger}\varepsilon^{\hat\alpha\hat\beta}\eta_{\hat\beta}^{+B}\bigg\}\\ = Q_+\eta + S_+\eta^*.&\end{aligned}$$

The final expression for the supercharges is[6]

$$Q_A^{\hat\alpha} = \mathrm{Tr}\bigg\{-\theta^{B\hat\alpha}\left(\frac{1}{R}X_{BA} - 2i\Pi_{BA}\right) - \sqrt{2}\varepsilon^{\hat\beta\hat\alpha}\theta_{B\hat\beta}^\dagger[X^{BC}, X_{CA}]\bigg\}, \quad (3)$$

$$S^{A\hat\alpha} = \mathrm{Tr}\bigg\{\theta_{B\hat\beta}^\dagger\left(\frac{1}{R}X^{BA} + 2i\Pi^{BA}\right) - \sqrt{2}[X_{BC}, X^{CA}]\theta_{\hat\beta}^B\bigg\}\varepsilon^{\hat\beta\hat\alpha}.$$

[5]In order to obtain the supercharges integrated over S^3, we used the properties of the spherical harmonics, as well as other properties of the Pauli matrices. These properties can be found in [26–28], and include $\bar\sigma^\mu i\sigma^2\sigma_\mu^T = (\bar\sigma^\mu)^T i\sigma^2\sigma_\mu = -2i\sigma^2$. In the same references one can find the expansion of spin 1 vector fields. We also used an identification between the radius of the sphere R and the Yang-Mills coupling constant g_{YM} such that $4\pi^2R^3/g_{\mathrm{YM}}^2 \to 1$. This prefactor shows up when obtaining the action of the plane-wave matrix theory action from $\mathcal{N}=4$ SYM action, and would also appear in the charges.

[6]Note that in our choice of basis the relation $S = Q^\dagger$ is not manifest.

III. THE $\mathfrak{su}(2 \mid 3)$ SUBSECTOR AND ITS RESTRICTION TO THE $\mathfrak{su}(2 \mid 2)$

We will continue by studying the sector $\mathfrak{su}(2 \mid 3)$, as in [20]. For that we reduce our fields as follows:

$$\theta^\alpha \equiv \theta^{4\hat\alpha}, \qquad \phi^a \equiv X^{a4}, \qquad \alpha = 1, 2; \qquad a = 1, 2, 3.$$

By construction we have $\bar\phi^a \equiv \phi_a$, and $\pi_a = \Pi_{4a}$, as well as $X^{BC} = \frac{1}{2}\varepsilon^{BCAD}X_{AD}$. The supercharges restricted to this sector can then be written as

$$\begin{aligned}Q_a^\alpha &= \mathrm{Tr}\bigg\{-\theta^{4\hat\alpha}\left(\frac{1}{R}X_{4a} - 2i\Pi_{4a}\right) - \sqrt{2}\theta_{4\hat\beta}^\dagger\varepsilon^{\hat\beta\hat\alpha}[X^{4C}, X_{Ca}]\bigg\}\\ &= \mathrm{Tr}\bigg\{\theta^\alpha\left(\frac{1}{R}\bar\phi_a + 2i\pi_a\right) - \sqrt{2}\theta_\beta^\dagger\varepsilon^{\alpha\beta}\varepsilon_{abc}[\phi^c, \phi^b]\bigg\};\end{aligned} \quad (4)$$

$$\begin{aligned}S^{a\alpha} &= \mathrm{Tr}\bigg\{\theta_{4\hat\beta}^\dagger\left(\frac{1}{R}X^{4a} + 2i\Pi^{4a}\right) - \sqrt{2}[X_{4C}, X^{Ca}]\theta_{\hat\beta}^4\bigg\}\varepsilon^{\hat\beta\hat\alpha}\\ &= \mathrm{Tr}\bigg\{\theta_\beta^\dagger\left(\frac{1}{R}\phi^a - 2i\bar\pi^a\right) - \sqrt{2}\varepsilon^{abc}[\bar\phi_c, \bar\phi_b]\theta^\gamma\varepsilon_{\gamma\beta}\bigg\}\varepsilon^{\beta\alpha}.\end{aligned} \quad (5)$$

In order to continue, we will need to rewrite the fields in terms of creation/annihilation operators. First identify $\frac{1}{R} = \frac{m}{6}$, i.e. exchange the parameter R by a mass parameter m [26]. Then consider the expansion of the six scalars/momenta X_i, Π_i:

$$\begin{cases} a_i = \sqrt{\frac{3}{m}}(i\Pi_i + \frac{m}{6}X_i),\\ a_i^\dagger = \sqrt{\frac{3}{m}}(-i\Pi_i + \frac{m}{6}X_i),\end{cases} \Rightarrow \begin{cases} X_i = \sqrt{\frac{3}{m}}(a_i + a_i^\dagger),\\ \Pi_i = \frac{1}{2i}\sqrt{\frac{m}{3}}(a_i - a_i^\dagger).\end{cases}$$

The bosons X_{AB} are a combination of two real scalar fields such that $X_{a4} = \frac{1}{2}(X_a + iX_{a+3})$, $a = 1, 2, 3$. If we now define the creation annihilation operators as $a^a \equiv a^a + ia^{a+3}$ and $b^{a\dagger} = a^{a\dagger} + ia^{a+3\dagger}$, with $a = 1, 2, 3$, we then have the following expansions for our (complex) fields:

$$\begin{aligned}\phi^a &\equiv X^{a4} = \sqrt{\frac{3}{m}}(a^a + b^{\dagger a});\\ \pi_a &\equiv \Pi_{4a} = \frac{1}{4i}\sqrt{\frac{m}{3}}(a_a^\dagger - b_a),\end{aligned} \quad (6)$$

with equivalent expressions for fields $\bar\phi_a$ and $\bar\pi^a$. Introducing also fermionic creation operators, the fermions become

$$\theta^{\dagger\alpha} = c^\alpha = \varepsilon^{\alpha\beta}c_\beta; \qquad \theta^\alpha = c^{\dagger\alpha}. \quad (7)$$

We will be interested in the action of the charges on the subspace of states that will only have excitations of $c^\dagger$ and $b^\dagger$, so we will drop the oscillators a, $a^\dagger$ in the bosonic fields. We find

$$Q_a^\alpha = \mathrm{Tr}\left\{\sqrt{\frac{m}{3}}c^{\dagger\alpha}b_a - \frac{3\sqrt{2}}{m}\varepsilon^{\alpha\beta}\varepsilon_{abc}[b^{\dagger c}, b^{\dagger b}]c_\beta\right\},$$
$$S_\alpha^a = \mathrm{Tr}\left\{-\sqrt{\frac{m}{3}}b^{\dagger a}c_\alpha - \frac{3\sqrt{2}}{m}\varepsilon_{\alpha\beta}\varepsilon^{abc}c^{\dagger\beta}[b_c, b_b]\right\}. \quad (8)$$

As expected, these results are similar to the ones in [27], up to a change of basis for the gamma matrices.

A. The $\mathfrak{su}(2\,|\,2)$ subsector: Vacuum and excitations

The study will focus on states that transform in the $\mathfrak{su}(2|3)$ sector and are single trace (gauge invariant) operators of the fields (3 bosons and 2 fermions). This *spin chain* arises from the large N limit of the gauge theory. In this sector the action of the algebra generators can be found in [20]. Consider now the vacuum as a long string of $Z \equiv \phi^3$ fields. In oscillator notation, we have $Z = b^{3\dagger}$, and the vacuum state can be written as

$$|0, J\rangle \equiv |Z^J\rangle \equiv \frac{1}{\sqrt{J}N^{J/2}}\,\mathrm{Tr}(b^{3\dagger J})|0\rangle.$$

A generalization of this vacuum consists of an infinitely long string of Z fields (the asymptotic regime, $J \to \infty$), as in [10]. The excitations are now the other fields of the $\mathfrak{su}(2|3)$ algebra, $\chi \in \{\psi^1, \psi^2|\phi^1, \phi^2\}$, which corresponds to the $\mathfrak{su}(2|2)$ subsector of the algebra. The excitations can move through the chain on Z's with some momentum p. Thus, in momentum space we can write

$$\chi = \sum_{n_k=1}^{N} e^{ip_k n_k}\chi(n_k) = \sum_{n_k=1}^{J} e^{ip_k n_k}\chi_k \equiv \chi(p_k),$$

where n denotes the position of the impurity/excitation χ on the vacuum string.

