Fusion rules and four-point functions in the AdS_3 Wess-Zumino-Novikov-Witten model

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We study the operator product expansion in the AdS_3 Wess-Zumino-Novikov-Witten (WZNW) model. The operator-product expansion of primary fields and their spectral flow images is computed from the analytic continuation of the expressions in the H_3^+ WZNW model, adding spectral flow. We argue that the symmetries of the affine algebra require a truncation which establishes the closure of the fusion rules on the Hilbert space of the theory. Although the physical mechanism determining the decoupling is not completely understood, we present several consistency checks on the results. A preliminary analysis of factorization allows to obtain some properties of four-point functions involving fields in generic sectors of the theory, to verify that they agree with the spectral flow selection rules and to show that the truncation must be realized in physical amplitudes for consistency.

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I. INTRODUCTION

String theory on AdS_3 with Neveu-Schwarz antisymmetric background field is one of the best understood string theories in curved geometries and it has been very useful for the analysis of black holes in two and three dimensions and of some cosmological spacetimes. It is so far the only case in which the AdS/CFT correspondence [1] can be examined beyond the supergravity approximation with control over the world sheet theory, and this property has allowed to show, in particular, the equivalence among three-point correlators of BPS observables in the superstring on $AdS_3 \times S^4 \times T^4$ and those of the dual conformal field theory (CFT) [2].

The world sheet of the bosonic string propagating on AdS_3 is described by the $SL(2,\mathbb{R})$ Wess-Zumino-Novikov-Witten (WZNW) model. The string spectrum is built from affine primaries of a product of left and right copies of the universal cover of $SL(2,\mathbb{R})$ and their spectral flow images [3]. It consists of long strings with the continuous energy spectrum arising from the principal continuous representation and its spectral flow images, and short strings with discrete physical spectrum resulting from the highest-weight discrete representation and its spectral flow images. A no ghost theorem for this spectrum was proved in [3] and verified in [4]. Amplitudes on the sphere were computed in [5], analytically continuing the expressions obtained for the Euclidean $H_3^+ = \frac{SL(2,\mathbb{C})}{SU(2)}$ WZNW model in [6,7]. Some subtleties of the analytic continuation relating the H_3^+ and AdS_3 models were clarified in [5] and this allowed one to construct, in particular, the four-point function of unflowed short strings. Integrating over the moduli space of the world sheet, it was shown that the string amplitude can be expressed as a sum of products of three-point functions with intermediate physical states, i.e., the structure of the factorization agrees with the Hilbert space of the theory.

A step-up towards a proof of consistency and unitarity of the theory involves the construction of four-point functions including states in different representations and the verification that only unitary states corresponding to long and short strings in agreement with the spectral flow selection rules are produced in the intermediate channels. To achieve this goal, the analytic and algebraic structure of the $SL(2,\mathbb{R})$ WZNW model should be explored further.

Most of the important progress achieved in [5] is based on the better though not yet completely understood Euclidean H_3^+ model. Together with the Liouville theory, these are examples of nonrational conformal field theories with continuous families of primary fields. The absence of singular vectors and the lack of chiral factorization in the relevant current algebra representations obstruct the use of the powerful techniques from rational conformal field theories. Nevertheless, a generalized conformal bootstrap approach was successfully applied in [6,7] to the H_3^+ model on the punctured sphere, allowing one to discuss the factorization of four-point functions. In principle, this method offers the possibility to unambiguously determine any $n >$ 3-point function in terms of two- and three-point functions once the operator product expansions of two operators and the structure constants are known. Aiming to carry out some initial steps towards developing this procedure for the more involved AdS_3 WZNW model, in these notes we examine the role of the spectral flow symmetry on the analytic continuation of the operator product expansion from H_3^+ to the relevant representations of $SL(2,\mathbb{R})$ and on the factorization properties of four-point functions.

While only contributions of the highest-weight states are usually written in an operator-product expansion (OPE), the descendants being neglected, a fundamental problem of the AdS_3 WZNW model is that the spectral flow operation maps primaries into descendants and vice versa. Thus, to

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complete this program it is necessary to learn more about the spectral flow representations and the secondary fields than is currently known. Nevertheless, based on previous work in [5–10], we are able to make some progress. We obtain the OPE of fields in all sectors of the theory and discuss some properties of the factorization limit of four-point functions.

The paper is organized as follows. In Sec. II we review some well-known results on the H_3^+ and AdS_3 WZNW models in order to setup the notations. In Sec. III we analytically continue the expressions obtained in [6,7] from the Euclidean to the Lorentzian model and we add spectral flow to obtain the OPE of primary fields and their spectral flow images. The extension of the OPE to generic descendants is discussed in Sec. IV where we show that the spectral flow symmetry requires a truncation of the fusion rules determining the closure of the operator algebra on the Hilbert space of the theory. In Sec. V we consider the factorization of four-point functions and study some of its properties. Finally, Sec. VI contains a summary and conclusions. Some technical details of the calculations are included in Appendices A 1 and A 3 and the relation of our results to certain conclusions in [5] is the content of Appendix A 2.

II. REVIEW OF THE H_3^+ AND AdS_3 WZNW MODELS

In this section we review some well-known results on the H_3^+ and the AdS_3 WZNW models in order to setup the notations.

A thorough study of the $H_3^+ = \frac{SL(2,\mathbb{C})}{SU(2)}$ WZNW model was presented in [6,7]. The Lagrangian formulation was developed in [11] and it follows from

$$\mathcal{L} = k(\partial\phi\bar{\partial}\phi + e^{2\phi}\bar{\partial}\gamma\partial\bar{\gamma}). \tag{2.1}$$

Normalizable operators $\Phi_j(x, \bar{x}; z, \bar{z})$, $x, z \in \mathbb{C}$, are labeled by the spin $j = -\frac{1}{2} + i\mathbb{R}_+$ of a principal continuous representation of $SL(2, \mathbb{C})$ and can be semiclassically identified with the expression

$$\Phi_j(x, \bar{x}; z, \bar{z}) = \frac{2j+1}{\pi}((\gamma - x)(\bar{\gamma} - \bar{x})e^{\phi} + e^{-\phi})^{2j}. \tag{2.2}$$

They satisfy the following OPE with the holomorphic $SL(2, \mathbb{C})$ currents

$$J^a(z)\Phi_j(x, \bar{x}; z', \bar{z}') \sim \frac{D^a\Phi_j(x, \bar{x}; z', \bar{z}')}{z - z'}, \qquad a = \pm, 3, \tag{2.3}$$

where $D^- = \partial_x$, $D^3 = x\partial_x - j$, $D^+ = x^2\partial_x - 2jx$, and they have the conformal weight $\tilde{\Delta} = -\frac{j(j+1)}{k-2}$. The asymptotic $\phi \to \infty$ expansion, given by

$$\Phi_j(x, \bar{x}|z, \bar{z}) \sim :e^{2(-1-j)\phi(z)}:\delta^2(\gamma(z) - x) + B(j):e^{2j\phi(z)}:|\gamma(z) - x|^{4j}, \tag{2.4}$$

fixes a normalization and determines the relation between Φ_j and Φ_{-1-j} as

$$\Phi_j(x, \bar{x}|z, \bar{z}) = B(j)\int_{\mathbb{C}} d^2x'|x - x'|^{4j}\Phi_{-1-j}(x', \bar{x}'; z, \bar{z}), \tag{2.5}$$

where the reflection coefficient $B(j)$ is given by

$$B(j) = \frac{k-2}{\pi}\frac{\nu^{1+2j}}{\gamma(-\frac{1+2j}{k-2})}, \qquad \nu = \pi\frac{\Gamma(1 - \frac{1}{k-2})}{\Gamma(1 + \frac{1}{k-2})}, \qquad \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}. \tag{2.6}$$

For our purposes, it is convenient to transform the primary fields to the m basis as

$$\Phi^j_{m,\bar{m}}(z, \bar{z}) = \int d^2x x^{j+m}\bar{x}^{j+\bar{m}}\Phi_{-1-j}(x, \bar{x}; z, \bar{z}), \tag{2.7}$$

where $m = \frac{n+is}{2}$, $\bar{m} = \frac{-n+is}{2}$, $n \in \mathbb{Z}$, $s \in \mathbb{R}$. The fields $\Phi^j_{m,\bar{m}}$ have the following OPE with the chiral currents

$$\begin{aligned} J^{\pm}(z)\Phi^j_{m,\bar{m}}(z', \bar{z}') &\sim \frac{\mp j + m}{z - z'}\Phi^j_{m\pm 1,\bar{m}}(z', \bar{z}'),\\ J^3(z)\Phi^j_{m,\bar{m}}(z', \bar{z}') &\sim \frac{m}{z - z'}\Phi^j_{m,\bar{m}}(z', \bar{z}'), \end{aligned} \tag{2.8}$$

and the relation between $\Phi^j_{m,\bar{m}}$ and $\Phi^{-1-j}_{m,\bar{m}}$ is given by

$$\begin{aligned} \Phi^j_{m,\bar{m}}(z, \bar{z}) &= B(-1-j)c^{-1-j}_{m,\bar{m}}\Phi^{-1-j}_{m,\bar{m}}(z, \bar{z})\\ &= \frac{\pi B(-1-j)}{\gamma(2+2j)}\\ &\quad\times\frac{\Gamma(1+j+m)\Gamma(1+j-\bar{m})}{\Gamma(-j+m)\Gamma(-j-\bar{m})}\Phi^{-1-j}_{m,\bar{m}}(z, \bar{z}). \end{aligned} \tag{2.9}$$

The following operator product expansion for any product $\Phi_{j_1}\Phi_{j_2}$ was determined in [6,7]:

$$\begin{aligned} &\Phi_{j_2}(x_2|z_2)\Phi_{j_1}(x_1|z_1)\\ &\quad= \int_{\mathcal{P}^+} dj_3 C(-j_1, -j_2, -j_3)|z_2 - z_1|^{-\tilde{\Delta}_{12}}\\ &\quad\quad\times\int_{\mathbb{C}} d^2x_3|x_1 - x_2|^{2j_{12}}|x_1 - x_3|^{2j_{13}}|x_2 - x_3|^{2j_{23}}\\ &\quad\quad\times\Phi_{-1-j_3}(x_3|z_1) + \text{descendants}. \end{aligned} \tag{2.10}$$

Here, the integration contour is $\mathcal{P}^+ = -\frac{1}{2} + i\mathbb{R}_+$, the structure constants $C(j_i)$ are given by

$$C(j_1,j_2,j_3) = -\frac{G(1-j_1-j_2-j_3)G(-j_{12})G(-j_{13})G(-j_{23})}{2\pi^2\nu^{j_1+j_2+j_3-1}\gamma(\frac{k-1}{k-2})G(-1)G(1-2j_1)G(1-2j_2)G(1-2j_3)}, \tag{2.11}$$

with $G(j) = (k-2)^{((j(1-j-k))/(2(k-2)))}\Gamma_2(-j|1,k-2)\times \Gamma_2(k-1+j|1,k-2)$, $\Gamma_2(x|1,w)$ being the Barnes double Gamma function, $\tilde{\Delta}_{12} = \tilde{\Delta}(j_1) + \tilde{\Delta}(j_2) - \tilde{\Delta}(j_3)$ and $j_{12} = j_1 + j_2 - j_3$, etc.

The OPE (2.10) holds for a range of values of j_1, j_2 given by

$$|\mathrm{Re}(j_{21}^{\pm})| < \tfrac{1}{2}, \qquad j_{21}^{+} = j_2 + j_1 + 1, \qquad j_{21}^{-} = j_2 - j_1. \tag{2.12}$$

This is the maximal region in which j_1, j_2 may vary such that none of the poles of the integrand hits the contour of integration over j_3. However, as long as the imaginary parts of $j_{21}^{\pm}$ do not vanish, Teschner [7] showed that (2.10) admits an analytic continuation to generic complex values of j_1, j_2, defined by deforming the contour $\mathcal{P}^+$. The deformed contour is given by the sum of the original one plus a finite number of circles around the poles leading to a finite sum of residue contributions to the OPE. When $j_{21}^{\pm}$ are real, one can give them a small imaginary part which is sent to zero after deforming the contour.

Inserting (2.10) into a four-point function gives an expansion of the correlator which takes the form of an integral with respect to the spin of the intermediate representation. The integrand factorizes into structure constants, two-point functions, and conformal blocks. Since these expressions are analytic in j's (up to delta functions), correlation functions involving states with arbitrary spin values may be obtained through an appropriate analytic continuation. This procedure was implemented in [5] to construct the four-point function of short strings in AdS_3.

The world sheet of the string propagating on AdS_3 is described by the $SL(2,\mathbb{R})$ WZNW model which shares the $\widehat{sl(2)}$ symmetries with the H_3^+ model though it differs in the allowed representations. The spectrum of the AdS_3 WZNW model was determined in [3] and it is constructed from a product of left and right copies of representations of the universal cover of $SL(2,\mathbb{R})$. It is built on products of conventional representations of the zero modes, i.e., the principal continuous representations $\mathcal{C}_j^\alpha \otimes \mathcal{C}_j^\alpha$ with $j = -\frac{1}{2} + i\mathbb{R}$, $\alpha = (0,1]$ and the lowest-weight discrete series $\mathcal{D}_j^+ \otimes \mathcal{D}_j^+$ with $j \in \mathbb{R}$ and $-\frac{k-1}{2} < j < -\frac{1}{2}$. It contains the current algebra descendants $\hat{\mathcal{C}}_j^\alpha \otimes \hat{\mathcal{C}}_j^\alpha$, $\hat{\mathcal{D}}_j^+ \otimes \hat{\mathcal{D}}_j^+$, and spectral flow images $\hat{\mathcal{C}}_j^{\alpha,w} \otimes \hat{\mathcal{C}}_j^{\alpha,w}$, $\hat{\mathcal{D}}_j^{+,w} \otimes \hat{\mathcal{D}}_j^{+,w}$, with the same value of j and the same amount of spectral flow on the left and right sectors. Throughout this paper, we deal with these representations of the universal cover of $SL(2,\mathbb{R})$, to which we refer as $SL(2,\mathbb{R})$ for short.

The spectral flow representations are generated by the following automorphism of the current algebra

$$\tilde{J}_n^3 = J_n^3 - \frac{k}{2}w\delta_{n,0}, \qquad \tilde{J}_n^{\pm} = J_{n\pm w}^{\pm}, \tag{2.13}$$

with $w \in \mathbb{Z}$, which gives a copy of the Virasoro algebra with

$$\tilde{L}_n = L_n + wJ_n^3 - \frac{k}{4}w^2. \tag{2.14}$$

Unlike in the compact $SU(2)$ case, different amounts of spectral flow give inequivalent representations of the current algebra of $SL(2,\mathbb{R})$.

An affine primary state in the unflowed sector is mapped by the automorphism (2.13) to a highest/lowest-weight state of the global $sl(2)$ algebra. We denote these fields in the spectral flow sector w as $\Phi_{m,\bar{m}}^{j,w}$. Their explicit expressions will not be needed below. It is only necessary to know that they verify the following OPE with the currents:

$$J^3(z)\Phi_{m,\bar{m}}^{j,w}(z',\bar{z}') \sim \frac{m+\frac{k}{2}w}{z-z'}\Phi_{m,\bar{m}}^{j,w}(z',\bar{z}'),$$
$$J^{\pm}(z)\Phi_{m,\bar{m}}^{j,w}(z',\bar{z}') \sim \frac{\mp j+m}{(z-z')^{\pm w}}\Phi_{m\pm1,\bar{m}}^{j,w}(z',\bar{z}') + \cdots$$

and $m-\bar{m} \in \mathbb{Z}$, $m+\bar{m} \in \mathbb{R}$.

Two- and three-point functions of the fields $\Phi_j(x|z)$ in the H_3^+ model were computed in [6,7]. Following [5,8,9,12], we assume that correlation functions of primary fields in the $SL(2,\mathbb{R})$ WZNW model are those of H_3^+ with j_i, m_i, $\bar{m}_i$ taking values in representations of $SL(2,\mathbb{R})$. The spectral flow operation is straightforwardly performed in the m basis where the only change in the w-conserving expectation values of fields $\Phi_{m,\bar{m}}^{j,w}$ in different w sectors is in the powers of the coordinates z_i, $\bar{z}_i$. Correlation functions may violate w conservation according to the following spectral flow selection rules:

$$-N_t+2 \le \sum_{i=1}^{N_t} w_i \le N_c - 2, \quad \text{at least one state in } \hat{\mathcal{C}}_j^{\alpha,w}\otimes\hat{\mathcal{C}}_j^{\alpha,w}, \tag{2.15}$$

$$-N_d+1 \le \sum_{i=1}^{N_t} w_i \le -1, \quad \text{all states in } \hat{\mathcal{D}}_j^{+,w}\otimes\hat{\mathcal{D}}_j^{+,w}, \tag{2.16}$$

with $N_t = N_c + N_d$ and N_c, N_d are the total numbers of operators in $\hat{\mathcal{C}}_j^{\alpha,w}\otimes\hat{\mathcal{C}}_j^{\alpha,w}$ and $\hat{\mathcal{D}}_j^{+,w}\otimes\hat{\mathcal{D}}_j^{+,w}$, respectively.

