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New spin(7) holonomy metrics admitting G_2 holonomy reductions and M-theory/type-IIA dualities

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As is well known, when $D6$ branes wrap a special Lagrangian cycle on a noncompact Calabi-Yau threefold in such a way that the internal string frame metric is a Kähler one there exists a dual description, which is given in terms of a purely geometrical 11-dimensional background with an internal metric of G_2 holonomy. It is also known that when $D6$ branes wrap a coassociative cycle of a noncompact G_2 manifold in the presence of a self-dual two-form strength the internal part of the string frame metric is conformal to the G_2 metric and there exists a dual description, which is expressed in terms of a purely geometrical 11-dimensional background with an internal noncompact metric of spin(7) holonomy. In the present work it is shown that any G_2 metric participating in the first of these dualities necessarily participates in one of the second type. Additionally, several explicit spin(7) holonomy metrics admitting a G_2 holonomy reduction along one isometry are constructed. These metrics can be described as R fibrations over a 6-dimensional Kähler metric, thus realizing the pattern spin(7) $\to G_2 \to$ (Kahler) mentioned above. Several of these examples are further described as fibrations over the Eguchi-Hanson gravitational instanton and, to the best of our knowledge, have not been previously considered in the literature.

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I. INTRODUCTION

Spaces of G_2 and spin(7) holonomy were the only two cases of the Berger list of the possible holonomy groups for Riemanian geometries [1] whose existence was not clear. This situation completely changed with the construction of explicit noncompact examples in [2,3] and the proof of the existence of compact ones given in [4,5]. Since the appearance of these works, further special holonomy metrics were found in [6–21]. The reduction of the holonomy from $SO(7)$ or $SO(8)$ to G_2 or spin(7) implies that these metrics are Ricci flat, that is, $R_{ij} = 0$ being R_{ij} the Ricci tensor constructed with the special holonomy metric under consideration. Another of their salient features is the presence of at least one covariantly constant Killing spinor η, that is, a globally defined spinor satisfying $D\eta = 0$ being D the standard covariant derivative in the representation of the field. If the holonomy is exactly G_2 or spin(7) there is only one of such spinors, in other cases the holonomy will be reduced to a smaller subgroup. In fact, the presence of a parallel spinor η makes these spaces relevant for constructing supersymmetric solutions of supergravity theories or even vacuum solutions of superstring theories. This comes from the general fact that the number of supersymmetries preserved by these solutions is related to the number of independent parallel spinors that the internal manifold admits. Compactifications of M theory (or its low energy limit, 11-dimensional supergravity) on G_2 or spin(7) spaces give $N = 1$ supersymmetric theories in four and three dimensions, respectively. Additionally, compactifications of heterotic string theory on these spaces also provide $N = 1$ supersymmetry in three and two dimensions, respectively [22].

From a phenomenological point of view, compactifications of 11-dimensional supergravity over G_2 holonomy spaces constitute an attractive possibility, as the resulting low energy theory is four dimensional. But if the internal space is smooth then the four-dimensional theory will be $N = 1$ supergravity coupled to Abelian vector fields, and no chiral matter on non-Abelian vector fields will appear. Nevertheless, nonperturbative effects arising by singularities may generate chiral matter and non-Abelian gauge fields in four dimensions [23]. For this reason special attention was paid to G_2 holonomy spaces developing conical singularities. Another motivation for studying special holonomy manifolds appears in the context of dualities, as the present understanding of the dynamics of $N = 1$ supersymmetric theories relies partially in the existence of dual realizations of a given theory. An old example was considered in [24–26] where it was shown that type IIA propagating on the deformed conifold with D branes and IIA on the resolved conifold with Ramond-Ramond (RR) fluxes are dual to each other. This duality has been derived by lifting both backgrounds to purely geometrical M-theory ones with two G_2 holonomy manifolds admitting a smooth interpolation. As D branes con-

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tain gauge fields, this allows to study infrared dynamics by means of M theory on G_2 manifolds. Other contexts in which these spaces appear are in [27–45].

New dual descriptions involving special holonomy manifolds were reported in [46]. In this reference the geometries corresponding to $D6$ branes wrapping a supersymmetric 3-cycle in noncompact Calabi-Yau (CY) threefold and to $D6$ branes wrapping a coassociative 4-cycle in a noncompact G_2 holonomy space were described from an 11-dimensional perspective. Both cases give $N = 1$ supersymmetry. For the CY case it was shown that if the ten-dimensional string frame is a wrapped product with an internal Kähler metric then the supersymmetry generator becomes a covariantly constant gauge spinor. This requirement, which is stronger than $N = 1$ supersymmetry, is known as a ''strong supersymmetry condition'' and is translated into a system of equations involving the dilaton and the RR two-form. These equations are a sort of ''monopole equations'' describing the special Lagrangian cycles the $D6$ branes wrap and the backreaction of the branes on the physical metric. The lift of these IIA solutions to 11 dimensions results in a purely geometrical background with an internal G_2 holonomy metric. Similarly, for $D6$ branes wrapping a coassociative cycle on a G_2 manifold with self-dual two-form field strength the ten-dimensional string frame is a wrapped product with an internal metric conformal to the G_2 holonomy one. As for the previous case, the lift of these IIA solutions to 11 dimensions results in a purely geometrical background, but the internal metric is now of spin(7) holonomy.

There are two inherent mathematical problems that arise in the context of the dualities mentioned above. One is the classification of the G_2 holonomy metrics possessing an isometry action preserving the closed G_2 structure in such a way that the quotient of the seven-dimensional structure by this action is a Kähler one; the other is the classification of the spin(7) holonomy metrics with a structure preserving isometry such a quotient of the 8-metric by this action gives a 7-metric conformal to a G_2 holonomy one. The former situation has been studied in more detail. In fact in [47], one of the authors has identified these G_2 metrics with the ones discovered independently in [20], and the relation between the generalized monopole equation and those classifying the G_2 geometry in [20] was pointed out explicitly. Besides, an infinite class of explicit examples were presented in [21]. All these examples are described as fibrations over hyper-Kähler metrics of the Gibbons-Hawking type. Instead, the situation corresponding to spin(7) manifolds is less understood and one of the purposes of the present work is to study it further. A natural question is whether or not the G_2 metrics participating in both dualities are related. In the present work it is shown that *any* G_2 metric admitting a six-dimensional Kähler reduction along an isometry can be obtained by a quotient of a closed spin(7) structure by a structure preserving isometry. We are not able to prove or reject the inverse statement. The conclusion is that the set of G_2 metrics obtained by reduction of a spin(7) holonomy manifold in the way described above is *equal or bigger* than the ones that admit six-dimensional Kähler reductions.

