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N-point functions in rolling tachyon background

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We study n-point boundary correlation functions in timelike boundary Liouville theory, relevant for open string multiproduction by a decaying unstable D brane. We give an exact result for the one-point function of the tachyon vertex operator and show that it is consistent with a previously proposed relation to a conserved charge in string theory. We also discuss when the one-point amplitude vanishes. Using a straightforward perturbative expansion, we find an explicit expression for a tachyon n-point amplitude for all n, however the result is still a toy model. The calculation uses a new asymptotic approximation for Toeplitz determinants, derived by relating the system to a Dyson gas at finite temperature.

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I. INTRODUCTION AND SUMMARY

String theory contains D branes of opposite charge, so one should be able to understand their annihilation process. A related problem is the decay of a single unstable brane, such as a D brane in bosonic string theory. A simple model for the D-brane decay describes a process starting from the infinite past, involving a spatially homogenous tachyon field rolling towards the true minimum of its effective potential [1,2]. A basic open problem is to calculate tree-level string scattering amplitudes in the rolling tachyon background, corresponding to production of multiple closed or open strings by the decaying brane. There are both conceptual and technical aspects to this problem. Because the background is time dependent, there are different ways to define the notion of vacuum and asymptotic states. A technical framework for the bosonic homogeneous brane decay is the timelike boundary Liouville theory (TBL) coupled to 25 free massless spacelike bosons, and the problem of computing n-point correlation functions [3]. Calculations are difficult since they involve complicated coupled integrals and/or nonintuitive analytic continuations.

In this paper we focus on calculating boundary n-point functions in TBL. The two-point function, associated to the rate of open string pair production by a decaying brane, has been investigated before [3,4], and also in a curved spacetime (AdS_3) in [5]. (Other string production work is found in [6–16].) A simple toy model is obtained by moving to the minisuperspace approximation, where strings are pointlike, and the problem reduces to a relatively simple quantum mechanical scattering problem. Returning back to the original setup, the standard prescription is to start from spacelike boundary Liouville theory (SBL), where the two-point and three-point functions have known well-defined analytic expressions [17–19], and then continue to the timelike theory by $b \to i$, $\phi \to iX^0$. However, the continuation must involve a prescription to avoid the accumulation of an infinite number of poles and zeroes which would render the answer ill defined. One way to motivate a prescription is by aiming to make contact with the minisuperspace analysis. This procedure gives a physically pleasant answer, exponentially suppressed open string pair production at high energies. However, some doubt remains, as the prescription for the analytic continuation was not unique and some of the steps involved are rather indirect. It is desirable to pursue alternative approaches; they may give further support to the previous analysis or lead to other reasonable prescriptions. Moreover, the previous method is difficult to extend beyond the two-point function.

An alternative method to compute correlation functions in TBL was given in [15]. Instead of indirect arguments, the method [15] is based on a straightforward perturbative expansion, and the observation that random matrix theory (RMT) [20] techniques become applicable to the ensuing integrals. This method was successfully applied to compute the bulk-boundary function [15,16]. On the other hand, the same problem was also considered by Liouville theory methods. The bulk-boundary function was calculated in spacelike Liouville theory in [21]. Reference [22] then investigated the analytic continuation from spacelike Liouville theory to timelike theory and found a result for the bulk-boundary function which is similar to that of [15].

In the method of [15], correlation functions are related to expectation values of periodic functions (Fisher-Hartwig symbols) in the circular ensemble of unitary matrices (CUE), also equivalent to Toeplitz determinants of Fourier coefficients. This observation was extended to n-point functions and superstrings in [16]. Alternatively, the n-point functions can be related to thermal expectation values in a classical log gas of unit charges in two dimen-

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sions, e.g., the Dyson gas. In [15], this observation was made at a formal level, while the problem of actually finding explicit answers for the correlation functions still remained.

In the present paper, we use the interpretation of the correlations functions as thermal Dyson gas expectation values and then use physical insight to find analytic expressions. We are able to derive an expression for an n-point amplitude. The virtue of our approach is that it is relatively straightforward, and it is powerful enough to for a first time yield an analytic expression for an n-point amplitude for all n. The downside is that at the moment we do not have quantitative control of our approximations by the time we compute the amplitude. Consequently, we do not yet know how to compare the result with the previous one for the open string pair creation amplitude. Nevertheless, we consider the techniques that we have developed to be a step forward towards full control of the scattering problem.

The paper is organized as follows. Section II begins with a review of some facts of TBL. We present some preliminary calculations and discuss the one-point function and the vanishing one-point amplitude. Next, we present a contour integration trick which is powerful in summing the series expansion for the correlation functions. We then use the one-point function formula to test a recently proposed master formula [23] for a string theoretic definition of a conserved charge. Section III reviews the relation of TBL to Dyson gas at finite temperature. In Sec. IV, we use this connection to derive an approximation for the integrals which appear as coefficients in the series expansion of an n-point amplitude, then use the approximation for the coefficients and the contour integration trick of Sec. II to derive a toy model result for the amplitudes. Some calculational details are left in Appendices A, B, and C.

II. BOUNDARY AMPLITUDES IN TIMELIKE LIOUVILLE THEORY

Let us first review some facts to identify the problem of interest. Full scattering amplitudes in bosonic string theory involve contributions from the timelike X^0 and the 25 spacelike directions $\vec{X} = (X^I)$. However, as discussed in [7], one can simplify the calculations by adopting a gauge where the string vertex operators factorize into a form

$$V = e^{i\omega X^0} V_{\rm sp}(\vec{X}), \tag{1}$$

so that all dependence on X^0 is in the simple exponential factor, while $V_{\rm sp}$ contains the more complicated polarization tensor factor and only depends on the spacelike directions $\vec{X}$. For a homogeneous rolling tachyon background depending only on X^0, all the complications arise from contractions in the X^0 direction between the background and the vertex operators, while contractions in the spatial directions give a simple contribution. Correspondingly, the n-point correlation functions in the homogeneous rolling tachyon background factorize into a product of an n-point function of $e^{i\omega_a X^0(\tau_a)}$ (where the label $a = 1, \ldots, n$) in the TBL theory and an n-point function of $V_{\rm sp}(\vec{k}_a; \vec{X}(\tau_a))$ in the theory of free spacelike bosons,

$$\left\langle \prod_{a=1}^{n} e^{i\omega_a X^0(\tau_a)} \right\rangle_{\rm TBL} \left\langle \prod_{a=1}^{n} V_{\rm sp}(\vec{k}_a; \vec{X}(\tau_a)) \right\rangle_{\rm free} \equiv e^{-i\sum_a \vec{k}_a\cdot\vec{x}} \left\langle \prod_{a=1}^{n} e^{i\omega_a X^0(\tau_a)} \right\rangle_{\rm TBL} F_{\rm free}[(\vec{k}_a); (\tau_a)], \tag{2}$$

where we separated the spacelike zero modes. The on-shell conditions $k_a^2 = -\omega_a^2 + \vec{k}_a^2 = -m_a^2$ can be satisfied for a range of values of ω_a. The problem of interest is to calculate n-point functions in TBL for generic ω_a. We will also try to compute the full scattering amplitude for n open string tachyons.

The action of the TBL theory is

$$S_{\rm TBL} = -\frac{1}{2\pi}\int_{\rm disk} \partial X^0 \bar{\partial} X^0 + \lambda \oint dt e^{X^0}. \tag{3}$$

Eventually we will be interested in the open string n-point tachyon amplitude

$$\mathcal{A}_n(\omega_1, \vec{k}_1; \ldots; \omega_n, \vec{k}_n) = \int d^p\vec{x} e^{-i\sum_a \vec{k}_a\cdot\vec{x}} \int \prod_{a=1}^{n} \frac{d\tau_a}{2\pi} \times F_{\rm free}[(\vec{k}_a); (\tau_a)] \int \mathcal{D}X^0 e^{-S_{\rm TBL}} \times \prod_{a=1}^{n} e^{i\omega_a X^0(\tau_a)}, \tag{4}$$

at tree level, where the momenta $\vec{k}_a$ are in the spacelike directions of the decaying p-dimensional brane, τ_a denote points on the boundary of the disk (unit circle).[1] For tachyons the contribution from the spacelike directions (with divergent self-contractions removed) is

$$F_{\rm free}[(\vec{k}_a); (\tau_a)] = \prod_{a<b} |e^{i\tau_a} - e^{i\tau_b}|^{2\vec{k}_a\cdot\vec{k}_b}, \tag{5}$$

with the on-shell condition $k_a^2 = -\omega_a^2 + \vec{k}_a^2 = 1$. The conservation of spatial momentum has been discussed, e.g., in [15]. As discussed in the introduction, different approaches have been used for the calculation. We will follow the approach of [1,15] and first expand $\mathcal{A}_n$ as a power series, in powers of the boundary interaction. We also separate out the overall zero mode x^0 dependence, so $\mathcal{A}_n$ becomes

[1] We could use the conformal Killing group (CKG) $PSL(2, R)$ to fix three of the vertex operator coordinates τ_a, but we have chosen to leave them unfixed and average over the locations.

$$\mathcal{A}_n = \delta_{0,\sum_a \vec{k}_a} \int \prod_{a=1}^{n} \frac{d\tau_a}{2\pi} F[\cdots] \times \int dx^0 e^{ix^0 \sum_{a=1}^{n} \omega_a} \sum_{N=0}^{\infty} \frac{(-2\pi\lambda e^{x^0})}{N!} \int \prod_{i=1}^{N} \frac{dt_i}{2\pi} \times \left\langle e^{X'^0(t_1)} \cdots e^{X'^0(t_N)} \prod_{a=1}^{n} e^{i\omega_a X'^0(\tau_a)} \right\rangle. \quad (6)$$

After the Wick contractions and substituting the Green's functions, the amplitude takes the form of a power series of coupled integrals.

