

Non-Fermi liquid from a charged black hole: A critical Fermi ball

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Using the anti-de Sitter/conformal field theory correspondence, we calculate a fermionic spectral function in a $2 + 1$ dimensional nonrelativistic quantum field theory which is dual to a gravitational theory in the AdS_4 background with a charged black hole. The spectral function shows no quasiparticle peak but the Fermi surface is still well-defined. Interestingly, all momentum points inside the Fermi surface are critical and the gapless modes are defined in a *critical Fermi ball* in the momentum space.

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Landau Fermi liquid theory is a low energy effective theory for degenerate many-body Fermi systems. In Fermi liquid states, the curvature of a Fermi surface suppresses nonforward scatterings, which makes it possible for well-defined quasiparticles to exist in the low energy limit [1,2]. Then noninteracting Fermi gases essentially capture the qualitative nature of interacting Fermi systems. Although the Fermi liquid theory provides correct descriptions for most metals, strongly correlated systems including the high temperature superconductors [3] and heavy fermion compounds [4,5] are not described by the Fermi liquid theory. These so called non-Fermi liquid states are different states of conducting matters and an understanding of those states is very important not only for applications but also from a fundamental physics point of view. However, theoretical understandings of non-Fermi liquid states are still limited. In particular, there is no general theoretical tool which enables one to make quantitative predictions. The fundamental difficulty lies in the fact that for strongly correlated non-Fermi liquid states the well-developed standard perturbation theories usually break down.

On the other hand, there have been significant developments in understanding a class of strongly coupled quantum field theories. In the AdS/CFT correspondence, a gravitational theory in the anti-de Sitter (AdS) space is dual to a strongly coupled conformal field theory (CFT) defined on the boundary of the AdS space [6–8]. In a large N limit, the gravitational theory is reduced to a classical gravity from which one can understand nontrivial strong coupling physics of the boundary CFT. Recently the AdS/CFT correspondence has been applied to various phenomena which arise in the context of condensed matter systems [9–16]. Therefore it is of interest to find a dual gravitational description for a non-Fermi liquid state. The goal of this paper is to study dynamical properties of a strongly interacting nonrelativistic quantum field theory by calculating a fermionic spectral function from the AdS/CFT correspondence. For this, we consider a gravitational background of a charged black hole where the charge induces a nonzero density of fermions in the boundary quantum field theory.

We consider the action

$$S = \frac{1}{\kappa_4^2} \int d^4x \sqrt{-g} \left[\frac{1}{4} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{3}{2} \right] + \int d^4x \sqrt{-g} \left[\bar{\psi} \gamma^\mu \left(\partial_\mu + \frac{1}{2} \omega_{bc}^{\mu} \Sigma_{bc} - i A_\mu \right) \psi - m \bar{\psi} \psi \right] + \int d^3x \sqrt{-g_\epsilon} \bar{\psi} \psi. \quad (1)$$

Here $x^\mu = (t, x, y, z)$ is the space-time coordinate with signature $(-1, 1, 1, 1)$. R is the scalar curvature and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor of a $U(1)$ gauge field A_μ . ψ is a four-component Dirac spinor, ω_{bc}^{μ} is the spin connection, and $\Sigma_{bc} = \frac{1}{4} [\Gamma_b, \Gamma_c]$ is the generator of the local Lorentz transformation with Γ_a , the gamma matrices. $\gamma^\mu = e_a^\mu \Gamma^a$ where e_a^μ is the tetrad. The action describes the $U(1)$ gauge field and the Dirac spinor coupled with gravity in the background with a negative cosmological constant which is set to be -1 in our unit. The last term in the action is a boundary term defined at $z = \epsilon$ where g_ϵ is the determinant of the induced metric on the $2 + 1D$ space. Although the boundary term does not affect the equation of motion in the bulk, it is important for obtaining a nontrivial dependence of the saddle point action on the boundary value of the spinor field [17].