It is tempting to connect the mathematical and physical aspects of the dualities described above by saying that whenever a G_2 metric provides a dual description for $D6$ branes wrapping a special Lagrangian cycle on a CY manifold satisfying the strong supersymmetry conditions it also describes a configuration of $D6$ branes wrapping a coassociative cycle inside the G_2 manifold in the presence of a self-dual RR two-form. The fact that the corresponding backgrounds are completely determined in terms of the G_2 metric may suggest that those $D6$ brane configurations are dual to each other, and the link between them is provided by the G_2 structure. This is a very interesting statement but we are still cautious for the following reason. A configuration of $D6$ branes wrapping a special Lagrangian cycle in a CY manifold will appear only if magnetic sources for the RR two-form F are present, that is, $dF = N\delta$. Consider the G_2 metric dual to one of such configurations. By use of the result of the present work, one can lift it to a spin(7) metric and construct an 11-dimensional background which is the direct sum of this metric with the Minkowski one in three dimensions. Clearly, the usual Kaluza-Klein reduction along an isometry gives a IIA background with a nontrivial dilaton and a self-dual RR two-form F', and the internal metric is conformal to the G_2 holonomy metric. If there are delta types of sources for F', then this configuration will correspond to $D6$ branes wrapping a coassociative cycle and the duality we are talking about seems to be plausible. But we did not find a formal argument which insures that such singularities will appear, even if they were present for the initial configurations. One can argue that $dF' = 0$ everywhere, but F' is nontrivial due to bad asymptotics at infinity. In our opinion this is not the case, up to possible pathological counterexamples. In any case, we suspect that our result encodes very interesting class dualities between $D6$ brane configurations.

In addition, we are able to find new G_2 holonomy examples not considered in [20] and their lift to spin(7) metrics. All these examples arise as fibrations over the Eguchi-Hanson gravitational instanton and the fiber quantities are defined over a complex submanifold of the Eguchi-Hanson space. In this situation all the fiber quantities are defined by the solution of a Laplace type equation on the curved instanton metric, otherwise the underlying problem becomes nonlinear and in consequence, harder to solve.

The present work is organized as follows. In Sec. II a system of equations is presented describing the lift of a G_2 holonomy metric to a spin(7) one, which is essentially the one considered in [46]. In addition, a brief characterization of the G_2 holonomy metrics which admit a Kähler reduction is given. In particular, it is shown that *any* of these G_2 metrics can be lifted to a spin(7) one by means of these equations, which is one of the main results of the present work. In Sec. III some known examples of these G_2 metrics [20,21] are presented and the lifting to spin(7) metrics is performed explicitly. In Sec. IV we review a method for constructing G_2 holonomy metrics admitting Kähler reductions in terms of an initial hyper-Kähler four-dimensional metric together with certain quantities defined over a complex submanifold of the hyper-Kähler manifold [20]. We show that this method linearizes the otherwise nonlinear system describing this geometry and converts it into a Laplace type equation on the curved hyper-Kähler metric. We find a nontrivial solution when the hyper-Kähler manifold is the Eguchi-Hanson gravitational instanton and construct the corresponding special holonomy metrics. In the last section we make a brief discussion of the presented results.

II. SPIN(7) METRICS ADMITTING G_2 REDUCTIONS

A. The defining equations

As was mentioned in the introduction, a configuration of $D6$ branes wrapping a coassociative submanifold of a G_2 manifold in such a way that the field strength F_{ab} satisfying the self-duality condition

$$4F_{ab} + c_{abcd}F_{cd} = 0, \tag{2.1}$$

is described in terms of a type IIA background with an internal 7-metric which, in the string frame, is conformal to a G_2 holonomy metric [46]. Here c_{abcd} are the duals of the octonion multiplication constants. Any of these IIA backgrounds can be lifted to a purely geometrical solution of 11-dimensional supergravity of the form

$$g_{11} = g_{(1,2)} + g_8, \tag{2.2}$$

being g_8 a spin(7) holonomy metric possessing a Killing vector preserving also the spin(7) calibration four-form. Here $g_{1,2}$ is the Minkowski metric in three dimensions. The purpose of the section is to clarify the relation between these G_2 and spin(7) holonomy metrics.

Consider an eight-dimensional space M_8 with metric

$$g_8 = e^{6f}(dz + A)^2 + e^{-2f}g_7 \tag{2.3}$$

such that the one-form A, the 7-metric g_7, and the function f are independent on the coordinate z. This condition means that $V = \partial_z$ is a local Killing vector, which induces a local decomposition $M_8 = M_7 \times R_z$ if z is noncompact or $M_8 = M_7 \times U(1)_z$ if z is an angular coordinate. In the following we will impose that g_8 is of spin(7) holonomy and that g_7 is of G_2 holonomy and we will derive the consequences of this statement, with the further assumption that $V = \partial_z$ also preserves the spin(7) structure.

By defining the one-form $e^8 = e^{3f}(dz + A)$ the spin(7) calibration four-form corresponding to g_8 can be decomposed in the following form:

$$\Omega_8 = e^8 \wedge \tilde{\Phi} + *\tilde{\Phi}. \tag{2.4}$$

Here $\tilde{\Phi}$ and $*\tilde{\Phi}$ are a pair of G_2 invariant three and four-forms for the metric $e^{-2f}g_7$. As the function f appearing in the expression (2.4) is z independent, it follows that the whole four-form (2.4) will be preserved by $V = \partial_z$. Furthermore $\tilde{\Phi} = e^{-3f}\Phi$ and $*\tilde{\Phi} = e^{-4f} * \Phi$ being Φ and $*\Phi$ certain G_2 invariant three and four-forms for the metric g_7. The four-form (2.4) can be expressed in terms of Φ and $*\Phi$ as

$$\Omega_8 = (dz + A) \wedge \Phi + e^{-4f} * \Phi. \tag{2.5}$$

As g_7, by assumption, has holonomy in G_2 it follows that $d\Phi = d * \Phi = 0$. Then the condition for spin(7) holonomy $d\Omega_8 = 0$ will be equivalent to the following system:

$$F \wedge \Phi + d(e^{-4f}) \wedge *\Phi = 0, \tag{2.6}$$

being $F = dA$. By construction F is a closed two-form.

In principle, if one starts with a closed G_2 structure and solves (2.6) then the result is a spin(7) holonomy metric. The problem is that, in general, it is not easy to find a nontrivial solution. In fact, if one starts with an arbitrary G_2 metric it can be a hard task to guess an anzatz for F and f in such a way that the resulting system of equations takes a manageable form. Let us also note that this system does not classify completely all the spin(7) metrics admitting a G_2 holonomy reduction. Even if $d\Phi = d * \Phi \neq 0$ there could exist a rotation of the tetrad frame of g_7 such that $d\Phi' = d * \Phi' = 0$ for certain new calibration forms. We will try not to classify all the possible solutions of (2.6), but instead we will find some particular ones. The G_2 metrics from which we will start are a special class of G_2 holonomy metrics which are defined by admitting Kähler reductions [20,21,46,47]. Fortunately, we will be able to solve (2.4) for all these metrics.