The amplitude $\mathcal{A}_n$ then becomes

$$\mathcal{A}_n(\xi_1, \ldots, \xi_n) = \delta_{0,\sum_a \vec{k}_a} \int dx^0 \exp\left[x^0 \sum_{a=1}^{n} \xi_a\right] \times \bar{\mathcal{A}}_n(2\pi\lambda e^{x^0}),$$

$$\text{where } \bar{\mathcal{A}}_n(z) = \sum_{N=0}^{\infty} (-z)^N I_{\xi_1,\ldots,\xi_n}(N), \quad (7)$$

where we have adopted the notation

$$z \equiv 2\pi\lambda e^{x^0}; \qquad \xi_a \equiv i\omega_a, \quad (8)$$

and the integrals

$$I_{\xi_1,\ldots,\xi_n}(N) = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{dt_i}{2\pi} \prod_{a=1}^{n} \frac{d\tau_a}{2\pi} \left[\prod_{1\le i<j\le N} |e^{it_i} - e^{it_j}|^2\right] \times \left[\prod_{i=1}^{N} \prod_{a=1}^{n} |e^{i\tau_a} - e^{it_i}|^{2\xi_a}\right] \times \left[\prod_{1\le a<b\le n} |e^{i\tau_a} - e^{i\tau_b}|^{2\xi_a\xi_b + 2\vec{k}_a\cdot\vec{k}_b}\right], \quad (9)$$

which include the spacelike contribution F. In order to do the sum over N we need to work out the t_i integrals for arbitrary N. When calculating the integrals it is often useful to assume that ξ_a are positive real numbers and continue to imaginary ξ_a, i.e., to real energies ω_a, in the end. This is not problematic since the t_i integrals converge for $\mathrm{Re}\xi_a > -1/2$ and thus define an analytic function of ξ_a in this region.

A. Some preliminary considerations

The simplest case to consider is $n = 1$, the one-point boundary amplitude. Invariance under translation requires the one-point function to vanish unless the operator at the boundary has zero conformal weight, rendering the case trivial. However, it turns out that some calculations will be useful for the nontrivial case $n > 1$. It is also known that in a noncompact conformal field theory (CFT) (integrated) one-point functions can be nonzero [23,24]. Reference [23] proposed a relation between a one-point function and a spacetime boundary term. In our case, we can use a TBL one-point function as a check of the master formula in [23] and find it to be consistent.

Let us postpone other discussions for a moment and just focus on a straightforward calculation. We consider the series that appears in (7), with $n = 1$,

$$\bar{\mathcal{A}}_1(z) = \sum_{N=0}^{\infty} (-z)^N \cdot I_\xi(N), \quad (10)$$

where now

$$I_\xi(N) = \frac{1}{N!} \int_0^{2\pi} \frac{d\tau}{2\pi} \int \left[\prod_{i=1}^{N} \frac{dt_i}{2\pi} |e^{i\tau} - e^{it_i}|^{2i\omega}\right] \times \left[\prod_{1\le i<j\le N} |e^{it_i} - e^{it_j}|^2\right] \quad (11)$$

$$= \frac{1}{N!} \int \left[\prod_{i=1}^{N} \frac{dt_i}{2\pi} |1 - e^{it_i}|^{2\xi}\right] \left[\prod_{1\le i<j\le N} |e^{it_i} - e^{it_j}|^2\right]. \quad (12)$$

Here we denoted $\xi = i\omega$, where ω is the energy of the open string.[2] It is interesting to note that the integrand is independent of τ, the coordinate of the vertex operator at the boundary, so that the τ integral is trivial. In other words, the integrand is invariant under translations along the boundary, independently of $\xi = i\omega$. However, the total one-point function also contains the contribution from the spacelike directions with a $\delta_{0,k}$ factor, which along with the on-shell condition will constrain ω. But let us focus back to the properties of the series (10).

The same series has been considered in the context of a general bulk-boundary amplitude, which has been calculated in closed form in [15,16]. The bulk-boundary amplitude involves

$$\hat{\mathcal{A}}_{1+1}(\omega_c, \omega_o) \equiv \int dx^0 e^{i(\omega_o+\omega_c)x^0} \sum_{N=0}^{\infty} (-z)^N I_{i\omega_o}(N), \quad (13)$$

where ω_c is the energy of the bulk operator $\exp\{i\omega_c X^0(z, \bar{z})\}$ and $I_{i\omega_o}(N)$ is the integral (11) evaluated at $\xi = i\omega_o$, where ω_o is the energy of the boundary operator.[3] First, the integral evaluates to the relatively simple expression

$$I_\xi(N) = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2\xi)}{\Gamma(j+\xi)^2} = \frac{G(\xi+1)^2}{G(2\xi+1)} \frac{G(N+2\xi+1)G(N+1)}{G(N+\xi+1)^2}, \quad (14)$$

[2] Note that (after removing the self-contractions in the spacelike directions) $F_{\text{free}} = 1$.

[3] The one-point amplitude is formally the limit $\omega_c \to 0$ of the bulk-boundary amplitude (13). We also omitted a δ-function term [see Subsection II B].

where G is the Barnes G function. After converting to integral representation of the Γ functions, the sum over N in (10) can be done [15,16], leading to the result

$$\hat{\mathcal{A}}_{1+1}(\omega_c, \omega_o) = -i\pi \frac{(2\pi\lambda)^{-i(\omega_c+\omega_o)}}{\sinh\pi(\omega_c+\omega_o)} \times \exp\left[\int_0^\infty \frac{dt(1-e^{-i\omega_o t})^2}{2t(1-\cosh t)} \times (1-e^{i(\omega_c+\omega_o)t})\right]. \tag{15}$$

We would first like to point out an interesting feature, which was not investigated in [15,16]. Let us write it in terms of the Barnes G functions, using the integral representation

$$\log G(z+1) = \int_0^\infty \frac{dt}{t} e^{-t}\left[\frac{z(z-1)}{2} - \frac{z}{1-e^{-t}} + \frac{1-e^{-zt}}{(1-e^{-t})^2}\right]; \qquad \mathrm{Re}(z) > -1. \tag{16}$$

We find

$$\hat{\mathcal{A}}_{1+1}(\omega_c, \omega_o) = -i\pi \frac{(2\pi\lambda)^{-i(\omega_c+\omega_o)}}{\sinh\pi(\omega_c+\omega_o)} J_{i\omega_o}(i(\omega_c+\omega_o)), \tag{17}$$

where

$$J_\xi(s) = \frac{G(\xi+1)^2}{G(2\xi+1)} \frac{G(2\xi-s+1)G(-s+1)}{G(\xi-s+1)^2}. \tag{18}$$

The asymptotic behavior [15,16] follows easily from (18),

$$J_{i\omega_o}(i(\omega_c+\omega_o)) \sim_{\omega_c\to\infty} \omega_c^{-\omega_o^2}, \tag{19}$$

by using the asymptotic series of the Barnes G function

$$\log G(z+1) = z^2\left(\frac{1}{2}\log z - \frac{3}{4}\right) + \frac{z}{2}\log 2\pi - \frac{1}{12}\log z + \zeta'(-1) + \mathcal{O}(1/z^2). \tag{20}$$

An interesting feature is that (18) is a natural continuation of (14) to noninteger values, replacing $N \to -s$, but (14) was the Nth coefficient in the series (13), while (18) is essentially the sum.[4] We will show how coefficients convert to the sum in the next Subsection II B, by a new contour integral trick which also allows a more controlled investigation of the convergence of the series (13). The other benefit of the calculation is that it can also be applied to n-point amplitudes. But let us first continue with the one-point function.

As seen by comparing (10) and (13), we can formally use the result (17) to obtain a formula for the Fourier transform of (10) by setting $\omega_c = 0$ and $\omega_o = \omega$, giving

$$\begin{aligned}\hat{\mathcal{A}}_1(\omega) &= \hat{\mathcal{A}}_{1+1}(0,\omega) \\ &= -i\pi \frac{(2\pi\lambda)^{-i\omega}}{\sinh\pi\omega} \exp\left[-\int_0^\infty dt \times \frac{(1-e^{-i\omega t})(1-\cos\omega t)}{t(1-\cosh t)}\right] \\ &= (2\pi\lambda)^{-i\omega}\Gamma(i\omega)\frac{G(i\omega+1)^3 G(2-i\omega)}{G(2i\omega+1)}.\end{aligned} \tag{21}$$

Notice that we will carefully rederive this formula in the next subsection. The singularities and zeroes of this function are listed in Appendix C. In particular, the zeroes are located at the imaginary axis, at $\omega = in$, where n is an integer, except at $n = 0, \pm 1$. Consider then the full one-point tachyon amplitude (the $n = 1$ case of (7))

$$\begin{aligned}\mathcal{A}_1(\omega) &= \delta_{0,\vec{k}} \int dx^0 \exp(i\omega x^0)\mathcal{A}_1(2\pi\lambda e^{x^0}) \\ &= \delta_{0,k}\hat{\mathcal{A}}_1(\omega).\end{aligned} \tag{22}$$

The momentum conservation condition $\vec{k} = 0$ along with $\omega^2 = -1 + \vec{k}^2$ demands $\omega = \pm i$ so that the amplitude involves the operator $\exp(\mp X^0)$. The result is

$$\mathcal{A}_1(\omega) = \delta_{0,\vec{k}}\tfrac{1}{2}(\pi\lambda)^{\pm 1}. \tag{23}$$

Note that the choice $\omega = -i$ is related to the disk partition function by

$$\hat{\mathcal{A}}_1(\omega = -i) = -\frac{1}{2\pi}\int dx^0 \frac{\partial}{\partial\lambda} Z_{\mathrm{disk},\lambda}(x^0) = \frac{1}{2\pi\lambda}, \tag{24}$$

where $Z_{\mathrm{disk},\lambda}(x^0) = \bar{\mathcal{A}}_0(x^0) = 1/(1+2\pi\lambda e^{x^0})$. Conversely, for $\omega \neq \pm i$, the on-shell condition requires $\vec{k} \neq 0$ so that the one-point amplitude vanishes. Even though the amplitude vanishes for generic ω, the expression (21) will be met again in the context of higher point amplitudes. It will be interesting to know its asymptotic behavior in the limit $|\omega| \to \infty$. It can be calculated to arbitrary order by using the asymptotics of Barnes G (20). The leading terms are

$$\hat{\mathcal{A}}_1(\omega) = -i\pi\frac{(2\pi\lambda)^{-i\omega}}{\sinh\pi\omega}\exp\left[\omega^2\left(\frac{i\pi}{2}\mathrm{sgn}(\mathrm{Re}\,\omega) + 2\log 2\right) - \frac{1}{4}\log(i\omega) - \frac{i\pi}{12}\mathrm{sgn}(\mathrm{Re}\,\omega) + \frac{1}{12}\log 2 + 3\zeta'(-1)\right][1+\mathcal{O}(\omega^{-2})], \tag{25}$$

where $\arg(\omega) \neq \pm\pi/2$.