The above action has an AdS_4 black hole solution given by

$$ds^2 = \frac{1}{z^2} \left[\alpha^2 (-f(z)^2 dt^2 + dx^2 + dy^2) + \frac{dz^2}{f(z)^2} \right], \quad (2)$$

$$A_0 = q\alpha(z - 1),$$

where $f(z) = \sqrt{1 + q^2 z^4 - (1 + q^2) z^3}$. This is a special case of a more general dyonic black hole solution considered in Ref. [9]. In this coordinate system, the horizon of the black hole is at $z = 1$. The metric describes the AdS_4 space near the boundary at $z = 0$. The Hawking temperature of the black hole is $T_H = \frac{\alpha}{4\pi} (3 - q^2)$ [9]. For nonzero q and α , the black hole carries a nonzero charge. Nonvanishing components of the spin connection are

$\omega_{\hat{t}}^{\hat{z}} = z^2(\frac{f}{z})'$ and $\omega_{\hat{x}}^{\hat{z}} = \omega_{\hat{y}}^{\hat{z}} = -f$, where $(\hat{t}, \hat{x}, \hat{y}, \hat{z})$ represents the local Lorentz coordinate.

According to the AdS/CFT correspondence, we can view this classical gravitational theory as a strongly coupled 2 + 1D quantum field theory in a large N limit. The Hawking temperature corresponds to the temperature of the boundary field theory. This theory can be motivated from the M theory defined in $\text{AdS}_4 \times S^7$ which describes the low energy physics of the 2 + 1D supersymmetric Yang-Mills theory with 16 supercharges. If the infinite tower of the Kaluza-Klein modes are truncated self consistently in the M theory, the resulting theory would contain the above theory with particular values of the fermionic mass. Here, we will not restrict ourselves to the M theory and we will regard the fermionic mass as a free parameter which characterizes the corresponding 2 + 1D quantum field theory. In particular, we will focus on the case with $m = 0$ where the chiral symmetry simplifies the calculation significantly.

Recently, it has been proposed that a 2 + 1 dimensional $U(N) \times U(N)$ Chern-Simons matter theory at level k is dual to the type IIA string theory on $\text{AdS}_4 \times CP^3$ in a 't Hooft limit with a fixed $\lambda = N/k$ [18]. The present gravitational theory may be related to the Chern-Simons matter theory in the strong coupling limit ($\lambda \gg 1$) at a finite chemical potential. In this paper, instead of attempting to establish a precise connection with a microscopic theory, we take the gravitational theory in Eq. (1) as our starting point and examine the dynamics of the fermion, in a hope that the gravity description may capture some universal features of strongly interacting fermions at finite density.

The Dirac spinor is a scalar field which is linearly coupled with a fermionic field in the boundary theory. The gauge field is coupled with a conserved $U(1)$ current which includes the current of the fermion. The electrostatic potential induces a nonzero density of the boundary fermions. The chemical potential of the boundary theory is given by $\mu = A_0(z=0) = -q\alpha$. This is crucial in obtaining a system of fermions with a finite density. It is noted that the fermionic field that couples with the Dirac spinor can be a composite field in the ultraviolet theory. In the following, we will calculate the ‘‘single particle’’ spectral function of the fermion which is possibly a composite particle.

For $m = 0$, the chiral symmetry enables us to focus on one chiral mode. Because of the 2 + 1 dimensional translational symmetry, we can assume a plane wave solution for the left chiral modes ψ_- and $\bar{\psi}_-$ which satisfy $\Gamma^5 \psi_- = -\psi_-$ and $\bar{\psi}_- \Gamma^5 = -\bar{\psi}_-$,

$$\begin{aligned} \psi(t, x, y, z) &= e^{-i(\omega t - \mathbf{k} \cdot \mathbf{r})} \psi_-(z), \\ \bar{\psi}(t, x, y, z) &= e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})} \bar{\psi}_-(z), \end{aligned} \quad (3)$$

where $\mathbf{r} = (x, y)$ and $\mathbf{k} = (k_x, k_y)$. In the chiral representation with

$$\Gamma^{\hat{0}} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

$$\Gamma^{\hat{i}} = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix},$$

where σ^i are the Pauli matrices with $i = x, y, z$, and I is the 2×2 identity matrix, the equation of motion for the two-component chiral spinors becomes