The spectral flow preserving two-point function is given by

$$\langle\Phi^{j,w}_{m,\bar m}(z,\bar z)\Phi^{j',-w}_{m',\bar m'}(z',\bar z')\rangle = \delta^2(m+m')(z-z')^{-2\Delta(j)} \times(\bar z-\bar z')^{-2\bar\Delta(j)}[\delta(j+j'+1) + B(-1-j)c^{-1-j}_{m,\bar m}\delta(j-j')], \tag{2.17}$$

where $\Delta(j)=\tilde\Delta(j)-wm-\frac{k}{4}w^2=-\frac{j(j+1)}{k-2}-wm-\frac{k}{4}w^2$. For states in discrete series, it is convenient to work with spectral flow images of both lowest- and highest-weight representations related by the identification $\hat{\mathcal D}^{+,w}_j\equiv\hat{\mathcal D}^{-,w+1}_{-(k/2)-j}$, which determines the range of values for the spin

$$-\frac{k-1}{2}<j<-\frac{1}{2}, \tag{2.18}$$

and allows one to obtain the (± 1) unit spectral flow two-point functions from (2.17).

Spectral flow conserving three-point functions are the following:

$$\left\langle\prod_{i=1}^3\Phi^{j_i,w_i}_{m_i,\bar m_i}(z_i,\bar z_i)\right\rangle = \delta^2\left(\sum m_i\right)C(1+j_i)W\left[\begin{array}{c} j_1,j_2,j_3\\ m_1,m_2,m_3\end{array}\right] \times\prod_{i<j}z_{ij}^{-\Delta_{ij}}\bar z_{ij}^{-\bar\Delta_{ij}}, \tag{2.19}$$

where $z_{ij}=z_i-z_j$ and $C(j_i)$ is given by (2.11). The function W is

$$W\left[\begin{array}{c} j_1,j_2,j_3\\ m_1,m_2,m_3\end{array}\right] = \int d^2x_1d^2x_2x_1^{j_1+m_1}\bar x_1^{j_1+\bar m_1}x_2^{j_2+m_2}\bar x_2^{j_2+\bar m_2} \times|1-x_1|^{-2j_{13}-2}|1-x_2|^{-2j_{23}-2} \times|x_1-x_2|^{-2j_{12}-2}, \tag{2.20}$$

and we omit the obvious $\bar m$ dependence in the arguments to lighten the notation. This integral was computed in [13].

The one unit spectral flow three-point function [5] is given by[1]

$$\left\langle\prod_{i=1}^3\Phi^{j_i,w_i}_{m_i,\bar m_i}(z_i,\bar z_i)\right\rangle = \delta^2\left(\sum m_i\pm\frac{k}{2}\right) \times\frac{\tilde C(1+j_i)\tilde W\left[\begin{array}{c} j_1,j_2,j_3\\ \pm m_1,\pm m_2,\pm m_3\end{array}\right]}{\gamma(j_1+j_2+j_3+3-\frac{k}{2})} \times\prod_{i<j}z_{ij}^{-\Delta_{ij}}\bar z_{ij}^{-\bar\Delta_{ij}}, \tag{2.21}$$

where $\sum_i w_i=\pm1$, the $\pm$ signs correspond to the $\pm$ signs in the right-hand side (r.h.s.),

$$\tilde C(j_i)\sim B(-j_1)C\left(\frac{k}{2}-j_1,j_2,j_3\right), \tag{2.22}$$

up to k-dependent factors, j-independent factors, and

$$\tilde W\left[\begin{array}{c} j_1,j_2,j_3\\ m_1,m_2,m_3\end{array}\right] = \frac{\Gamma(1+j_1+m_1)}{\Gamma(-j_1-\bar m_1)}\frac{\Gamma(1+j_2+\bar m_2)}{\Gamma(-j_2-m_2)} \times\frac{\Gamma(1+j_3+\bar m_3)}{\Gamma(-j_3-m_3)}. \tag{2.23}$$

For discrete states, this expression is related to the $\sum_i w_i=\pm 2$ three-point function through $\hat{\mathcal D}^{+,w}_j\equiv\hat{\mathcal D}^{-,w+1}_{-(k/2)-j}$.

In the following sections, we shall use these results to study the analytic continuation of the OPE (2.10) from the H_3^+ to the AdS_3 WZNW model. Then, we shall discuss some aspects of the factorization of four-point functions.

III. OPERATOR ALGEBRA IN THE $SL(2,\mathbb{R})$ WZNW MODEL

A nontrivial check on the OPE (2.10) and structure constants (2.11) of the H_3^+ WZNW model is that the well-known fusion rules of degenerate representations [15] are exactly recovered by analytically continuing j_i, $i=1,2$ [6]. On the other hand, it was argued in [5–9,12] that correlation functions in the H_3^+ and AdS_3 WZNW models are related by analytic continuation and moreover, the $k\to\infty$ limit of the OPE of unflowed fields computed along these lines in [8,9] exhibits complete agreement with the classical tensor products of representations of $SL(2,\mathbb{R})$ [16]. It seems then natural to conjecture that the OPE of all fields in the spectrum of the AdS_3 WZNW model can be obtained from (2.10) analytically continuing j_1, j_2 from the range (2.12).

However, the spectral flowed fields do not belong to the spectrum of the H_3^+ model and moreover, the spectral flow symmetry transforms primaries into descendants. Thus, a better knowledge of these representations seems necessary in order to obtain the fusion rules in the AdS_3 model. Nevertheless, we will show that it is possible to obtain them from the H_3^+ model by analytic continuation and by taking into account the w-violating structure constants in addition to (2.11). In this section, we explore this possibility in order to get the OPE of primary fields and their spectral flow images in the $SL(2,\mathbb{R})$ WZNW model.

To deal with highest/lowest weight and spectral flow representations, it is convenient to work in the m basis. We have to keep in mind that when j is real, new divergences appear in the transformation from the x basis and it must be performed for certain values of m_i, $\bar m_i$, $i=1,2$. Indeed, to transform the OPE (2.10) to the m basis using (2.7), the integrals over x_1, x_2 in the r.h.s. must be interchanged with the integral over j_3 and this process does not commute in general if there are divergences. However, restricting j_1, j_2 to the range (2.12), one can check that

[1] For an independent calculation of three-point functions using the free field approach, see [14].

the integrals commute and are regular when $|m_i| < \frac{1}{2}$ and $|\bar{m}_i| < \frac{1}{2}$, $i = 1, 2, 3$, where $m_3 = m_1 + m_2$, $\bar{m}_3 = \bar{m}_1 + \bar{m}_2$. For other values of m_i, $\bar{m}_i$ the OPE must be defined, as usual, by analytic continuation of the parameters. Therefore, after performing the x_1, x_2 integrals, the OPE (2.10) in the m basis is found to be

$$\Phi^{j_1}_{m_1,\bar{m}_1}(z_1,\bar{z}_1)\Phi^{j_2}_{m_2,\bar{m}_2}(z_2,\bar{z}_2)|_{w=0} = \int_{\mathcal{P}} dj_3 |z_{12}|^{-2\tilde{\Delta}_{12}} Q^{w=0}\begin{bmatrix} j_1, j_2, j_3 \\ m_1, m_2, m_3 \end{bmatrix} \Phi^{j_3}_{m_3,\bar{m}_3}(z_1,\bar{z}_2) + \text{descendants}, \tag{3.1}$$

where we have defined

$$Q^{w=0}\begin{bmatrix} j_1, j_2, j_3 \\ m_1, m_2, m_3 \end{bmatrix} = C(1+j_1, 1+j_2, -j_3) \times W\begin{bmatrix} j_1, j_2, -1-j_3 \\ m_1, m_2, -m_3 \end{bmatrix}. \tag{3.2}$$

It is easy to see that the integrand is symmetric under $j_3 \to -1 - j_3$ using the identity [9]

$$\frac{W\begin{bmatrix} j_1, j_2, j_3 \\ m_1, m_2, m_3 \end{bmatrix}}{W\begin{bmatrix} j_1, j_2, -1-j_3 \\ m_1, m_2, m_3 \end{bmatrix}} = \frac{C(1+j_1, 1+j_2, -j_3)}{C(1+j_1, 1+j_2, 1+j_3)} \times B(-1-j_3) c^{-1-j_3}_{m_3,\bar{m}_3}, \tag{3.3}$$

and as a consequence of (2.9). In the x basis, every pole in (2.10) appears duplicated, one over the real axis and another one below, and the $j_3 \to -1 - j_3$ symmetry implies that the integral may be equivalently performed either over $\mathrm{Im} j_3 > 0$ or over $\mathrm{Im} j_3 < 0$ [7]. In the m basis, the (j_1, j_2)-dependent poles are also duplicated, but the m-dependent poles are not. The $j_3 \to -1 - j_3$ symmetry is still present, as we discussed above, because of poles and zeros in the normalization of $\Phi^j_{m,\bar{m}}$. The integral must be extended to the full axis $\mathcal{P} = -\frac{1}{2} + i\mathbb{R}$ before performing the analytic continuation in m_1, m_2 because the m-dependent poles fall on the real axis. The maximal regions in which m_1, m_2 may vary such that none of the poles hit the contour of integration are $|m_1 + m_2| < \frac{1}{2}$ and $|\bar{m}_1 + \bar{m}_2| < \frac{1}{2}$.

Since the w-conserving structure constants of operators $\Phi^{j,w}_{m,\bar{m}} \in \mathcal{C}^{\alpha,w}_j$ or $\mathcal{D}^{+,w}_j$ in different w sectors do not change in the m basis,[2] the OPE (3.1) should also hold for fields obtained by spectral flowing primaries to arbitrary w sectors, as long as they satisfy $w_1 + w_2 = w_3$. But this OPE would yield an incorrect zero answer if used to compute a w-violating three-point function. It then seems natural to additionally take into account the spectral flow nonpreserving structure constants and consider the following OPE[3]:

$$\Phi^{j_1,w_1}_{m_1,\bar{m}_1}(z_1,\bar{z}_1)\Phi^{j_2,w_2}_{m_2,\bar{m}_2}(z_2,\bar{z}_2) = \sum_{w=-1}^{1}\int_{\mathcal{P}} dj_3 Q^w z_{12}^{-\Delta_{12}}\bar{z}_{12}^{-\bar{\Delta}_{12}} \times \Phi^{j_3,w_3}_{m_3,\bar{m}_3}(z_2,\bar{z}_2) + \cdots, \tag{3.4}$$

with $w = w_3 - w_1 - w_2$, $m_3 = m_1 + m_2 - \frac{k}{2}w$, $\bar{m}_3 = \bar{m}_1 + \bar{m}_2 - \frac{k}{2}w$, and

$$Q^{w=\pm1}(j_i; m_i, \bar{m}_i) = \tilde{W}\begin{bmatrix} j_1, j_2, j_3 \\ \mp m_1, \mp m_2, \pm m_3 \end{bmatrix} \frac{\tilde{C}(j_i+1)}{B(-1-j_3)c^{-j_3-1}_{m_3,\bar{m}_3}\gamma(j_1+j_2+j_3+3-\frac{k}{2})} \sim \frac{\Gamma(\pm\bar{m}_3 - j_3)}{\Gamma(1+j_3\mp m_3)}\prod_{a=1}^{2}\frac{\Gamma(1+j_a\mp m_a)}{\Gamma(-j_a\pm\bar{m}_a)}\frac{C(\frac{k}{2}-1-j_1, 1+j_2, 1+j_3)}{\gamma(j_1+j_2+j_3+3-\frac{k}{2})}. \tag{3.5}$$

For completeness, according to the spectral flow selection rules (2.16), we should also include terms with $w = \pm 2$ in the sum. However, we shall show in the next section that these do not affect the results of the OPE. The integrand is symmetric under $j_3 \to -1 - j_3$. This follows from (3.3) and the analogous identity

$$\frac{\tilde{W}\begin{bmatrix} j_1, j_2, j_3 \\ m_1, m_2, m_3 \end{bmatrix}}{\tilde{W}\begin{bmatrix} j_1, j_2, -1-j_3 \\ m_1, m_2, m_3 \end{bmatrix}} = \frac{\tilde{C}(1+j_1, 1+j_2, -j_3)\gamma(j_1+j_2+j_3+3-\frac{k}{2})}{\tilde{C}(1+j_1, 1+j_2, 1+j_3)\gamma(j_1+j_2-j_3+2-\frac{k}{2})} B(-1-j_3)c^{-1-j_3}_{m_3,\bar{m}_3}, \tag{3.6}$$

together with the reflection relation

$$\Phi^{j,w}_{m,\bar{m}}(z,\bar{z}) = B(-1-j)c^{-1-j}_{m,\bar{m}}\Phi^{-1-j,w}_{m,\bar{m}}(z,\bar{z}). \tag{3.7}$$

The dots in (3.4) stand for spectral flow images of current algebra descendants with the same J^3_0 eigenvalues m_3, $\bar{m}_3$. This expression is valid for j_1, j_2 in the range (2.12) and the restrictions on m_1, m_2 depend on Q^w. The maximal regions in which they may vary such that none of the poles hit the contour of integration are, other than $|m_1 + m_2| < \frac{1}{2}$ and

[2]We denote the series containing the highest/lowest-weight states obtained by spectral flowing primaries as $\mathcal{C}^{\alpha,w}_j$, $\mathcal{D}^{+,w}_j$.

[3]A similar expression was proposed in [10] and some supporting evidence was presented from the relation between the H_3^+ model and Liouville theory.

$|\bar{m}_1 + \bar{m}_2| < \frac{1}{2}$ for $Q^{w=0}$, $\min\{m_1 + m_2, \bar{m}_1 + \bar{m}_2\} < -\frac{k-1}{2}$ for $Q^{w=-1}$ and $\max\{m_1 + m_2, \bar{m}_1 + \bar{m}_2\} > \frac{k-1}{2}$ for $Q^{w=+1}$. For other values of j_1, j_2 and m_1, m_2 the OPE must be defined by analytic continuation. In the rest of this section we perform this continuation.

To specifically display the contributions to (3.4), we have to study the analytic structure of Q^w. We first consider the simpler case $w = \pm 1$ and we refer to the terms proportional to $Q^{w=\pm 1}$ as *spectral flow nonpreserving contributions* to the OPE. Then, we investigate $Q^{w=0}$ and obtain the *spectral flow preserving contributions*.

A. Spectral flow nonpreserving contributions

Let us study the analytic structure of $Q^{w=\pm 1}$ in (3.5). The m-independent poles arising from the last factor are the same for both $w = \pm 1$ sectors and are explicitly given by

$$\begin{aligned} j_3 &= \pm j_{21}^- + \frac{k}{2} - 1 + p + q(k-2), \\ j_3 &= \pm j_{21}^- - \frac{k}{2} - p - q(k-2), \\ j_3 &= \pm j_{21}^+ + \frac{k}{2} - 1 + p + q(k-2), \\ j_3 &= \pm j_{21}^+ - \frac{k}{2} - p - q(k-2), \end{aligned} \tag{3.8}$$

with $p, q = 0, 1, 2, \ldots$ The m-dependent poles, instead, vary according to the spectral flow sector. However they are connected through $(m, \bar{m}) \leftrightarrow (-m, -\bar{m})$ and thus going from $w = -1$ to $w = +1$ involves the change $\mathcal{D}_{j_i}^{-,w_i} \otimes \mathcal{D}_{j_i}^{-,w_i} \leftrightarrow \mathcal{D}_{j_i}^{+,w_i} \otimes \mathcal{D}_{j_i}^{+,w_i}$. Therefore we concentrate on the contributions from $w = -1$.

By abuse of notation, from now on we denote the states by the representations they belong to and we write only the holomorphic sector for short, e.g., when $\Phi_{m_i,\bar{m}_i}^{j_i,w_i} \in \mathcal{D}_{j_i}^{+,w_i} \otimes \mathcal{D}_{j_i}^{+,w_i}$, $i = 1, 2$, we write the set of all possible operator products $\Phi_{m_1,\bar{m}_1}^{j_1,w_1} \Phi_{m_2,\bar{m}_2}^{j_2,w_2}$ for generic quantum numbers within these representations as $\mathcal{D}_{j_1}^{+,w_1} \times \mathcal{D}_{j_2}^{+,w_2}$.

Let us study the OPE of fields in all different combinations of representations. First consider the case $\Phi_{m_i,\bar{m}_i}^{w_i,j_i} \in \mathcal{C}_{j_i}^{\alpha_i,w_i} \otimes \mathcal{C}_{j_i}^{\alpha_i,w_i}$, $i = 1, 2$.