B. A contour integral method

Next we calculate the integrated amplitude using a contour integration trick which allows us to sum the series over

[4] A similar observation has been made in the case of bulk amplitudes in spacelike Liouville theory in [25].

N in (7) and analytically continue the resulting amplitude to the region where the defining sum is not convergent. The essential required feature of the coefficients $I_\xi(N)$ is that they should not diverge too fast for large N. For concreteness and simplicity we will first consider the series (10) and (13). However, our method can also be applied to higher point functions as we will discuss in Sec. IV. More precisely, the calculation can be generalized to the case of the n-point amplitude (7) if we use a suitable approximate form for the integral coefficients $I_{\xi_1,\ldots,\xi_n}(N)$. As the contour integration method enables us to control the convergence of the sum and the integral it is more rigorous than the original calculation in [15,16].

We begin by studying the analytic structure of $J_\xi(s)$ of (18) and the asymptotics of $I_\xi(N)$ for large N. We will first consider the case where ξ is real and positive. Recall the continuation of the coefficient formula (14) to noninteger values of $N = -s$, given by (18). From the asymptotic formula of the Barnes G funtion (20) it immediately follows that $J_\xi(s)$ has a powerlike behavior for large s,

$$J_\xi(s) = \frac{G(\xi+1)^2}{G(2\xi+1)}(-s)^{\xi^2}\left[1+\mathcal{O}\left(\frac{1}{s}\right)\right]; \qquad \arg s \neq 0. \tag{26}$$

In addition, since $G(z+1)$ is an entire function with zeroes at $z = -1, -2, \ldots$, the poles of $J_\xi(s)$ are located at[5] $s = \xi+1, \xi+2, \ldots$.

Thus in the region $|z| < 1$, where the sum in (13) converges, the asymptotic behavior of J_ξ in (26) enables us to write the sum as

$$\bar{\mathcal{A}}_1(z) = \sum_{N=0}^{\infty}(-z)^N \cdot I_\xi(N) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{\pi z^{-s}}{\sin\pi s} J_\xi(s) ds, \tag{27}$$

where the contour $\mathcal{C}$ wraps around the negative real s axis as depicted in Fig. 1, picking up the residues at the poles of $1/\sin(\pi s)$ at $s = 0, -1, -2, \ldots$ which produce the terms in the series. Note that the zeroes of $G(1-s)$ in $J_\xi(s)$ cancel the poles of $1/\sin(\pi s)$ for $s = 1, 2, 3, \ldots$ Since $1/\sin(\pi s)$ vanishes exponentially for large imaginary s, we may deform the contour (keeping $|z| < 1$) in (27) to

$$\bar{\mathcal{A}}_1(z) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\pi z^{-s}}{\sin\pi s} J_\xi(s) ds, \tag{28}$$

where $0 < \gamma < \xi + 1$. This integral converges everywhere except for negative real z (if the principal branch of z^{-s} with $|\arg z| < \pi$ is used) and thus defines the analytic continuation of $\bar{\mathcal{A}}_1(z)$ to $|z| \geq 1$, $|\arg z| < \pi$. Moreover, for $|z| > 1$ we can continue to deform the contour to

$$\bar{\mathcal{A}}_1(z) = \frac{1}{2\pi i}\oint_{\mathcal{C}'} \frac{\pi z^{-s}}{\sin\pi s} J_\xi(s) ds, \tag{29}$$

[5]For $\xi = 1, 2, \ldots$ the poles are found at $s = 2\xi, 2\xi + 1, \ldots$.

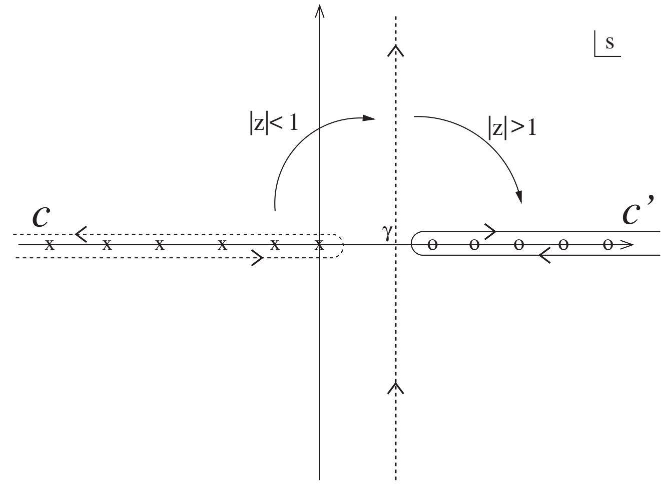

FIG. 1. The different integration contours on the s plane that define the analytic continuation of $\bar{\mathcal{A}}_1(z)$ for all values of z. Integration over the contours $\mathcal{C}$, $\gamma + i\mathbf{R}$, and $\mathcal{C}'$, converge for $|z| < 1$, $|\arg z| < \pi$, and $|z| > 1$, respectively. The x's and the o's denote the poles of $J_\xi(s)/\sin\pi s$.

where $\mathcal{C}'$ wraps around the positive real s axis as shown in Fig. 1. The integral is convergent for all $|z| > 1$ so there are no singularities in this region but a logarithmic branch cut ending at $z = \infty$ which arises from the factor z^{-s}. The residue contributions at the poles of $J_\xi(s)$ at $s = \xi + 1, \xi + 2, \ldots$ give the $1/z$ expansion

$$\bar{\mathcal{A}}_1(z) = (C_\xi + D_\xi \log z) z^{-\xi-1}\left[1+\mathcal{O}\left(\frac{1}{z}\right)\right], \tag{30}$$

where the constants C_ξ, D_ξ can be calculated using (18).

To summarize, from the different integral representations (27)–(29) it follows that the only singular points of $\bar{\mathcal{A}}_1(z)$ are $z = -1$ and $z = \infty$. In particular, on the integration path in the one-point amplitude

$$\hat{\mathcal{A}}_1(\xi) = \int_{-\infty}^{\infty} dx^0 e^{\xi x^0} \bar{\mathcal{A}}_1(x^0) \tag{31}$$

i.e., $z = 2\pi\lambda e^{x^0} = 0\ldots\infty$, $\bar{\mathcal{A}}_1(z)$ has no singularities. Using the series in (27) and in (30) we see that the integrand vanishes exponentially

$$e^{\xi x^0}\bar{\mathcal{A}}_1(x^0) \sim_{x^0\to\infty} e^{-x^0}, \qquad e^{\xi x^0}\bar{\mathcal{A}}_1(x^0) \sim_{x^0\to-\infty} e^{\xi x^0} \tag{32}$$

for large $\pm x^0$ so the integral over x^0 in (31) is convergent. Moreover, note that inserting the definition of z in (28) we find

$$\bar{\mathcal{A}}_1(x^0) = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\pi(2\pi\lambda)^{-s}}{2\pi i}\frac{e^{-sx^0}}{\sin\pi s} J_\xi(s) ds \tag{33}$$

which defines the inverse of the (bilateral) Laplace transform. The inverse relation then gives the master formula for the one-point amplitude in terms of $J_\xi(s)$,

$$\widetilde{\mathcal{A}}(s) \equiv \int_{-\infty}^{\infty} dx^0 e^{sx^0} \bar{\mathcal{A}}_1(x^0) = \frac{\pi(2\pi\lambda)^{-s}}{\sin\pi s} J_\xi(s). \quad (34)$$

The steps from (27) to (34) show how the analytic continuation of the coefficients of the series ends up as its sum. From the asymptotics of $\bar{\mathcal{A}}_1(x^0)$ we see that (34) converges for $0 < s < \xi + 1$, as expected from the positions of poles of $J_\xi(s)/\sin\pi s$ [and the choice of γ in (28)]. In particular,

$$\hat{\mathcal{A}}_1(\xi) = \widetilde{\mathcal{A}}(s=\xi) = \frac{\pi(2\pi\lambda)^{-\xi}}{\sin\pi\xi} J_\xi(\xi) = (2\pi\lambda)^{-\xi}\Gamma(\xi)\frac{G(1+\xi)^3 G(2-\xi)}{G(2\xi+1)} \quad (35)$$

reproducing the result (21) above.

In the end, we want to continue the result (35) for the integrated amplitude for imaginary $\xi = i\omega$. For imaginary ξ the above analysis is not essentially changed: the poles of $J_\xi(s)$ move to $s = i\omega + 1, i\omega + 2, \ldots$ but still lie to the right of the imaginary axis, so that $\bar{\mathcal{A}}_1(x^0)$ vanishes exponentially $\bar{\mathcal{A}}_1(x^0) \sim e^{-x^0}$ for $x^0 \to \infty$. However, after inserting $s = i\omega$ in (34) the convergence in the opposite direction $x^0 \to -\infty$ is lost. We find instead

$$e^{i\omega x^0}\bar{\mathcal{A}}_1(x^0) \sim_{x^0\to-\infty} e^{i\omega x^0} \quad (36)$$

which signals the presence of a δ function. Indeed, the integral can be interpreted as[6]

$$\hat{\mathcal{A}}_1(\omega) = \pi\delta(\omega) + (2\pi\lambda)^{-i\omega}\Gamma(i\omega) \times \frac{G(1+i\omega)^3 G(2-i\omega)}{G(2i\omega+1)} = (2\pi\lambda)^{-i\omega}\Gamma(i(\omega - i\epsilon))\frac{G(1+i\omega)^3 G(2-i\omega)}{G(2i\omega+1)}, \quad (37)$$

where the $i\epsilon$ changes the value of ω slightly to that direction where the x^0 integral is convergent.

C. The one-point function as a boundary term in spacetime

As discussed in [23], one difference between CFTs in compact and noncompact target spacetimes is that in the latter case boundary terms can spoil the holomorphicity of the stress tensor. This modifies its operator-product expansion (OPE) with other operators, and lead [23] to derive a master formula relating the one-point function (on a sphere or at the boundary of a disk) to a boundary term in spacetime, so as to give a string theoretic definition for a conserved charge, as an extension from field theory. For a disk one-point function, the master formula is

$$\langle \mathcal{O}(z,\bar{z})\rangle = \widetilde{\mathcal{N}} \int d^D x \partial_\mu \Big\{ \int_{D_2} d^2 z' e^{2\omega(z',\bar{z}')} \Big[\Big(\frac{z'+z}{2z}\Big)(z'-z) \times \langle \partial X^\mu(z',\bar{z}')\mathcal{O}(z,\bar{z})\rangle_{D_2} + \Big(\frac{\bar{z}'+\bar{z}}{2\bar{z}}\Big)(\bar{z}'-\bar{z}) \times \langle \bar\partial X^\mu(z',\bar{z}')\mathcal{O}(z,\bar{z})\rangle_{D_2} \Big]\Big\}, \quad (38)$$

where $\widetilde{\mathcal{N}}$ is a normalization factor, the metric on the disk is $ds^2 = e^{2\omega(z,\bar{z})}dzdz'$, and $\mathcal{O}(z,\bar{z})$ is a local boundary operator in the CFT with D-dimensional target space.