A general state with K impurities can then be written as

$$|\chi_1, \ldots, \chi_K; J\rangle = \sum_{n_1,\ldots,n_K=1} e^{ip_1n_1+\cdots+ip_Kn_K} \times |Z\cdots Z\chi_1 Z\cdots\chi_2\cdots\chi_K\ldots Z\rangle.$$

For an asymptotic state ($J \to \infty$) we consider the dilute gas approximation, where the positions $n_1, \ldots, n_k$ of the impurities obey $n_1 \ll n_2 \ll \cdots \ll n_K$.

We should note that on shell the physical states are cyclic (property of the trace), and so we must have $\sum_{k=1}^K p_k = 0$.

Now that we have defined the states that the supercharges will be acting on, we can determine their action. The first step will be to check what the charges do to just one excitation on the vacuum. Then one can generalize to multiexcitation states of the $\mathfrak{su}(2|2)$ subsector of $\mathfrak{su}(2|3)$. Once we have the action of the charges on a multiexcitation state, we can determine the commutator of two supercharges, as a check of our results.

In this subsector the charges (8) become

$$Q_a^\alpha = \sqrt{\frac{m}{3}}\mathrm{Tr}\left\{\psi^\alpha\frac{\partial}{\partial\phi^a} - \left(\sqrt{\frac{3}{m}}\right)^3\sqrt{2}\varepsilon_{ab}\varepsilon^{\alpha\beta}[Z, \phi^b]\frac{\partial}{\partial\psi^\beta}\right\},$$
$$S_\alpha^a = \sqrt{\frac{m}{3}}\mathrm{Tr}\left\{-\phi^a\frac{\partial}{\partial\psi^\alpha} - \left(\sqrt{\frac{3}{m}}\right)^3\sqrt{2}\varepsilon^{ab}\varepsilon_{\alpha\beta}\psi^\beta \times\left[\frac{\partial}{\partial Z}, \frac{\partial}{\partial\phi^b}\right]\right\}, \quad (9)$$

where we chose a coherent state basis, such that

$$c^{\dagger\alpha} \to \psi^\alpha; \qquad c_\alpha \to \frac{\partial}{\partial\psi^\alpha};$$
$$b^{\dagger a} \to \phi^a; \qquad b_a \to \frac{\partial}{\partial\phi^a}.$$

For $a = 3$, we have the identification $\phi^3 \equiv Z$. The factor $\sqrt{\frac{m}{3}}$ will appear as an overall factor in every charge calculated, and will be dropped, as we know that the quadratic terms come from the free theory $g_{\mathrm{YM}} = 0$.

We now proceed to determine the action of the supercharge Q (and equivalently S) on a *single excitation* state $|\chi; J\rangle = \sum_n e^{ipn}|Z^{n-1}\chi Z^{J-n+1}\rangle$. If the excitation is bosonic, $\chi_\ell = \phi^\ell$, then

$$Q_a^\alpha|\chi; J\rangle = \sum_n e^{ipn}\delta_a^\ell|Z^{n-1}\psi^\alpha(n)Z^{J-n+1}; J\rangle,$$

while if the excitation is fermionic, $\chi^\beta = \psi^\beta$, we have

$$\begin{aligned}
Q_a^\beta|\chi; J\rangle &= -\left(\sqrt{\frac{3}{m}}\right)^3\sqrt{2}\varepsilon_{ab}\varepsilon^{\alpha\beta}\sqrt{\frac{J+1}{J}}N^{1/2}|Z^{n-1}[Z, \phi^b]Z^{J-n+1}; J+1\rangle \\
&= -\left(\sqrt{\frac{3}{m}}\right)^3\sqrt{2}\varepsilon_{ab}\varepsilon^{\alpha\beta}\sqrt{\frac{J+1}{J}}N^{1/2}|Z^n\phi^b(n+1)Z^{J-n+1}; J+1\rangle + \left(\sqrt{\frac{3}{m}}\right)^3 \\
&\quad\times\sqrt{2}\varepsilon_{ab}\varepsilon^{\alpha\beta}\sqrt{\frac{J+1}{J}}N^{1/2}|Z^{n-1}\phi^b(n)Z^{J-n+2}; J+1\rangle \\
&\approx -\left(\sqrt{\frac{3}{m}}\right)^3\sqrt{2}\varepsilon_{ab}\varepsilon^{\alpha\beta}\sum_n e^{ipn}(e^{-ip}-1)N^{1/2}|Z^{n-1}\phi^b(n)Z^{J-n+2}; J+1\rangle.
\end{aligned}$$

It can be seen from the expression above that the insertion of a Z field before the excitation changes its phase by e^{-ip}, while the insertion after the excitation leaves that phase untouched. This is a property of the asymptotic state, for which an infinite number of Z fields exist after the (last) excitation. This was seen in [10] as being equivalent to "opening" the trace. In the above expression we also kept only the first order in $\frac{1}{J}$.

From the results shown above, we can easily determine the generalization to a *multiexcitation* state. First, rewrite the state as

$$|\chi; J\rangle \equiv |\chi_1 \ldots \chi_K; J\rangle = \sum_{\{l_i\}} e^{ip_1 l_1 + \cdots + ip_K l_K} \chi_1^\dagger \chi_2^\dagger \cdots \chi_K^\dagger |0; J\rangle. \tag{10}$$

The action of one charge on such state is (zeroth order in $\frac{1}{J}$)

$$\begin{aligned} Q_a^\alpha |\chi_1 \ldots \chi_K; J\rangle &= \sum_{k=1}^{K} \sum_{\{l_i\}} e^{ip_1 l_1 + \cdots + ip_K l_K} \bigg(\prod_{m=1}^{k-1} (-1)^{F(m)} \bigg) \chi_1^\dagger \chi_2^\dagger \cdots (Q_a^\alpha \chi_k^\dagger) \cdots \chi_K^\dagger |0; J\rangle \\ &= \sum_{k=1}^{K} \sum_{\{l_i\}} e^{ip_1 l_1 + \ldots + ip_K l_K} \bigg(\prod_{m=1}^{k-1} (-1)^{F(m)} \bigg) \bigg\{ \delta(\chi_k^\dagger, \phi^b) \delta_a^b \chi_1^\dagger \chi_2^\dagger \cdots \psi^\alpha(l_k) \cdots \chi_K^\dagger |0; J\rangle \\ &\quad - \frac{\sqrt{2N}}{M^3} \delta(\chi_k^\dagger, \psi^\beta) \bigg(\prod_{m=k+1}^{K} e^{-ip_m} \bigg) (e^{-ip_k} - 1) \varepsilon_{ab} \varepsilon^{\alpha\beta} \chi_1^\dagger \chi_2^\dagger \cdots \phi^b(l_k) \cdots \chi_K^\dagger |0; J+1\rangle \bigg\}, \end{aligned} \tag{11}$$

and similarly for the S charge (noticing that the action of S on a bosonic excitation returns an extra factor of N). In here $\delta(\chi_k^\dagger, \phi^b)$ means that the excitation $\chi(l_k)$ is bosonic ϕ^b, while in $\delta(\chi_k^\dagger, \psi^\beta)$ the excitation $\chi(l_k)$ is fermionic ψ^β. The factor $(-1)^{F(m)}$ is equal to 1 if χ_m is bosonic and -1 if χ_m is fermionic. Finally we defined $M = \sqrt{\frac{m}{3}}$. When χ_k is a fermionic excitation, one gets the expected factor of $(e^{-ip_k} - 1)$, which already showed up in the single excitation case, but one also gets an extra factor of $\prod_{m=k+1}^{K} e^{-ip_m}$. This last factor can also be explained by the insertion of the Z field. In fact in the single excitation case we saw that Z changed the momentum if inserted before the excitation on the chain of fields. But now the field Z gets inserted before all of the excitations χ_m with $m > k$, hence the change of momenta of all these excitations.

The results of the action of Q and S on a multiexcitation state will be summarized next using a nonlocal notation (see also [9]).