1. $\mathcal{C}_{j_1}^{\alpha_1,w_1} \times \mathcal{C}_{j_2}^{\alpha_2,w_2}$

The pole structure of $Q^{w=-1}$ is represented in Fig. 1(a) for $\min\{m_1 + m_2, \bar{m}_1 + \bar{m}_2\} < -\frac{k-1}{2}$. Recalling that $m_3 = m_1 + m_2 + \frac{k}{2}$, then $\min\{m_3, \bar{m}_3\} < \frac{1}{2}$, and therefore the poles from the factor $\frac{\Gamma(-j_3-\bar{m}_3)}{\Gamma(1+j_3+m_3)}$ are to the right of the integration contour. Moreover, given that all m-independent poles are to the right of the axis $\frac{k}{2} - 1$ or to the left of $-\frac{k}{2}$, we conclude that the OPE $\mathcal{C}_{j_1}^{\alpha_1,w_1} \times \mathcal{C}_{j_2}^{\alpha_2,w_2}$ receives no spectral flow violating contributions from discrete representations when $\min\{m_1 + m_2, \bar{m}_1 + \bar{m}_2\} < -\frac{k-1}{2}$.

Some poles cross the integration contour when $\min\{m_1 + m_2, \bar{m}_1 + \bar{m}_2\} > -\frac{k-1}{2}$. They are sketched in Fig. 1(b) and indicate contributions from the discrete series $\mathcal{D}_{j_3}^{+,w_3=w_1+w_2-1}$ with $j_3 = -\min\{m_3, \bar{m}_3\} + n$, $n = 0, 1, 2, \ldots$, and such that $j_3 < -\frac{1}{2}$. Since $Q^{w=\pm 1}$ does not vanish for $j_3 = -\frac{1}{2} + i\mathbb{R}$ and m_3 is not correlated with j_3, there are terms from $\mathcal{C}_{j_3}^{\alpha_3,w_3=w_1+w_2-1}$ in this OPE as well. Therefore we get

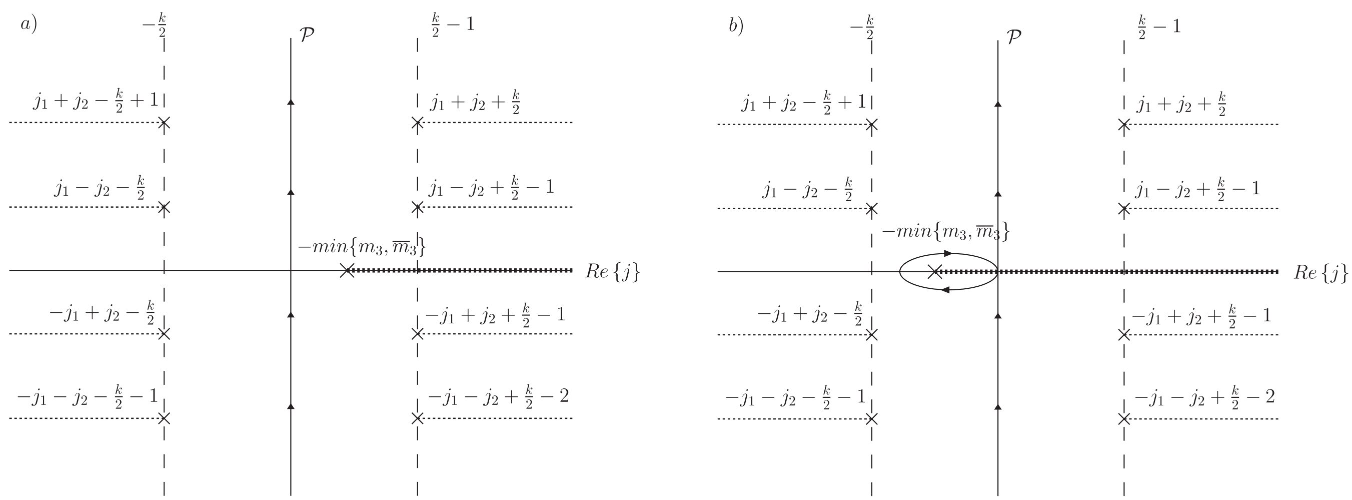

FIG. 1. Case $\mathcal{C}_{j_1}^{\alpha_1,w_1} \times \mathcal{C}_{j_2}^{\alpha_2,w_2}$. The solid line indicates the integration contour $\mathcal{P} = -\frac{1}{2} + i\mathbb{R}$ in the j_3 complex plane. The dots above or below the real axis represent the (j_1, j_2)-dependent poles and those on the real axis correspond to the m-dependent poles. The crosses are the positions of the first poles in the series. (a) When $m_1 + m_2 < -\frac{k-1}{2}$ or $\bar{m}_1 + \bar{m}_2 < -\frac{k-1}{2}$, there are no poles crossing the contour of integration. (b) When $m_1 + m_2 > -\frac{k-1}{2}$ and $\bar{m}_1 + \bar{m}_2 > -\frac{k-1}{2}$, poles from the factor $\frac{\Gamma(-j_3-\bar{m}_3)}{\Gamma(1+j_3+m_3)}$ cross the contour, indicating the contribution to the OPE from states in discrete representations.

$$\mathcal{C}^{\alpha_1,w_1}_{j_1}\times\mathcal{C}^{\alpha_2,w_2}_{j_2}|_{|w|=1}=\sum_{j_3<-(1/2)}\mathcal{D}^{+,w_3=w_1+w_2-1}_{j_3}+\sum_{j_3<-(1/2)}\mathcal{D}^{-,w_3=w_1+w_2+1}_{j_3}+\sum_{w=-1,1}\int_{\mathcal{P}}dj_3\mathcal{C}^{\alpha_3,w_3=w_1+w_2+w}_{j_3}+\cdots,\tag{3.9}$$

where $|_{|w|=1}$ denotes that only spectral flow nonpreserving contributions are displayed in the right-hand side.

2. $\mathcal{C}^{\alpha_1,w_1}_{j_1}\times\mathcal{D}^{\pm,w_2}_{j_2}$

To analyze this case, we need to perform the analytic continuation for j_2 away from $-\frac{1}{2}+is_2$. When is_2 is continued to the real interval $(-\frac{k-2}{2},0)$, the series of m-independent poles changes as shown in Fig. 2. It is easy to see that these poles do not cross the contour of integration. For instance, $\mathrm{Re}\{j_1+j_2+\frac{k}{2}\}>0$, $\mathrm{Re}\{j_1-j_2+\frac{k}{2}-1\}>\frac{k}{2}-1$, etc. Similarly, as in the previous case, only poles from $\frac{\Gamma(-j_3-\bar m_3)}{\Gamma(1+j_3+m_3)}$ can cross the contour, but due to the factor $\frac{\Gamma(1+j_2+\bar m_2)}{\Gamma(-j_2-m_2)}$ there are contributions from the discrete series just for $\Phi^{j_2,w_2}_{m_2,\bar m_2}\in\mathcal{D}^{-,w_2}_{j_2}\otimes\mathcal{D}^{-,w_2}_{j_2}$. Therefore we get

$$\mathcal{C}^{\alpha_1,w_1}_{j_1}\times\mathcal{D}^{\pm,w_2}_{j_2}|_{|w|=1}=\int_{\mathcal{P}}dj_3\mathcal{C}^{\alpha_3,w_3=w_1+w_2\pm1}_{j_3}+\sum_{j_3<-(1/2)}\mathcal{D}^{\mp,w_3=w_1+w_2\pm1}_{j_3}+\cdots.\tag{3.10}$$

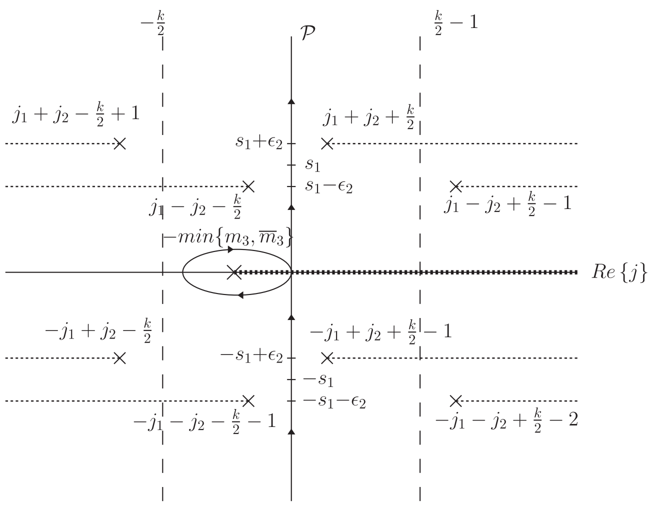

FIG. 2. Case $\mathcal{C}^{\alpha_1,w_1}_{j_1}\times\mathcal{D}^{\pm,w_2}_{j_2}$. Only m-dependent poles can cross the contour of integration. This occurs when both m_1+m_2 and $\bar m_1+\bar m_2$ are larger than $-\frac{k-1}{2}$. We have given j_2 an infinitesimal imaginary part, ϵ_2, to better display the (j_1,j_2)-dependent series of poles.

3. $\mathcal{D}^{\pm,w_1}_{j_1}\times\mathcal{D}^{\pm,w_2}_{j_2}$ *and* $\mathcal{D}^{\pm,w_1}_{j_1}\times\mathcal{D}^{\mp,w_2}_{j_2}$

Let us first analytically continue both j_1 and j_2 to the interval $(-\frac{k-1}{2},-\frac{1}{2})$, which is shown in Fig. 3. The correct way to do this is to consider that both j_1 and j_2 have an infinitesimal imaginary part, ϵ_1 and ϵ_2, respectively, which is sent to zero after computing the integral.

The m-independent poles cross the contour of integration only when $j_1+j_2<-\frac{k+1}{2}$. However, due to the factors $\frac{\Gamma(1+j_1+m_1)}{\Gamma(-j_1-\bar m_1)}\frac{\Gamma(1+j_2+\bar m_2)}{\Gamma(-j_2-m_2)}$ in $Q^{w=-1}$, the contributions from these poles only survive when the quantum numbers of both $\Phi^{j_1,w_1}_{m_1,\bar m_1}$ and $\Phi^{j_2,w_2}_{m_2,\bar m_2}$ are in $\mathcal{D}^{-,w_i}_{j_i}\otimes\mathcal{D}^{-,w_i}_{j_i}$, $i=1,2$. In this case, the poles at $j_3=j_1+j_2+\frac{k}{2}+n$ give contributions from $\mathcal{D}^{-,w_3=w_1+w_2-1}_{j_3}$. This may be seen noticing that $j_3=m_1+m_2+\frac{k}{2}+n_3=\bar m_1+\bar m_2+\frac{k}{2}+\bar n_3$, with $n_3=n+n_1+n_2$ and $\bar n_3=n+\bar n_1+\bar n_2$, or using $m_3=m_1+m_2+\frac{k}{2}$, $\bar m_3=\bar m_1+\bar m_2+\frac{k}{2}$, so that $j_3=m_3+n_3=\bar m_3+\bar n_3$. Instead, the contributions from the poles at $j_3=-j_1-j_2-\frac{k}{2}-1-n$ seem to cancel due to the factor $\frac{\Gamma(-j_3-\bar m_3)}{\Gamma(1+j_3+m_3)}$. However, these zeros are cancelled because the operator diverges. In fact, using (3.7) and relabeling $j_3\to-1-j_3$, it is straightforward to recover exactly the same contribution from the poles at $j_3=j_1+j_2+\frac{k}{2}+n$. Obviously, this was expected as a consequence of the symmetry $j_3\leftrightarrow-1-j_3$ of the integrand in (3.4).

Finally, the m-dependent poles give contributions from $\mathcal{D}^{+,w_3=w_1+w_2-1}_{j_3}$. Actually, when $\min\{m_3,\bar m_3\}>\frac{1}{2}$ some of the m-dependent poles cross the contour. Using m conservation it is not difficult to check that these contributions fall inside the range (2.18).

Let us continue the analysis, considering the OPE $\mathcal{D}^{\mp,w_1}_{j_1}\times\mathcal{D}^{\pm,w_2}_{j_2}$. For instance, take the limiting case $j_1=m_1+n_1+i\epsilon_1$ and $j_2=-m_2+n_2+i\epsilon_2$ with $\epsilon_1,\epsilon_2\to0$. The factor $\frac{\Gamma(1+j_2+\bar m_2)}{\Gamma(-j_2-m_2)}$ vanishes as a simple zero. However, some poles from the series $j_3=j_2-j_1-\frac{k}{2}-n$ will overlap with the m-dependent poles. But because the m-independent simple poles are outside the contour of integration, in the limit $\epsilon_i\to0$ they may cancel the simple zeros. The way to compute this limit is determined by the definition of the three-point function. We assume that a finite and nonzero term remains in the limit.[4]

[4]In the limit ϵ_1, $\epsilon_2\to0$, $\mathrm{Res}(Q^{w=-1})\sim\frac{\epsilon_2}{\epsilon_2-\epsilon_1}$. The same ambiguity appears in the three-point function including $\Phi^{j_1,w_1}_{m_1,\bar m_1}\in\mathcal{D}^{-,w_1}_{j_1}\otimes\mathcal{D}^{-,w_1}_{j_1}$, $\Phi^{j_2,w_2}_{m_2,\bar m_2}\in\mathcal{D}^{+,w_2}_{j_2}\otimes\mathcal{D}^{+,w_2}_{j_2}$, with $n_1\le n_2$ such that $j_3=j_1-j_2-\frac{k}{2}-\mathbb{Z}_{n\ge0}$. The resolution of this ambiguity requires an interpretation of the divergences. The w-selection rules allow one to assume that a finite term survives in the limit. For instance, consider a generic three-point function $\langle\mathcal{D}^{-,w_1}_{j_1}\mathcal{D}^{+,w_2}_{j_2}\mathcal{D}^{+,w_3}_{j_3}\rangle$ with $w_1+w_2+w_3=-1$. According to (2.16) this is nonvanishing (for certain values of j_i, not determined from the w-selection rules). Indeed, the divergence from the $\delta^2(\sum_i m_i-\frac{k}{2})$ in (2.21) cancels the zero from $\Gamma(-j_3-m_3)$ and then the pole in $C(1+j_i)\sim\frac{1}{\epsilon_2-\epsilon_1}$ must cancel the zero from $\frac{\Gamma(1+j_2+\bar m_2)}{\Gamma(-j_2-\bar m_2)}\sim\epsilon_2$, leaving a finite and nonvanishing contribution.

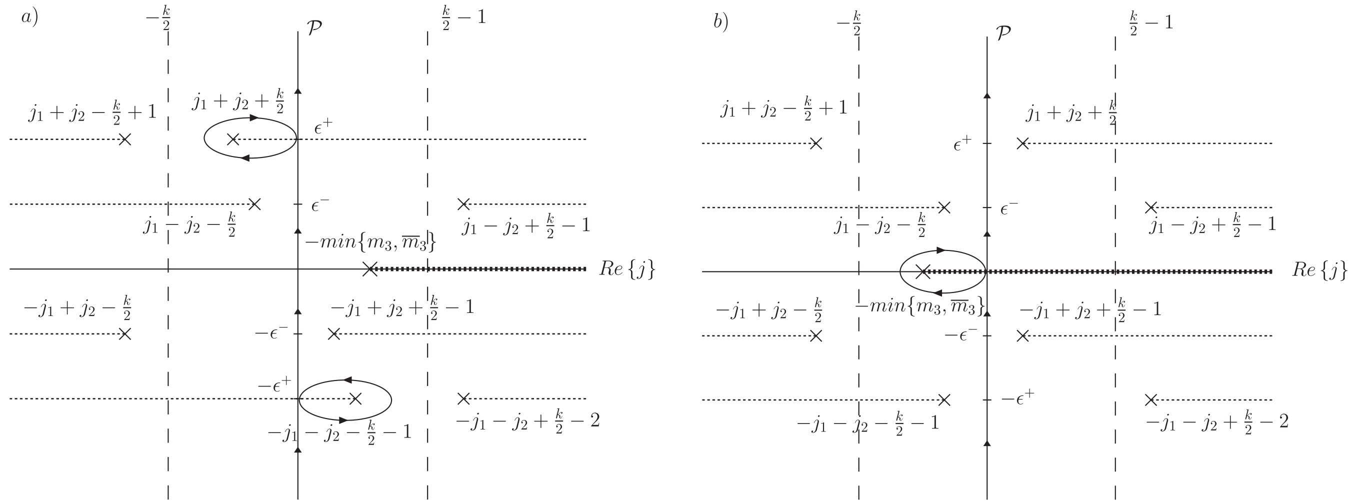

FIG. 3. Case $\mathcal{D}_{j_1}^{w_1}\times\mathcal{D}_{j_2}^{w_2}$. Both m-dependent and m-independent poles can cross the contour of integration. There are two possibilities: (1) $\mathcal{D}_{j_1}^{-,w_1}\times\mathcal{D}_{j_2}^{-,w_2}$. When $j_1+j_2<-\frac{k+1}{2}$, only m-independent poles can cross the contour, as shown in Fig. 3(a) and when $j_1+j_2>-\frac{k-1}{2}$, only m-dependent poles can cross as shown in Fig. 3(b). (2) $\mathcal{D}_{j_1}^{\mp,w_1}\times\mathcal{D}_{j_2}^{\pm,w_2}$. Both m-dependent and m-independent poles can cross the contour but only the former survive after taking the limit ϵ^+, $\epsilon^-\to 0$, where $\epsilon^\pm=\epsilon_1\pm\epsilon_2$.