Reference [23] considered various applications where open or closed string background gauge fields or gravitational field were turned on. The open string rolling tachyon background gives a nice new nontrivial example to test the master formula (38). The world sheet action is nonpolynomial, and the master formula involves two-point functions in the interacting theory. We choose the local boundary operator to be the exponential, $\mathcal{O} = \exp\{i\omega X^0\}$, inserted at[7] $z = e^{i\tau}$. Its one-point amplitude is (21), which already is a (space)time integral. So we need to show that the integrand $\bar{\mathcal{A}}_1(x^0)$ can be rewritten as a total derivative as in (38). Our convention for the metric of the disk is $ds^2 = dwd\bar{w}$, so the relation to check is

$$\bar{\mathcal{A}}_1(x^0) = \frac{\partial\mathcal{B}(x^0)}{\partial x^0}, \quad (39)$$

where

$$\mathcal{B}(x^0) = \widetilde{\mathcal{N}} \int_{\rm disk} d^2 w \Big[\frac{w^2 - e^{2i\tau}}{2e^{i\tau}} \langle \partial X^0(w,\bar{w}) e^{i\omega X^0(\tau)}\rangle'_{\rm TBL} + \frac{\bar{w}^2 - e^{-2i\tau}}{2e^{-i\tau}} \langle \bar\partial X^0(w,\bar{w}) e^{i\omega X^0(\tau)}\rangle'_{\rm TBL}\Big], \quad (40)$$

where the primes indicate that we have separated the zero mode x^0. To show that this relation holds, we evaluate the right-hand side. The details of this calculation are relegated to Appendix A, in part because they involve a step that is discussed in the next Sec. III. The end result is that (39) holds, so that the one-point function is consistent with the general expectation from (38).

III. ON COULOMB GAS RELATION

The TBL is related to a statistical mechanical system, the Dyson gas of particles on a unit circle [26,27].[8] The key property is that the two-dimensional Green's functions can be interpreted as coming from two interacting Coulomb gas particles confined on a circle,

[6]The result can be checked explicitly by writing $\bar{\mathcal{A}}_1(x^0) = \bar{\mathcal{A}}_1(x^0)|_{\omega=0} + [\bar{\mathcal{A}}_1(x^0) - \bar{\mathcal{A}}_1(x^0)|_{\omega=0}]$ where the first term is simple to integrate and the latter does not contribute to the singularity.

[7]The one-point function is eventually independent of the location.

[8]The analogy has recently been extended to full S brane (or timelike boundary sine-Gordon theory) [28] and to non-BPS half S brane [29].

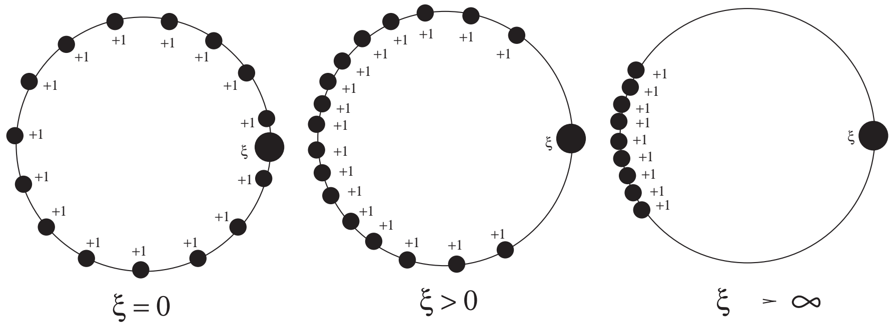

FIG. 2. Depicted is the interpretation of (48). On the unit circle, embedded in a heat bath, there are N positive unit charges and an additional positive charge ξ. As the charge strength ξ increases, the repulsive force acting on the unit charges wins over their mutual repulsion, forcing the unit charges closer to each other on the other side of the circle.

$$V(e^{it_i}, e^{it_j}) = -\log|e^{it_i} - e^{it_j}|, \tag{41}$$

where t_i, t_j are the respective angles. The perturbation expansion in λ of (3) becomes related to the grand canonical ensemble of unit charges on the circle,

$$Z_G = \sum_{N=0}^{\infty} \frac{z^N}{N!} \int \bigg[\prod_{i=1}^{N} \frac{dt_i}{2\pi}\bigg] e^{-\beta H}, \tag{42}$$

where the inverse temperature is fixed to $\beta = 2$, z is the fugacity, and $N!$ accounts for identical particles. The Hamiltonian contains only a potential energy term[9]

$$H = \sum_{\text{pairs}} V(t_i, t_j) = -\sum_{1\le i<j\le N} \log|e^{it_i} - e^{it_j}|. \tag{43}$$

In this paper we focus only on the canonical ensemble. As discussed in [27], correlators in TBL are related to adding additional particles into the ensemble. The one-point function (10) requires one additional particle with an arbitrary charge ξ at an angle τ. The Hamiltonian becomes

$$H_\xi = -\sum_{1\le i<j\le N} \log|e^{it_i} - e^{it_j}| - \xi \sum_{i\le i<j<N} \log|e^{i\tau} - e^{it_i}|, \tag{44}$$

and the canonical partition function is

$$Z_\xi = \frac{1}{N!} \int \frac{d\tau}{2\pi} \int \bigg[\prod_i \frac{dt_i}{2\pi}\bigg] e^{-\beta H_\xi} \tag{45}$$

$$= \frac{1}{N!} \int \frac{d\tau}{2\pi} \bigg[\prod_i \frac{dt_i}{2\pi}\bigg] \prod_{i<j} |e^{it_i} - e^{it_j}|^2 \prod_i |e^{i\tau} - e^{it_i}|^{2\xi}. \tag{46}$$

The integrand does not depend on the angle τ, hence it can be consistently set to zero. We recognize $Z_\xi = I_\xi(N)$ of (11).

[9]See the discussion on the physical interpretation in the original paper by F. Dyson [26].

We can now draw insight from the physical interpretation to better understand the integrals and their various extensions. As an example, consider the integral corresponding to the canonical ensemble expectation value

$$\bigg\langle \sum_{i=1}^{N} \cos(\tau - t_i) \bigg\rangle_{\text{can}} \equiv \frac{1}{Z_\xi} \cdot \frac{1}{N!} \int \prod_i \frac{dt_i}{2\pi} \prod_{i<j} |e^{it_i} - e^{it_j}|^2 \times \prod_i |e^{i\tau} - e^{it_i}|^{2\xi} \sum_i \cos(\tau - t_i), \tag{47}$$

which corresponds to the sum of the projected relative distances of the original charges to the additional charge. In part by inspired guesswork we have found a result

$$\bigg\langle \sum_{i=1}^{N} \cos(\tau - t_i) \bigg\rangle_{\text{can}} = -\frac{N\xi}{N + \xi} \tag{48}$$

for the integral. We have not constructed a proof for this formula, but have checked it for $\xi = 0, 1, 2, 3, 4$ and for any N, and a consistency check will be given in Appendix B.[10] We can visualize the $\xi \to \infty$ limit (at finite N) of the result (48) easily in Fig. 2: as the additional charge becomes stronger, it forces the unit charges further towards the antipodal point of the circle.

IV. THE *n*-POINT BOUNDARY AMPLITUDE

The full n-point amplitude (7) is very complicated and so are the integral coefficients (9) even at small N, $n(>1)$. In this section we will consider an approximation or a toy model version of a full calculation. We begin by studying the integrals (9). We interpret them as Toeplitz determinants. One can then consider a known approximation in the large N limit and try to improve it to be good enough to be used in the series expansion (7) at every N, while hoping for it to be simple enough so that the series can be summed. We use the Coulomb gas analogue and find a physically

[10]After the first version of this work was finished, we were informed by H. Schomerus that he has constructed a proof [30] of this formula. We thank him for bringing this to our attention.

motivated improved asymptotic approximation of (9). This approximation agrees with the previously known asymptotics at leading order in $1/N$, but reproduces the next-to-leading $1/N$ corrections to the asymptotics of the integrals better than the old result (but not exactly). Even more importantly, it is found to work well for small values of N even up to $N = 0$, which contribute significantly in the final amplitude in the end. However, the approximation is still simple enough to sum the series in N to calculate the integrated n-point amplitude. In the approximation, essentially the "interactions" between the ξ_a insertions can be neglected. In our end result, the n-point amplitude factorizes to a product of n independent one-point amplitudes. We also present a simple example that helps to understand and motivate the derivation in Appendix B.

A. Large N asymptotics

To start with, the t_i integrals in (9) can be done [15,16] giving

$$I_{\xi_1,\ldots,\xi_n}(N) = \int \prod_{a=1}^{n} \frac{d\tau_a}{2\pi} \bigg[\prod_{1\le a<b\le n} |e^{i\tau_a} - e^{i\tau_b}|^{2\xi_a\xi_b + 2\vec{k}_a\cdot\vec{k}_b} \bigg] \times \det T_N[f], \tag{49}$$

where $\det T_N[f]$ is the $N \times N$ Toeplitz determinant of Fourier coefficients of the function

$$f_{\tau_1,\ldots,\tau_n}(t) = \prod_{a=1}^{n} |e^{i\tau_a} - e^{it}|^{2\xi_a}, \tag{50}$$

see [15,16] for more details.