1. Twisted vs nonlocal notations

The supercharges Q and S acting on a general state $|\chi; J\rangle$ can be written in a *nonlocal notation*:

$$\begin{aligned} Q_a^\alpha |\chi; J\rangle = \sum_{k=1}^{K} \{ & a_k \delta_a^b \delta(\chi_k^\dagger, \phi^b) |\chi_1 \cdots \psi^\alpha \cdots \chi_K; J\rangle \\ & + b_k \varepsilon_{ab} \varepsilon^{\alpha\beta} \delta(\chi_k^\dagger, \psi^\beta) \\ & \times |\chi_1 \cdots \phi^b \cdots \chi_K; J+1\rangle \}, \end{aligned} \tag{12}$$

$$\begin{aligned} S_\alpha^a |\chi; J\rangle = \sum_{k=1}^{K} \{ & c_k \varepsilon^{ab} \varepsilon_{\alpha\beta} \delta(\chi_k^\dagger, \phi^b) \\ & \times |\chi_1 \cdots \psi^\beta \cdots \chi_K; J-1\rangle \\ & + d_k \delta_\alpha^\beta \delta(\chi_k^\dagger, \psi^\beta) |\chi_1 \cdots \phi^a \cdots \chi_K; J\rangle \}, \end{aligned} \tag{13}$$

where the coefficients are given by

$$\begin{aligned} a_k &= \prod_{m=1}^{k-1} (-1)^{F(m)}, \\ b_k &= \frac{\sqrt{2N}}{M^3} \bigg[\prod_{m=1}^{k-1} (-1)^{F(m)} \bigg] (1 - e^{-ip_k}) \bigg[\prod_{m=k+1}^{K} e^{-ip_m} \bigg] \\ &= \frac{\sqrt{2N}}{M^3} e^{-iP} (e^{ip_k} - 1) \bigg[\prod_{m=1}^{k-1} (-1)^{F(m)} e^{ip_m} \bigg], \\ c_k &= \frac{\sqrt{2N}}{M^3} \bigg[\prod_{m=1}^{k-1} (-1)^{F(m)} \bigg] (e^{ip_k} - 1) \bigg[\prod_{m=k+1}^{K} e^{ip_m} \bigg] \\ &= \frac{\sqrt{2N}}{M^3} e^{iP} (1 - e^{-ip_k}) \bigg[\prod_{m=1}^{k-1} (-1)^{F(m)} e^{-ip_m} \bigg], \\ d_k &= -\prod_{m=1}^{k-1} (-1)^{F(m)}. \end{aligned} \tag{14}$$

There is one other notation, introduced by Beisert in [9], called the *twisted notation*. In this local notation we have

$$\begin{aligned} Q_{a,k}^\alpha |\cdots \phi_k^b \cdots\rangle &= a_k' \delta_a^b |\cdots \mathcal{Y}^+ \psi_k^\alpha \cdots\rangle, \\ Q_{a,k}^\alpha |\cdots \psi_k^\beta \cdots\rangle &= b_k' \varepsilon^{\alpha\beta} \varepsilon_{ab} |\cdots \mathcal{Z}^+ \mathcal{Y}^- \phi_k^b \cdots\rangle, \\ S_{\alpha,k}^a |\cdots \phi_k^b \cdots\rangle &= c_k' \varepsilon^{ab} \varepsilon_{\alpha\beta} |\cdots \mathcal{Z}^- \mathcal{Y}^+ \psi_k^\beta \cdots\rangle, \\ S_{\alpha,k}^a |\cdots \psi_k^\beta \cdots\rangle &= d_k' \delta_\alpha^\beta |\cdots \mathcal{Y}^- \phi_k^a \cdots\rangle. \end{aligned} \tag{15}$$

We notice the presence of the markers $\mathcal{Z}^\pm$ and $\mathcal{Y}^\pm$. These markers have a simple explanation, up to one loop. The marker $\mathcal{Y}^\pm$ marks the position on the string of fields (the state) where a fermion field was inserted ($\mathcal{Y}^+$) or removed ($\mathcal{Y}^-$). In the twisted notation we are only given the action of the supercharge on the field k of the string. But in order for a supercharge to act on such field it will have to pass by the previous ones. If these are bosonic fields nothing happens, but if they are fermionic, a minus sign will appear (for each fermionic field it passes). Thus, it is important to know where the supercharge acted, which is done by the marker. The marker is shifted around as follows:

$$|\cdots \chi_k \mathcal{Y}^{\pm} \cdots\rangle = (\xi_k)^{\pm 1} |\cdots \mathcal{Y}^{\pm} \chi_k \cdots\rangle,$$

where

$$\xi_k = (-1)^{F(k)} = \begin{cases} 1 & \text{if } \chi_k \text{ bosonic,} \\ -1 & \text{if } \chi_k \text{ fermionic.} \end{cases} \tag{16}$$

The marker $\mathcal{Z}^{\pm}$ marks a position where an extra Z field was inserted in the string. This changes the length of the vacuum spin chain, reflecting a change in the momenta of the excitation fields. But this change in momenta only affects the excitation fields after the position of the marker. The marker has the property

$$|\cdots \chi_k \mathcal{Z}^{\pm} \cdots\rangle = \frac{x_k^{\pm}}{x_k^{\mp}} |\cdots \mathcal{Z}^{\pm} \chi_k \cdots\rangle, \quad \text{where } \frac{x_k^{\pm}}{x_k^{\mp}} = e^{\pm i p_k}, \tag{17}$$

with p_k being the momenta of the excitation χ_k, as before.

In summary, the twisted notation is a local notation, since it only provides the action of the supercharge on the excitation field χ_k, plus a set of markers that allow us to rewrite it in a nonlocal notation, as found in (12) and (13). We can go from the twisted notation to the nonlocal one by removing the markers from the first, i.e., shifting them so that they will be at the right (or left) of all the excitation fields.

In the local twisted notation we have[7]

$$a_k' = -d_k' = 1, \qquad b_k' = \frac{\sqrt{2N}}{M^3}(1 - e^{-ip_k}),$$
$$c_k' = -\frac{\sqrt{2N}}{M^3}(1 - e^{ip_k}).$$

2. Comparison with Beisert at 1 loop

One can find the all-loop version of these coefficients in [9], for both the nonlocal and the twisted notation. In fact we can expand the (nonlocal) coefficients given in that reference to order $\mathcal{O}(g)$, and compare them to our results. These coefficients are

$$a_k = \gamma_k \prod_{j=1}^{k-1} (-1)^{F(j)},$$

$$b_k = g\frac{\alpha}{\gamma_k}(1 - e^{ip_k}) \prod_{j=1}^{k-1} (e^{ip_k}(-1)^{F(j)}),$$

$$c_k = i\frac{\gamma_k}{\alpha x_k^+} \prod_{j=1}^{k-1} (e^{-ip_k}(-1)^{F(j)}),$$

$$d_k = g\frac{x_k^+}{i\gamma_k}(1 - e^{-ip_k}) \prod_{j=1}^{k-1} (-1)^{F(j)}.$$

We used the identifications (16) and (17) into the transcribed coefficients, and also made a rescaling of the parameter $\gamma_k \rightarrow \sqrt{g}\gamma_k$. The expansion in g is hidden in the dependence of x^+, x^- on the coupling constant:

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}. \tag{18}$$

This last equation, together with (17), allows us to solve for $x^+(g)$:

$$x^+ = i\frac{1 + \sqrt{1 + 16g^2 \sin^2(p/2)}}{2g(1 - e^{-ip})}.$$

Then by expanding this expression up to order $\mathcal{O}(g)$, we obtain exact agreement with (14), as long as we identify $\gamma_k = (-1)^{F(k)}$ and $\alpha = e^{-iP}$. Note that the relation between the normalized 't Hooft coupling g and the Yang-Mills coupling constant $g_{\rm YM}$ is $g = \frac{g_{\rm YM}}{4\pi}\sqrt{N_c}$, from the gauge group $SU(N_c)$.

The other charges that we are interested in determining are the Hamiltonian H and the central charges of the extended algebra P, K. These charges arise from commutation relations between the supercharges, which will be determined next.

B. Commutation relations

At this moment we have calculated only the supercharges of the full extended algebra $\mathfrak{su}(2|2)$, up to $\mathcal{O}(g)$. We are interested in having the complete set of charges at this order, which comprises also the rotations generators L, R, the dilatation operator H, and also the central charges of the extended algebra P, K (bosonic generators of momentum and boosts, which have zero eigenvalues when applied to physical states). All of these generators can be obtained to $\mathcal{O}(g)$ from the commutation relations of the supercharges.