$$\begin{aligned} \left[\frac{i\omega z}{\alpha f} + \frac{iqz(z-1)}{f} + \frac{i\mathbf{k} \cdot \boldsymbol{\sigma}}{\alpha} z \right. \\ \left. + \left(\frac{zf'}{2} - \frac{3f}{2} + zf\partial_z \right) \sigma^z \right] \psi_-(z) = 0, \\ \bar{\psi}_-(z) \left[\frac{i\omega z}{\alpha f} + \frac{iqz(z-1)}{f} - \frac{i\mathbf{k} \cdot \boldsymbol{\sigma}}{\alpha} z \right. \\ \left. + \left(\frac{zf'}{2} - \frac{3f}{2} + \tilde{\partial}_z z f \right) \sigma^z \right] = 0. \end{aligned} \quad (4)$$

At $T = 0$ ($q^2 = 3$), the solution near the horizon ($z \rightarrow 1$) reads $\psi_-(z) \sim e^{-i\omega(1/(6\alpha(1-z)))\sigma^z} C$, $\bar{\psi}_-(z) \sim \bar{C} e^{-i\omega(1/(6\alpha(1-z)))\sigma^z}$ in the leading order of $(1-z)$ for some fixed spinors C, \bar{C} . To maintain the causality in the boundary theory, we impose the ingoing boundary condition

$$\begin{aligned} \psi_-(z) &\sim e^{i\omega(1/(6\alpha(1-z)))} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ \bar{\psi}_-^T(z) &\sim e^{-i\omega(1/(6\alpha(1-z)))} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned} \quad (5)$$

as $z \rightarrow 1$. At a finite temperature with $q^2 < 3$, the boundary condition is modified to be

$$\psi_- \sim e^{-i\omega((\ln(1-z))/(\alpha(3-q^2)))} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$\bar{\psi}_-^T \sim e^{i\omega((\ln(1-z))/(\alpha(3-q^2)))} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Therefore there is a family of solutions parametrized by a complex number for each ψ_- and $\bar{\psi}_-$ that satisfies the ingoing boundary condition near the horizon. Near the boundary of the AdS space ($z \rightarrow 0$), the spinors behave as $\psi_-(z) \sim z^{3/2} \chi$ and $\bar{\psi}_-(z) \sim z^{3/2} \bar{\chi}$, where χ and $\bar{\chi}$ should be chosen so that the solution satisfies the ingoing boundary condition near the horizon. We represent the solution near the boundary as

$$\begin{aligned} \psi_-(z) &\sim \left(\frac{z}{\alpha} \right)^{3/2} \begin{pmatrix} P(\omega, \mathbf{k}) \\ 1 \end{pmatrix} \eta, \\ \bar{\psi}_-^T(z) &\sim \left(\frac{z}{\alpha} \right)^{3/2} \begin{pmatrix} Q(\omega, \mathbf{k}) \\ 1 \end{pmatrix} \eta^*, \end{aligned} \quad (6)$$

where η and η^* are Grassmann numbers (not spinors) that we use to impose boundary data. Once the second compo-

nents of the spinors are chosen to be η and η^* , the complex functions $P(\omega, k)$ and $Q(\omega, k)$ are uniquely determined from the boundary condition near the horizon. It is noted that we could have chosen the boundary condition near $z = 0$ in different ways. This is because the two components of the spinors decay in the same power [19]. Different boundary conditions may correspond to different field theories on the boundary [20]. However, we note that Eq. (6) is a natural choice for the following reasons. First, the vector $(0, 1)$ along which we impose boundary data is an eigenvector of σ^z , the generator of the rotation in the $x - y$ plane. Therefore, this prescription is independent of momentum direction, which guarantees that the propagator that we will calculate below is invariant under the rotation. Second, once the second component of ψ_- is chosen as boundary data, it is natural to choose the second component of $\bar{\psi}_-$ as boundary data. This can be seen by turning on a small fermion mass which mixes ψ_- and ψ_+ , and identifying $\bar{\psi}_- \sim \psi_+^\dagger$. Although not shown here, if we choose the $(1, 0)$ component as our boundary data, which is another possible choice consistent with the above conditions, we obtain the same spectral function.