The determinant is too complicated to allow us to sum the series (7). However, Toeplitz determinants are known to simplify at large N. In particular, the large N asymptotics of the determinant $\det T_N[f]$ is known for (50). It reads [31,32] (see also [33])

$$\det T_N[f] = N^{\sum_{a=1}^{n}\xi_a^2} \prod_{1\le a<b\le n} |e^{i\tau_a} - e^{i\tau_b}|^{-2\xi_a\xi_b} \times \prod_{a=1}^{n} \frac{G(\xi_a+1)^2}{G(2\xi_a+1)} \bigg[1 + \mathcal{O}\bigg(\frac{1}{N}\bigg)\bigg]. \tag{51}$$

Moreover, the asymptotic behavior of (49) factorizes

$$\begin{aligned} T_{\xi_1,\ldots,\xi_n}(N) &\equiv \det T_N[f] \prod_{1\le a<b\le n} |e^{i\tau_a} - e^{i\tau_b}|^{2\xi_a\xi_b} \\ &= \prod_{a=1}^{n} N^{\xi_a^2} \frac{G(\xi_a+1)^2}{G(2\xi_a+1)} \bigg[1 + \mathcal{O}\bigg(\frac{1}{N}\bigg)\bigg] \\ &= \prod_{a=1}^{n} \frac{G(\xi_a+1)^2}{G(2\xi_a+1)} \frac{G(N+2\xi_a+1)G(N+1)}{G(N+\xi_a+1)^2} \times \bigg[1 + \mathcal{O}\bigg(\frac{1}{N}\bigg)\bigg] \\ &= \prod_{a=1}^{n} T_{\xi_a}(N) \bigg[1 + \mathcal{O}\bigg(\frac{1}{N}\bigg)\bigg], \end{aligned} \tag{52}$$

where $T_{\xi_1,\ldots,\xi_n}(N)$ is the asymptotically τ_a independent factor of $\det T_N[f]$ and we used (20) to write the asymptotics in terms of Barnes G functions (see also (26)). Here $T_\xi(N) = I_\xi(N)$ is the one-point function discussed above in Sec. II.

The asymptotic formula (52) has a nice physical interpretation in terms of the classical Coulomb gas on a circle, where $T_{\xi_1,\ldots,\xi_n}(N)$ is the partition function for N identical unit charges at the inverse temperature $\beta = 2$, with n additional particles having charges $\xi_1, \ldots, \xi_n$ at fixed angles $\tau_1, \ldots, \tau_n$. Let us assume for a moment that all ξ_a are positive integers.[11] Then each particle with charge ξ_a can be thought to be a cluster of ξ_a unit charges. We can then imagine constructing a typical configuration of the gas with the n test charges, from a gas of $N + \sum_a \xi_a$ unit charges, by clustering unit charges at distinct locations to form the test charges ξ_a. For $n < \sum_{a=1}^{n} \xi_a \ll N$, the typical separation of unit charges is $\sim 1/N$, much less than the typical separation between the test charges/charge clusters. Now, we can first interpret the $N^{\xi_a^2}$ factors in (52) arising from the self-energies of the charge clusters. For a cluster with charge ξ_a, the self-energy is given by[12]

$$E_{\rm self} = -\sum_{1\le i<j\le \xi_a} \log|x_i - x_j| \simeq \sum_{1\le i<j\le \xi_a} \log N \simeq \frac{\xi_a^2}{2} \log N \tag{53}$$

giving the contribution

$$e^{2E_{\rm self}} \sim N^{\xi_a^2} \tag{54}$$

to the partition function. Second, the factorization of (52) can be understood as the absence of intercluster interactions at this level of approximation. A heuristic argument could be the following. Consider a large number of unit charges on the real axis (a piece of the unit circle after magnification) with a typical separation $d \simeq 2\pi/N$. Choose $\xi \ll N$ charges at $x_1, \ldots, x_\xi$ near the origin (so that $x_\xi \sim d$) and perturb their locations, $x_k \to x_k - \delta_k$, by $\delta_k \sim d$ symmetrically such that $\sum_{k=1}^{\xi} \delta_k = 0$, to create a cluster of charge ξ. The change in the electrostatic potential after creating the cluster is then

$$\Delta V(x) = -\sum_{k=1}^{\xi} \log\bigg[1 + \frac{\delta_k}{x - x_k}\bigg] \tag{55}$$

as felt at point x outside the cluster, $x > x_k$. For $x \gg d$ we find that the deformation of the potential vanishes rapidly,

[11] This is potentially a dangerous assumption, since eventually we want to set $\xi_a = i\omega_a$ where typically ω_a are real, and naive continuation from integers to complex plane is known to be problematic—see, e.g., the discussion in [15]. We will return to this issue in the end of the section.

[12] In fact, one obtains $E_{\rm self} \simeq \xi_a(\xi_a - 1)\log N/2$, but the term $-\xi_a \log N/2$ cancels against a change in the $1/N!$ factors which are discussed below.

$$\Delta V \sim \xi\left(\frac{d}{x}\right)^2, \tag{56}$$

and the contribution to the change in the total energy in the leading order must thus come from the interaction of the cluster between the unit charges within the region $x \sim d \sim 1/N$. However, at the distance to the neighboring clusters, the change is negligible, so the intercluster interactions are suppressed.

The analysis suggests a natural way to try to improve the asymptotic formula (52). As explained above, in (52) intercluster interactions are absent. However, the $1/N$ corrections due to these interactions can be added almost completely in a very simple manner. Naturally, each of the ξ_a charges must feel the Coulomb force of the N unit charges *and* the other clusters with ξ_b charges, $a \neq b$. An easy modification of (52) to accommodate these is the following. Increase the number of background unit charges acting on the ξ_a cluster from N to $\tilde{N}_a = N + \sum_{b\neq a}\xi_b$ in the asymptotic formula (52). This indeed replaces the Coulomb force of each ξ_b charge as a cluster of ξ_b separate unit charges. The total effective number of unit charges in the gas then becomes $\tilde{N} = \tilde{N}_a + \xi_a = N + \sum_{a=1}^n \xi_a$. We write the improved asymptotic formula for a renormalized Toeplitz determinant $\hat{T}_{\xi_1,\ldots,\xi_n}(N)$, which is simply related to $T_{\xi_1,\ldots,\xi_n}(N)$ of (52). The modification is needed since $T_{\xi_1,\ldots,\xi_n}(N)$ contains the normalization factor $1/N!$ which we want to be replaced by $1/\tilde{N}!$:

$$\begin{aligned}\hat{T}_{\xi_1,\ldots,\xi_n}(N) &= \frac{N!}{\Gamma(\tilde{N}+1)}T_{\xi_1,\ldots,\xi_n}(N)\\ &= \frac{1}{\Gamma(\tilde{N}+1)}\Big[\prod_{1\le a<b\le n}|e^{i\tau_a}-e^{i\tau_b}|^{2\xi_a\xi_b}\Big]\\ &\quad\times\int\prod_{i=1}^{N}\frac{dt_i}{2\pi}\Big[\prod_{1\le i<j\le N}|e^{it_i}-e^{it_j}|^2\Big]\\ &\quad\times\Big[\prod_{i=1}^{N}\prod_{a=1}^{n}|e^{i\tau_a}-e^{it_i}|^{2\xi_a}\Big].\end{aligned} \tag{57}$$

Following the discussion above, we replace the asymptotic formula (52) by an improved formula for (57),

$$\hat{T}_{\xi_1,\ldots,\xi_n}(N) = \prod_{a=1}^{n}\hat{T}_{\xi_a}(\tilde{N}_a)\Big[1+\mathcal{O}\Big(\frac{1}{N}\Big)\Big], \tag{58}$$

where

$$\begin{aligned}\hat{T}_{\xi_a}(\tilde{N}_a) &= \frac{\Gamma(\tilde{N}_a+1)}{\Gamma(\tilde{N}+1)}T_{\xi_a}(\tilde{N}_a)\\ &= \frac{1}{\Gamma(\tilde{N}+1)}\int\prod_{i=1}^{\tilde{N}_a}\frac{dt_i}{2\pi}\Big[\prod_{1\le i<j\le\tilde{N}_a}|e^{it_i}-e^{it_j}|^2\Big]\\ &\quad\times\Big[\prod_{i=1}^{\tilde{N}_a}|1-e^{it_i}|^{2\xi_a}\Big]\end{aligned} \tag{59}$$

is the properly normalized partition function for an ξ_a charge in the background of $\tilde{N}_a$ unit charges. Inverting the relation (57), we can rewrite (58) as an improved approximation for $T_{\xi_1,\ldots,\xi_n}$,

$$\begin{aligned}T_{\xi_1,\ldots,\xi_n}(N) &\approx T_{\rm norm}\prod_{a=1}^{n}T_{\xi_a}(\tilde{N}_a)\\ &= T_{\rm norm}\prod_{a=1}^{n}\frac{G(\tilde{N}+\xi_a+1)G(\tilde{N}-\xi_a+1)}{G(\tilde{N}+1)^2}\\ &\quad\times\frac{G(\xi_a+1)^2}{G(2\xi_a+1)}\\ &\equiv T^{\rm aprx}_{\xi_1,\ldots,\xi_n}(N),\end{aligned} \tag{60}$$

where we introduced the notation $T^{\rm aprx}_{\xi_1,\ldots,\xi_n}(N)$ for the improved asymptotics and the normalization factor reads

$$T_{\rm norm} = \frac{\prod_{a=1}^{n}\Gamma(\tilde{N}_a+1)}{\Gamma(\tilde{N}+1)^{n-1}N!}. \tag{61}$$

Note that $T^{\rm aprx}_{\xi_1,\ldots,\xi_n}$ reduces to (52) for $N\to\infty$ and still has $1/N$ corrections, but they are expected to be essentially smaller than for (52). In Appendix B we discuss the simplest nontrivial example $(\xi_1,\xi_2)=(2,2)$, where the exact results are known [34,35] and find that the improved asymptotics (64) reduces the deviation from the exact result by more than an order of magnitude at large N. The new asymptotics continues to be a very good approximation to the exact result even for small values of N. Moreover, note that setting, e.g., $\xi_n = 1$ in (64) correctly reproduces $T^{\rm aprx}_{\xi_1,\ldots,\xi_{n-1}}(N+1)$.