The central charges of the extended algebra receive no loop corrections, and as such, can be obtained exactly by the anticommutation relations $\{Q, Q\} \sim P$ and $\{S, S\} \sim K$, by knowing the zeroth order of the supercharges. The other generators will be obtained from the last anticommutator $\{Q, S\} \propto R + L + H$, but while the zeroth order supercharges will be enough to determine rotation generators

[7] The coupling constant $M^6 = (\frac{m}{3})^3$ is related to the Yang-Mills coupling constant $g_{\rm YM}$ in the following way:

$$\frac{1}{M^6} = \frac{g_{\rm YM}^2}{32\pi^2}.$$

This relation comes from matching the prefactor of the reduced SYM action with the prefactor of the matrix model action. In fact we had $m = \frac{6}{R}$, where R was the radius of S^3. Taking the radius small corresponds to $m \gg 1$ and consequently $g_{\rm YM} \ll 1$.

L and R, the central charge H will only be known correctly up to $\mathcal{O}(g)$, as we will see below.

In the anticommutator of any two supercharges the only terms that will not vanish are the ones where the two supercharges are applied to the same excitation. The anticommutator of two Q charges is

$$\{Q_b^\beta, Q_a^\alpha\}|\chi_1 \ldots \chi_K; J\rangle = \sum_{k=1}^{K} |\chi_1 \cdots (\{Q_b^\beta, Q_a^\alpha\}\chi_k) \cdots \chi_K; J\rangle = \frac{\sqrt{2N}}{M^3} \sum_{k=1}^{K} \bigg[(1 - e^{-ip_k}) \prod_{l=k+1}^{K} e^{-ip_l}\bigg] |\chi_1 \ldots \chi_K; J+1\rangle$$
$$= \frac{\sqrt{2N}}{M^3}(1 - e^{-i\sum_{k=1}^{K} p_k})|\chi_1 \ldots \chi_K; J+1\rangle.$$

This is just the action of the central charge $\{Q, Q\} \propto P$ of the extended algebra on a multiexcitation state. The action of the other central charge of the extended algebra is obtained from $\{S, S\} \propto K$:

$$\{S_\beta^b, S_\alpha^a\}|\chi_1 \ldots \chi_K; J\rangle = \frac{\sqrt{2N}}{M^3}(1 - e^{i\sum_{k=1}^{K} p_k}) \times |\chi_1 \ldots \chi_K; J-1\rangle.$$

We know from [9] that there is an outer automorphism relating H and the central charges of the extended algebra P, K, which corresponds to an $\mathfrak{sl}(2)$ algebra. Closure of this algebra on the original commutation relations of the supercharges requires that

$$H^2 - PK = \tfrac{1}{4}. \tag{19}$$

This relation should only hold when we consider the all loop H, and not only when we consider the first two orders. Using nonlocal notation, we find that the product PK is given by

$$PK = -\frac{2N}{M^6}(e^{-i\sum_{k=1}^{K} p_k} - 1)(e^{+i\sum_{k=1}^{K} p_k} - 1) = \frac{8N}{M^6}\sin^2\bigg(\sum_{k=1}^{K}\frac{p_k}{2}\bigg) = \frac{8N}{M^6}\sin^2\bigg(\frac{p}{2}\bigg),$$

and so $H^2 = \frac{1}{4} + PK = \frac{1}{4} + \frac{8N}{M^6}\sin^2(\frac{p}{2})$, which implies

$$H = \pm\frac{1}{2}\sqrt{1 + \frac{32N}{M^6}\sin^2\bigg(\frac{p}{2}\bigg)} = \pm\frac{1}{2}\sqrt{1 + \frac{g_{\rm YM}^2 N}{\pi^2}\sin^2\bigg(\frac{p}{2}\bigg)}.$$

This is the result expected at one loop. The identification of the matrix model mass parameter with the Yang-Mills coupling coupling holds at one loop but some mismatches were seen to appear at higher loop calculations, implying some kind of BMN scaling breakdown, and a substitution of the factor $\frac{32N}{M^6}$ for a function $f(\frac{N}{M^6})$ [31].

We now calculate the anticommutator of Q and S, which will be proportional to $\mathcal{L}_\beta^\alpha$, R_b^a and the Hamiltonian H:

$$\{Q_b^\beta, S_\alpha^a\}|\chi_1 \ldots \chi_K; J\rangle = \sum_{k=1}^{K}\{c_k b_k \varepsilon^{aa'}\varepsilon_{bb'}\delta_\alpha^\beta \delta(\chi_k^\dagger, \phi^{a'})|\chi_1 \cdots \phi^{b'} \cdots \chi_K; J\rangle$$
$$+ c_k b_k \varepsilon^{\beta\beta'}\varepsilon_{\alpha\alpha'}\delta_b^a \delta(\chi_k^\dagger, \psi^{\beta'})|\chi_1 \cdots \psi^{\alpha'} \cdots \chi_K; J\rangle + a_k d_k \delta_\alpha^\beta \delta(\chi_k^\dagger, \phi^b)|\chi_1 \cdots \phi^a \cdots \chi_K; J\rangle$$
$$+ a_k d_k \delta_b^a \delta(\chi_k^\dagger, \psi^\alpha)|\chi_1 \cdots \psi^\beta \cdots \chi_K; J\rangle\}.$$

From Eqs. (14) we have that

$$a_k d_k = -1;$$
$$b_k c_k = \frac{4N}{M^6}(1 - e^{-ip_k})(e^{ip_k} - 1) = -\frac{16N}{M^6}\sin^2\bigg(\frac{p_k}{2}\bigg).$$

Also, we know from the algebra (1) that

$$\mathcal{L}_\alpha^\beta|\psi^\gamma\rangle = \delta_\alpha^\gamma|\psi^\beta\rangle - \tfrac{1}{2}\delta_\alpha^\beta|\psi^\gamma\rangle,$$
$$R_b^a|\phi^c\rangle = \delta_b^c|\phi^a\rangle - \tfrac{1}{2}\delta_b^a|\phi^c\rangle.$$

For multiparticle states this generalizes to

$$\mathcal{L}_\alpha^\beta|\chi_1 \ldots \chi_K; J\rangle = \sum_{k=1}^{K}\chi_1^\dagger \cdots \mathcal{L}_\alpha^\beta(\chi_k^\dagger) \cdots \chi_K^\dagger|0; J\rangle,$$

with a similar result for the charge R_b^a.[8]

One can now easily see that

[8] The charges $\mathcal{L}$ and R are the generators of the algebra that correspond to rotations of the ψ^γ $\mathfrak{su}(2)$ algebra and of the ϕ^a $\mathfrak{su}(2)$ algebras, respectively. As such, $\mathcal{L}_\alpha^\beta|\phi^c\rangle = 0$, and $R_b^a|\psi^\gamma\rangle = 0$.

$$\{Q^\beta_b, S^a_\alpha\}|\chi_1\ldots\chi_K;J\rangle = \delta^\beta_\alpha R^a_b|\chi_1\ldots\chi_K;J\rangle + \delta^a_b\mathcal{L}^\beta_\alpha|\chi_1\ldots\chi_K;J\rangle + \delta^\beta_\alpha\delta^a_b\sum_{k=1}^K\Big(\frac{1}{2}a_kd_k + b_kc_k\Big)|\chi_1\ldots\chi_K;J\rangle - \delta^\beta_\alpha\sum_{k=1}^K b_kc_k\delta(\chi^\dagger_k,\phi^b)|\chi_1\cdots\phi^a\cdots\chi_K;J\rangle - \delta^a_b\sum_{k=1}^K b_kc_k\delta(\chi^\dagger_k,\psi^\alpha)|\chi_1\cdots\psi^\beta\cdots\chi_K;J\rangle. \quad (20)$$

If we compare with the expected results from commutation relations given in (1), the last two terms seem to be extra. But in fact this is the exact result. We (anti-)commuted only the order g^0 and order g^1 of the supercharges. That is, we calculated the nonzero anticommutators $\{Q_0, S_0\} \propto R + L + H_0$ and $\{Q_1, S_1\}$. This last anticommutator contributes to order g^2 of the Hamiltonian, H_2, but there will be another contribution to H_2: the two-loop terms of the supercharges, Q_2 and S_2, will have nonzero commutation relations with S_0 and Q_0, respectively, and contribute to $\mathcal{O}(g^2)$. So H_2 (the energy central charge of order g^2) will be fully determined by

$$H_2 \propto \{S_1, Q_1\} + \{S_2, Q_0\} + \{S_0, Q_2\}. \quad (21)$$

Only by considering all of the above anticommutators we will get the correct result for the H_2. For calculations see Appendix A, and also [20].