Clearly, the spin(7) metrics presented above can be extended to a purely geometrical background of the form (2.2). This background can be rewritten in the IIA form

$$g_{11} = e^{-\phi}g_{10} + e^{2\phi}(dz + H_3)^2, \tag{2.7}$$

$V = \partial_z$ being the corresponding Killing vector. The usual reduction to IIA supergravity gives

$$g_{\mathrm{IIA}} = \eta^{1/3}g_{(1,2)} + \eta^{-1/9}g_7, \qquad F = \omega_3, \tag{2.8}$$

g_7 being the G_2 holonomy metric and where the dilaton ϕ is defined through the relation $e^{2\phi} = \eta^{-2/3}$. The seven-dimensional internal part of the background (2.8) is then *conformal* to the G_2 metric, in accordance with [46] and our previous discussion.

B. G_2 holonomy metrics admitting Kähler reductions

The next step is to find nontrivial solutions of the system (2.6), or equivalently, to construct nontrivial spin(7) metrics possessing an isometry such that the orbits of the Killing vector induce a seven-dimensional metric conformal to a G_2 holonomy metric. As we will show below, the system (2.6) can be solved for the large class of G_2 metrics considered in [20] and independently in [46]. These metrics always possess a Killing vector which preserves the whole G_2 structure such that the induced metric by taking the quotient with respect to this isometry is a six-dimensional Kähler metric. In [46] the local form of these metrics is described in terms of "generalized monopole equations" while in [20] the description is given in terms of a nonlinear system that we will describe below. In addition, the analysis of [47] shows that both descriptions are equivalent. The reason for choosing the second formalism is that, as we will see, it considerably simplifies the lifting equation (2.6).

In general, if a G_2 holonomy metric possesses an isometry preserving the whole G_2 structure, then the orbits of the Killing vector induce an $SU(3)$ structure with generically nonzero torsion classes [48]. But if the associated $SU(3)$ structure is Kähler, then the initial G_2 has another isometry which commutes with the former one [20]. Therefore any of such G_2 metrics is toric from the very beginning. In addition, it is possible to make a further reduction with respect to the second isometry and describe the G_2 and Kähler metrics as fibrations over a certain Kähler four-dimensional metric which we will specify below.[1]

Let us describe schematically the local form of the G_2 holonomy metrics in question; further details can be found in the original reference [20]. Their local form is

$$g_7 = \frac{(d\alpha + H_2)^2}{\mu^2} + \mu\left(ud\mu^2 + \frac{(d\beta + H_1)^2}{u} + g_4(\mu)\right). \tag{2.9}$$

All the quantities defining g_7 are independent on the coordinates α and β, therefore (2.9) is toric with Killing vectors $V_1 = \partial_\alpha$ and $V_2 = \partial_\beta$. The metric $g_4(\mu)$ is Kähler and defined over a four manifold M and it depends on μ as a parameter. It also admits a complex μ-independent symplectic two-form $\Omega = \omega_2 + i\omega_3$, where being "symplectic" means that it is closed, $d\Omega = 0$. On the other hand, being "complex" implies that

$$\omega_2 \wedge \omega_2 = \omega_3 \wedge \omega_3, \qquad \omega_2 \wedge \omega_3 = 0, \tag{2.10}$$

and that the equation

$$\omega_2(J_1\cdot, \cdot) = \omega_3(\cdot, \cdot) \tag{2.11}$$

defines a complex structure J_1. In other words, the Niejenhuis tensor of J_1 vanishes identically or equivalently J_1 is integrable. The two-form $\tilde{\omega}_1(\mu)$ constructed by lowering the indices of J_1 with $g_4(\mu)$ is in general μ dependent and closed on M. It is also orthogonal to ω_2 and ω_3 with respect to the wedge product, that is

$$\tilde{\omega}(\mu) \wedge \omega_2 = \tilde{\omega}(\mu) \wedge \omega_3 = 0. \tag{2.12}$$

The function u in (2.9) depends on the coordinates of M and on the parameter μ, and it is defined through the relation

$$2\mu\tilde{\omega}_1(\mu) \wedge \tilde{\omega}_1(\mu) = u\Omega \wedge \bar{\Omega}. \tag{2.13}$$

This function always exists because the wedge products in (2.13) are proportional to the volume form $V(g_4)$ of $g_4(\mu)$. In fact

$$\tilde{\omega}_1(\mu) \wedge \tilde{\omega}_1(\mu) = V(g_4).$$

The forms H_1 and H_2 are defined on $M \times \mathbf{R}_\mu$ and M, respectively, by the equations

$$dH_1 = (d^c_M u) \wedge d\mu + \frac{\partial\tilde{\omega}_1}{\partial\mu}, \qquad dH_2 = -\omega_2, \tag{2.14}$$

with $d^c_M = J_1 d_M$. The last equation can always be solved locally as the forms $\tilde{\omega}_1$ and ω_2 are closed. The integrability condition associated to the first (2.14) is the evolution equation

$$\frac{\partial^2\tilde{\omega}_1}{\partial^2\mu} = -d_M d^c_M u. \tag{2.15}$$

Now a theorem given in [20] insures that if the system of equations described above are satisfied then the metric (2.9) are of G_2 holonomy. This statement is not difficult to see. The calibration three-form corresponding to the metrics (2.9) is

$$\Phi = \tilde{\omega}_1(\mu) \wedge (d\alpha + H_2) + d\mu \wedge (d\beta + H_1) \wedge (d\alpha + H_2) + \mu(\omega_2 \wedge (d\beta + H_1) + u\omega_3 \wedge d\mu), \tag{2.16}$$

and the dual form $*\Phi$ corresponding to (2.16) is given by [21]

$$*\Phi = \mu^2\tilde{\omega}_1(\mu) \wedge d\mu \wedge (d\beta + H_1) + u\omega_2 \wedge (d\alpha + H_2) \wedge d\mu + \omega_3 \wedge (d\beta + H_1) \wedge (d\alpha + H_2) + \mu^2\tilde{\omega}_1(\mu) \wedge \tilde{\omega}_1(\mu). \tag{2.17}$$

By means of (2.14), (2.15), and (2.13) it follows that $d\Phi = d * \Phi = 0$.

The G_2 metrics (2.9) are fibered over the six-dimensional metric

$$g_6 = ud\mu^2 + \frac{(d\beta + H_1)^2}{u} + g_4(\mu), \tag{2.18}$$

which is Kähler with Kähler form

[1] Note that the lifting of these metrics to eight dimensions by (2.6) will give a spin(7) metric with three commuting isometries, as the initial G_2 metric is toric.

$$K = (d\beta + H_1) \wedge d\mu + \tilde{\omega}_1. \tag{2.19}$$

The converse of these statements are also true. That is, for given a G_2 holonomy manifold Y with a metric g_7 possessing a Killing vector that preserves the calibration forms Φ and $*\Phi$ and such that the six-dimensional metric g_6 obtained from the orbits of the Killing vector is Kähler, there exists a coordinate system in which g_7 takes the form (2.9) $g_4(\mu)$ being a one-parameter four-dimensional metric admitting a complex symplectic structure Ω and a complex structure J_1, the quantities appearing in this expression being related by (2.11) and the conditions (2.14), (2.15), and (2.13). This is the most involved part of the proofs and we refer the reader to the original Ref. [20].