Finally, we collect our results in a new asymptotic approximation for the Toeplitz determinant:

$$\begin{aligned}\det T_N[f] \approx &\prod_{1\le a<b\le n}|e^{i\tau_a}-e^{i\tau_b}|^{-2\xi_a\xi_b}\frac{\Gamma(N+\sum_a\xi_a+1)}{\Gamma(N+1)}\\ &\cdot\prod_{a=1}^{n}\frac{\Gamma(N-\xi_a+\sum_b\xi_b+1)}{\Gamma(N+\sum_b\xi_b+1)}\frac{G(\xi_a+1)^2}{G(2\xi_a+1)}\\ &\cdot\frac{G(N+\xi_a+\sum_b\xi_b+1)G(N-\xi_a+\sum_b\xi_b+1)}{G(N+\sum_b\xi_b+1)^2}.\end{aligned} \tag{62}$$

If we substitute this to (49), we note that the integrals over τ_a give

$$\begin{aligned}I_{\xi_1=0,\ldots,\xi_n=0}(N) &= \int\prod_{a=1}^{n}\frac{d\tau_a}{2\pi}\Big[\prod_{1\le a<b\le n}|e^{i\tau_a}-e^{i\tau_b}|^{2\vec{k}_a\cdot\vec{k}_b}\Big]\\ &\equiv \mathcal{N}[(\vec{k}_a)],\end{aligned} \tag{63}$$

and the result for the integral becomes

$$I_{\xi_1,\ldots,\xi_n}(N) \approx I^{\rm aprx}_{\xi_1,\ldots,\xi_n}(N) \equiv \mathcal{N}[(\vec{k}_a)]\frac{\Gamma(N+\sum_a \xi_a+1)}{\Gamma(N+1)} \cdot \prod_{a=1}^n \frac{\Gamma(N-\xi_a+\sum_b \xi_b+1)}{\Gamma(N+\sum_b \xi_b+1)}\frac{G(\xi_a+1)^2}{G(2\xi_a+1)} \cdot \frac{G(N+\xi_a+\sum_b \xi_b+1)G(N-\xi_a+\sum_b \xi_b+1)}{G(N+\sum_b \xi_b+1)^2}. \tag{64}$$

The k_a^I dependence thus completely factorizes into the normalization factor $\mathcal{N}$.

There is, however, a caveat in the above derivation: the result (64) only makes sense when the normalization integral $\mathcal{N}$ is convergent. From the definition (63) we see that the integral is singular whenever any of the products $\vec{k}_a \cdot \vec{k}_b \to -1/2$, which can easily occur for physical momentum values. These singularities are unphysical and they are absent in the original integral of (9). What happens is that for $\vec{k}_a \cdot \vec{k}_b \to -1/2$ the τ integrals become heavily peaked at $\tau_a \simeq \tau_b$. More precisely, the dominant contribution to the integral comes from the region where $\tau_b - \tau_a \sim 1/N$. In this region the large N limit does not reproduce the τ dependence correctly: for positive $\xi_a\xi_b$ the integrand vanishes more rapidly than suggested by the large N limit as $\tau_a - \tau_b \to 0$, which creates a cutoff for the normalization integral $\mathcal{N}$.

To avoid this caveat we shall assume that $\vec{k}_a \cdot \vec{k}_b > -1/2$ which can be satisfied together with momentum conservation only if all spatial momenta are small. Even when this condition is not met, the result (64) may work as a reasonable model for the ξ and N dependencies of (49). We are planning to study the τ dependence of the Toeplitz determinant more closely in a forthcoming publication.

B. A model amplitude

Let us now study what can be said about the integrated amplitude

$$\mathcal{A}_n = \delta_{0,\sum_a \vec{k}_a}\int dx^0 \exp\Big[x^0\sum_{a=1}^n \xi_a\Big]\bar{\mathcal{A}}_n(2\pi\lambda e^{x^0}) = \delta_{0,\sum_a \vec{k}_a}\int dx^0 \exp\Big[x^0\sum_{a=1}^n \xi_a\Big]\sum_{N=0}^\infty (-z)^N I_{\xi_1,\ldots,\xi_n}(N) \tag{65}$$

based on the asymptotic formula (64). Notice that since the coefficients $I_{\xi_1,\ldots,\xi_n}(N)$ are asymptotically equal to a product of one-point functions they also exhibit a powerlike behavior for large N [see (26)]. This fact strongly suggests that the analysis of Subsection II B can be extended to higher point functions, which requires that there is an analytic continuation $J_{\xi_1,\ldots,\xi_n}(s)$ of $I_{\xi_1,\ldots,\xi_n}(N)$ to complex values of $N = -s$ that has a powerlike behavior for $s \to \infty$ in all sectors of the complex s plane. At least, as we shall see below, the continuation exists for the asymptotic formula (64) (and also (52)). Also, we calculated $I_{\xi_1,\ldots,\xi_n}(N)$ for sets of small positive integers ξ_a and for $\vec{k}_a \cdot \vec{k}_b = 0$ in [34], and found that they are, in fact, polynomials of N. See also Appendix B where we treat a simple case, $(\xi_1, \xi_2) = (2, 2)$, as an example.

This motivates us to check what is the result if one simply inserts the improved asymptotic formula (64) to (7) and to repeat the analysis of Subsection II B. For the sake of concreteness, we discuss the two-point function.[13] As explained above, it is required that there is such an analytic continuation of $I^{\rm aprx}_{\xi_1,\xi_2}(N)$ of (64) to complex $s = -N$ that does not blow up exponentially for $|s| \to \infty$. Remarkably, the simplest continuation of (64) works:

$$J^{\rm aprx}_{\xi_1,\xi_2}(s) = \frac{\mathcal{N}[\vec{k}_1]}{\Gamma(1-s)\Gamma(\xi_1+\xi_2-s+1)}\prod_{a=1}^2 \frac{G(\xi_a+1)^2}{G(2\xi_a+1)} \times \frac{G(-s+\xi_1+\xi_2+\xi_a+1)G(2-s+\xi_a)}{G(-s+\xi_1+\xi_2+1)^2} \tag{66}$$

indeed has a powerlike asymptotic behavior for large $|s|$ as can be verified using the formulae (64) and (52) above. Further, we need to check that the singularities of $J^{\rm aprx}_{\xi_1,\xi_2}(s)$ do not conflict with the contour deformations of Subsection II B. If $\xi_{1,2} > 0$ the poles of $J^{\rm aprx}_{\xi_1,\xi_2}(s)$ are located at $s = \xi_1 + \xi_2 + 1, \xi_1 + \xi_2 + 2, \ldots$ As for the one-point amplitude in Subsection II B, they are to the right of $s = \xi_1 + \xi_2$, where we will evaluate $J^{\rm aprx}_{\xi_1,\xi_2}(s)$ in the end [see (68) below]. As discussed in Subsection II B, this means that the model two-point function

$$\bar{\mathcal{A}}_2^{\rm aprx}(z) = \sum_{N=0}^\infty (-z)^N I^{\rm aprx}_{\xi_1,\xi_2}(N) \tag{67}$$

has no singularities for $z > 0$ and vanishes sufficiently fast for $x^0 \to \infty$ to make the integral over x^0 in (65) convergent. Notice that this is not the case if the "naive" asymptotic formula (52) is used instead of (64).[14] In particular, as discussed in Appendix B, (67) has the correct asymptotic behavior for $z \to +\infty$ for integer (ξ_1, ξ_2) only if one uses (64).

The above checks ensure that following the analysis in Subsection II B (see Eqs. (34), (35), and (37)) we can sum and integrate the approximated integrals $I^{\rm aprx}_{\xi_1,\xi_2}$ of (64). The result is a model two-point amplitude,

[13]For the two-point function spatial momentum conservation and on-shell conditions actually fix $\omega_1 = \omega_2$.

[14]However, also the naive formula leads to a well-defined integral for imaginary $\xi_a = i\omega_a$ which we will need in the end.

$$\mathcal{A}_2 \approx \delta_{0,\vec{k}_1+\vec{k}_2} \frac{-i\pi(2\pi\lambda)^{-i(\omega_1+\omega_2)}}{\sinh\pi(\omega_1+\omega_2)} J^{\text{aprx}}_{i\omega_1,i\omega_2}(i\omega_1+i\omega_2)$$
$$= \delta_{0,\vec{k}_1+\vec{k}_2} \mathcal{N}[\vec{k}_1](2\pi\lambda)^{-i(\omega_1+\omega_2)}\Gamma(i\omega_1+i\omega_2) \times \prod_{a=1}^{2} \frac{G(i\omega_a+1)^3 G(2-i\omega_a)}{G(2i\omega_a+1)}, \quad (68)$$

where we already rotated to imaginary $\xi_a = i\omega_a$ and omitted a δ-term. Notice the similarity to (21) which stems from the factorized form of the asymptotics (64).

Similarly as the one-point amplitude in (37), the final result (68) is expected to include a term $\propto \delta(\omega)$. The delta term arises in the x^0 integration of $\bar{\mathcal{A}}_2(x^0)$ from the oscillations in the region $x^0 \to -\infty$: indeed, for imaginary $\xi = \xi_1 + \xi_2 = i\omega_1 + i\omega_2$ the integrand $e^{i(\omega_1+\omega_2)x^0}\bar{\mathcal{A}}_2(x^0)$ continues to vanish exponentially for $x^0 \to +\infty$, but for $x^0 \to -\infty$ the function $\bar{\mathcal{A}}_2$ approaches a constant, which leads to oscillating behavior. The resulting δ contribution can be isolated as follows. Write

$$\mathcal{A}_2 = \delta_{0,\vec{k}_1+\vec{k}_2} \int dx^0 e^{i(\omega_1+\omega_2)x^0} \bar{\mathcal{A}}^{\text{aprx}}_2(x^0)$$
$$= \delta_{0,\vec{k}_1+\vec{k}_2} \Big\{ I^{\text{aprx}}_{i\omega_1,i\omega_2}(N=0) \int dx^0 e^{i(\omega_1+\omega_2)x^0} \bar{\mathcal{A}}_1(x^0)|_{\omega=0} + \int dx^0 e^{i(\omega_1+\omega_2)x^0} [\bar{\mathcal{A}}^{\text{aprx}}_2(x^0) - I^{\text{aprx}}_{i\omega_1,i\omega_2}(N=0) \times \bar{\mathcal{A}}_1(x^0)|_{\omega=0}] \Big\}, \quad (69)$$

where $\bar{\mathcal{A}}_1(x^0)|_{\omega=0} = \bar{\mathcal{A}}_0(x^0) = 1/(1+2\pi\lambda e^{x^0})$. Then the integrand of the first term has a simple form and oscillates for $x^0 \to -\infty$ while that of the second one is complicated but vanishes exponentially in both directions $x^0 \to \pm\infty$. Hence the δ contribution comes solely from the first term which can be integrated exactly, while for the second integral is well defined even for imaginary $\xi_1 + \xi_2$ and the analytic continuation of (68) from the region of $\text{Re}(\xi_1+\xi_2) > 0$ can be trusted. The δ-term that adds to (68) is seen to be[15]

$$\mathcal{A}_{2,\delta} = \pi\delta_{0,\vec{k}_1+\vec{k}_2} I^{\text{aprx}}_{i\omega_1,i\omega_2}(N=0)\delta(\omega_1+\omega_2). \quad (70)$$

Naively, since we effectively replace $N \to -i(\omega_1+\omega_2)$ in the asymptotic expansion (64), one would expect the improved asymptotic formula (68) to be a good estimate for large energies for which $1/N$ is small. Unfortunately, the correction to (64) is likely to include terms which are $\sim\omega_a/N$ (or worse) and become (at least) $\mathcal{O}(1)$ at $N \sim \omega_a$. Thus the result (68) only serves as a model of the exact result for any values of the energies. However, note that the calculations of Appendix B suggest that the correction terms are small: the improved approximation is seen to work well also for small values of N and also when continued to negative (but small) values of N.