C. Supercharges as operators in momentum space

We now present a description of the supercharges in terms of operators in momentum space. Consider as before an infinite chain of fields Z. The vacuum state, written before as $|0;J\rangle = \mathrm{Tr}(Z^J)|0\rangle$, can be rewritten, in the "Hamiltonian formalism" introduced in [4] as $|0;J\rangle = (b^\dagger_z)^J|0\rangle$, where $b^\dagger_z$ creates an extra Z field in the string.[9] Then we can write a state with K impurities as

$$|\Psi\rangle = \sum_{n_1,\ldots,n_K} e^{ip_jn_j}b^\dagger(n_1)\cdots b^\dagger(n_K)|0;J\rangle = b^\dagger(p_1)\cdots b^\dagger(p_K)|0;J\rangle.$$

We are imposing dilute gas approximation, in which we consider $n_1 \ll n_2 \ll \cdots \ll n_K$. We will now assume $p_1 < p_2 < \cdots < p_K$.

In the last expression for $|\Psi\rangle$ we used the creation operators $b^\dagger(n) = (b^\dagger_z)^n b^\dagger(b_z)^n$, which create a boson b at position n in the string of Z's. One can also introduce $c^\dagger(n) = (b^\dagger_z)^n c^\dagger(b_z)^n$ as a creation operator for a fermion at position n. The action of the Hamiltonian in this framework can be found in [4], and a further comparison with lattice strings can be found in [36].

To write the action of the supercharges in terms of these operators, we also need to introduce a partial momentum operator

$$\hat{\mathcal{P}}(p) = \int_0^p dp'p'[b^\dagger(p')b(p') + c^\dagger(p')c(p')],$$

or the discrete momentum version

$$\hat{\mathcal{P}}(p) = \sum_{k=0}^{p-1} k[b^\dagger(k)b(k) + c^\dagger(k)c(k)].$$

The total momentum operator is just $\hat{P} = \hat{\mathcal{P}}(p_{\max})$, where $p_{\max}$ is either ∞ in the continuum case, or finite (but large) in the lattice. Also, define an operator $\hat{\Theta}$ conjugate to the "R-charge operator" $\hat{\mathcal{J}}$. In the spin-chain formalism, $\hat{\mathcal{J}}$ effectively measures the length of the chain of Z fields, and $\hat{\Theta}$ changes that length:

$$\hat{\mathcal{J}}e^{\pm i\hat{\Theta}}|0,J\rangle = (J\pm 1)e^{\pm i\hat{\Theta}}|0,J\rangle.$$

We can now proceed to the action of the supercharges. In momentum space, they become

$$Q^\beta_b = -\frac{\sqrt{2N}}{M^3}\varepsilon_{bb'}\varepsilon^{\beta\beta'}e^{i\hat{\Theta}}e^{-i\hat{P}}\sum_p b^{b'\dagger}(p)(e^{ip}-1)\times e^{i\hat{\mathcal{P}}(p)}c_{\beta'}(p) + \sum_p c^{\beta\dagger}(p)b_b(p);$$
$$S^a_\alpha = \frac{\sqrt{2N}}{M^3}\varepsilon_{\alpha\alpha'}\varepsilon^{aa'}e^{-i\hat{\Theta}}e^{i\hat{P}}\sum_p c^{\alpha'\dagger}(p)(1-e^{-ip})\times e^{-i\hat{\mathcal{P}}(p)}b_{a'}(p) - \sum_p b^{a\dagger}(p)c_\alpha(p). \quad (22)$$

It is not hard to check that these definitions give us the results obtained in the previous section. In the above expression the sum over momenta has increments of $\frac{2\pi}{J}$.[10]

Commuting two central charges Q will give us the central charge $\mathcal{P}$:

$$\{Q,Q\} = e^{i\hat{\Theta}}e^{-i\hat{P}}\sum_p b^\dagger(p)(e^{ip}-1)e^{i\hat{\mathcal{P}}(p)}b(p) + e^{i\hat{\Theta}}e^{-i\hat{P}}\sum_p c^\dagger(p)(e^{ip}-1)e^{i\hat{\mathcal{P}}(p)}c(p) = \mathcal{P}. \quad (23)$$

[9] The subscript z is used in this section to distinguish the creation operator $b^\dagger_z$ for the boson Z from the creation operator $b^{a\dagger}$ for the two bosonic impurities.

[10] The operator $e^{\pm i\hat{\Theta}}$ does not commute with the sum over the momenta, as it changes the increments in the sum. But in the very large limit of J, this change will be negligible.

One can show that the central charge takes the much more common form[11]:

$$\mathcal{P} = e^{i\hat{\Theta}} e^{-i\hat{P}}(e^{i\hat{P}} - 1) = e^{i\hat{\Theta}}(1 - e^{-i\hat{P}}). \tag{24}$$

To summarize, we found expressions for the supercharges as operators in momentum space, as well as for their commutation relations, in the large J limit. These expressions once applied to states with K impurities will result in the expressions obtained in the previous section.

IV. SUSY GENERATORS IN $AdS_5 \times S^5$

This section will be devoted to determining the action of the supercharges from the string action in $AdS_5 \times S^5$ on a lattice string, followed by a comparison of its structure to supercharge actions obtained in the previous Sec. III C.

We now turn to the action of the supercharges from the $AdS_5 \times S^5$ perspective. We start from the results of [16,19]. In Appendix B we find a summary of those results, and their restriction to the $\mathfrak{su}(2|2)$ subsector. The fermionic supercharges Q and S are given by

$$S_a^\alpha = -\frac{1}{2}\int d\sigma e^{-(i/2)x_-}(i\theta^\alpha(2P^Y + iY)_a - \epsilon^{\alpha\beta}\epsilon_{ab}\theta_\beta^\dagger Y'^b),$$
$$Q_\alpha^a = \frac{1}{2}\int d\sigma e^{(i/2)x_-}(i\theta_\alpha^\dagger(2P^Y - iY)^a + \epsilon_{\alpha\beta}\epsilon^{ab}\theta^\beta Y_b').$$

Before continuing, let us notice that the coordinate $x_-(\sigma)$ obeys

$$x_-(\sigma) = \int_{-r}^{\sigma} d\sigma' x_-'(\sigma') + x_-(-r) = \int_{-r}^{\sigma} d\sigma' \pi_{ws}(\sigma') + x_-^0,$$

where $x_-' = \pi_{ws}(\sigma)$ is the world sheet momentum density. The total world sheet momentum is given by $p_{ws} = \int_{-r}^{r} d\sigma \pi_{ws}(\sigma)$.