Including the contributions from continuous representations, we get the following results:

$$\begin{aligned}\mathcal{D}_{j_1}^{\pm,w_1}\times\mathcal{D}_{j_2}^{\pm,w_2}|_{|w|=1} &= \int_{\mathcal{P}^+} dj_3 C_{j_3}^{\alpha_3,w_3=w_1+w_2\pm1} \\ &+\sum_{-j_1-j_2-(k/2)\leq j_3<-(1/2)}\mathcal{D}_{j_3}^{\mp,w_3=w_1+w_2\pm1} \\ &+\sum_{j_1+j_2+(k/2)\leq j_3<-(1/2)}\mathcal{D}_{j_3}^{\pm,w_3=w_1+w_2\pm1} \\ &+\cdots. \end{aligned}\tag{3.11}$$

$$\begin{aligned}\mathcal{D}_{j_1}^{+,w_1}\times\mathcal{D}_{j_2}^{-,w_2}|_{|w|=1} = &\sum_{j_3<j_2-j_1-(k/2)}\mathcal{D}_{j_3}^{-,w_3=w_1+w_2+1} \\ &+\sum_{j_3<j_1-j_2-(k/2)}\mathcal{D}_{j_3}^{+,w_3=w_1+w_2-1} \\ &+\cdots. \end{aligned}\tag{3.12}$$

B. Spectral flow preserving contributions

The analytic structure of $Q^{w=0}(j_i;m_i,\bar{m}_i)$ in (3.2) was studied in [9]. Here we present the analysis mainly to discuss some subtleties which are crucial to perform the analytic continuation of m_i, $\bar{m}_i$. $i=1,2$. Although our treatment of the m-dependent poles differs from that followed in [9], we show in this section that the results coincide.

The function $C(1+j_i)$ has zeros at $j_i=\frac{j-1}{2}$, $i=1,2,3$, and poles at $j=-j_1-j_2-j_3-2$, $-1-j_1-j_2+j_3$, $-1-j_1-j_3+j_2$, or $-1-j_2-j_3+j_1$ where $j:=p+q(k-2)$, $-(p+1)-(q+1)(k-2)$, $p,\ q=0,1,2,\cdots$. To explore the behavior of the function W, we use the expression [9]

$$W\begin{bmatrix} j_1,j_2,j_3\\ m_1,m_2,m_3\end{bmatrix}=(i/2)^2[C^{12}\bar{P}^{12}+C^{21}\bar{P}^{21}],\tag{3.13}$$

with $(i/2)^2P^{12}=s(j_1+m_1)s(j_2+m_2)C^{31}-s(j_2+m_2)\times s(m_1-j_2+j_3)C^{13}$,

$$\begin{aligned}C^{12}&=\frac{\Gamma(-N)\Gamma(1+j_3-m_3)}{\Gamma(-j_3-m_3)}\\&\quad\times G\begin{bmatrix}-m_3-j_3,-j_{13},1+m_2+j_2\\ -m_3-j_1+j_2+1,m_2-j_1-j_3\end{bmatrix},\\ C^{31}&=\frac{\Gamma(1+j_3+m_3)\Gamma(1+j_3-m_3)}{\Gamma(1+N)}\\&\quad\times G\begin{bmatrix}1+N,1+j_1+m_1,1-m_2+j_2\\ j_3+j_2+m_1+2,j_1+j_3-m_2+2\end{bmatrix},\end{aligned}$$

$$\begin{aligned}G\begin{bmatrix}a,b,c\\ e,f\end{bmatrix}&=\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(e)\Gamma(f)}F\begin{bmatrix}a,b,c\\ e,f\end{bmatrix}\\&=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c+n)}{\Gamma(e+n)\Gamma(f+n)\Gamma(n+1)},\end{aligned}\tag{3.14}$$

and $N=1+j_1+j_2+j_3$, $s(x)=\sin(\pi x)$. $\bar{P}^{ab}(\bar{C}^{ab})$ is obtained from $P^{ab}(C^{ab})$ by replacing $(m_i\to\bar{m}_i)$ and $P^{ba}(C^{ba})$ from $P^{ab}(C^{ab})$ by changing $(j_1,\ m_1\leftrightarrow j_2,\ m_2)$ and

$$F\begin{bmatrix}a,b,c\\ e,f\end{bmatrix}={}_3F_2(a,b,c;e,f;1).$$

An equivalent expression for W, which will be useful below, is the following [9]

$$\begin{aligned}W\begin{bmatrix} j_1,j_2,j_3\\ m_1,m_2,m_3\end{bmatrix}&=D_1C^{12}\bar{C}^{12}+D_2C^{21}\bar{C}^{21}\\&\quad+D_3[C^{12}\bar{C}^{21}+C^{21}\bar{C}^{12}],\end{aligned}\tag{3.15}$$

where

$$
\begin{aligned}
D_1 &= \frac{s(j_2+m_2)s(j_{13})}{s(j_1-m_1)s(j_2-m_2)s(j_3+m_3)}[s(j_1+m_1)\\
&\quad\times s(j_1-m_1)s(j_2+m_2)-s(j_2-m_2)\\
&\quad\times s(j_2-j_3-m_1)s(j_2+j_3-m_1)],\\
D_2 &= D_1(j_1,m_1\leftrightarrow j_2,m_2),\\
D_3 &= -\frac{s(j_{13})s(j_{23})s(j_1+m_1)s(j_2+m_2)s(j_1+j_2+m_3)}{s(j_1-m_1)s(j_2-m_2)s(j_3+m_3)}.
\end{aligned}
\tag{3.16}
$$

Studying the analytic structure of $Q^{w=0}$ is a difficult task as a consequence of the complicated form of W. The analysis greatly simplifies when analytically continuing the quantum numbers of one operator to those of a discrete representation. Indeed, when $j_1=-m_1+n_1=-\bar m_1+\bar n_1$, and $n_1,\bar n_1=0,1,2,\cdots$,

$$W\begin{bmatrix} j_1,j_2,j_3\\ m_1,m_2,m_3\end{bmatrix}$$

reduces to $W_1=D_1C^{12}\bar C^{12}$ [9], i.e.,

$$
\begin{aligned}
W_1\begin{bmatrix} j_1,j_2,j_3\\ m_1,m_2,m_3\end{bmatrix} &= \frac{(-)^{m_3-\bar m_3+\bar n_1}\pi^2\gamma(-N)}{\gamma(-2j_1)\gamma(1+j_{12})\gamma(1+j_{13})}\frac{\Gamma(1+j_3-m_3)\Gamma(1+j_3-\bar m_3)}{\Gamma(1+j_3-m_3-n_1)\Gamma(1+j_3-\bar m_3-\bar n_1)}\\
&\quad\times\prod_{i=2,3}\frac{\Gamma(1+j_i+m_i)}{\Gamma(-j_i-\bar m_i)}F\begin{bmatrix} -n_1,-j_{12},1+j_{23}\\ -2j_1,1+j_3-m_3-n_1\end{bmatrix}F\begin{bmatrix} -\bar n_1,-j_{12},1+j_{23}\\ -2j_1,1+j_3-\bar m_3-\bar n_1\end{bmatrix}.
\end{aligned}
\tag{3.17}
$$

It is easy to see that

$$
\begin{aligned}
&\frac{\Gamma(1+j_3-m_3)}{\Gamma(1+j_3-m_3-n_1)}F\begin{bmatrix} -n_1,-j_{12},1+j_{23}\\ -2j_1,1+j_3-m_3-n_1\end{bmatrix}\\
&=\sum_{n=0}^{n_1}\frac{(-)^n n_1!}{n!(n_1-n)!}\frac{\Gamma(n-j_{12})}{\Gamma(-j_{12})}\frac{\Gamma(n+1+j_{23})}{\Gamma(1+j_{23})}\\
&\quad\times\frac{\Gamma(-2j_1)}{\Gamma(n-2j_1)}\frac{\Gamma(1+j_3-m_3)}{\Gamma(n+1+j_3-m_3-n_1)}.
\end{aligned}
\tag{3.18}
$$

Recall that the OPE involves the function

$$W\begin{bmatrix} j_1,j_2,j_3\\ m_1,m_2,-m_3\end{bmatrix}$$

and then the change $(m_3,\bar m_3)\to(-m_3,-\bar m_3)$ is required in the above expressions to analyze $Q^{w=0}$. Thus, for generic $2j_i\notin\mathbb{Z}$, the poles and zeros of $Q^{w=0}(j_i;m_i,\bar m_i)$ are contained in

$$
\begin{aligned}
C(1+j_i)&\frac{\gamma(-1-j_1-j_2-j_3)}{\gamma(1+j_{12})\gamma(1+j_{13})}\\
&\times\frac{\Gamma(1+m_2+j_2)\Gamma(-m_3-j_3)}{\Gamma(-\bar m_2-j_2)\Gamma(1+\bar m_3+j_3)},
\end{aligned}
\tag{3.19}
$$

plus possible additional zeros in (3.18) and its antiholomorphic equivalent expression (see Appendix A). The (j_1,j_2)-dependent poles in (3.19) are at $j_3=\pm j_{21}^{\pm}+p+(q+1)(k-2)$, $\pm j_{21}^{\pm}-(p+1)-q(k-2)$, $\mp j_{21}^{\pm}+p+q(k-2)$, $\mp j_{21}^{\pm}-(p+1)-(q+1)(k-2)$. There are also zeros at $1+2j_i=p+q(k-2)$, $-(p+1)-(q+1)(k-2)$, $i=1,2,3$.

Let us first consider $\Phi^{w_1,j_1}_{m_1,\bar m_1}\in\mathcal{D}^{+,w_1}_{j_1}\otimes\mathcal{D}^{+,w_1}_{j_1}$ and note that when $\Phi^{w_1,j_1}_{m_1,\bar m_1}\in\mathcal{D}^{-,w_1}_{j_1}\otimes\mathcal{D}^{-,w_1}_{j_1}$ the OPE follows directly using the symmetry of the spectral flow conserving two- and three-point functions under $(m_i,\bar m_i)\leftrightarrow(-m_i,-\bar m_i)$, $\forall\ i=1,2,3.$[5]

1. $\mathcal{D}^{\pm,w_1}_{j_1}\times\mathcal{C}^{\alpha_2,w_2}_{j_2}$

Consider $j_1=-m_1+n_1+i\epsilon_1$ with $n_i\in\mathbb{Z}_{\geq 0}$ and ϵ_1 an infinitesimal positive number, and $j_2=-\frac{1}{2}+is_2$ not correlated with m_2. In this case,

$$W\begin{bmatrix} j_1,j_2,j_3\\ m_1,m_2,m_3\end{bmatrix}\approx W_1\begin{bmatrix} j_1,j_2,j_3\\ m_1,m_2,m_3\end{bmatrix}.$$

The m-independent poles are to the right or to the left of the contour of integration as sketched in Fig. 4(a). If $\min\{m_3,\bar m_3\}<\frac{1}{2}$, none of the m-dependent poles cross the contour, implying that only continuous series contribute to the spectral flow conserving terms of the OPE $\mathcal{D}^{+,w_1}_{j_1}\times\mathcal{C}^{\alpha_2,w_2}_{j_2}$. On the other hand if $\min\{m_3,\bar m_3\}>\frac{1}{2}$, this OPE also receives contributions from $\mathcal{D}^{+,w_3=w_1+w_2}_{j_3}$. Note that when $j_1\approx m_1+n_1$,

$$W\begin{bmatrix} j_1,j_2,j_3\\ m_1,m_2,m_3\end{bmatrix}\approx W_1\begin{bmatrix} j_1,j_2,j_3\\ -m_1,-m_2,-m_3\end{bmatrix},$$

which implies that the spectral flow conserving terms in the OPE $\mathcal{D}^{-,w_1}_{j_1}\times\mathcal{C}^{\alpha_2,w_2}_{j_2}$ contain contributions from the continuous representations as well as from $\mathcal{D}^{-,w_3}_{j_3}$ when $\max\{m_3,\bar m_3\}<-\frac{1}{2}$. So we find

[5]This symmetry follows directly from the integral expression for

$$W\begin{bmatrix} j_1,j_2,j_3\\ m_1,m_2,m_3\end{bmatrix}$$

performing the change of variables $(x_i,\bar x_i)\to(x_i^{-1},\bar x_i^{-1})$ in (2.20).

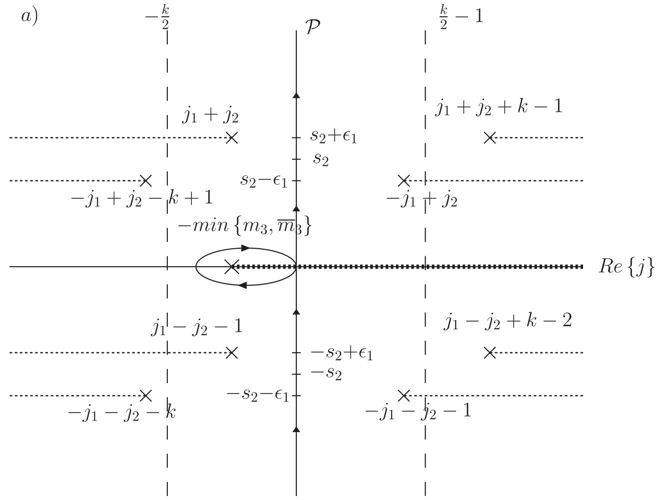

FIG. 4. Analytic continuation of $Q^{w=0}$ for (j_1, j_2) values away from the axis $-\frac{1}{2}+i\mathbb{R}$, using W_1 instead of W. In Fig. 4(a) $j_2 = -\frac{1}{2}+is_2$ and only m-dependent poles can cross the contour of integration. In Fig. 4(b) $-\frac{k-1}{2} < j_2 < -\frac{1}{2}$ was considered. While m-independent poles only cross the contour when $j_2 < j_1$, m-dependent poles can cross independently of the values of j_1, j_2, but they are annihilated unless $j_2 > j_1$.

$$\mathcal{D}_{j_1}^{\pm,w_1} \times \mathcal{C}_{j_2}^{\alpha_2,w_2}|_{w=0} = \int_{\mathcal{P}} dj_3 \mathcal{C}_{j_3}^{\alpha_3,w_3=w_1+w_2} + \sum_{j_3<-1/2} \mathcal{D}_{j_3}^{\pm,w_3=w_1+w_2} + \cdots. \tag{3.20}$$

2. $\mathcal{D}_{j_1}^{\pm,w_1} \otimes \mathcal{D}_{j_2}^{\mp,w_2}$ and $\mathcal{D}_{j_1}^{\mp,w_1} \otimes \mathcal{D}_{j_2}^{\mp,w_2}$

When j_2 is continued to $(-\frac{k-1}{2}+i\epsilon_2, -\frac{1}{2}+i\epsilon_2)$, ϵ_2 being an infinitesimal positive number, W is again well approximated by W_1 as long as $j_2 \neq -m_2+n_2+i\epsilon_2$, $-\bar{m}_2+\bar{n}_2+i\epsilon_2$. Otherwise, one also has to consider $W_2 \equiv D_2 C^{21}\bar{C}^{21}$, but the result coincides exactly with the one obtained using W_1, so we are restricted to this. Two m-independent series of poles may cross the contour of integration: $j_3 = j_1 - j_2 - 1 - p - q(k-2)$ and $j_3 = j_2 - j_1 + p + q(k-2)$, both with $q=0$. The former has $j_3 > -\frac{1}{2}$ and the latter, $j_3 < -\frac{1}{2}$. The m-dependent poles in $Q^{w=0}$ arise from $\frac{\Gamma(-j_3-\bar{m}_3)}{\Gamma(1+j_3+m_3)}$. When $j_2 = -m_2+n_2+i\epsilon_2$, because of the factor $\Gamma(-j_2-m_2)^{-1}$, only m-dependent poles give contributions from discrete series. To see this, consider the m-independent poles at $j_3 = j_1 + j_2 - p - q(k-2)$. These are outside the contour of integration and in the limit ϵ_1, $\epsilon_2 \to 0$ some of them may overlap with the m-dependent ones. Again, one may argue that this limit leaves a finite and nonvanishing factor.