The class metrics presented in this section include as particular cases the G_2 metrics which are dual to $D6$ branes wrapping a special Lagrangian cycle and satisfying the strong supersymmetric conditions, i.e., the conditions for the supersymmetry generator to be a covariantly constant gauge spinor. These conditions will hold only if in the string frame the IIA string metric has a Kähler internal part, which forces the G_2 dual metric to be in our class. Note that the Killing vector fields preserve the metric and Φ, thus it preserves $*\Phi$ and the whole G_2 structure. Another interesting fact is that

$$*\,\Phi|_M = V(g_4),$$

therefore the Kähler base g_4 is a coassociative submanifold. In the same way for a fixed value of the coordinates of g_4 one obtains from (2.9) the three-dimensional metric

$$g_3 = \frac{d\alpha^2}{\mu^2} + u d\mu^2 + \mu\frac{d\beta^2}{u} \tag{2.20}$$

defined on certain space M_3, and it follows that

$$*\,\Phi|_{M_3} = V(g_3),$$

therefore M_3 is an associative submanifold. These are calibrated submanifolds [49] and are supersymmetric from the physical point of view [50].

C. Uplifting to spin(7) metrics

In the present subsection it will be shown that any of the G_2 holonomy metrics (2.9) described above can be lifted to a spin(7) holonomy one by means of the lifting formula (2.6). The two-form $F = dA$ appearing in this formula must be closed. From the fact that $\Omega = \omega_2 + i\omega_3$ is symplectic and that $dH_2 = -\omega_2$ (see (2.14)) the most natural anzatz is

$$F = dA = -\omega_3. \tag{2.21}$$

The system (2.6) reduces in this case to

$$\omega_3 \wedge \Phi = d(e^{-4f}) \wedge *\Phi. \tag{2.22}$$

From (2.17) it follows that the right-hand side of (2.22) is

$$\begin{aligned} d(e^{4f}) \wedge *\Phi = {} & d(e^{-4f}) \wedge (\mu^2\tilde{\omega}_1(\mu) \wedge d\mu \wedge (d\beta + H_1) \\ & + u\omega_2 \wedge (d\alpha + H_2) \wedge d\mu + \omega_3 \\ & \wedge (d\beta + H_1) \wedge (d\alpha + H_2) \\ & + \mu^2\tilde{\omega}_1(\mu) \wedge \tilde{\omega}_1(\mu)). \end{aligned} \tag{2.23}$$

The left-hand side is obtained from (2.16) and is

$$\begin{aligned} \omega_3 \wedge \Phi = {} & \tilde{\omega}_1(\mu) \wedge (d\alpha + H_2) \wedge \omega_3 + d\mu \wedge (d\beta + H_1) \\ & \wedge (d\alpha + H_2) \wedge \omega_3 + \mu(\omega_2 \wedge (d\beta + H_1) \\ & + u\omega_3 \wedge d\mu) \wedge \omega_3. \end{aligned} \tag{2.24}$$

But from (2.13) we see that $\omega_2 \wedge \omega_3 = \tilde{\omega}_1 \wedge \omega_3 = 0$ and also that

$$u\omega_3 \wedge \omega_3 = \mu\tilde{\omega}_1 \wedge \tilde{\omega}_1.$$

Taking into account these relations the formula (2.24) is simplified to

$$\begin{aligned} \omega_3 \wedge \Phi = {} & d\mu \wedge ((d\beta + H_1) \wedge (d\alpha + H_2) \wedge \omega_3 \\ & + \mu^2\tilde{\omega}_1 \wedge \tilde{\omega}_1). \end{aligned} \tag{2.25}$$

Equating (2.25) to (2.23) gives the equation

$$\begin{aligned} & d(e^{-4f}) \wedge d\mu \wedge (\mu^2\tilde{\omega}_1(\mu) \wedge (d\beta + H_1) - u\omega_2 \\ & \quad \wedge (d\alpha + H_2)) + d(e^{-4f}) \wedge (\omega_3 \wedge (d\beta + H_1) \\ & \quad \wedge (d\alpha + H_2) + \mu^2\tilde{\omega}_1(\mu) \wedge \tilde{\omega}_1(\mu)) \\ & = d\mu \wedge (\omega_3 \wedge (d\beta + H_1) \wedge (d\alpha + H_2) \\ & \quad + \mu^2\tilde{\omega}_1(\mu) \wedge \tilde{\omega}_1(\mu)). \end{aligned} \tag{2.26}$$

The solution of this system is immediate. If $d(e^{-4f}) = d\mu$ the two first terms of the left-hand side vanishes and the two last ones are equal to the right-hand side. We choose then $e^{-4f} = \mu + b$ and our spin(7) metrics become

$$g_8 = \frac{(dz + H_3)^2}{(\mu + b)^{3/2}} + (\mu + b)^{1/2} g_7 \tag{2.27}$$

being $dH_3 = -\omega_3$ and g_7 the G_2 holonomy metrics described in the previous section. Thus, we have found the spin(7) metrics we were looking for. The reason for which the family is infinite is because the G_2 family over which are fibered is also infinite [20,21].

III. EXPLICIT SPIN(7) EXAMPLES

The local expression (2.27) describes an infinite family of spin(7) metrics admitting G_2 reductions. In this section we found the metrics corresponding to known G_2 holonomy cases [21]. The lift in this case presents no difficulties but serves as warm-up for the next section, in which less trivial examples will be worked out.

A. Two different general solutions

Any of the spin(7) metrics (2.27) are fibrations over a G_2 holonomy metric g_7 of the type described in Sec. II B, which are constructed in terms of solutions of the Eqs. (2.11), (2.12), (2.13), and (2.14). The one-form H_3 satisfies $dH_3 = -\omega_3$. A simple example is obtained by assuming that the function u defined in (2.13) does not vary when we move on M but depends on the coordinate μ. Then Eq. (2.15) gives that $\tilde{\omega}_1 = (c\mu + d)\omega_1$ being ω_1 independent on μ. In addition $\tilde{\omega}_1$ is closed on M_4 from where it follows that $d_4\omega_1 = 0$. This means that if one starts with a hyper-Kähler triplet ω_i of some hyper-Kähler manifold M all the conditions (2.9), (2.10), (2.11), (2.12), (2.13), and (2.14) are solved except (2.13), which becomes then an algebraic equation defining u. The solution is $u = \mu(c\mu + d)^2$. Also $g_4(\mu) = (c\mu + d)\bar{g}_4$ being $\bar{g}_4$ the hyper-Kähler metric corresponding to ω_i. The resulting 7-metrics (2.9) have the following expression:

$$g_7 = \frac{(d\alpha + H_2)^2}{\mu^2} + \frac{(d\beta + H_1)^2}{(c\mu^2 + d)^2} + \mu^2(c\mu + d)^2 d\mu^2 + \mu(c\mu + d)\bar{g}_4. \quad (3.1)$$

Moreover Eqs. (2.14) are in this case

$$dH_1 = \omega_1, \qquad dH_2 = -\omega_2. \quad (3.2)$$

These metrics are usually well behaved away from the point $\mu = 0$ or $\mu = -b$ if $b < 0$.