We now want to perform a mode expansion. To do so we will follow the notation of [17]. For the bosonic fields we have

$$\begin{aligned} Y_a &= \frac{1}{\sqrt{\omega}}(A_a + B_a^\dagger); \\ P^a &= \frac{\sqrt{\omega}}{4i}(A^{\dagger a} - B^a); \\ Y^a &= \bar{Y}_a = \frac{1}{\sqrt{\omega}}(A^{\dagger a} + B^a); \\ P_a &= \bar{P}^a = i\frac{\sqrt{\omega}}{4}(A_a - B_a^\dagger), \end{aligned} \tag{25}$$

where $\omega = \sqrt{1 + \frac{1}{2}\tilde{\lambda}\partial_\sigma^2}$, and $\tilde{\lambda}$ is the effective coupling constant (light-cone gauge) in the pp-wave limit $\tilde{\lambda} \equiv \frac{4\lambda}{P_+^2}$, kept finite when P_+, $\lambda \to \infty$. For the fermionic fields we have

$$\theta^\alpha = \sqrt{\frac{1}{2}\left(1 + \frac{1}{\omega}\right)} c^\alpha; \qquad \theta_\alpha^\dagger = \sqrt{\frac{1}{2}\left(1 + \frac{1}{\omega}\right)} c_\alpha^\dagger. \tag{26}$$

With these expansions, we get the following results:

$$i\theta^\alpha(2P^Y + iY)_a = -\sqrt{\frac{\omega + 1}{2\omega}} c^\alpha \times \left\{\frac{\sqrt{\omega}}{2}(A_a - B_a^\dagger) + \frac{1}{\sqrt{\omega}}(A_a + B_a^\dagger)\right\},$$
$$\theta_\beta^\dagger Y'^b = \sqrt{\frac{\omega + 1}{2\omega}} c_\alpha^\dagger \frac{\sqrt{\tilde{\lambda}}\partial_\sigma}{\sqrt{2\omega}}(A^{\dagger b} + B^b).$$

We will be keeping $Y \approx B^\dagger$, dropping the oscillators A, $A^\dagger$. Then up to order $\mathcal{O}(\sqrt{\tilde{\lambda}})$,

$$Q_\alpha^a = \frac{1}{4}\int d\sigma e^{(i/2)x_-}(c_\alpha^\dagger B^a + \sqrt{2}\epsilon_{\alpha\beta}\epsilon^{ab} c^\beta \sqrt{\tilde{\lambda}}\partial_\sigma B_b^\dagger). \tag{27}$$

The same can be done for the supercharge S, which then becomes

$$S_a^\alpha = \frac{1}{4}\int d\sigma e^{-(i/2)x_-}(c^\alpha B_a^\dagger + \sqrt{2}\epsilon^{\alpha\beta}\epsilon_{ab} c_\beta^\dagger \sqrt{\tilde{\lambda}}\partial_\sigma B^b). \tag{28}$$

For a comparison with the super Yang-Mills supercharges found in Sec. III C, we need to discretize the above results. To do so recall that $r = P_+/2$, and $\int_{-r}^{r} d\sigma = P_+$. Then the lattice version of Q is

$$\begin{aligned} Q_\alpha^a &= \frac{1}{4}\sum_{\ell=1}^{P_+} e^{ix_-^0/2}\left(\prod_{k=0}^{\ell} e^{(i/2)\pi(k)}\right)\{c_\alpha^\dagger(\ell)B^a(\ell) \\ &\quad + \sqrt{2}\epsilon_{\alpha\beta}\epsilon^{ab} c^\beta(\ell)\sqrt{\tilde{\lambda}}(B_b^\dagger(\ell) - B_b^\dagger(\ell - 1))\} \\ &= \frac{1}{4}\sum_{\ell=1}^{P_+} e^{(i/2)x_-^0} e^{(i/2)p(\ell)}\{c_\alpha^\dagger(\ell)B^a(\ell) \\ &\quad + \sqrt{2}\epsilon_{\alpha\beta}\epsilon^{ab}\sqrt{\tilde{\lambda}}(B_b^\dagger(\ell) - B_b^\dagger(\ell - 1))c^\beta(\ell)\}, \end{aligned} \tag{29}$$

where $p(\ell) = \sum_{k=1}^{\ell} \pi(k)$.

To continue, we need to write what $p(\ell)$ does to an excitation:

$$e^{(i/2)p(\ell)}\chi(\ell_k)e^{-(i/2)p(\ell)} = \begin{cases} \chi(\ell_k) & \ell_k < \ell \\ \chi(\ell_k + 1) & \ell_k > \ell \end{cases}.$$

By performing the following change of variables, $c_\alpha^\dagger(\ell) \to e^{-(i/2)x_-^0} e^{-(i/2)p(\ell)} c_\alpha^\dagger(\ell)$, the charge becomes

[11] It can be proven by using the property (valid for any power n, proven by induction, and for χ fermionic or bosonic)

$$\hat{\mathcal{P}}^n(p)\chi^\dagger(p') = \chi^\dagger(p')[\theta(p - p')p' + \hat{\mathcal{P}}(p)]^n.$$

$$Q^a_\alpha = \frac{1}{4}\sum_{\ell=1}^{P_+}\{c^\dagger_\alpha(\ell)B^a(\ell) + \sqrt{2}\epsilon_{\alpha\beta}\epsilon^{ab}\sqrt{\tilde{\lambda}}e^{(i/2)x^0_-}e^{(i/2)p(\ell)}c^\beta(\ell)e^{(i/2)p(\ell)}e^{(i/2)x^0_-}(B^\dagger_b(\ell) - B^\dagger_b(\ell-1))\}$$
$$= \frac{1}{4}\sum_{\ell=1}^{P_+}\{c^\dagger_\alpha(\ell)B^a(\ell) + \sqrt{2}\epsilon_{\alpha\beta}\epsilon^{ab}\sqrt{\tilde{\lambda}}e^{ix^0_-}(B^\dagger_b(\ell) - B^\dagger_b(\ell-1))e^{ip(\ell)}c^\beta(\ell)\}. \tag{30}$$

The other supercharge S^α_a can also be determined to be

$$S^\alpha_a = \frac{1}{4}\sum_{\ell=1}^{P_+} e^{-(i/2)x^0_-}e^{-(i/2)p(\ell)}(c^\alpha(\ell)e^{(i/2)p(\ell)}e^{(i/2)x^0_-}B^\dagger_a(\ell) + \sqrt{2}\epsilon^{\alpha\beta}\epsilon_{ab}e^{-(i/2)x^0_-}e^{-(i/2)p(\ell)}c^\dagger_\beta(\ell)\sqrt{\tilde{\lambda}}(B^b(\ell) - B^b(\ell-1)))$$
$$= \frac{1}{4}\sum_{\ell=1}^{P_+}(B^\dagger_a(\ell)c^\alpha(\ell) + \sqrt{2}\epsilon^{\alpha\beta}\epsilon_{ab}\sqrt{\tilde{\lambda}}e^{-ix^0_-}e^{-ip(\ell)}c^\dagger_\beta(\ell)(B^b(\ell) - B^b(\ell-1))). \tag{31}$$

If we wrote these charges in momentum space, we would obtain the exact structure for the supercharges (22), as long as we make the correspondence that the conjugate pair $(x^0_-, P_+) \leftrightarrow (\hat{\Theta}, \hat{\mathcal{J}})$. In the above expressions, x^0_- plays the part of the length changing operator, as it is the conjugate variable to P_+, the total light-cone momentum, which is in its turn related to the width of the world sheet cylinder. For closed strings the total world sheet momentum $p_{\rm ws}$ has to vanish (on shell)—-level-matching condition. If we relax this condition (off shell) and take $P_+ \to \infty$, then we obtain the centrally extended algebra with extra central charges C, C^* added to the Hamiltonian H (the same as the generators of translations P and boosts K).

One other way of checking the results is by writing the supercharges in the first quantized framework. Choosing again a state such that

$$|\chi_1\cdots\chi_K; P_+\rangle = \sum_{\{m_i\}=0}^{P^+} e^{ip_1m_1+\cdots+ip_Km_K}\chi_1(m_1)\cdots\chi_K(m_K)|0;P_+\rangle,$$

where $\chi_i(m_i) = b_z^{m_i}\chi_i b_z^{-m_i}$, with b_z being the oscillators equivalent to the field Z. Then

$$Q^a_\alpha|\chi_1\cdots\chi_K;P_+\rangle = \frac{1}{4}\sum_{k=1}^{K}\left(\prod_{m=1}^{k-1}(-1)^{F(m)}\right)\Big\{\delta(\chi_k, B^\dagger_b)\delta^a_b|\chi_1\cdots c^\dagger_\alpha(k)\cdots\chi_K;P_+\rangle$$
$$+ \sqrt{2\tilde{\lambda}}\delta(\chi_k, c^\dagger_\beta)\epsilon^{ab}\epsilon_{\alpha\beta}\left(\prod_{l=k+1}^{K}e^{ip_l} - \prod_{l=k}^{K}e^{ip_l}\right)|\chi_1\cdots B^\dagger_b(k)\cdots\chi_K;P_+ + 1\rangle\Big\}. \tag{32}$$

Doing the same calculation for the S generator, one gets

$$S^\alpha{}_a|\chi_1\cdots\chi_K;P_+\rangle = \frac{1}{4}\sum_{k=1}^{K}\left(\prod_{m=1}^{k-1}(-1)^{F(m)}\right)\Big\{\delta(\chi_k, c^\dagger_\beta)\delta^\alpha_\beta\chi_1(m_1)\cdots(B^\dagger_a(k))\cdots\chi_K(m_K)|0;P_+\rangle$$
$$+ \sqrt{2\tilde{\lambda}}\delta(\chi_k, B^\dagger_b)\epsilon^{\alpha\beta}\epsilon_{ab}\left(\prod_{l=k+1}^{K}e^{-ip_l} - \prod_{l=k}^{K}e^{-ip_l}\right)|\chi_1\cdots c^\dagger_\beta(k)\cdots\chi_K;P_+ - 1\rangle\Big\}. \tag{33}$$

From this we can again see that the actions of the supercharges Q and S have a similar structure at one loop, on both sides of the correspondence. But while the results presented in this section are perturbative in $\tilde{\lambda}$ (BMN limit), the results presented in the previous section are perturbative in the 't Hooft coupling λ, so one cannot perform a direct comparison.