From the AdS/CFT dictionary, the Green's function of the fermion in the boundary theory is given by $G(\omega, \mathbf{k}) = i \frac{\delta^2 S[\eta^*, \eta]}{\delta \eta \delta \eta^*}$, where $S[\eta^*, \eta]$ is the gravity action evaluated for the saddle configuration of the spinor fields which

satisfy the boundary conditions, Eqs. (5) and (6). The bulk spinor action in Eq. (1) vanishes at saddle points. Only the boundary term contributes to the action and we obtain the Green's function

$$G(\omega, \mathbf{k}) = i(P(\omega, \mathbf{k})Q(\omega, \mathbf{k}) + 1). \quad (7)$$

The quantity of physical importance is the spectral function, $A(\omega, \mathbf{k}) = \lim_{\delta \rightarrow 0^+} \text{Im}G(\omega + i\delta, \mathbf{k})$ which measures how much spectral weight a fermion with momentum \mathbf{k} has at energy ω .

We numerically integrate the equation of motion (4) to obtain the spectral function as a function of ω and $k = \sqrt{k_x^2 + k_y^2}$. Because of the rotational symmetry, $A(\omega, \mathbf{k})$ does not depend on momentum direction. In Fig. 1, we show the spectral function at zero temperature. For a large momentum, the spectral function as a function of energy shows a broad peak which is centered at a negative energy ω_0 . The broad peak does not disperse significantly as momentum changes. However, the width of the broad peak becomes larger as momentum increases and the edges of the broad peak disperse as $\omega_{\text{edge}} \approx \pm k + \omega_0$. There is no quasiparticle peak, which implies that the fermions are in a non-Fermi liquid state.

Although there is no delta function peak, the spectral function shows sharp peaks at zero energy for momenta smaller than a critical momentum k_c . To closely examine the low energy structure, we display the spectral function as a function of energy for fixed values of momentum in Fig. 2. The sharp peak appears to have an algebraic singularity at $\omega = 0$. In the next paragraph, we will provide numerical evidence that there exists a true singularity at zero energy. We emphasize that this is not a quasiparticle peak. As is shown in Fig. 2, the zero energy peak is more pronounced at a smaller momentum and the size of the peak decreases as momentum increases. The critical momentum k_c above which the zero energy peak disappears coincides with the momentum at which the edge of the broad peak crosses the Fermi energy $\omega = 0$. Therefore, we interpret k_c as Fermi momentum. The most striking feature is that the algebraic singularities at zero energy exist for all momenta below the Fermi momentum. Namely, all momentum points inside a two dimensional disk with $|\mathbf{k}| < k_c$ has the singular peak at zero energy. We call the set of these momentum points a critical Fermi ball. It is noted that possibilities of obtaining flat bands which are similar to the Fermi ball have been previously considered in condensed matter systems [21]. Although not shown here, the Fermi momentum and the absolute value of the energy of the broad peak increases as α increases. If we switch the sign of q , the broad peak is centered at a positive energy. However, the position of the critical Fermi ball does not change.

To show that there is true singularity at $\omega = 0$, the spectral function at a momentum below k_c is shown as a function of imaginary frequency in Fig. 3. Indeed, the