It is straightforward to check that the analysis of Subsection II B can be similarly applied to the approximate asymptotic formula (64) for $n > 2$. The consequent generalization of (68) can be simplified to read

$$\mathcal{A}_n \approx \delta_{0,\sum_a \vec{k}_a} \mathcal{N}[(\vec{k}_a)](2\pi\lambda)^{-i\sum_a \omega_a} \Gamma\Big(i\sum_a \omega_a\Big) \times \prod_{a=1}^{n} \frac{G(i\omega_a+1)^3 G(2-i\omega_a)}{G(2i\omega_a+1)}. \quad (71)$$

As discussed in Subsection IVA the $\vec{k}_a$ dependence factorizes into the factor $\mathcal{N}[(\vec{k}_a)]$ at least for small $\vec{k}_a$. We give the singularities and the asymptotic behavior of the model amplitude in Appendix C.

Notice that our approximation is not restricted to only positive integer valued ξ_a. The original asymptotic formula (52) is valid (see [33]) for $\text{Re}\xi_a > -1/2$. While our improved formula (64) was motivated using integer ξ_a, it approaches the original one (52) at $N \to \infty$ for any set of (complex) ξ_a's. In Appendix B it was demonstrated that the improved asymptotics works much better than (52) for sets of integer ξ_a. There is no reason to believe why it should fail to be an improvement also for imaginary $\xi_a = i\omega_a$.

Final comments. We conclude with some final thoughts. We have presented a method to calculate n-point boundary functions. It would be important to develop similar methods for n-point bulk correlators. The main obstacle for a straightforward generalization of our calculations is the following. The boundary operators correspond to test charges that we constructed from unit charges of the Dyson gas. However, the bulk operators cannot similarly be made of the unit charges on the boundary—a different trick must be found for the bulk correlation function calculations. Another important issue is to develop a clear estimate how good an estimate (C5) is for the amplitude. A promising way to test our method would be to work directly in spacelike boundary Liouville theory, use our method to compute the boundary two-point function, and then compare with the exact known result of [17–19].

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[15]Naturally, a similar term also appears in the exact amplitude and is given by eliminating the superscript "aprx" of I in (70).

APPENDIX A: ON KRS RELATION

In this appendix we check (39). Following [23], Eq. (4.7), we need to calculate

$$\mathcal{B} = \int_{\rm disk} d^2w\left[\frac{w^2 - e^{2i\tau}}{2e^{i\tau}}\langle\partial X^0(w,\bar w)e^{i\omega X^0(\tau)}e^{-S_{\rm bdry}}\rangle' + \frac{\bar w^2 - e^{-2i\tau}}{2e^{-i\tau}}\langle\bar\partial X^0(w,\bar w)e^{i\omega X^0(\tau)}e^{-S_{\rm bdry}}\rangle'\right], \quad (A1)$$

where the primes of the expectation values indicate that the zero mode x^0 is left unintegrated. We start from (straightforward use of Wick theorem)

$$\mathcal{C} = \left\langle e^{i\omega_c X^0(w,\bar w)}e^{i\omega X^0(\tau)}\prod_{i=1}^N e^{X^0(t_i)}\right\rangle' = |1 - w\bar w|^{-\omega_c^2/2}|w - e^{i\tau}|^{-2\omega\omega_c}\prod_{i<j}|e^{it_i} - e^{it_j}|^2 \times \prod_i |e^{i\tau} - e^{it_i}|^{2i\omega}|w - e^{it_i}|^{2i\omega_c} \quad (A2)$$

which is to be integrated over $t_i = 0\ldots 2\pi$ and summed over N. Notice that

$$\left\langle\partial X^0(w,\bar w)e^{\xi X^0(\tau)}\prod_i e^{X^0(t_i)}\right\rangle' = -i\frac{\partial^2}{\partial w\partial\omega_c}\mathcal{C}|_{\omega_c=0} = \left[\sum_i\frac{1}{w - e^{it_i}} + \frac{\xi}{w - e^{i\tau}}\right]\prod_{i<j}|e^{it_i} - e^{it_j}|^2 \times \prod_i|e^{i\tau} - e^{it_i}|^{2\xi} \quad (A3)$$

and similarly for the term containing $\bar\partial$ in (A1). Recall that $\xi = i\omega$. Let us do the w integration first. The w-dependent part reads

$$I_w = \int d^2w\left[\frac{w^2 - e^{2i\tau}}{2e^{i\tau}}\left(\sum_i\frac{1}{w - e^{it_i}} + \frac{\xi}{w - e^{i\tau}}\right) + \text{H.c.}\right], \quad (A4)$$

where the H.c. assumes real ξ. Developing the integrand at $w, \bar w = 0$ we see that only the constant term survives,

$$I_w = \pi\xi + \frac{\pi}{2}\sum_i(e^{i\tau - it_i} + e^{-i\tau + it_i}) = \pi\left[\xi + \sum_i\cos(\tau - t_i)\right]. \quad (A5)$$

Let us then do the t_i integrations. The integral $\propto \xi$ is the $I_\xi(N)$ discussed above, and for the second term we use (48). Putting the results together,

$$\mathcal{B} = \pi e^{\xi x^0}\sum_N(-z)^N\left[\xi - \frac{N\xi}{N + \xi}\right]I_\xi(N) = \pi\xi^2 e^{\xi x^0}\sum_N\frac{(-z)^N}{N + \xi}I_\xi(N) = \frac{\pi\xi^2}{z^\xi}\int_0^z dz'(z')^{\xi - 1}\bar{\mathcal{A}}_1(z'). \quad (A6)$$

Moreover, the x^0 dependencies of the $1/z^\xi$ and the $e^{\xi x^0}$ exactly cancel, whence after derivating with respect to x^0 [23]

$$\frac{\partial\mathcal{B}}{\partial x^0} = \pi\xi^2\bar{\mathcal{A}}_1. \quad (A7)$$

We have thus checked the formula (3.14) of [23] in this special case. The ξ^2 in the proportionality constant arises from the conformal dimension of the operator $e^{\xi X^0}$, which is included in the normalization factor $\widetilde{\mathcal{N}}$ of Subsection II C.

APPENDIX B: A SPECIAL CASE OF THE *n*-POINT AMPLITUDE

To clarify the involved derivation of the model for the n-point amplitude in Sec. IV, we consider here the simplest nontrivial example, $(\xi_1, \xi_2) = (2, 2)$ and $\vec k_1 \cdot \vec k_2 = 0$,[16] that can be calculated also exactly. We have also checked other cases of sets of small integers, and found similar results.

Let us start with the results for the integral $I_{2,2}(N)$ of (9) which equals $I_2(N, 2)/N!$ of [34]. Hence we have

$$\begin{aligned} I_{2,2}(N) &= \frac{2}{8!}\left[35\frac{(N+8)!}{N!} + 77\frac{(N+7)!}{(N-1)!} + 27\frac{(N+6)!}{(N-2)!} + \frac{(N+5)!}{(N-3)!}\right] \\ &= \frac{35N^3 + 467N^2 + 2046N + 2940}{5040}\prod_{k=1}^5(N+k)I_{2,2}^{\rm asymp}(N) = \left[\frac{(N+3)(N+2)^2(N+1)}{12}\right]^2 \\ &= \frac{(N+3)^2(N+2)^4(N+1)^2}{144}I_{2,2}^{\rm aprx}(N) = \frac{(N+2)(N+1)}{(N+4)(N+3)}\left[\frac{(N+5)(N+4)^2(N+3)}{12}\right]^2 \\ &= \frac{(N+5)(N+4)^2}{144}\prod_{k=1}^5(N+k), \end{aligned} \quad (B1)$$

[16] Notice that the condition $\vec k_1 \cdot \vec k_2 = 0$ eliminates all dependence on spatial momentum. It actually conflicts with momentum conservation.

where the first, the second, and third formula are the exact result, the one obtained from the asymptotic formula (52), and the improved asymptotic formula (64), respectively. For $N \to \infty$ we find

$$\frac{I_{2,2}^{\rm asymp}(N)}{I_{2,2}(N)} = 1 - \frac{432}{35N} + \frac{142\,384}{1225N^2} + \mathcal{O}\left(\frac{1}{N^3}\right)\frac{I_{2,2}^{\rm aprx}(N)}{I_{2,2}(N)} = 1 - \frac{12}{35N} + \frac{2594}{1225N^2} + \mathcal{O}\left(\frac{1}{N^3}\right). \quad \text{(B2)}$$

Some of the values of the integrals are tabulated in Table I. The improved formula works much better, in particular, for low values of N. For higher values of ξ_a the improvement is even more drastic, basically since the difference between the effective number of unit charges $\tilde{N} = N + \sum_{a=1}^{n} \xi_a$, which is used in the improved asymptotics, and the actual number of unit charges N increases.

The two-point function

$$\bar{\mathcal{A}}_2(x^0)|_{\xi_1=\xi_2=2} = \sum_{N=0}^{\infty} (-z)^N I_{2,2}(N), \quad \text{(B3)}$$

where $z = 2\pi\lambda e^{x^0}$, can be calculated explicitly for all the results (B1). In particular, for large x^0 we have

$$\bar{\mathcal{A}}_2(x^0) = -\frac{2}{(2\pi\lambda)^6}e^{-6x^0} + \frac{72}{(2\pi\lambda)^7}e^{-7x^0} + \mathcal{O}(e^{-8x^0}),$$
$$\bar{\mathcal{A}}_2^{\rm asymp}(x^0) = -\frac{1}{(2\pi\lambda)^4}e^{-4x^0} + \frac{36}{(2\pi\lambda)^5}e^{-5x^0} + \mathcal{O}(e^{-6x^0}),$$
$$\bar{\mathcal{A}}_2^{\rm aprx}(x^0) = -\frac{10}{3(2\pi\lambda)^6}e^{-6x^0} + \frac{90}{(2\pi\lambda)^7}e^{-7x^0} + \mathcal{O}(e^{-8x^0}). \quad \text{(B4)}$$

The improved asymptotic formula produces also the large x^0 asymptotics nicely: $\bar{\mathcal{A}}_2^{\rm aprx}$ is correct up to the proportionality constant for $x^0 \to \infty$.