V. CONCLUSIONS

In this paper we studied in detail the Q, S generators of the extended algebra $\mathfrak{su}(2|2)$ in the plane-wave matrix theory formalism. By using a coherent basis we determined the supercharges in the nonlocal notation of Beisert [9] (as well as in the local twisted notation), and determined some of the coefficients in this notation up to order $\mathcal{O}(g_{\rm YM})$.

We also determined the anticommutation relations of these supercharges and obtained the expected results for the central charges P, K, and H. We saw that we needed to know the Hamiltonian up to two loops in order to have a closed (anti-)commutation relation between Q and S.

We finally wrote a first quantized formulation of the supercharges obtained directly from the sigma-model action for the string. Having the supercharges written in that way allowed us to compare their structure with what we had previously calculated from the gauge side.

The evidence seems to point to $\mathcal{N}=4$ SYM and IIB superstring theory being integrable models in the 't Hooft limit. We also said that the scattering matrix is completely defined by the underlying symmetry algebra $\mathfrak{psu}(2,2|4)$. One finds that the S matrix actually retains a symmetry algebra that is two copies of a central extension of the $\mathfrak{psu}(2|2)$ algebra, in particular, $\mathfrak{psu}(2|2)\ltimes\mathbb{R}^3=\mathfrak{su}(2|2)\ltimes\mathbb{R}^2$. This symmetry of the S matrix is expected to be a Yangian [37–39], and have an underlying Hopf algebra [40,41]. (See also [42–44].) Having these new developments in mind, it would be interesting to apply the methods used in this paper to the study of the Hopf algebra related to the central extension and get some results on the corresponding Yangian generators.

The sector of near 1/2 BPS operators in $\mathcal{N}=4$ super Yang-Mills theory has been well studied by the use of collective methods [35,36], and the same methods can be used to study the elements of the algebra in the 1/4 BPS sector (work in progress).

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APPENDIX A: COMMUTING THE $\mathfrak{su}(2|2)$ SUPERCHARGES UP TO TWO LOOPS

The expressions found here are restrictions to the $\mathfrak{su}(2|2)$ subsector of the full sector $\mathfrak{su}(2|3)$ found in [20]. The supercharges at order g^0, Q_0, and S_0, at order g^1, Q_1, and S_1, and at order g^2, Q_2, and S_2 in the dilute gas approximation can be written as follows:

$$(Q_0)^b_\beta=\begin{Bmatrix} b\\ \beta\end{Bmatrix},\qquad (S_0)^\alpha_a=\begin{Bmatrix} \alpha\\ a\end{Bmatrix};$$

$$(Q_1)^b_\beta=\frac{A}{\sqrt{2}}\varepsilon_{\beta\beta'}\varepsilon^{bb'}\left(\begin{Bmatrix} \beta'\\ b'3\end{Bmatrix}-\begin{Bmatrix} \beta'\\ 3b'\end{Bmatrix}\right),$$
$$(S_1)^\alpha_a=\frac{A}{\sqrt{2}}\varepsilon_{aa'}\varepsilon^{\alpha\alpha'}\left(\begin{Bmatrix} a'3\\ \alpha'\end{Bmatrix}-\begin{Bmatrix} 3a'\\ \alpha'\end{Bmatrix}\right);$$

$$(Q_2)^b_\beta=\left(\frac{A^2}{4}-\frac{i}{2}\gamma_3+\frac{i}{2}\gamma_4\right)\left(\begin{Bmatrix} b3\\ \beta3\end{Bmatrix}+\begin{Bmatrix} 3b\\ 3\beta\end{Bmatrix}\right)+\left(-\frac{A^2}{4}-i\gamma_1\right)\left(\begin{Bmatrix} b3\\ 3\beta\end{Bmatrix}+\begin{Bmatrix} 3b\\ \beta3\end{Bmatrix}\right),$$
$$(S_2)^\alpha_a=\left(\frac{A^2}{4}+\frac{i}{2}\gamma_3-\frac{i}{2}\gamma_4\right)\left(\begin{Bmatrix} \alpha3\\ a3\end{Bmatrix}+\begin{Bmatrix} 3\alpha\\ 3a\end{Bmatrix}\right)+\left(-\frac{A^2}{4}+i\gamma_1\right)\left(\begin{Bmatrix} \alpha3\\ 3a\end{Bmatrix}+\begin{Bmatrix} 3\alpha\\ a3\end{Bmatrix}\right).$$

We will be using the notation used in [20]. The index 3 above means an insertion of a field Z. The action of $\begin{Bmatrix} \alpha bc\\ c\alpha b\end{Bmatrix}$ on a state looks for a sequence of a fermion followed by two bosons, and permutes them in the order second boson-fermion–first boson. As an example in $\mathfrak{su}(2|3)$, where indices 1, 2, and 3 correspond to bosons and indices 4, 5 correspond to fermions, we have

$$\begin{Bmatrix} \alpha bc\\ c\alpha b\end{Bmatrix}|142\,334\,452\rangle=|134\,234\,452\rangle+|242\,334\,415\rangle.$$

Determining the anticommutation relations, we have

$$\frac{2}{A^2}\{(S_1)^\alpha_a,(Q_1)^b_\beta\}=\delta^b_a\delta^\alpha_\beta\frac{1}{A^2}H_2-\delta^b_a\left[2\begin{Bmatrix} \alpha\\ \beta\end{Bmatrix}-\begin{Bmatrix} 3\alpha\\ \beta3\end{Bmatrix}-\begin{Bmatrix} \alpha3\\ 3\beta\end{Bmatrix}\right]-\delta^\alpha_\beta\left[2\begin{Bmatrix} b\\ a\end{Bmatrix}-\begin{Bmatrix} 3b\\ a3\end{Bmatrix}-\begin{Bmatrix} b3\\ 3a\end{Bmatrix}\right];$$

$$\frac{2}{A^2}\{(S_2)^\alpha_a,(Q_0)^b_\beta\}+\frac{2}{A^2}\{(S_0)^\alpha_a,(Q_2)^b_\beta\}=2\left[\delta^\alpha_\beta\begin{Bmatrix} b\\ a\end{Bmatrix}+\delta^b_a\begin{Bmatrix} \alpha\\ \beta\end{Bmatrix}\right]-\delta^\alpha_\beta\left[\begin{Bmatrix} 3b\\ a3\end{Bmatrix}-\begin{Bmatrix} b3\\ 3a\end{Bmatrix}\right]-\delta^b_a\left[\begin{Bmatrix} 3\alpha\\ \beta3\end{Bmatrix}+\begin{Bmatrix} \alpha3\\ 3\beta\end{Bmatrix}\right].$$

Then the sum of these anticommutators gives

$$\{(S_1)^\alpha_a,(Q_1)^b_\beta\}+\{(S_2)^\alpha_a,(Q_0)^b_\beta\}+\{(S_0)^\alpha_a,(Q_2)^b_\beta\}=\tfrac{1}{2}\delta^b_a\delta^\alpha_\beta H_2,$$

where the two-loop contribution for the Hamiltonian (dilute gas approx) is

$$\frac{1}{A^2}H_2 = 2\begin{Bmatrix} a \\ a \end{Bmatrix} + 2\begin{Bmatrix} \alpha \\ \alpha \end{Bmatrix} - \begin{Bmatrix} a3 \\ 3a \end{Bmatrix} - \begin{Bmatrix} 3a \\ a3 \end{Bmatrix} - \begin{Bmatrix} \alpha 3 \\ 3\alpha \end{Bmatrix} - \begin{Bmatrix} 3\alpha \\ \alpha 3 \end{Bmatrix}.$$

From the results presented above, we can see that we can only get the complete order g^2 of the Hamiltonian from the commutation of the supercharges if we consider their two-loop contributions.