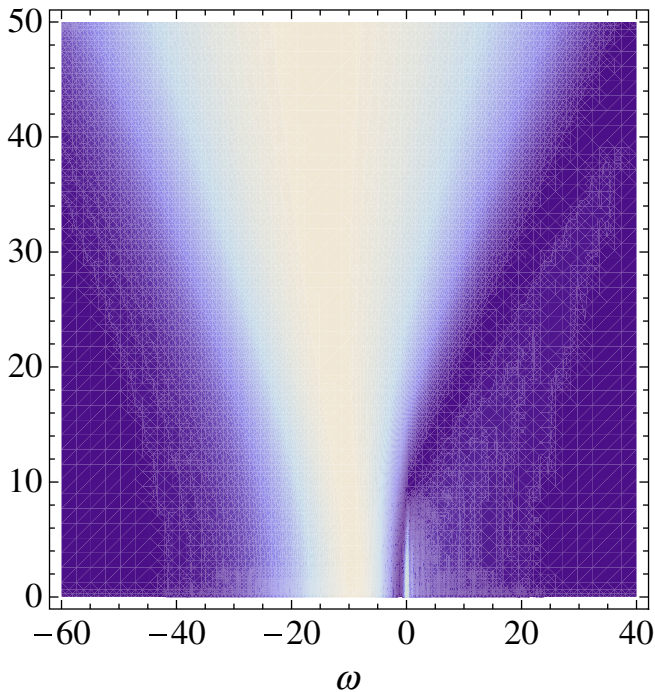


FIG. 1 (color online). The contour plot of the zero temperature spectral function as a function of energy and momentum for $q = -\sqrt{3}$ and $\alpha = 10$. The darkest region represents the area with no spectral weight and the brightest region, the highest value of the spectral function.

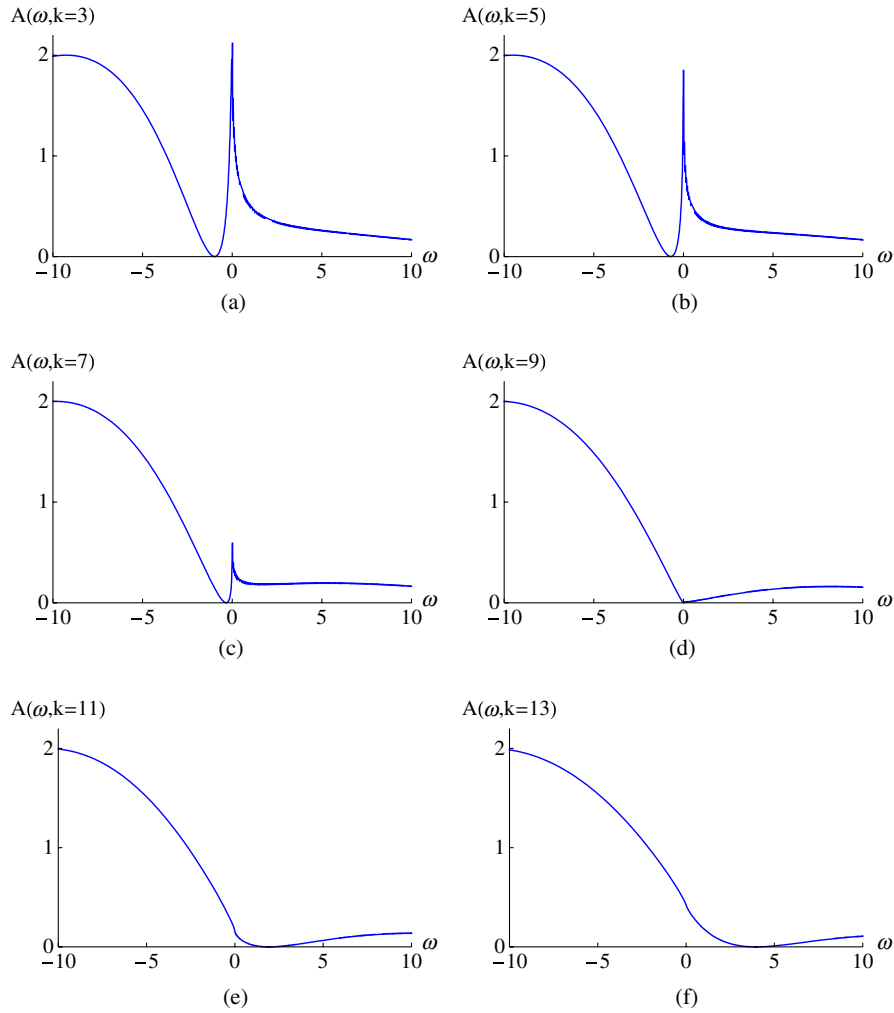


FIG. 2 (color online). The energy distribution curves of the spectral function for $q = -\sqrt{3}$ and $\alpha = 10$ at momenta (a) $k = 3$, (b) $k = 5$, (c) $k = 7$, (d) $k = 9$, (e) $k = 11$, and (f) $k = 13$.

spectral function has a strong singularity near $\omega = 0$ along the imaginary axis. A careful reader may note that the height of the peak is still finite and the position of the peak is slightly away from $\omega = 0$. These are artifacts that originated from the fact that Eq. (4) has been numerically integrated over a range $[0, 1 - \epsilon]$ with a small but nonzero ϵ to avoid the divergence in the equations at $z = 1$. As a smaller ϵ is used, the position of the peak moves to $\omega = 0$ and the singularity is enhanced systematically. This implies that the singularity at $\omega = 0$ is genuine.