The analytic continuation of $I_{2,2}(N)$, $J_{2,2}(s)$ is found by letting $N \to -s$ in (B1). The result for the integrated amplitude is then obtained by applying (68) to the three different cases of (B1), which gives

$$\mathcal{A}_2|_{\xi_1=\xi_2=2} = \lim_{s\to 4}\frac{\pi(2\pi\lambda)^{-4}}{\sin\pi s}J_{2,2}(s) = -\frac{1}{70(2\pi\lambda)^4}, \infty, 0, \quad \text{(B5)}$$

where the first, the second, and the third numbers are the exact result, the result for the naive asymptotic formula (52), and the result for the improved formula (64), respectively. Both the asymptotic approximations give an incorrect result by an infinite factor. However, the numerical factor $-1/70$ of the exact result is extremely small when compared, e.g., to the series coefficients of Table I, whence the zero result obtained by the improved asymptotic formula should be, in fact, considered as a good approximation. It would be interesting to be able to compare our model amplitude to the exact result for more physical, noninteger values of ξ_a where no accidental zeroes or infinities are expected to occur.

We end this appendix by an encouraging observation in a more general setup. It is straightforward to check that, in fact, for any sets of integer $(\xi_1, \ldots \xi_n)$ the expected exact asymptotic behavior [34]

$$\bar{\mathcal{A}}_n(x^0) \sim \exp\left[-x^0\sum_a \xi_a - x^0 \max\{\xi_a\}\right], \quad \text{(B6)}$$

is reproduced by $\bar{\mathcal{A}}_n^{\rm aprx}$ similarly as for $(\xi_1, \xi_2) = (2, 2)$ in (B4). We denote the analytic continuation of $I_{\xi_1,\ldots\xi_n}^{\rm aprx}(N)$ of (64) to complex $s = -N$ by $J_{\xi_1,\ldots\xi_n}^{\rm aprx}(s)$ in analogue to (18), (66) above. Using the behavior of Barnes G near its zeroes from Appendix C, a lengthy calculation shows that the first pole of $J_{\xi_1,\ldots\xi_n}^{\rm aprx}(s)/\sin\pi s$ on the positive real axis occurs at $s = \sum_a \xi_a + \max_a\{\xi_a\}$ in the special case of integer ξ_a. Hence, for the n-point function and integer ξ_a, (30) indeed becomes

$$\bar{\mathcal{A}}_n^{\rm aprx}(x^0) \sim \exp\left[-x^0\sum_a \xi_a - x^0 \max\{\xi_a\}\right]. \quad \text{(B7)}$$

In other words, the corresponding poles of $J_{\xi_1,\ldots\xi_n}^{\rm aprx}$ and the exact analytic continuation $J_{\xi_1,\ldots\xi_n}$ coincide. Note that these poles lie at positive s, i.e., negative N, while $J_{\xi_1,\ldots\xi_n}^{\rm aprx}$ results from an asymptotic formula (64) for large positive N. This observation gives more confidence to the model amplitude of (68) and (71) which was derived using (64).

APPENDIX C: SINGULARITIES AND ASYMPTOTICS OF THE MODEL AMPLITUDE

Barnes $G(z)$ is an entire function and has its zeroes on the negative real axis,

$$G(z+1) = (-1)^{n(n-1)/2}G(n+1) \times (z+n)^n[1 + \mathcal{O}(z+n)], \quad \text{(C1)}$$

for $n = 1, 2, \ldots$ Hence all the special points (zeroes or

TABLE I. Exact and approximated values of the integral $I_{2,2}(N)$. The first tabulated row is the exact result of the integral, while the two others are given by the asymptotic formulae (52) and (64), written explicitly in (B1).

N	0	1	2	3	5	10	100
$I_{2,2}(N)$	70	784	4590	18 968	175 320	7 514 650	91 680 976 745 020
$I_{2,2}^{\rm asymp}(N)$	1	36	400	2500	38 416	2 944 656	81 349 594 398 801
$I_{2,2}^{\rm aprx}(N)$	200/3	750	4410	54 880/3	170 100	7 357 350	91 384 995 374 400

singularities) of the one-point amplitude $\hat{\mathcal{A}}_1(\omega)$ of (21) lie on the imaginary axis,

$$\hat{\mathcal{A}}_1 = \frac{(-1)^{n(n-1)/2}(2\pi\lambda)^{-n}G(n+1)^4}{G(2n+1)} \times (i\omega - n)^{n-1}[1 + \mathcal{O}(i\omega - n)];$$
$$\hat{\mathcal{A}}_1 = \frac{(-1)^{n(n-1)/2}(2\pi\lambda)^{n}G(n+1)^4}{2^{2n}G(2n+1)} \times (i\omega + n)^{n-1}[1 + \mathcal{O}(i\omega + n)];$$
$$\hat{\mathcal{A}}_1 = -\frac{\pi(2\pi\lambda)^{n+1/2}G(1/2-n)^3G(3/2+n)}{2^{2n+1}G(2n+2)} \times (i\omega + n + 1/2)^{-2n-1}[1 + \mathcal{O}(i\omega + n + 1/2)] \tag{C2}$$

for any $n = 0, 1, 2, \ldots$ In particular, poles are found at $\omega = 0$ (where $\hat{\mathcal{A}}_1 \sim 1/i\omega$) and at $\omega = i/2, 3i/2, 5i/2, \ldots$

The singularities of the model n-point amplitude

$$\hat{\mathcal{A}}_n \approx (2\pi\lambda)^{-i\sum_a \omega_a}\Gamma\Big(i\sum_a \omega_a\Big) \times \prod_{a=1}^{n} \frac{G(i\omega_a + 1)^3 G(2 - i\omega_a)}{G(2i\omega_a + 1)} \tag{C3}$$

arise similarly from the poles of $\Gamma(i\sum_a \omega_a)$ and from the zeroes of each $G(2i\omega_a + 1)$. As above, the hat denotes that we dropped the $\vec{k}_a$ dependent terms. At the possible singularities $\sum_a \omega_a \simeq -m$, $i\omega_b \simeq -m$, $i\omega_b \simeq m+1$, and $i\omega_b \simeq -m - 1/2$, we find

$$\hat{\mathcal{A}}_n = \frac{(-2\pi\lambda)^m}{m!}\prod_{a=1}^{n}\frac{G(i\omega_a+1)^3G(2-i\omega_a)}{G(2i\omega_a+1)}\Bigg|_{i\sum_a\omega_a=-m}\frac{1}{i\sum_a\omega_a+m}\Big[1+\mathcal{O}\Big(i\sum_a\omega_a+m\Big)\Big];$$
$$\hat{\mathcal{A}}_n = \frac{(-1)^{m(m+1)/2}(2\pi\lambda)^m(2\pi\lambda)^{-i\sum_{a\neq b}\omega_a}\Gamma(i\sum_{a\neq b}\omega_a - m)G(m+1)^3G(2+m)}{2^{2m}G(2m+1)}\prod_{a\neq b}\frac{G(i\omega_a+1)^3G(2-i\omega_a)}{G(2i\omega_a+1)} \times (i\omega_b+m)^m[1+\mathcal{O}(i\omega_b+m)];$$
$$\hat{\mathcal{A}}_n = \frac{(-1)^{m(m+1)/2}(2\pi\lambda)^{-i\sum_{a\neq b}\omega_a}\Gamma(i\sum_{a\neq b}\omega_a + m + 1)G(m+2)^3G(m+1)}{(2\pi\lambda)^{m+1}G(2m+3)}\prod_{a\neq b}\frac{G(i\omega_a+1)^3G(2-i\omega_a)}{G(2i\omega_a+1)} \times (i\omega_b - m - 1)^m \times [1+\mathcal{O}(i\omega_b - m - 1)];$$
$$\hat{\mathcal{A}}_n = \frac{(-1)^m(2\pi\lambda)^{m+1/2-i\sum_{a\neq b}\omega_a}\Gamma(i\sum_{a\neq b}\omega_a - m - 1/2)G(1/2-m)^3G(5/2+m)}{2^{2m+1}G(2m+2)}\prod_{a\neq b}\frac{G(i\omega_a+1)^3G(2-i\omega_a)}{G(2i\omega_a+1)} \times (i\omega_b + m + 1/2)^{-2m-1}[1+\mathcal{O}(i\omega_b + m + 1/2)] \tag{C4}$$

respectively, for each $m = 0, 1, 2, \ldots$ and $b = 1, 2, \ldots, n$, and assuming that the singularities are distinct (which requires $n > 1$). We thus find poles only at negative integer values of $i\sum_a \omega_a$ and at negative half-integer values of each $i\omega_b$ whereas the amplitude vanishes for almost all integer values of each $i\omega_b$.

Since the model amplitude has a factorized form, its asymptotics can be immediately written down based on the one-point result (25). It reads

$$\hat{\mathcal{A}}_n = (2\pi\lambda)^{-i\sum_a\omega_a}\Gamma\Big(i\sum_{a=1}^{n}\omega_a\Big)\prod_{a=1}^{n}\Gamma(1-i\omega_a)\exp\Big\{\sum_{a=1}^{n}\Big[\omega_a^2\Big(\frac{i\pi}{2}\mathrm{sgn}(\mathrm{Re}\,\omega_a) + 2\log 2\Big) - \frac{1}{4}\log(i\omega_a) - \frac{i\pi}{12}\mathrm{sgn}(\mathrm{Re}\,\omega_a) + \frac{1}{12}\log 2 + 3\zeta'(-1)\Big]\Big\}\prod_{a=1}^{n}[1+\mathcal{O}(\omega_a^{-2})]. \tag{C5}$$

Note that the prefactor vanishes exponentially for any $\omega_a \to \infty$ and thus does not affect the leading asymptotic behavior $\sim \exp[\omega_a^2]$, which is similar to that of the one-point amplitude in (25).

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