APPENDIX B: THE Q SUPERCHARGES IN THE $\mathfrak{su}(2|2)$ SECTOR, ON THE STRING SIDE

As can be seen in [16,19], we can write the charges as

$$Q_{\mathcal{M}} = \int d\sigma e^{i\alpha x_-}\chi(B_1(x,p) + \zeta B_3(x,p) + \cdots) + \mathcal{O}(\chi^3), \tag{B1}$$

where we only kept the term linear in fermion fields, and kept all the bosonic terms of the expansion [$B_n(x,p)$ is the term with a product of n bosonic fields].

The next step is to determine the Poisson brackets of two charges with $\alpha_1 = \alpha_2 = 1$. For example (see the appendix of [19]),

$$\{Q_a^\alpha, Q_b^\beta\} \sim \epsilon^{\alpha\beta}\epsilon_{ab}\int_{-r}^{r} d\sigma e^{-ix_-}\left(x'_- + \frac{d}{d\sigma}f(x,p)\right),$$

where $f(x,p)$ is a local function of the transverse fields. The result for $\{\bar{Q}_\alpha^a, \bar{Q}_\beta^b\}$ can be obtained by conjugation. Integrating this expression, we get

$$\{Q^\alpha{}_a, Q^\beta{}_b\} \sim \epsilon^{\alpha\beta}\epsilon_{ab}\int_{-r}^{r} d\sigma \frac{d}{d\sigma}e^{ix_-} = \epsilon^{\alpha\beta}\epsilon_{ab}e^{-ix_-(-r)}(e^{-i[x_-(r)-x_-(-r)]} - 1).$$

We know that $p_{\rm ws} = x_-(r) - x_-(-r)$. We also impose the boundary condition $x_-(-r) = x_-^0$, which is the zero mode of x_-, conjugate to P_+. Then

$$\{Q^\alpha{}_a, Q^\beta{}_b\} \sim \frac{1}{\zeta}\epsilon^{\alpha\beta}\epsilon_{ab}e^{-ix_-^0}(e^{-ip_{\rm ws}} - 1),$$

$$\{\bar{Q}^a{}_\alpha, \bar{Q}^b{}_\beta\} \sim \frac{1}{\zeta}\epsilon^{ab}\epsilon_{\alpha\beta}e^{ix_-^0}(e^{ip_{\rm ws}} - 1),$$

and consequently, the central charges are c, c^* with

$$c = \frac{1}{\zeta}e^{-ix_-^0}(1 - e^{ip_{\rm ws}})e^{-ip_{\rm ws}}. \tag{B2}$$

Looking at the value of the central charge here and the one obtained from the spin-chain formalism, we can conclude that the results are correct up to an overall phase $e^{\pm ip_{\rm ws}}$, as long as we match $\{Q, \bar{Q}\} \leftrightarrow \{S, Q\}$ and $\{C, C^\dagger\} \leftrightarrow \{K, P\}$. This overall phase is natural, as different boundary conditions for x_- will differ from each other by such a phase. Also the algebra (1) allows a $U(1)$ automorphism, which means we can always multiply all supercharges by some phase that can depend on all central charges.

1. Some comments

In the case of P_+ infinite, the zero mode x_-^0 vanishes, but the same is not true for finite light-cone momentum. This brings some problems, as for P_+ (which is effectively the length of the string) finite, the transverse fields do not have to vanish at the string points, and the symmetry algebra is thus changed.

At the quantum level, both $p_{\rm ws}$ and x_-^0 are promoted to operators $\mathbf{P}$, $\mathbf{X}_-^0$, and the central charges are

$$\mathbf{C} = \frac{1}{\zeta}e^{-i\mathbf{X}_-^0}(e^{-i\mathbf{P}} - 1), \tag{B3}$$

and its conjugate $\mathbf{C}^\dagger$. $\mathbf{X}_-^0$ is the conjugate quantum operator of $\mathbf{P}_+$. If we consider a state $\mathbf{P}_+|p_+\rangle = p_+|p_+\rangle$, then a state $e^{i\alpha\mathbf{X}_-^0}|p_+\rangle$ obeys

$$\mathbf{P}_+e^{i\alpha\mathbf{X}_-^0}|p_+\rangle = (\alpha + p_+)e^{i\alpha\mathbf{X}_-^0}|p_+\rangle. \tag{B4}$$

Because P_+ acts as the length of the string, the operator $e^{i\alpha\mathbf{X}_-^0}$ will be the length changing operator. The Hilbert space of the theory will be a direct sum, $\mathcal{H} = \bigoplus_{p_+}\mathcal{H}_{p_+}$, of spaces of each of the eigenvalues of $\mathbf{P}_+$.

2. $\mathfrak{su}(2|2)$ subsector and mode expansion

The explicit form of the charges $Q_{\mathcal{M}}$ was determined in [19]. The algebra $\mathcal{J}$ includes two $\mathfrak{psu}(2|2)$ subalgebras. We will be focusing on the $\mathfrak{psu}(2|2)_R$.

The leading quadratic order of (B1) can be read from the results in [19]. The fermionic charges are at leading order:

$$\begin{aligned} Q^\alpha{}_a = -\frac{1}{2}\int d\sigma e^{-(i/2)x_-}[&i\theta^\alpha(2P^Y + iY)_a + (2P^Z - iZ)^\alpha\eta_a^\dagger - \theta^{\dagger\alpha}Y'_a - iZ'^\alpha\eta_a \\ &+ \epsilon^{\alpha\beta}\epsilon_{ab}(i\theta_\beta(2P^Y + iY)^b + (2P^Z - iZ)_\beta\eta^{\dagger b} \\ &- \theta_\beta^\dagger Y'^b - iZ'_\beta\eta^b)], \end{aligned} \tag{B5}$$

$$\begin{aligned} \bar{Q}_\alpha{}^a = \frac{1}{2}\int d\sigma e^{(i/2)x_-}[&i\theta_\alpha^\dagger(2P^Y - iY)^a - (2P^Z + iZ)_\alpha\eta^a \\ &+ \theta_\alpha Y'^a - iZ'_\alpha\eta^{\dagger a} + \epsilon_{\alpha\beta}\epsilon^{ab}(i\theta^{\dagger\beta}(2P^Y - iY)_b \\ &- (2P^Z + iZ)^\beta\eta_b + \theta^\beta Y'_b - iZ'^\beta\eta_b^\dagger)] \neq (Q^\alpha{}_a)^\dagger. \end{aligned} \tag{B6}$$

We want to restrict ourselves to the $\mathfrak{su}(2|2)$ subsector of [10]. This corresponds to keeping only the 2 complex coordinates Y^a and the respective conjugate momenta P^y. These will correspond, in the SYM side, to our bosonic excitations ϕ^a, with $a = 1, 2$. In terms of the fermions we will be interested in only keeping θ^α, $\theta_\alpha^\dagger$, which will correspond to the 2 fermionic fields ψ_α, $\psi^{\dagger\alpha}$ from SYM.

The vacuum of the fields Z in Yang-Mills theory will, in this case, correspond to [4]

$$\frac{1}{\sqrt{J}N^{J/2}}\,\mathrm{Tr}(Z^J) \leftrightarrow |0, p^+\rangle.$$

With these restrictions, the fermionic supercharges (B5) and (B6) become

$$S_a^\alpha = -\frac{1}{2}\int d\sigma e^{-(i/2)x_-}(i\theta^\alpha(2P^Y + iY)_a - \epsilon^{\alpha\beta}\epsilon_{ab}\theta_\beta^\dagger Y'^b),$$

$$Q_\alpha^a = \frac{1}{2}\int d\sigma e^{(i/2)x_-}(i\theta_\alpha^\dagger(2P^Y - iY)^a + \epsilon_{\alpha\beta}\epsilon^{ab}\theta^\beta Y_b').$$

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