Unlike the Fermi liquid state or a non-Fermi liquid state with a critical Fermi surface [22] where low energy excitations exist only near a Fermi surface, in the present non-Fermi liquid state all momentum points below the Fermi surface are important at low energies. Therefore we expect that the low energy properties of this state to be drastically different from a Fermi liquid state or a non-Fermi liquid state with a critical Fermi surface. For example, some low temperature thermodynamic properties of this 2 + 1D non-Fermi liquid state with a critical Fermi

ball will behave like a 3 + 1D critical Fermi surface. In a sense, this “dimensional lift” is not surprising because the 2 + 1D non-Fermi liquid theory is described by the 3 + 1D gravity.

What would be the origin of the non-Fermi liquid behavior? In the most trivial scenario, the non-Fermi liquid behavior can be caused by a composite nature of the

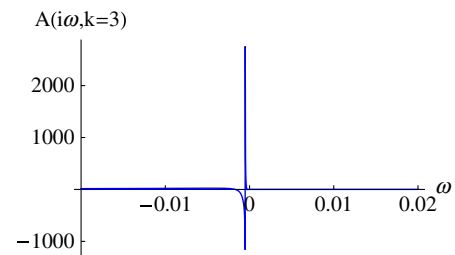


FIG. 3 (color online). The spectral function at $k = 3$ as a function of imaginary frequency with the same parameters used in Fig. 2.

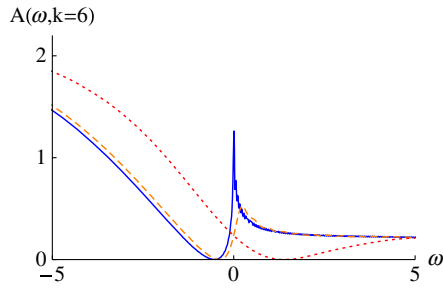


FIG. 4 (color online). Temperature dependence of the spectral function at $k = 6$. With a fixed $\alpha = 10$, q is changed to tune temperatures to $T = 0$ (solid line), $T = \frac{1}{4\pi}$ (dashed line), and $T = \frac{10}{4\pi}$ (dotted line).

fermion field. If the fermion is a composite of weakly interacting fields, it will decay into multiple modes and the spectral function will show a broad feature without a quasiparticle peak. However, the sharp zero energy peak in the spectral function suggests that the fermion field is not a mere composite of weakly interacting fields. The fermion is a rather well-defined excitation at low energies irrespective of whether it is a fundamental or composite particle. Then the non-Fermi liquid behavior can be due to strong interactions between the fermions.

Although the occurrence of the critical Fermi ball is somewhat counter-intuitive, one may understand it as a consequence of strong interactions. Since the gravitational description is valid in the strong coupling limit, where the interaction energy scale is presumably larger than the

Fermi energy, even those fermions which are deep inside the Fermi surface can participate in the low energy physics, overcoming the kinetic energy penalty.

Until now, we have examined the zero temperature spectral function. In Fig. 4, we compare the spectral function at zero temperature and finite temperatures. As expected, the singular zero energy peak is rounded at finite temperatures due to thermal fluctuations. If the temperature is high enough, the sharp peak completely disappears. At finite temperatures, the Fermi momentum is not sharply defined, but the position of the broad peak is not sensitive to temperature.

In summary, we solved the Dirac equation in a charged black hole background to extract a fermionic spectral function of a $2 + 1$ dimensional strongly coupled field theory at finite chemical potentials. The spectral function revealed a critical Fermi ball in the momentum space where all momentum points inside the Fermi ball are critical.

In the future, it would be interesting to study the physical properties of the critical Fermi ball in more detail, such as possible instabilities and thermodynamic/transport properties. Because of the presence of extensive gapless modes, it is expected that there are infinitely many singular channels of particle-hole and particle-particle excitations with different momenta which compete with each other.

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