When $j_2 = m_2 + n_2 + i\epsilon_2$, at first sight there are no zeros. If $j_2 - j_1 < -\frac{1}{2}$, some poles with $q=0$ in the series $j_3 = j_2 - j_1 + p + q(k-2)$ and $j_3 = j_1 - j_2 - 1 - p - q(k-2)$ cross the contour, as shown in Fig. 4(b). Using the relation between j_i and m_i and m conservation, it follows that the former poles can be rewritten as $j_3 = m_3 + n_3 = \bar{m}_3 + \bar{n}_3$, where $n_3 = n_2 - n_1 + p$ and $\bar{n}_3 = \bar{n}_2 - \bar{n}_1 + p$. Obviously, if $n_2 \geq n_1$ and $\bar{n}_2 \geq \bar{n}_1$, all the residues picked up by the contour deformation imply contributions to the OPE from $\mathcal{D}_{j_3}^{-,w_3=w_1+w_2}$. When $n_2 < n_1$ or $\bar{n}_2 < \bar{n}_1$, only those values of p for which both n_3 and $\bar{n}_3$ are non-negative integers remain after taking the limit ϵ_1, $\epsilon_2 \to 0$. This is because of extra zeros appearing in W_1 which are not explicit in (3.17) (see Appendix A 1). Using the results in the Appendix and the identity (3.7) it is straightforward to see that the latter series of poles give the same contributions.

The poles at $j_3 = -\min\{m_3, \bar{m}_3\} + n_3$ may cross the contour. If this happens, they overlap with the m-independent poles. But there are double zeros cancelling these contributions.

If $j_2 - j_1 > -\frac{1}{2}$, only m-dependent poles may cross the contour. But they give contributions only if they do not overlap with the poles at $j_3 = j_1 - j_2 - 1 - n$, again because of the presence of double zeros. Therefore, these contributions remain only for $j_3 \geq j_1 - j_2$.

Putting all together we get

$$\begin{aligned}\mathcal{D}_{j_1}^{+,w_1} \times \mathcal{D}_{j_2}^{-,w_2}|_{w=0} = &\int_{\mathcal{P}} dj_3 \mathcal{C}_{j_3}^{\alpha_3,w_3=w_1+w_2} \\ &+ \sum_{j_2-j_1\leq j_3<-(1/2)} \mathcal{D}_{j_3}^{-,w_3=w_1+w_2} \\ &+ \sum_{j_1-j_2\leq j_3<-(1/2)} \mathcal{D}_{j_3}^{+,w_3=w_1+w_2} \\ &+ \cdots,\end{aligned} \tag{3.21}$$

$$\mathcal{D}_{j_1}^{\pm,w_1} \times \mathcal{D}_{j_2}^{\pm,w_2}|_{w=0} = \sum_{j_3\leq j_1+j_2} \mathcal{D}_{j_3}^{\pm,w_3=w_1+w_2} + \cdots. \tag{3.22}$$

3. $\mathcal{C}_{j_1}^{\alpha_1,w_1}\times\mathcal{C}_{j_2}^{\alpha_2,w_2}$

The zero and pole structure of $Q^{w=0}$ is given by

$$Q^{w=0}(j_i;m_i,\bar m_i)\sim C(1+j_i)\frac{\gamma(-N)}{s(\bar m_3+j_3)}G\begin{bmatrix} m_3-j_3,-j_{13},1+m_2+j_2\\ m_3-j_1+j_2+1,m_2-j_1-j_3\end{bmatrix}\times\left\{s(\bar m_1+j_1)G\begin{bmatrix}1+N,1+\bar m_1+j_1,1-\bar m_2+j_2\\ 2+\bar m_1+j_2+j_3,2-\bar m_2+j_1+j_3\end{bmatrix}-s(\bar m_1-j_2+j_3)G\begin{bmatrix}1+N,1+\bar m_2+j_2,1-\bar m_1+j_1\\ 2+\bar m_2+j_1+j_3,2-\bar m_1+j_2+j_3\end{bmatrix}\right\}+(j_1,m_1,\bar m_1)\leftrightarrow(j_2,m_2,\bar m_2).$$

$$G\begin{bmatrix}a,b,c\\ e,f\end{bmatrix}$$

has simple poles at a, b, $c=0,-1,-2,\ldots$ as well as at $u=e+f-a-b-c=0,-1,-2,\cdots$, if a, b, $c\neq 0,-1,-2,\cdots$. So, a direct analysis leads to the conclusion that, when $j_i=-\frac{1}{2}+is_i$, $i=1,2$, the poles are contained in the following expression:

$$C(1+j_i)\Gamma(-N)[\Gamma(-j_{12})]^2\Gamma(-j_{13})\Gamma(-j_{23})\times\Gamma(-j_3+m_3)\Gamma(-j_3-\bar m_3)\Gamma(1+j_3+\bar m_3). \quad (3.23)$$

Instead, if one looks for poles in $Q^{w=0}$ using (3.15), they seem to be those contained in

$$C(1+j_i)[\Gamma(-N)\Gamma(-j_{12})\Gamma(-j_{13})\Gamma(-j_{23})]^2\times\Gamma(-j_3+m_3)\Gamma(-j_3-\bar m_3)\Gamma(1+j_3+\bar m_3). \quad (3.24)$$

These different behaviors in the (j_1,j_2)-dependent poles suggest that one must be very careful when analyzing the analytic structure of $Q^{w=0}$. The $(m_3,\bar m_3)$-dependent poles coincide in both expressions. However, the symmetries of W imply that for generic j_1, j_2 and m_1, m_2, the m_3-dependent poles must be symmetric under $m_3\leftrightarrow\bar m_3$ as well as under $(m_3,\bar m_3)\leftrightarrow(-m_3,-\bar m_3)$, and this does not seem to be the case in the expressions above.

This puzzle is a consequence of the intricate functional form of W. Extra zeros may be hidden. Actually, let us show that the correct behavior of $Q^{w=0}$ must be of the form[6]

$$Q^{w=0}\sim\frac{\Gamma(-j_3-m_3)\Gamma(-j_3+\bar m_3)}{\Gamma(1+j_3-m_3)\Gamma(1+j_3+\bar m_3)}, \quad (3.25)$$

for generic j_1, j_2 and for m_1, m_2 not correlated with them, up to regular and nonvanishing contributions for $j_3=\pm m_3+n_3=\pm\bar m_3+\bar n_3$, with $n_3,\bar n_3\in\mathbb{Z}$.

To check (3.25), let us consider $j_3=-m_3+n_3+i\epsilon_3=-\bar m_3+\bar n_3+i\epsilon_3$, with ϵ_3 an infinitesimal number. Using (3.15) with the relabeling $1\leftrightarrow 3$, only a term like D_1 remains in W because the other terms behave as ϵ_3 and there are no extra divergences to cancel the zeros when $\epsilon_3\to 0$. Then, W behaves as W_1 in (3.17), with the relabeling discussed above. The factor

$$\frac{\Gamma(1+j_1-m_1)}{\Gamma(1+j_1-m_1-n_3)}F\begin{bmatrix}-n_3,-j_{23},1+j_{12}\\ -2j_3,1+j_1-m_1-n_3\end{bmatrix}$$

and the similar antiholomorphic one have no poles or zeros when j_1 and m_1 are not correlated. So, we conclude that for $j_3=-m_3+q_3+i\epsilon_3=-\bar m_3+\bar q_3+i\epsilon_3$, W has no m_3-dependent poles or zeros, and then

$$Q^{w=0}\sim C(1+j_i)\gamma(-N)\gamma(-j_{23})\gamma(-j_{13})\times\frac{\Gamma(-j_3-m_3)\Gamma(-j_3+\bar m_3)}{\Gamma(1+j_3-m_3)\Gamma(1+j_3+\bar m_3)}. \quad (3.26)$$

Using the symmetry $(m_i,\bar m_i)\leftrightarrow(-m_i,-\bar m_i)$ of W, it is straightforward to deduce that the same behavior is obtained for $j_3=m_3+n_3+i\epsilon_3=\bar m_3+\bar n_3+i\epsilon_3$.

We may now analyze the OPE $\mathcal{C}_{j_1}^{\alpha_1,w_1}\times\mathcal{C}_{j_2}^{\alpha_2,w_2}$. A sum over continuous representations appears because $Q^{w=0}$ does not vanish for $j_3=-\frac{1}{2}+is_3$ when s_3 is a real number. On the other hand, the expression (3.25) shows that there are no contributions from discrete representations provided $\min\{m_1+m_2,\bar m_1+\bar m_2\}<\frac{1}{2}$ and $\max\{m_1+m_2,\bar m_1+\bar m_2\}>-\frac{1}{2}$. Obviously both bounds cannot be violated at the same time. When the first one is violated, operators belonging to spectral flow images of lowest-weight representations contribute to the OPE. On the contrary, when the second bound is not satisfied, operators in spectral flow images of highest-weight representations appear in the OPE.

Extra poles could possibly appear in the m basis, implying contributions from operators not belonging to $\mathcal{C}_j^{\alpha,w}$ or $\mathcal{D}_j^{\pm,w}$ representations. However, the poles of ${}_3F_2$ are well known and no other than those in (3.23) and (3.24) appear in W. Instead, there could be extra zeros cancelling certain poles as a consequence of particular combinations of the arguments in ${}_3F_2$. As we have shown, these possible zeros cannot cancel the m_3-dependent poles. This information supports the conclusion that the OPE is closed among $\mathcal{C}_j^{\alpha,w}$ and $\mathcal{D}_j^{\pm,w}$ representations.

[6]The pole structure of this expression is obviously symmetric under $m_3\leftrightarrow\bar m_3$ as well as under $(m_3,\bar m_3)\leftrightarrow(-m_3,-\bar m_3)$.

Finally, we want to remark the importance of a relation like (3.25), because the other expressions (3.23) and (3.24) do not admit a definition of the OPE as analytic continuation since the m_3-dependent poles do not seem to begin (or end) at a given point.

Therefore, we conclude that the w-conserving contributions to the OPE of two continuous representations are the following:

$$\mathcal{C}_{j_1}^{\alpha_1,w_1}\times\mathcal{C}_{j_2}^{\alpha_2,w_2}|_{w=0}\sim\int_{\mathcal{P}}dj_3\mathcal{C}_{j_3}^{\alpha_3,w_3=w_1+w_2}+\sum_{j_3<-(1/2)}\mathcal{D}_{j_3}^{+,w_3=w_1+w_2}+\sum_{j_3<-(1/2)}\mathcal{D}_{j_3}^{-,w_3=w_1+w_2}, \qquad (3.27)$$

up to descendants. Note that, in a particular OPE with m_i, $\bar{m}_i$ fixed, only one of the discrete series contributes, depending on the signs of m_3, $\bar{m}_3$.

Collecting all the results, the OPE for primary fields and their spectral flow images in the spectrum of the $SL(2,\mathbb{R})$ WZNW model are the following:

$$\mathcal{D}_{j_1}^{\pm,w_1}\times\mathcal{D}_{j_2}^{\pm,w_2}=\sum_{j_3\leq j_1+j_2}\mathcal{D}_{j_3}^{\pm,w_3=w_1+w_2}+\sum_{-j_1-j_2-(k/2)\leq j_3<-(1/2)}\mathcal{D}_{j_3}^{\mp,w_3=w_1+w_2\pm1}+\sum_{j_1+j_2+(k/2)\leq j_3<-(1/2)}\mathcal{D}_{j_3}^{\pm,w_3=w_1+w_2\pm1}+\int_{\mathcal{P}}dj_3\mathcal{C}_{j_3}^{\alpha_3,w_3=w_1+w_2\pm1}+\cdots. \qquad (3.28)$$

$$\mathcal{D}_{j_1}^{+,w_1}\times\mathcal{D}_{j_2}^{-,w_2}=\sum_{j_1-j_2\leq j_3<-(1/2)}\mathcal{D}_{j_3}^{+,w_3=w_1+w_2}+\sum_{j_2-j_1\leq j_3<-(1/2)}\mathcal{D}_{j_3}^{-,w_3=w_1+w_2}+\sum_{j_3\leq j_2-j_1-(k/2)}\mathcal{D}_{j_3}^{-,w_3=w_1+w_2+1}+\sum_{j_3\leq j_1-j_2-(k/2)}\mathcal{D}_{j_3}^{+,w_3=w_1+w_2-1}+\int_{\mathcal{P}}dj_3\mathcal{C}_{j_3}^{\alpha_3,w_3=w_1+w_2}+\cdots, \qquad (3.29)$$

$$\mathcal{D}_{j_1}^{\pm,w_1}\times\mathcal{C}_{j_2}^{\alpha_2,w_2}=\sum_{w=0}^{1}\int_{\mathcal{P}}dj_3\mathcal{C}_{j_3}^{\alpha_3,w_3=w_1+w_2\pm w}+\sum_{j_3<-(1/2)}\mathcal{D}_{j_3}^{\pm,w_3=w_1+w_2}+\sum_{j_3<-(1/2)}D_{j_3}^{\mp,w_3=w_1=w_2\pm1}+\cdots, \qquad (3.30)$$

$$\mathcal{C}_{j_1}^{\alpha_1,w_1}\times\mathcal{C}_{j_2}^{\alpha_2,w_2}=\sum_{w=0}^{1}\sum_{j_3<-(1/2)}(\mathcal{D}_{j_3}^{+,w_3=w_1+w_2-w}+\mathcal{D}_{j_3}^{-,w_3=w_1+w_2+w})+\sum_{w=-1}^{1}\int_{\mathcal{P}}dj_3\mathcal{C}_{j_3}^{\alpha_3,w_3=w_1+w_2+w}+\cdots. \qquad (3.31)$$

In order to analyze these results, let us first restrict to the spectral flow conserving contributions for $w_i=0$, $i=1,2$. In this case, exactly the same results were obtained in [9] using the following prescription for the OPE of $w=0$ primary fields $\Phi_{m_1,\bar{m}_1}^{j_1}$ $\Phi_{m_2,\bar{m}_2}^{j_2}$ [7]:

$$\Phi_{m_1,\bar{m}_1}^{j_1}(z_1,\bar{z}_1)\Phi_{m_2,\bar{m}_2}^{j_2}(z_2,\bar{z}_2)\underset{z_1\to z_2}{\sim}\sum_{j_3}|z_{12}|^{-2\tilde{\Delta}_{12}}\times Q^{w=0}(j_i;m_i,\bar{m}_i)\Phi_{m_1+m_2,\bar{m}_1+\bar{m}_2}^{j_3}(z_2,\bar{z}_2), \qquad (3.32)$$

where $Q^{w=0}$ was obtained using the standard procedure, i.e., multiplying both sides of (3.32) by a fourth field in the $w=0$ sector and taking expectation values. The formal symbol $\sum_{j_3}$ denotes integration over $\mathcal{D}_{j_3}^{\pm}$ and $\mathcal{C}_{j_3}^{\alpha_3}$, namely,

$$\sum_{j_3}=\int_{\mathcal{P}^+}dj_3+\delta_{\mathcal{D}_{j_3}^{\pm}}\oint_{\mathcal{C}}dj_3. \qquad (3.33)$$

The integration over $\mathcal{P}^+$ stands for summation over $\mathcal{C}_j^{\alpha}$. The contour integral along $\mathcal{C}$ encloses the poles from $\mathcal{D}_{j_3}^{\pm}$ and $\delta_{\mathcal{D}_{j_3}^{\pm}}$ means that j_3 is picked up from the poles in $Q^{w=0}$ by the contour $\mathcal{C}$ only when it belongs to a discrete representation. The range of j_3 is $\mathrm{Re}j_3\leq-\frac{1}{2}$ and $\mathrm{Im}j_3\geq0$, consistently with the argument which determined $Q^{w=0}$ because $\sum_{j_3}$ picks up only one term in (2.17). This prescription to deal with the j- and m-independent poles was shown to be compatible with the one suggested in [7] for the H_3^+ model. The strategy designed in (3.33) for the treatment of m-dependent poles, which were absent in [7], aimed to reproducing the classical tensor product of representations of $SL(2,\mathbb{R})$ in the limit $k\to\infty$.[8] This

[7]See [8] for previous work involving highest-weight representations.

[8]We thank Y. Satoh for comments on this point.

proposal for the OPE includes in addition the requirement that poles with divergent residues should not be picked up.