A second type of metrics are obtained with a function u which depends on μ and also varies on M. This case is more difficult to deal with but still we will find below several explicit examples. Consider as before a hyper-Kähler structure ω_i with its Ricci flat metric $\bar{g}_4$ and deform one of the Kähler two-forms, say ω_1, to a new one $\tilde{\omega}_1(\mu)$ of the form

$$\tilde{\omega}_1(\mu) = \omega_1 - d_4 d_4^c G, \quad (3.3)$$

G being a function on $M \times R_\mu$. Then the compatibility conditions (2.10), (2.11), and (2.12) are satisfied by (3.3). Inserting (3.3) into the evolution equation (2.15) gives

$$\partial_\mu^2 G = 2u, \quad (3.4)$$

therefore u is completely determined in terms of G. The equation for G is found from (2.14) as follows. The relation

$$\tilde{\omega}_1(\mu) \wedge \tilde{\omega}_1(\mu) = (\omega_1 - d_4 d_4^c G) \wedge (\omega_1 - d_4 d_4^c G) = M(G)\omega_1 \wedge \omega_1 \quad (3.5)$$

defines a nonlinear operator $M(G)$. This operator always exists as all the terms in (3.5) are proportional to the volume form of $g_4(\mu)$. Then the insertion of (3.4) into (2.14) gives

$$2\mu M(G) = \partial_\mu^2 G, \quad (3.6)$$

which is the equation we were looking for. Also, from (2.15) it follows that

$$H_1 = -d_4^c \partial_\mu G. \quad (3.7)$$

Note that in general, the metric tensor $g_4(\mu)$ in (2.9) is *not* the hyper-Kähler metric $\bar{g}_4$ in general. If K denotes the Kähler potential corresponding to ω_1 then the metric $g_4(\mu)$ is the one which corresponds to the modified Kähler potential $\bar{K} = K - G$. This metric is obviously Kähler, but not necessarily hyper-Kähler. Equations (3.4), (3.5), (3.6), and (3.7) define a new family of G_2 metrics and all the objects defining the metric are related essentially to a single function G satisfying (3.6).

To find the general solution of the previous equations is extremely complicated because $M(G)$ is a nonlinear operator. The source of nonlinearity is given by the term $d_4 d_4^c G \wedge d_4 d_4^c G$ in (3.5). Nevertheless, there exist special cases in which

$$d_4 d_4^c G \wedge d_4 d_4^c G = 0. \quad (3.8)$$

In these situations the operator $M(G)$ reduces to the linear operator [20]

$$M(G) = 1 + \Delta_4 G$$

Δ_4 being the Laplacian over the starting hyper-Kähler metric $\bar{g}_4$. In fact, the full system describing the G_2 geometry is linear in these cases. The condition (3.8) is satisfied when the function G is defined over a complex submanifold on the hyper-Kähler manifold M [51]. This statement means the following. The starting hyper-Kähler structure ω_i is obviously Kähler, thus M is complex and parametrized in terms of certain complex coordinates $(z_1, z_2, \bar{z}_1, \bar{z}_2)$ which diagonalize J_1. Equation (3.8) will be satisfied when the function G is of the form $G = G(w, \bar{w})$ w being a single complex function of the z_i and $\bar{w}$ its complex conjugate. In particular, Eq. (3.6) defining the G_2 geometry will be reduced to

$$2\mu(1 + \Delta_4 G) = \partial_\mu^2 G. \quad (3.9)$$

This is an important simplification, although the task of finding solutions of a Laplace equation in a curved space is not easy in general. In the next section we will find an explicit solution by taking the Eguchi-Hanson gravitational instanton as our initial hyper-Kähler structure.

B. Simple known examples

The solution generating techniques described in the previous subsection require an initial hyper-Kähler structure. The simplest hyper-Kähler manifold is R^4 with its flat metric $g_4 = dx^2 + dy^2 + dz^2 + d\varsigma^2$ and with the closed hyper-Kähler triplet

$$\omega_1 = d\varsigma \wedge dy - dz \wedge dx, \qquad \omega_2 = d\varsigma \wedge dx - dy \wedge dz, \qquad \omega_3 = d\varsigma \wedge dz - dx \wedge dy.$$

This innocent looking case is indeed very interesting. Let us construct the metrics (3.1) corresponding to this structure. The forms H_i such that $dH_i = \omega_i$ are simply

$$H_1 = -xdz + yd\varsigma, \qquad H_2 = -ydz + xd\varsigma, \qquad H_3 = -ydx + zd\varsigma \tag{3.10}$$

and by selecting $c = 1$ and $d = 0$ in (3.1) the resulting G_2 metric is

$$g_7 = \frac{(d\alpha - xdz + yd\varsigma)^2}{\mu^2} + \frac{(d\beta - ydz - xd\varsigma)^2}{\mu^2} + \mu^4 d\mu^2 + \mu^2(dx^2 + dy^2 + dz^2 + d\varsigma^2). \tag{3.11}$$

The metrics (3.11) have been already obtained in the physical literature [19] and are interpreted in terms of domain wall configurations. Even in this simple case and though the base 4-metric has trivial holonomy, it has been shown that (3.11) is irreducible and has holonomy exactly G_2, not a subgroup.

Turning on the attention to the second ramification, a possible choice of complex coordinates for R^4 is $z_1 = x + iy$, $z_2 = z + i\varsigma$ and complex conjugates. If the functional dependence of the function G is assumed to be $G = G(\mu, z_1, \bar{z}_1)$ then the operator $M(G)$ reduces to the Laplacian operator in flat space

$$G'' + \mu(\partial_{xx}G + \partial_{yy}G) = 2\mu. \tag{3.12}$$

The separable solutions in the variable μ are of the form

$$G = \tfrac{1}{3}\mu^3 + V(x, y)K(\mu).$$

By introducing $G = G(\mu, x, y)$ into (3.12) it follows that $K(\mu)$ and $V(x, y)$ are solutions of the equations

$$K''(\mu) = p\mu K(\mu), \qquad \partial_{xx}V + \partial_{yy}V + pV = 0, \tag{3.13}$$

p being a parameter. By defining the $\tilde{\mu} = \mu/p^{1/3}$ the first of Eqs. (3.13) reduce to the Airy equation. The second is reduced to find eigenfunctions of the two-dimensional Laplace operator, which is a well-known problem in electrostatics. For $p > 0$ periodical solutions are obtained and for $p < 0$ there will appear exponential solutions.