In this section, we have followed a different path. We have treated the j- and m-dependent poles alike. However, although the equivalence between both prescriptions is not obvious *a priori*, we obtained the same results for the OPE of unflowed primary fields.[9] Indeed, notice that poles in $Q^{w=0}$ at values of quantum numbers in $\mathcal{C}_j^\alpha$ or $\mathcal{D}_{j_3}^\pm$ would not contribute to the OPE determined by (3.4) if they do not cross the contour $\mathcal{P}$, unlike to (3.32). On the other hand, contributions from operators in other representations, i.e., neither in $\mathcal{C}_j^\alpha$ nor in $\mathcal{D}_{j_3}^\pm$, could have appeared in (3.28), (3.29), (3.30), and (3.31), but they did not. Moreover, by a careful analysis of the analytic structure of $Q^{w=0}$ we have shown that there are no double poles, so that the regularization proposed in [9] is not really necessary.[10]

In the case $w_1 = w_2 = 0$, $k \to \infty$, the w-conserving contributions to the OPE of representations of the zero modes in (3.28), (3.29), (3.30), and (3.31) reproduce the classical tensor products of representations of $SL(2,\mathbb{R})$ obtained in [16]. Continuous series appear twice in the product of two continuous representations due to the existence of two linearly independent Clebsh-Gordan coefficients. As noted in [9], this is in agreement with the fact that both terms C^{12} and C^{21} in (3.13) contribute to $Q^{w=0}$ in the fusion of two continuous series. Moreover, it was also observed that the analysis can be applied for finite k without modifications. The results are given by replacing $\mathcal{D}_j^\pm$, $\mathcal{C}_j^\alpha$ in (3.28), (3.29), (3.30), and (3.31) by the corresponding affine representations $\hat{\mathcal{D}}_j^\pm$, $\hat{\mathcal{C}}_j^\alpha$. It is easy to see that this OPE of unflowed fields in the spectrum of the $SL(2,\mathbb{R})$ WZNW model is not closed, i.e., it gets contributions from discrete representations with $j_3 < -\frac{k-1}{2}$. When spectral flow is turned on, incorporating all the relevant representations of the theory and the complete set of structure constants as we have done in this section, the OPE still does not close, namely, there are contributions from discrete representations outside the range (2.12). In particular, this feature of the OPE of fields in discrete representations differs from the results in [5] where the factorization limit of the four-point function of $w = 0$ short strings was shown to be in accord with the Hilbert space of the theory.

In the following section we will show that assuming the OPE (3.28), (3.29), (3.30), and (3.31) holds for states in representations of the full current algebra, i.e., replacing $\mathcal{D}_j^{\pm,w}$, $\mathcal{C}_j^{\alpha,w}$ by $\hat{\mathcal{D}}_j^{\pm,w}$, $\hat{\mathcal{C}}_j^{\alpha,w}$, leads to inconsistencies unless a truncation is performed.

IV. TRUNCATION OF THE OPERATOR ALGEBRA AND FUSION RULES

The analysis of the previous section involved primary operators and their spectral flow images. Then, the OPE (3.28), (3.29), (3.30), and (3.31) explicitly includes some descendant fields. Assuming the appearance of spectral flow images of primary states in the fusion rules indicates that there are also contributions from descendants not obtained by spectral flowing primaries but descendants with the same J_0^3 eigenvalue, namely, replacing $\mathcal{D}_j^{\pm,w}$, $\mathcal{C}_j^{\alpha,w}$ by $\hat{\mathcal{D}}_j^{\pm,w}$, $\hat{\mathcal{C}}_j^{\alpha,w}$ in the r.h.s. of (3.28), (3.29), (3.30), and (3.31), some interesting conclusions can be drawn.

For instance, consider the spectral flow nonpreserving terms in the OPE $\mathcal{D}_{j_1}^{-,w_1} \times \mathcal{D}_{j_2}^{-,w_2}$, (3.28). If they are extended to the affine series, using the spectral flow symmetry they may be identified as

$$\sum_{-((k-1)/2) < \tilde{j}_3 \leq j_1 + j_2} \hat{\mathcal{D}}^{+,w_3 = w_1 + w_2 - 1}_{-\frac{k}{2} - \tilde{j}_3} \equiv \sum_{-((k-1)/2) < j_3 \leq j_1 + j_2} \hat{\mathcal{D}}^{-,w_3 = w_1 + w_2}_{j_3}. \tag{4.1}$$

This reproduces the spectral flow conserving terms in the first sum in (3.28). However, there is an important difference: here j_3 is automatically restricted to the region (2.18).

Analogously, applying the spectral flow symmetry to the discrete series contributing to the OPE $\mathcal{D}_{j_1}^{+,w_1} \times \mathcal{D}_{j_2}^{-,w_2}|_{|w|=1}$ in (3.12) leads to contributions from $\sum_{j_2 - j_1 \leq j_3} \hat{\mathcal{D}}_{j_3}^{-,w_3 = w_1 + w_2}$ as well as from $\sum_{j_1 - j_2 \leq j_3} \hat{\mathcal{D}}_{j_3}^{+,w_3 = w_1 + w_2}$, which were found among the spectral flow conserving terms with the extra condition $j_3 < -\frac{1}{2}$.

In order to see further implications of the spectral flow symmetry on the OPE (3.28), (3.29), (3.30), and (3.31), let us now consider operator products of descendants. Take the OPE $\hat{\mathcal{D}}_{j_1}^{+,w_1=0} \otimes \hat{\mathcal{D}}_{j_2}^{-,w_2=1}$.[11] Equation (3.29) gives spectral flow conserving contributions from $\hat{\mathcal{D}}_{j_3}^{-,w_3=1}$, for certain $m_i, \bar{m}_i$, $i = 1, 2$, with j_3 verifying (2.18). Using the spectral flow symmetry, one might infer that the contributions from $\hat{\mathcal{D}}_{j_3}^{+,w_3=0}$ to the OPE $\hat{\mathcal{D}}_{j_1}^{+,w_1=0} \otimes \hat{\mathcal{D}}_{j_2}^{+,w_2=0}$ in (3.28) would also be within the region (2.18). On the contrary, we found

[9] More generally, it can be shown that a generalization of the *ansatz* (3.32) for fields $\Phi_{m_1,\bar{m}_1}^{j_1,w_1}$ $\Phi_{m_2,\bar{m}_2}^{j_2,w_2}$, by adding the contributions from terms proportional to $Q^{w=\pm 1}$ and replacing $\delta_{D_{j_3}^\pm}$ by $\delta_{D_{j_3}^{\pm,w_3}}$, leads to the same results (3.28), (3.29), (3.30), and (3.31).

[10] This is very important because the double poles discussed in [9] would lead to inconsistencies in the analytic continuation of the OPE from H_3^+ that we have performed in this section. In particular, they would give divergent contributions to the OPE $\mathcal{D}_j^+ \times \mathcal{D}_j^-$ and, in addition, this OPE would be incompatible with $\mathcal{D}_j^- \times \mathcal{D}_j^+$, in contradiction with expectations from the symmetries of the function W.

[11] We use the tensor product symbol $\otimes$ to denote the OPE of fields in representations of the current algebra, to distinguish it from that of highest/lowest-weight fields.

terms in $\hat{\mathcal{D}}_{j_3}^{+,w_3=0}$ with $j_3 < -\frac{k-1}{2}$. Moreover, using the spectral flow symmetry again, these terms can be identified with contributions from $\hat{\mathcal{D}}_{j_3}^{-,w_3=1}$ with $j_3 > -\frac{1}{2}$ to the OPE $\hat{\mathcal{D}}_{j_1}^{+,w_1=0} \otimes \hat{\mathcal{D}}_{j_2}^{-,w_2=1}$, in contradiction with our previous result.

Similar puzzles are found identifying $\sum_{j_3<-(1/2)} \hat{\mathcal{D}}_{j_3}^{+,w_3=w_1+w_2-1} = \sum_{-((k-1)/2)<j_3} \hat{\mathcal{D}}_{j_3}^{-,w_3=w_1+w_2}$ in (3.30), which gives some of the spectral flow conserving contributions. It is interesting to note that only the states within the region (2.18) contribute in both cases, explicitly $j_3 = j_1 + \alpha_2 + n$, with $n \in \mathbb{Z}$ such that $-\frac{k-1}{2} < j_3 < -\frac{1}{2}$. It is also important to stress the following observation. For given j_1, m_1 and j_2, m_2 the spectral flow conserving part of the OPE (3.30) receives contributions from states with $\tilde{j}_3$, $\tilde{m}_3$ verifying $\tilde{j}_3 = \tilde{m}_3 + \tilde{n}_3$ with $\tilde{n}_3 = 0, 1, \cdots, \tilde{n}_3^{\max}$, $\tilde{n}_3^{\max}$ being the maximum integer such that $\tilde{j}_3 < -\frac{1}{2}$. On the other hand, the spectral flow nonconserving terms get contributions from $j_3 = -m_3 + n_3$ with $n_3 = 0, 1, 2, \ldots, n_3^{\max}$ and here $n_3^{\max}$ is the maximal nonnegative integer such that $j_3 < -\frac{1}{2}$. So, identifying both series implies considering $\tilde{j}_3 = -\frac{k}{2} - j_3$ and now $n_3^{\max}$ (which is the same as before) has to be the maximal nonnegative integer for which $\tilde{j}_3 > -\frac{k-1}{2}$. There is just one operator appearing in both contributions to the OPE. It has $\tilde{n}_3 = 0$ in the former and $n_3 = 0$ in the latter. This is a consequence of the relation $\Phi_{m=\bar{m}=-j}^{j,w=0} = \frac{\nu^{(k/2)-1}}{(k-2)} \times \frac{1}{B(-1-j')} \Phi_{m'=\bar{m}'=j'}^{j',w'=1}$ with $j' = -\frac{k}{2} - j$ [5]. One can check that the w-conserving three-point functions containing $\Phi_{m=\bar{m}=-j}^{j,w=0}$ reduce to the w-non-conserving ones involving $\Phi_{m'=\bar{m}'=j'}^{j',w'=1}$. This result can be generalized for arbitrary w sectors in the m basis, i.e., $\Phi_{m=\bar{m}=-j}^{j,w} \sim \Phi_{m'=\bar{m}'=j'}^{j',w'=w+1}$ up to a regular normalization for j in the region (2.18). For instance, one can reduce a spectral flow conserving three-point function including $\Phi_{m=\bar{m}=-j}^{j,w}$ to a one unit violating amplitude containing $\Phi_{m'=\bar{m}'=j'}^{j',w+1}$ using the identity

$$C(1+j_1, 1+j_2, 1+j_3) = \frac{\nu^{k-2}\gamma(k-2-j_{23})\gamma(2-k-2j_1)C(k+j_1-1, 1+j_2, 1+j_3)}{(k-2)\gamma(1+2j_1)\gamma(-N)\gamma(-j_{12})\gamma(-j_{13})}, \tag{4.2}$$

which is a consequence of the relation $G(j) = (k-2)^{1+2j}\gamma(-j)G(j-k+2)$.

The OPE (3.30) was obtained for states in $\mathcal{C}_j^{\alpha,w}$ and $\mathcal{D}_j^{\pm,w}$. When replacing operators in, say $\mathcal{D}_j^{-,w}$ by those in $\hat{\mathcal{D}}_j^{-,w}$, the latter can be interpreted as having been obtained by performing w units of spectral flow on primaries of $\hat{\mathcal{D}}_j^{-,w=0}$ or $w-1$ units of spectral flow on primaries of $\mathcal{D}_{-(k/2)-j}^{+}$, that is w units of spectral flow from $\mathcal{D}_{-(k/2)-j}^{+,w=-1}$, which in turn may be thought of as the highest-weight field in $\hat{\mathcal{D}}_j^{-,w=0}$ (see Fig. 5). Only the spectral flowed primary of highest-weight appears in both sets of contributions, i.e., the one with $n_3 = \tilde{n}_3 = 0$. This behavior was observed in all other cases, namely, the same discrete series arising in the OPE from $Q^{w=0}$ can also be seen to arise from $Q^{w=1}$ or $Q^{w=-1}$, but only one operator appears in both simultaneously.

Thus, even if the calculations involved operators in the series $\mathcal{D}_j^{\pm,w}$ and $\mathcal{C}_j^{\alpha,w}$, we collect here the results for the fusion rules[12] assuming $\Phi_{m_i,\bar{m}_i}^{j_i,w_i}(z_i,\bar{z}_i) \in \hat{\mathcal{D}}_{j_i}^{\pm,w_i}$ or $\hat{\mathcal{C}}_{j_i}^{\alpha_i,w_i}$, $i = 1, 2, 3$. Using the spectral flow symmetry to identify $\hat{\mathcal{D}}_j^{-,w} = \hat{\mathcal{D}}_{-(k/2)-j}^{+,w-1}$, we obtain:

$$(1)\ \hat{\mathcal{D}}_{j_1}^{+,w_1} \otimes \hat{\mathcal{D}}_{j_2}^{+,w_2} = \int_{\mathcal{P}} dj_3 \hat{\mathcal{C}}_{j_3}^{\alpha_3,w_3=w_1+w_2+1} \oplus \sum_{-((k-1)/2)<j_3\le j_1+j_2} \hat{\mathcal{D}}_{j_3}^{+,w_3=w_1+w_2} \oplus \sum_{j_1+j_2+(k/2)\le j_3<-(1/2)} \hat{\mathcal{D}}_{j_3}^{+,w_3=w_1+w_2+1},$$

$$(2)\ \hat{\mathcal{D}}_{j_1}^{+,w_1} \otimes \hat{\mathcal{C}}_{j_2}^{\alpha_2,w_2} = \sum_{-((k-1)/2)<j_3<-(1/2)} \hat{\mathcal{D}}_{j_3}^{+,w_3=w_1+w_2} \oplus \sum_{w=0}^{1} \int_{\mathcal{P}} dj_3 \hat{\mathcal{C}}_{j_3}^{\alpha_3,w_3=w_1+w_2+w},$$

$$(3)\ \hat{\mathcal{C}}_{j_1}^{\alpha_1,w_1} \otimes \hat{\mathcal{C}}_{j_2}^{\alpha_2,w_2} = \sum_{w=-1}^{0} \sum_{-((k-1)/2)<j_3<-(1/2)} \times \hat{\mathcal{D}}_{j_3}^{+,w_3=w_1+w_2+w} \oplus \sum_{w=-1}^{1} \times \int_{\mathcal{P}} dj_3 \hat{\mathcal{C}}_{j_3}^{\alpha_3,w_3=w_1+w_2+w}.$$

We have truncated the spin of the contributions from discrete representations following the criterion that processes related through the identity $\hat{\mathcal{D}}_j^{+,w} \equiv \hat{\mathcal{D}}_{-(k/2)-j}^{-,w+1}$ must be equal, i.e., equivalent operator products should get the same contributions. Indeed, one finds contradictions unless the OPE is truncated to keep j_3 within the region (2.18). As we have seen through some examples, extending the OPE (3.28), (3.29), (3.30), and (3.31) to representations of the current algebra, discrepancies occur both when comparing

[12] Actually, the fusion rules for two representations determine the exact decomposition of their tensor products. These not only contain information on the conformal families appearing in the r.h.s of the OPE, but also on their multiplicities. We shall not attempt to determine the latter here.

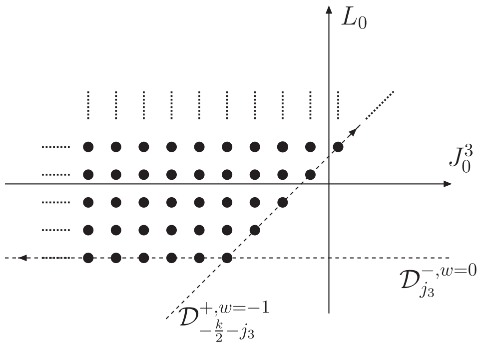

FIG. 5. Weight diagram of $\hat{\mathcal{D}}_{j_3}^{-,w=0}$. The lines with arrows indicate the states in $\mathcal{D}_{j_3}^{-,w=0}$ and $\mathcal{D}_{-(k/2)-j_3}^{+,w=-1}$. Consider a state in $\hat{\mathcal{D}}_{\tilde{j}}^{+,w=0}$, at level $\tilde{N}$ and weight $\tilde{m} = -\tilde{j} + \tilde{n}$. It follows from (2.13) and (2.14) that after spectral flowing by (-1) unit, this state maps to a state in $\hat{\mathcal{D}}_{j}^{-,w=0}$, with $j = -\frac{k}{2} - \tilde{j}$, level $N = \tilde{n}$ and weight $m = j - n$, with $n = \tilde{N}$. For instance primary states in $\hat{\mathcal{D}}_{-(k/2)-j_3}^{+,w=0}$, denoted simply by $\mathcal{D}_{-(k/2)-j_3}^{+,w=0}$, map to highest-weight states in $\hat{\mathcal{D}}_{j_3}^{-,w=0}$. So, only one state in $\mathcal{D}_{-(k/2)-j_3}^{+,w=-1}$ coincides with one in $\mathcal{D}_{j_3}^{-,w=0}$, namely, that with $\tilde{n} = 0$.

w-conserving with nonconserving contributions as well as when comparing w-conserving terms among themselves. So the truncation is imposed by self-consistency.

A strong argument in support of the fusion rules (1)–(3) is that only operators violating the bound (2.18) must be discarded. Indeed, the cut amounts to keeping just contributions from states in the spectrum,[13] i.e., it implies that the operator algebra is closed on the Hilbert space of the theory. However, the spectrum involves irreducible representations and there are no singular vectors to decouple states like in $SU(2)$ [17].[14] We do not have an understanding of the physical process determining the truncation. Moreover, the cut cannot be directly implemented in the analysis performed in the previous section because it would break analyticity. Therefore, either the prescription (3.4) must be modified to be consistent with the spectral flow symmetry or there is a yet to be discovered physical mechanism decoupling states. In other words, the OPE in the H_3^+ and the AdS_3 WZNW models do not seem to be just related by analytic continuation, at least not in the way we have implemented here.

Nevertheless, the results listed in items (1)–(3) above are supported by several consistency checks. First, the limit $k \to \infty$ contains the classical tensor products of representations of $SL(2,\mathbb{R})$ [16] when restricted to $w = 0$ fields. Second, as mentioned in the previous paragraph, once the OPE is truncated to keep only contributions from the spectrum, one can verify full consistency. In particular, the OPE $\hat{\mathcal{D}}_{j_1}^{+,w_1} \otimes \hat{\mathcal{D}}_{j_2}^{+,w_2}$ is consistent with the results in [5] (see the discussion in Appendix A 2). Finally, based on the spectral flow selection rules (2.15) and (2.16), the following alternative analysis can be performed. Let us consider, for instance, the operator product $\hat{\mathcal{D}}_{j_1}^{+,w_1} \otimes \hat{\mathcal{D}}_{j_2}^{+,w_2}$. Applying Eq. (2.16) to correlators involving three discrete states in $\hat{\mathcal{D}}_{j}^{+,w}$ requires either (i) $w_3 = -w_1 - w_2 - 1$ or (ii) $w_3 = -w_1 - w_2 - 2$. Therefore, together with m conservation, (i) implies that the three-point function $\langle \hat{\mathcal{D}}_{j_1}^{+,w_1} \hat{\mathcal{D}}_{j_2}^{+,w_2} \hat{\mathcal{D}}_{j_3}^{+,w_3=-w_1-w_2-1} \rangle$ will not vanish as long as the OPE $\hat{\mathcal{D}}_{j_1}^{+,w_1} \otimes \hat{\mathcal{D}}_{j_2}^{+,w_2}$ contains a state in $\hat{\mathcal{D}}_{j_3}^{-,w=w_1+w_2+1}$, which is equivalent to $\hat{\mathcal{D}}_{\tilde{j}_3}^{+,w=w_1+w_2}$. Indeed, this contribution appeared above. Similarly, (ii) implies that in order for $\langle \hat{\mathcal{D}}_{j_1}^{+,w_1} \hat{\mathcal{D}}_{j_2}^{+,w_2} \times \hat{\mathcal{D}}_{j_3}^{+,w_3=-w_1-w_2-2} \rangle$ to be nonvanishing, the OPE $\hat{\mathcal{D}}_{j_1}^{+,w_1} \otimes \hat{\mathcal{D}}_{j_2}^{+,w_2}$ must have contributions from $\hat{\mathcal{D}}_{j_3}^{-,w_3=w_1+w_2+2} \equiv \hat{\mathcal{D}}_{\tilde{j}_3}^{+,w_3=w_1+w_2+1}$, which in fact were found. Finally, when the third state involved in the three-point function is in the series $\hat{\mathcal{C}}_{j_3}^{\alpha_3,w_3}$, Eq. (2.15) leaves only one possibility, namely $w_3 = -w_1 - w_2 - 1$, and thus the OPE must include terms in $\hat{\mathcal{C}}_{j_3}^{\alpha_3,w_3=w_1+w_2+1}$, which actually appear in the list above. Although this analysis based on the spectral flow selection rules does not allow one to determine either the range of j_3 values or the OPE coefficients, it is easy to check that the series content in (1)–(3) is indeed completely reproduced in this way.

As mentioned in the previous section, in principle $w = \pm 2$ three-point functions should have been considered. However, the contributions from these terms are already contained in our results. If they gave contributions from discrete representations outside the spectrum, they should be truncated since the equivalent terms listed above do not include them. Contributions from operators in $\hat{\mathcal{D}}_{j_3}^{-,w_3=w_1+w_2+2}$ can only appear in case (1), namely, $\hat{\mathcal{D}}_{j_1}^{+,w_1} \otimes \hat{\mathcal{D}}_{j_2}^{+,w_2}$, for $j_3 = -k - j_1 - j_2 - n$. These correspond to the terms denoted as Poles_2 in [5], where they could not be interpreted in terms of physical string states and were then truncated. See Appendix A 2 for a detailed discussion.

In conclusion, the results presented in this section are in agreement with the spectral flow selection pattern (2.15) and (2.16); they are consistent with the results in [5] and determine the closure of the operator algebra when properly treating the spectral flow symmetry. The full consistency of the OPE should follow from a proof of factorization and crossing symmetry of the four-point functions, but closed expressions for these amplitudes are

[13]It is important to stress that the truncation is not discarding contributions from the *microstates* associated to the (j_1, j_2)-dependent poles that were found in [7]. Only m-dependent poles which are absent in the x basis present inconsistencies with the spectral flow symmetry.

[14]The spectral flow operators $\Phi_{\pm(k/2),\pm(k/2)}^{-(k/2)}$ have null descendants. Even though they are excluded from the range (2.18), they are necessary auxiliary fields to construct the states in spectral flow representations. Although the physical mechanism is not clear to us, these operators might play a role in the decoupling.

not known, even in the simpler H_3^+ model. In order to make some preliminary progress in this direction, in the next section we discuss certain properties of the factorization of four-point amplitudes involving states in different representations of the $SL(2,\mathbb{R})$ WZNW model, constructed along the lines in [7].

V. COMMENTS ON THE FACTORIZATION OF FOUR-POINT FUNCTIONS

Although a complete description of the contributions of descendant operators is not available to complete the bootstrap program, in this section we display some interesting properties of the amplitudes that can be useful to achieve a resolution of the theory. We first summarize known results on the s-channel factorization of four-point functions in the H_3^+ model and show that an alternative expression can be written in the AdS_3 WZNW model if the correlators in both models are related through analytic continuation. Then, we perform a qualitative study of the contributions of primaries and flowed primaries in the intermediate channels of the amplitudes and finally, we discuss the consistency of the factorization with the spectral flow selection rules.

A decomposition of the four-point function in the Euclidean model was worked out in [6,7] using the OPE (2.10) for pairs of primary operators $\Phi_{j_1}\Phi_{j_2}$ and $\Phi_{j_3}\Phi_{j_4}$. The s-channel factorization was written as follows:

$$\begin{aligned}&\langle\Phi_{j_1}(x_1|z_1)\Phi_{j_2}(x_2|z_2)\Phi_{j_3}(x_3|z_3)\Phi_{j_4}(x_4|z_4)\rangle\\&=|z_{34}|^{2(\tilde\Delta_2+\tilde\Delta_1-\tilde\Delta_4-\tilde\Delta_3)}|z_{14}|^{2(\tilde\Delta_2+\tilde\Delta_3-\tilde\Delta_4-\tilde\Delta_1)}|z_{24}|^{-4\tilde\Delta_2}\\&\quad\times|z_{13}|^{2(\tilde\Delta_4-\tilde\Delta_1-\tilde\Delta_2-\tilde\Delta_3)}\int_{\mathcal{P}^+}dj\,\mathcal{A}(j_i,j)\mathcal{G}_j(j_i,z,\bar z,x_i,\bar x_i)\\&\quad\times|z|^{2(\Delta_j-\Delta_1-\Delta_2)}.\end{aligned}\tag{5.1}$$

Here

$$\mathcal{A}(j_i,j)=C(-j_1,-j_2,-j)B(-j-1)C(-j,-j_3,-j_4)\tag{5.2}$$

and

$$\begin{aligned}\mathcal{G}_j(j_i,z,\bar z,x_i,\bar x_i)=&\sum_{n,\bar n=0}^{\infty}z^n\bar z^{\bar n}D^{(n)}_{x,j}(j_i,x_i)\bar D^{\bar n}_{\bar x,j}(j_i,\bar x_i)\\&\times G_j(j_i,x_i,\bar x_i),\end{aligned}\tag{5.3}$$

where $D^{(n)}_{x,j}(j_i,x_i)$ are differential operators containing the contributions from intermediate descendant states and

$$\begin{aligned}G_j(j_i,x_i,\bar x_i)=&|x_{12}|^{2(j_1+j_2-j)}|x_{34}|^{2(j_3+j_4-j)}\int d^2x d^2x'\\&\times|x_1-x|^{2(j_1+j-j_2)}|x_2-x|^{2(j_2+j-j_1)}\\&\times|x_3-x'|^{2(j_3+j-j_4)}|x_4-x'|^{2(j_4+j-j_3)}\\&\times|x-x'|^{-4j-4},\end{aligned}\tag{5.4}$$

which may be rewritten as

$$\begin{aligned}G_j(j_i,x_i,\bar x_i)=&\frac{\pi^2}{(2j+1)^2}|x_{34}|^{2(j_4+j_3-j_2-j_1)}|x_{24}|^{4j_2}\\&\times|x_{14}|^{2(j_4+j_1-j_2-j_3)}|x_{13}|^{2(j_3+j_2+j_1-j_4)}\\&\times\Big\{|F_j(j_i,x)|^2\\&+\frac{\gamma(1+j+j_4-j_3)\gamma(1+j+j_3-j_4)}{\gamma(2j+1)\gamma(j_1-j_2-j)\gamma(j_2-j_1-j)}\\&\times|F_{-1-j}(j_i,x)|^2\Big\},\end{aligned}$$

with $F_j(j_i,x)\equiv x_2^{j_1+j_2-j}F_1(j_1-j_2-j,j_4-j_3-j;-2j;x)$ and $x=\frac{x_{12}x_{34}}{x_{13}x_{24}}$.

The properties of (5.1) under $j\to-1-j$ allow one to extend the integration contour from $\mathcal{P}^+$ to the full axis $\mathcal{P}=-\frac{1}{2}+i\mathbb{R}$ and rewrite it in a holomorphically factorized form. Crossing symmetry follows from similar properties of a five-point function in Liouville theory and it amounts to establishing the consistency of the H_3^+ WZNW model [18].

Expression (5.1) is valid for external states Φ_{j_1}, Φ_{j_2} in the range (2.12) and similarly for Φ_{j_3}, Φ_{j_4}. In particular, it holds for operators in continuous representations of the $SL(2,\mathbb{R})$ WZNW model. The analytic continuation to other values of j_i was performed in [5]. In this process, some poles in the integrand cross the integration contour and the four-point function is defined as (5.1) plus the contributions of all these poles. This procedure allowed the authors to analyze the factorization of four-point functions of $w=0$ short strings in the boundary conformal field theory, obtained from primary states in discrete representations $\mathcal{D}^{w=0}_j\otimes\mathcal{D}^{w=0}_j$, by integrating over the world sheet moduli. It is important to stress that the aim in [5] was to study the factorization in the boundary conformal field theory with coordinates x_i, $\bar x_i$, so the x basis was found convenient. The conformal blocks were expanded in powers of the cross ratios x, $\bar x$ and then integrated over the world sheet coordinates z, $\bar z$. To study the factorization in the $SL(2,\mathbb{R})$ WZNW model instead, we expand the conformal blocks in powers of z, $\bar z$, and in order to consider the various sectors, we find convenient to translate (5.1) to the m basis.

To this purpose, one can verify that the integral over j commutes with the integrals over x_i, $\bar x_i$, $i=1,\ldots,4$ and that it is regular for $j^{\pm}_{21}$ and $j^{\pm}_{43}$ in the range (2.12) and for all of $|m|$, $|\bar m|$, $|m_i|$, $|\bar m_i|<\frac{1}{2}$, where we have introduced $m=m_1+m_2=-m_3-m_4$, $\bar m=\bar m_1+\bar m_2=-\bar m_3-\bar m_4$. Integrating in addition over x and x' in (5.4), we get

$$\begin{aligned}&\langle\Phi^{j_1}_{m_1,\bar m_1}\Phi^{j_2}_{m_2,\bar m_2}\Phi^{j_3}_{m_3,\bar m_3}\Phi^{j_4}_{m_4,\bar m_4}\rangle\\&=|z_{34}|^{2(\tilde\Delta_2+\tilde\Delta_1-\tilde\Delta_4-\tilde\Delta_3)}|z_{14}|^{2(\tilde\Delta_2+\tilde\Delta_3-\tilde\Delta_4-\tilde\Delta_1)}|z_{24}|^{-4\tilde\Delta_2}\\&\quad\times|z_{13}|^{2(\tilde\Delta_4-\tilde\Delta_1-\tilde\Delta_2-\tilde\Delta_3)}\int_{\mathcal{P}^+}dj\,\mathbb{A}^{w=0}_j(j_i;m_i,\bar m_i)\\&\quad\times|z|^{2(\tilde\Delta_j-\tilde\Delta_1-\tilde\Delta_2)}+\cdots,\end{aligned}\tag{5.5}$$

where

$$\mathbb{A}_j^{w=0}(j_i;m_i,\bar{m}_i)=\delta^{(2)}(m_1+\ldots+m_4)C(1+j_1,1+j_2,1+j)\times W\begin{bmatrix} j_1,j_2,j \\ m_1,m_2,-m\end{bmatrix}\frac{1}{B(-1-j)c_{m,\bar{m}}^{-1-j}}\times C(1+j_3,1+j_4,1+j)W\begin{bmatrix} j_3,j_4,j \\ m_3,m_4,m\end{bmatrix}. \tag{5.6}$$

An alternative representation of (5.6) was found in [19] in terms of higher generalized hypergeometric functions ${}_4F_3$. This new identity among hypergeometric functions is an interesting by-product of the present result.

The dots in (5.5) refer to higher powers of z, $\bar{z}$ corresponding to the integration of terms of the form $\mathbb{A}_j^{N,w=0}|z|^{2(\Delta_j^{(N)}-\tilde{\Delta}_1-\tilde{\Delta}_2)}$, where $\mathbb{A}_j^{N,w=0}$, $N=1,2,3,\ldots$ stand for contributions from descendant operators at level N with conformal weights $\Delta_j^{(N)}=\tilde{\Delta}_j+N$.

Notice that the symmetry under $j\leftrightarrow -1-j$ in (5.6), which can be easily checked by using the identity (3.3), allows one to extend the integral to the full axis $\mathcal{P}=-\frac{1}{2}+i\mathbb{R}$.

Given that correlation functions in the $SL(2,\mathbb{R})$ WZNW model in the m basis depend on the sum of w_i numbers, except for the powers of the coordinates z_i, $\bar{z}_i$, if the Lorentzian and Euclidean theories are simply related by analytic continuation, this result should hold, in particular, for states in continuous representations in arbitrary spectral flow sectors (with $|m_i|$, $|\bar{m}_i|$, $|m|<\frac{1}{2}$), as long as $\sum_i w_i=0$, i.e.,

$$\langle\Phi^{j_1,w_1}_{m_1,\bar{m}_1}\Phi^{j_2,w_2}_{m_2,\bar{m}_2}\Phi^{j_3,w_3}_{m_3,\bar{m}_3}\Phi^{j_4,w_4}_{m_4,\bar{m}_4}\rangle_{\sum_{i=1}^4 w_i=0} = z_{34}^{\Delta_2+\Delta_1-\Delta_4-\Delta_3}z_{14}^{\Delta_2+\Delta_3-\Delta_4-\Delta_1}z_{13}^{\Delta_4-\Delta_1-\Delta_2-\Delta_3}z_{24}^{-2\Delta_2}\times \text{c.c.}\times\int_{\mathcal{P}}dj\mathbb{A}_j^{w=0}(j_i;m_i,\bar{m}_i)z^{\Delta_j-\Delta_1-\Delta_2}\bar{z}^{\bar{\Delta}_j-\bar{\Delta}_1-\bar{\Delta}_2}+\cdots, \tag{5.7}$$

where $\Delta_j=-\frac{j(j+1)}{k-2}-m(w_1+w_2)-\frac{k}{4}(w_1+w_2)^2$ and c.c. stands for the obvious antiholomorphic $\bar{z}_i$ dependence. For other values of $j_1,\cdots,j_4,m_1,\ldots,\bar{m}_4$ the integral may diverge and must be defined by analytic continuation.