A simple example is obtained with the eigenfunction $V = q\sin(px)$, q being a constant. A solution of the Airy equation is given by

$$K = Ai(\tilde{\mu}) = \frac{1}{3}\tilde{\mu}^{1/2}(J_{1/3}(\tau) + J_{-1/3}(\tau)), \qquad \tau = i\frac{2\mu^{3/2}}{3p^{1/2}}.$$

Then the function G is

$$G = \frac{1}{3}\mu^3 + q\sin(px)Ai\left(\frac{\mu}{p^{1/3}}\right).$$

From (3.7) it is obtained that

$$H_1 = -pqAi(\tilde{\mu})'\cos(px)dy, \qquad u = \mu(1 + pqAi(\tilde{\mu})\sin(px)), \tag{3.14}$$

$$g_4(\mu) = \frac{u}{\mu}(dx^2 + dy^2) + dz^2 + d\varsigma^2.$$

In terms of the quantities defined above the generic G_2 holonomy metric (2.9) becomes [20]

$$g_7 = \frac{(d\chi - xdz + yd\varsigma)^2}{\mu^2} + \frac{(dv - pqAi(\tilde{\mu})'\cos(px)dy)^2}{H} + \mu(Hdx^2 + Hdy^2 + dz^2 + d\varsigma^2) + \mu^2 H d\mu^2, \tag{3.15}$$

where the function $H(\mu, x, y) = (1 + pqAi(\tilde{\mu})\sin(px))$ has been introduced. As before, the holonomy is *exactly* G_2 [20].

It is straightforward to construct from (3.15) or (3.11) a pair of holonomy spin(7) metrics, which are obtained from (2.27) and (3.10). The result is

$$g_8 = \frac{(dz + ydx - zd\varsigma)^2}{(\mu + b)^{3/2}} + (\mu + b)^{1/2} g_7 \tag{3.16}$$

g_7 being any of (3.15) or (3.11). The curvature tensor is irreducible for these metrics and the holonomy is not reduced to a subgroup.

IV. TWO FIBRATIONS OVER THE EGUCHI-HANSON GRAVITATIONAL INSTANTON

A. The Eguchi-Hanson metric as an ALE space

In this section the solutions of the two ramifications described above will be worked in the situation in which the Eguchi-Hanson gravitational instanton is the initial hyper-Kähler metric [52]. As is well known, this metric is preserved by an isometry which also preserves its Kähler forms ω_i, namely

$$\mathcal{L}_K\omega_1 = \mathcal{L}_K\omega_2 = \mathcal{L}_K\omega_3 = 0$$

K being the corresponding Killing vector. Such Killing vector K is called triholomorphic and it also preserves the complex structures J_i defined by the Kähler forms and the metric, thus it is also tri-Hamiltonian. For any four-dimensional hyper-Kähler structure with this property there exists a local system of coordinates in which $K = \partial_t$ and for which the metric takes generically the Gibbons-Hawking form [53]

$$g = V^{-1}(dt + A)^2 + V dx_i dx_j \delta^{ij}, \tag{4.1}$$

with a one-form A and a function V satisfying the linear system of equations

$$\nabla V = \nabla \times A. \tag{4.2}$$

In addition the hyper-Kähler triplet in this coordinate is given by

$$\omega_1 = (dt + A) \wedge dx - V dy \wedge dz,$$
$$\omega_2 = (dt + A) \wedge dy - V dz \wedge dx, \qquad (4.3)$$
$$\omega_3 = (dt + A) \wedge dz - V dx \wedge dy,$$

which is actually t independent. The Eguchi-Hanson solution corresponds to two monopoles on the z axis. Without losing generality, it can be considered that the monopoles are located in the positions $(0, 0, \pm c)$. The potentials for this configurations are

$$V = \frac{1}{r_+} + \frac{1}{r_-},$$
$$A = A_+ + A_- = \left(\frac{z_+}{r_+} + \frac{z_-}{r_-}\right) d\arctan(y/x),$$
$$r_\pm^2 = x^2 + y^2 + (z \pm c)^2.$$

The resulting metric (4.1) in Cartesian coordinates is

$$g = \left(\frac{1}{r_+} + \frac{1}{r_-}\right)^{-1}\left(d\tau + \left(\frac{z_+}{r_+} + \frac{z_-}{r_-}\right) d\arctan(y/x)\right)^2 + \left(\frac{1}{r_+} + \frac{1}{r_-}\right)(dx^2 + dy^2 + dz^2), \qquad (4.4)$$

where $z_\pm = z \pm c$. In order to recognize the Eguchi-Hanson metric in its standard form it is convenient to introduce a new parameter $a^2 = 8c$ and the elliptic coordinates defined by [54]

$$x = \frac{r^2}{8}\sqrt{1 - (a/r)^4}\sin\varphi\cos\theta,$$
$$y = \frac{r^2}{8}\sqrt{1 - (a/r)^4}\sin\varphi\sin\theta,$$
$$z = \frac{r^2}{8}\cos\varphi.$$

It is not difficult to check that in these coordinates

$$r_\pm = \frac{r^2}{8}(1 \pm (a/r)^2\cos\varphi),$$
$$z_\pm = \frac{r^2}{8}(\cos\varphi \pm (a/r)^2),$$
$$V = \frac{16}{r^2}(1 - (a/r)^4\cos^2\varphi)^{-1},$$
$$A = 2(1 - (a/r)^4\cos^2\varphi)^{-1}(1 - (a/r)^4)\cos\varphi d\theta,$$

and, with the help of these expressions, it is found that

$$g = \frac{r^2}{4}(1 - (a/r)^4)(d\theta + \cos\varphi d\tau)^2 + (1 - (a/r)^4)^{-1}dr^2 + \frac{r^2}{4}(d\varphi^2 + \sin^2\varphi d\tau). \qquad (4.5)$$

This is actually a more familiar expression for the Eguchi-Hanson instanton indeed. Its isometry group is $U(2) = U(1) \times SU(2)/\mathbf{Z}_2$. The holomorphic Killing vector is ∂_τ. This space is asymptotically locally Euclidean (ALE), which means that it asymptotically approaches the Euclidean metric; and therefore the boundary at infinity is locally S^3. However, the situation is rather different in what regards its global properties. This can be seen by defining the new coordinate

$$u^2 = r^2(1 - (a/r)^4)$$

for which the metric is rewritten as

$$g = \frac{u^2}{4}(d\theta + \cos\varphi d\tau)^2 + (1 + (a/r)^4)^{-2}du^2 + \frac{r^2}{4}(d\varphi^2 + \sin^2\varphi d\tau). \qquad (4.6)$$

The apparent singularity at $r = a$ has been moved now to $u = 0$. Near the singularity, the metric looks like

$$g \simeq \frac{u^2}{4}(d\theta + \cos\varphi d\tau)^2 + \frac{1}{4}du^2 + \frac{a^2}{4}(d\varphi^2 + \sin^2\varphi d\tau),$$

and, at fixed τ and φ, it becomes

$$g \simeq \frac{u^2}{4}d\theta^2 + \frac{1}{4}du^2.$$

This expression "locally" looks like the removable singularity of $\mathbf{R}^2$ that appears in polar coordinates. However, for actual polar coordinates, the range of θ covers from 0 to 2π, while in spherical coordinates in $\mathbf{R}^3$, $0 \le \theta < \pi$. This means that the opposite points on the geometry turn out to be identified and thus the boundary at infinity is the lens space $S^3/\mathbf{Z}_2$.

B. The first type of metrics

The task to find the G_2 metrics (3.1) that correspond to the Eguchi-Hanson instanton was already solved in [21]. The explicit expression of the one-forms H_i satisfying $dH_i = \omega_i$, ω_i being the Kähler forms (4.3), is the following:

$$H_1 = -x d\tau + (\log(r_+ + z_+) + \log(r_- + z_-))dy - 2axd\arctan(y/x), \qquad (4.7)$$

$$H_2 = +y d\tau + (\log(r_+ + z_+) + \log(r_- + z_-))dx + 2ayd\arctan(y/x), \qquad (4.8)$$

$$H_3 = -z d\tau - a(r_+ + r_-)d\arctan(y/x). \qquad (4.9)$$

These expressions are defined up to a redefinition by a total differential and, together with (3.1), define the following G_2 metric:

$$g_7 = \frac{(d\alpha + H_2)^2}{\mu^2} + \frac{(d\beta + H_1)^2}{(c\mu^2 + d)^2} + \mu^2(c\mu + d)^2 d\mu^2 + \mu(c\mu + d)\bar{g}_4, \qquad (4.10)$$

where $\bar{g}_4$ the Eguchi-Hanson metric. In addition, a spin(7) holonomy metric is obtained directly from (4.10) and (2.27).

C. The new examples

In this subsection we construct a family of G_2 and spin (7) metrics fibered over the Eguchi-Hanson instanton which, to our knowledge, have not been previously considered. For this it is convenient to define a new radial coordinate $\rho = r^2/4$ for Eguchi-Hanson metric (4.5) while keeping the angular coordinates unchanged. In terms of these coordinates (4.5) takes the form

$$g_{\rm EH} = \frac{\rho}{\rho^2 - a^2} d\rho^2 + \rho(\sigma_1^2 + \sigma_2^2) + \frac{\rho^2 - a^2}{\rho}\sigma_3^2 \quad (4.11)$$

being

$$\sigma_1 = \tfrac{1}{2}(\cos\theta d\varphi + \sin\theta \sin\varphi d\tau),$$
$$\sigma_2 = \tfrac{1}{2}(-\sin\theta d\varphi + \cos\theta \sin\varphi d\tau),$$
$$\sigma_3 = \tfrac{1}{2}(d\theta + \cos\varphi d\tau).$$

The Kähler forms are given by

$$\omega_i = e^0 \wedge e^i - \epsilon_{ijk} e^j \wedge e^k, \quad (4.12)$$

e^i being the tetrad basis

$$e^0 = \sqrt{\frac{\rho}{\rho^2 - a^2}} d\rho, \qquad e^{1,2} = \sqrt{\rho}\sigma_{1,2},$$
$$e^3 = \sqrt{\frac{\rho^2 - a^2}{\rho}}\sigma_3.$$

As usual, the hyper-Kähler structure (4.11) and (4.12) will be the starting point for constructing a G_2 holonomy metric. This is achieved with the help of a function G satisfying the Laplace type equation (3.9) together with the condition (3.8). As the last condition implies that G is defined on a complex submanifold of the hyper-Kähler space, it is necessary to find a complex coordinate system for (4.11). A well-known coordinate system is the one which diagonalizes the complex structure J_3 corresponding to the Kähler form ω_3. These coordinates are [55]

$$z_1 = (\rho^2 - a^2)^{1/4} \cos\left(\frac{\varphi}{2}\right) \exp\left(i\frac{\theta + \tau}{2}\right),$$
$$z_2 = (\rho^2 - a^2)^{1/4} \sin\left(\frac{\varphi}{2}\right) \exp\left(i\frac{\theta - \tau}{2}\right). \quad (4.13)$$

The hyper-Kähler metric (4.11) is expressed in this coordinate as

$$g_{1\bar{1}} = \frac{\rho^2 |z_2|^2 + \eta^2 |z_1|^2}{\rho\eta^2}, \quad (4.14)$$

$$g_{2\bar{2}} = \frac{\rho^2 |z_1|^2 + \eta^2 |z_2|^2}{\rho\eta^2}, \quad (4.15)$$

$$g_{1\bar{2}} = \frac{\eta^2 - \rho^2}{\rho\eta^2} z_2 \bar{z}_1, \quad (4.16)$$

which is symmetric under the interchange $z_1 \leftrightarrow z_2$. We have denoted $\eta = |z_1|^2 + |z_2|^2 = \sqrt{\rho^2 - a^2}$. The advantages of considering this coordinate are clear when calculating the Laplacian

$$\Delta_{\rm EH} = \frac{1}{\sqrt{\det(g)}} \partial_i (\sqrt{\det(g)} g^{ij} \partial_j).$$

In this coordinate $\det(g) = 1$ and the inverse metric is simply

$$g^{1\bar{1}} = g_{2\bar{2}}, \qquad g^{2\bar{2}} = g_{1\bar{1}}, \qquad g^{1\bar{2}} = -g_{2\bar{1}}.$$

Moreover, after certain calculation is obtained,

$$\partial_1 (g^{1\bar{1}}) = -\partial_2 (g^{2\bar{1}}), \qquad \partial_{\bar{1}} (g^{\bar{1}1}) = -\partial_{\bar{2}} (g^{\bar{2}1}).$$

The last equalities are more easily checked with *Mathematica* than by hand. From them it follows that the action of the Laplacian acting on a function $U(z_1, \bar{z}_1)$ is simply

$$\Delta_{\rm EH} U = g^{1\bar{1}} \partial_1 \partial_{\bar{1}} U. \quad (4.17)$$

As we have explained, Eq. (3.6) will become linear if the dependence of G with respect to the complex coordinates is $G = G(w, \bar{w})$, $w = w(z_1, z_2)$ being a holomorphic function of z_1 and z_2 and $\bar{w}$ its complex conjugate. If additionally $w(z_1, z_2) = z_1$ then the action of the Laplacian will be simply (4.17). For this reason we assume that $G = G(\mu, z_1, \bar{z}_1)$. Equation (3.9) is simplified with this ansatz to

$$\mu(1 + g^{1\bar{1}} \partial_1 \partial_{\bar{1}} G) = \partial_\mu^2 G. \quad (4.18)$$

But the component $g^{1\bar{1}}$ is a function of z_2 and G, by our assumption, is not. This observation together with (4.18) implies that

$$\partial_1 \partial_{\bar{1}} G = 0, \qquad \partial_\mu^2 G = \mu. \quad (4.19)$$

The most general solution of (4.19) is simply

$$G = \frac{\mu^3}{3} + \mu(F(z_1) + \bar{F}(\bar{z}_1)) + H(z_1) + \bar{H}(\bar{z}_1), \quad (4.20)$$

F and H being functions on the complex coordinate z_1 and $\bar{F}$ and $\bar{H}$ their complex conjugated.

The function G found above determines completely a family of special holonomy metrics given by (2.9) and (2.27). The fiber quantities in these expressions are obtained as follows. From (3.4) and (4.20) it follows that $2u = \mu$. The expression of the exterior derivatives over the Eguchi-Hanson manifold in our complex coordinates is

$$d_4 = \partial_{z_i} dz^i + \partial_{\bar{z}_i} d\bar{z}^i, \qquad d_4^c = i\partial_{z_i} dz^i - i\partial_{\bar{z}_i} d\bar{z}_i$$

and their action over (4.20) gives

$$d_4 d_4^c G = 0.$$

From (3.3) and the last equation it is obtained that $\tilde{\omega}_1(\mu) = \omega_1$ which means that $g_4 = \bar{g}_4$ is the Eguchi-Hanson metric g_{EH}. Note that this equality is accidental, for other initial hyper-Kähler structures it may not hold. From (3.7) it is obtained that

$$H_1 = i\mu(F' dz^1 - \bar{F}' d\bar{z}_1) + i(H' dz^1 - \bar{H}' d\bar{z}_1) = \Im((\mu F' + H') dz_1) \tag{4.21}$$

and is clear that it takes real values. Here $'$ means the derivative with respect to the argument. The corresponding G_2 and spin(7) metrics are easily constructed from (2.9) and (2.27) and the quantities defined above, the result is

$$g_7 = \frac{\mu^2}{2} d\mu^2 + (d\beta + \Im(F) d\mu)^2 + \frac{(d\alpha + H_2)^2}{\mu^2} + \mu g_{\mathrm{EH}} \tag{4.22}$$

for the G_2 holonomy metrics and

$$g_8 = \frac{(dz + H_3)^2}{(\mu + b)^{3/2}} + (\mu + b)^{1/2}\bigg(\frac{\mu^2}{2} d\mu^2 + 2(d\beta + \Im(F) d\mu)^2 + \frac{(d\alpha + H_2)^2}{\mu^2} + \mu g_{\mathrm{EH}}\bigg), \tag{4.23}$$

for the spin(7) holonomy ones. Here $d\beta$ has been redefined by adding a total differential and H_1 and H_2 are the forms (4.7) or (4.8).

A simple inspection shows that neither of the special holonomy metrics constructed above has a signature change problem. By defining the proper coordinate $\tau = \mu^2/2$ the G_2 metric (4.22) becomes

$$g_7 = d\tau^2 + \frac{(\tau d\beta + \Im(F) d\tau)^2}{\tau} + \frac{(d\alpha + H_2)^2}{\tau} + \tau^{1/2} g_{\mathrm{EH}}, \tag{4.24}$$

and we see from the square root that τ takes positive values and there is no change in the signature. Also, by selecting $b = 0$ in the spin(7) metric and defining $\eta = \mu^{9/4}$ the following expression is obtained:

$$g_8 = d\eta^2 + \frac{(\eta^{5/9} d\beta + \Im(F) d\eta)^2}{\eta^{8/9}} + \frac{(dz + H_3)^2}{\eta^{2/3}} + \frac{(d\alpha + H_2)^2}{\eta^{2/3}} + \eta^{2/3} g_{\mathrm{EH}} \tag{4.25}$$

for (4.23) and in this case the powers of η are all even and there is not signature change as η goes from positive to negative values.

The class of metrics (4.23) and (4.22) depend on an arbitrary choice of a holomorphic function $F(z_1)$. This is the only freedom to construct them. In fact both metrics arise as an R_β fibration and their quotient by the β isometry gives the same six or seven-dimensional metric. The function F indicates the way the lift of these six or seven-dimensional metrics to a G_2 or spin(7) holonomy one is performed. Therefore (4.23) and (4.22) describe an infinite family of special holonomy metrics.

V. DISCUSSION

The main result of the present work is the prove that *any* metric of G_2 holonomy with an isometry action preserving the metric and the calibration three and four-forms in such a way that the quotient of the G_2 structure by this action is *Kähler* admits a lift to a closed spin(7) structure. As the initial seven-metric always possesses two commuting Killing vectors, the underlying spin(7) holonomy metric will possess three commuting Killing vectors. All these special holonomy manifolds are noncompact in view of the description given in [20] for the G_2 case. Additionally, we have constructed several families of special holonomy metrics and we remark (4.25) and (4.22) which, in our opinion, are new examples. We were unable to show whether or not a G_2 holonomy metric obtained by the quotient of a closed spin(7) structure by an isometry preserving it admits a further reduction to a Kähler six-dimensional structure. This is an open question.

Although our results are formulated in mathematical terms, they have physical consequences. The class of G_2 holonomy metrics admitting Kähler reductions include the examples which realize a dual description of configurations of $D6$ branes wrapping a special Lagrangian cycle in a noncompact CY threefold and satisfying the strong supersymmetry conditions, which are the conditions that convert the supersymmetry generator into a covariantly constant gauge spinor. The presence of the $D6$ branes is due to sources for the RR two-form F, that is, $dF \sim N\delta$. The dual description is achieved in terms of an 11-dimensional background and is the direct sum of the G_2 metric plus the flat Minkowski metric in four dimensions. As our results imply that any of such G_2 metrics can be lifted to one of spin(7) holonomy, one can construct as well a purely geometrical background in 11 dimensions which is the direct sum of the spin(7) metric plus the flat Minkowski metric in three dimensions. The usual Kaluza-Klein reduction along one of the isometries will give a new IIA background with a nontrivial dilaton and a self-dual RR two-form F' in such a way that the 10-dimensional metric in the string frame contains an internal part conformal to the G_2 metric. It seems reasonable to postulate that delta sources for the RR two-form F' will be present as well, and the new IIA background will describe a configuration of $D6$ branes wrapping a coassociative 4-cycle of the G_2 holonomy manifold. Unfortunately, this statement concerning the singularities of the RR two-form does not follow directly from our analysis. One can argue that there may be no delta singularities for the new configuration but instead, the new RR two-form is nontrivial due to a bad behavior at infinity. We believe that this will not be the case when the initial configuration corresponds to $D6$ branes, but we do not have proof. This is an important problem for the following reason. The IIA backgrounds described above are completely determined in terms of the G_2 holonomy metric. If a given G_2 metric is determining completely a pair of such $D6$ brane configurations, then there may be a duality connecting the corresponding supersymmetric theories, and the link is provided by the G_2 structure. In our opinion, this possibility deserves further attention, as this duality will connect theories in three and four dimensions.

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