That a generic w-conserving four-point function involving primaries or highest/lowest-weight states in $\mathcal{C}_j^{\alpha,w}$ or $\mathcal{D}_j^{\pm,w}$ should factorize as in (5.7), if the amplitude with four $w=0$ states is given by (5.5), can be deduced from the relation [10]:

$$\left\langle\prod_{i=1}^n\Phi^{j_i,w_i}_{m_i,\bar{m}_i}(z_i,\bar{z}_i)\right\rangle_{\sum_{i=1}^n w_i=0}=\kappa\bar{\kappa}\left\langle\prod_{i=1}^n\Phi^{j_i,\tilde{w}_i=0}_{m_i,\bar{m}_i}(z_i,\bar{z}_i)\right\rangle, \tag{5.8}$$

where $\kappa=\prod_{i<j}z_{ij}^{-w_im_j-w_jm_i-(k/2)w_iw_j}$, $\bar{\kappa}=\prod_{i<j}z_{ij}^{-w_i\bar{m}_j-w_j\bar{m}_i-(k/2)w_iw_j}$, after Taylor expanding around $z=0$ the r.h.s. of the following identity:

$$\kappa z_{34}^{\tilde{\Delta}_2+\tilde{\Delta}_1-\tilde{\Delta}_4-\tilde{\Delta}_3}z_{14}^{\tilde{\Delta}_2+\tilde{\Delta}_3-\tilde{\Delta}_4-\tilde{\Delta}_1}z_{24}^{-2\tilde{\Delta}_2}z_{13}^{\tilde{\Delta}_4-\tilde{\Delta}_1-\tilde{\Delta}_2-\tilde{\Delta}_3}z^{\tilde{\Delta}_j-\tilde{\Delta}_1-\tilde{\Delta}_2} = z_{34}^{\Delta_2+\Delta_1-\Delta_4-\Delta_3}z_{14}^{\Delta_2+\Delta_3-\Delta_4-\Delta_1}z_{13}^{\Delta_4-\Delta_1-\Delta_2-\Delta_3}z_{24}^{-2\Delta_2}\times z^{\Delta_j-\Delta_1-\Delta_2}(1-z)^{-m_2w_3-m_3w_2-(k/2)w_2w_3}. \tag{5.9}$$

The conclusion is that, if the H_3^+ and AdS_3 models are simply related by analytic continuation, then (5.7) and its analytic continuation should hold for generic w-conserving four-point functions of fields in $\mathcal{C}_j^{\alpha,w}$ or $\mathcal{D}_j^{\pm,w}$.[15] However, expression (5.7) appears to be in contradiction with the factorization *ansatz* and the OPE found in Sec. III for the $SL(2,\mathbb{R})$ WZNW model, because it seems to contain just w-conserving channels. Actually, directly applying the factorization *ansatz* based on the OPE (3.4) would give the following expression for both w-conserving and violating four-point functions:

$$\langle\Phi^{j_1,w_1}_{m_1,\bar{m}_1}\Phi^{j_2,w_2}_{m_2,\bar{m}_2}\Phi^{j_3,w_3}_{m_3,\bar{m}_3}\Phi^{j_4,w_4}_{m_4,\bar{m}_4}\rangle \sim z_{34}^{\Delta_2+\Delta_1-\Delta_4-\Delta_3}z_{14}^{\Delta_2+\Delta_3-\Delta_4-\Delta_1}z_{13}^{\Delta_4-\Delta_1-\Delta_2-\Delta_3}z_{24}^{-2\Delta_2}\times\text{c.c.}\,\delta^2\left(\sum_{i=1}^4 m_i+\frac{k}{2}w_i\right)\sum_{w=-1}^{1}\int_{\mathcal{P}}djQ^wQ^{-w-\sum_{i=1}^4 w_i}\times B(-1-j)c_{m,\bar{m}}^{-1-j}z^{\Delta_j-\Delta_1-\Delta_2}\bar{z}^{\bar{\Delta}_j-\bar{\Delta}_1-\bar{\Delta}_2}+\cdots \tag{5.10}$$

with $m=m_1+m_2-\frac{k}{2}w=-m_3-m_4-\frac{k}{2}w$, $\bar{m}=\bar{m}_1+\bar{m}_2-\frac{k}{2}w=-\bar{m}_3-\bar{m}_4-\frac{k}{2}w$, and $\Delta_j=-\frac{j(j+1)}{k-2}-m(w_1+w_2+w)-\frac{k}{4}(w_1+w_2+w)^2$ (similarly for $\bar{\Delta}_j$). Actually, in the m basis, the starting point for the w-conserving four-point function would have been (5.7) plus an analogous contribution involving one unit spectral flow three-point functions, i.e., (5.7) rewritten in terms of $\mathbb{A}_j^{w=1}$ or $\mathbb{A}_j^{w=-1}$ instead of $\mathbb{A}_j^{w=0}$, where

$$\mathbb{A}_j^{w=\pm1}(j_i;m_i,\bar{m}_i)=\delta^{(2)}\left(\sum_{i=1}^4 m_i\right)\frac{\tilde{C}(1+j_1,1+j_2,1+j)}{\gamma(j_1+j_2+j+3-\frac{k}{2})}\tilde{W}\begin{bmatrix} j_1,j_2,j \\ \mp m_1,\mp m_2,\pm m\end{bmatrix}\frac{1}{B(-1-j)c_{m,\bar{m}}^{-1-j}}\times\frac{\tilde{C}(1+j_3,1+j_4,1+j)}{\gamma(j_3+j_4+j+3-\frac{k}{2})}\tilde{W}\begin{bmatrix} j_3,j_4,j \\ \pm m_3,\pm m_4,\pm m\end{bmatrix}. \tag{5.11}$$

[15]See Appendix A 3 for an alternative discussion directly in the m basis, independent of the x basis.

But if correlation functions in this model are to be obtained from those in the H_3^+ model [5–9,12], spectral flow conserving and nonconserving channels should give the same result for the w-conserving four-point functions. This does not imply that $\mathbb{A}_j^{w=0}$ and $\mathbb{A}_j^{w=\pm1}$ carry the same amount of information.[16] In general, if both expressions for the four-point functions were equivalent, one would expect that part of the information in $\mathbb{A}_j^{w=0}$ was contained in $\mathbb{A}_j^{w=\pm1}$ and the rest in the contributions from descendants in $\mathbb{A}_j^{N,w=\pm1}$.

A proof of this statement would require making explicit the higher order terms and possibly some contour manipulations, which we shall not attempt. Nevertheless there are several indications supporting this claim. A similar proposition was advanced in [10] for the H_3^+ model and some evidence was given that these possibilities might not be exclusive, depending on which correlator the OPE is inserted in. Furthermore, $w = 1$ long strings were found in the s-channel factorization of the four-point amplitude of $w = 0$ short strings in [5] starting from the holomorphically factorized expression for (5.1), rewriting the integrand and moving the integration contour. Moreover, in the m basis, spectral flow nonconserving channels can be seen to appear naturally from (5.7) in certain special cases, as we now show.

Identities among different expansions of four-point functions containing at least one field in discrete representations can be generated using the spectral flow symmetry. In particular, w-conserving four-point functions involving the fields $\Phi^{j_1,w_1}_{m_1=\bar m_1=-j_1}$ and $\Phi^{j_3,w_3}_{m_3=\bar m_3=j_3}$ coincide [up to $B(j_1)$, $B(j_3)$ factors] with the w-conserving amplitudes involving $\Phi^{j_1'=-(k/2)-j_1,w_1'=w_1+1}_{m_1'=\bar m_1'=j_1'}$ and $\Phi^{j_3'=-(k/2)-j_3,w_3'=w_3-1}_{m_3'=\bar m_3'=-j_3'}$.[17] This allows one to expand the four-point amplitude in two alternative ways, namely,

$$\int_{\mathcal{P}} dj\mathbb{A}_j^{w=0}(j_1, j_2, j_3, j_4; m_1, \ldots, \bar m_3, \bar m_4) \times z^{\Delta(j)-\Delta(j_1)-\Delta(j_2)}\bar z^{\bar\Delta(j)-\bar\Delta(j_1)-\bar\Delta(j_2)} + \cdots \tag{5.12}$$

or

$$\beta_{1,3}\int_{\mathcal{P}} dj\mathbb{A}_j^{w=0}(j_1', j_2, j_3', j_4; m_1', \ldots, \bar m_3', \bar m_4) \times z^{\Delta'(j)-\Delta(j_1')-\Delta(j_2)}\bar z^{\bar\Delta'(j)-\bar\Delta(j_1')-\bar\Delta(j_2)} + \cdots, \tag{5.13}$$

where $\beta_{1,3} \equiv \frac{B(-1-j_3)}{B(-1-j_1')}$ and the dots refer to contributions from descendants and, in addition, to residues at poles in $\mathbb{A}_j^{w=0}$ crossing $\mathcal{P}$ after analytic continuation of j_i ($i = 1, 3$ and eventually 2, 4) to the region (2.18). Explicitly, $\mathbb{A}_j^{w=0}(j_1', j_2, j_3', j_4; m_1', \ldots, \bar m_3', \bar m_4)$ is given by

$$\begin{aligned} &C(1+j_1', 1+j_2, 1+j)C(1+j_3', 1+j_4, 1+j)\\ &\times \frac{\pi^3\gamma(2+2j)}{B(-1-j)}\frac{\gamma(j-j_1'-j_2)\gamma(j_2-j_1'-j)}{\gamma(2+j_1'+j_2+j)\gamma(-2j_1')}\\ &\times \frac{\gamma(j-j_3'-j_4)\gamma(j_4-j_3'-j)}{\gamma(2+j_3'+j_4+j)\gamma(-2j_3')}\\ &\times \frac{\Gamma(1+j_2-m_2)\Gamma(1+j_4+\bar m_4)}{\Gamma(-j_2+\bar m_2)\Gamma(-j_4-m_4)}\\ &\times \frac{\Gamma(-j-\bar m)\Gamma(1+j-m)}{\Gamma(1+j+m)\Gamma(-j+\bar m)}. \end{aligned}$$

Using (4.2) and rewriting this expression in terms of j_i, m_i, the following equivalence can be shown

$$(5.13) = \int_{\mathcal{P}} dj\mathbb{A}_j^{w=1}(j_1, j_2, j_3, j_4; m_1, \ldots, \bar m_3, \bar m_4) \times z^{\Delta(j)-\Delta(j_1)-\Delta(j_2)}\bar z^{\bar\Delta(j)-\bar\Delta(j_1)-\bar\Delta(j_2)} + \ldots \tag{5.14}$$

Notice that not only the coefficient $\mathbb{A}_j^{w=1}$ but also the z_i, $\bar z_i$ dependence are as expected. In fact, $\Delta(j_1') = \tilde\Delta(j_1') - m_1'w_1' - \frac{k}{4}w_1'^2 = \tilde\Delta(j_1) - m_1w_1 - \frac{k}{4}w_1^2 = \Delta(j_1)$ and $\Delta'(j) = \tilde\Delta(j) - (m_1' + m_2)(w_1' + w_2) - \frac{k}{4}(w_1' + w_2)^2 = \tilde\Delta(j_1) - mw - \frac{k}{4}w^2 = \Delta(j_1)$, where $m = m_1 + m_2 - \frac{k}{2}$ and $w = w_1 + w_2 + 1$. Therefore, we have seen, in a particular example, that spectral flow conserving and violating channels can give the same result for four-point functions. This is a nontrivial result showing that the spectral flow symmetry allows one to exhibit w-nonconserving channels that are not equivalent to other w-conserving ones in expressions constructed as sums over w-conserving exchanges.

In Appendix A 3, we show that the terms explicitly displayed in both (5.7) and (5.14) are solutions of the Knizhnik-Zamolodchikov (KZ) equations. However, these equations do not give enough information to confirm that the full expressions (5.7) and (5.14) are equivalent.

The factorization of four-point functions reproduces the field content of the OPE. Therefore, the truncation imposed on the operator algebra by the spectral flow symmetry must be realized in physical amplitudes. Again, to confirm this would require more information on the contributions from descendant fields and studying crossing symmetry. Here, we just illustrate this point with one example. Take for instance the following four-point function[18]:

$$\langle \mathcal{D}^{+,w_1=0}_{j_1}\mathcal{D}^{+,w_2=-1}_{j_2}\mathcal{D}^{-,w_3=0}_{j_3}\mathcal{D}^{-,w_4=-1}_{j_4}\rangle, \tag{5.15}$$

in the particular case with $n_i = 0$, $\forall i$ (where $m_i = \pm j_i \mp$

[16] In other words, both expressions seem to give the same contribution in w-conserving four-point functions. However, one cannot always use either one of them. In particular, this is not expected to hold for w-violating amplitudes.

[17] This is a consequence of the identities discussed in the paragraph containing Eq. (4.2) in the previous section.

[18] Here, as in the previous section, we denote the states by the representations they belong to and we omit the antiholomorphic part for short.

n_i) and $j_1 + j_2 = j_3 + j_4 < -\frac{k-1}{2}$. The OPE (3.28) implies one intermediate state in the s channel in $\mathcal{D}_j^{+,w=-1}$, with $j = j_1 + j_2 = -m$ as well as exchanges of states in $\mathcal{D}_j^{+,w=0}$ if $j_1 + j_2 = j_3 + j_4 < -\frac{k+1}{2}$ with $j = j_1 + j_2 + \frac{k}{2} + n$, $n = 0, 1, 2, \ldots$ such that $j < -\frac{1}{2}$, and also of continuous states in $\mathcal{C}_j^{\alpha,w=0}$. The unique state found in $\mathcal{D}_j^{+,w=-1}$ is equivalent to the highest-weight state in $\mathcal{D}_{\tilde{j}}^{-,w=0}$ with $\tilde{j} = -\frac{k}{2} - j > -\frac{1}{2}$.

This four-point function must coincide with the following one:

$$\langle \mathcal{D}_{j_1}^{+,w_1=0} \mathcal{D}_{\tilde{j}_2}^{-,w_2=0} \mathcal{D}_{\tilde{j}_3}^{-,w_3=0} \mathcal{D}_{\tilde{j}_4}^{+,w_4=0} \rangle, \tag{5.16}$$

where as usual $\tilde{j}_i = -\frac{k}{2} - j_i$ (notice that this holds without "hats" because $n_i = 0$, $\forall i$). Now $\tilde{j}_2 - j_1 = \tilde{j}_4 - j_3 > -\frac{1}{2}$. Therefore, (3.29) implies that only states from $\mathcal{C}_j^{\alpha,w=0}$ as well as from $\mathcal{D}_j^{+,w=0}$ with $j = j_1 - \tilde{j}_2 + n = j_1 + j_2 + \frac{k}{2} + n$ propagate in the intermediate s channel, the latter requiring the extra condition $\tilde{j}_2 - j_1 = \tilde{j}_4 - j_3 > \frac{1}{2}$, i.e., $j_1 + j_2 = j_3 + j_4 < -\frac{k+1}{2}$. The important remark is that no intermediate states from $\mathcal{D}_{\tilde{j}}^{-,w=0}$ appear in the factorization. This behavior was discussed in the previous section when studying the consequences of the spectral flow symmetry on the OPE. However, we have considered this case carefully here because it explicitly displays the fact that the same four-point function factorizes in two different ways and the unique difference is an extra state violating the bounds (2.18). Recall that we are only considering primaries and their spectral flow images. We expect that some consistency requirements, such as crossing symmetry, will automatically realize the OPE displayed in the previous section in physical amplitudes.

An indication in favor of the bootstrap approach to this nonrational CFT is that the expressions reproduce the spectral flow selection rules (2.15) and (2.16) for four-point functions in different sectors. Indeed, let us analyze this feature in a four-point function involving only external discrete states or their spectral flow images. The bounds (2.16) require $-3 \le \sum_{i=1}^{4} w_i \le -1$, in agreement with the factorization of this amplitude in any channel. Indeed, consider for instance

$$\langle \hat{\mathcal{D}}_{j_1}^{+,w_1} \hat{\mathcal{D}}_{j_2}^{+,w_2} \hat{\mathcal{D}}_{j_3}^{+,w_3} \hat{\mathcal{D}}_{j_4}^{+,w_4} \rangle. \tag{5.17}$$

The OPE $\hat{\mathcal{D}}_{j_1}^{+,w_1} \otimes \hat{\mathcal{D}}_{j_2}^{+,w_2}$ computed in the previous section (and similarly for j_3, j_4) requires either $w_1 + w_2 = -w_3 - w_4 - 1$ or $w_1 + w_2 = -w_3 - w_4 - 2$ or $w_1 + w_2 = -w_3 - w_4 - 3$ for discrete intermediate states and $w_1 + w_2 = -w_3 - w_4 - 2$ for continuous intermediate states, and similarly in the other channels.

Repeating this analysis for four-point functions involving fields in different representations, it is straightforward to conclude that the spectral flow selection rules for four-point functions in different sectors can be obtained from those for two- and three-point functions, or equivalently from the OPE found in Sec. IV.

VI. SUMMARY AND CONCLUSIONS

We have studied the OPE in the AdS_3 WZNW model. Performing the analytic continuation of the expressions in the Euclidean H_3^+ WZNW model proposed in [6,7] and adding spectral flow, i.e., considering the full set of structure constants, we obtained the OPE of spectral flow images of primary fields in the Lorentzian theory. Assuming the results also hold for affine descendants, we have argued that a truncation is necessary in order to avoid contradictions, and we have shown that a consistent cut amounts to the closure of the operator algebra on the Hilbert space of the theory. Indeed, the spectral flow symmetry implies that only operators outside the physical spectrum must be discarded and moreover, every physical state contributing to a given OPE is also found to appear in all possible equivalent operator products. The fusion rules obtained in this way are consistent with results in [5], deduced from the factorization of four-point functions of $w = 0$ short strings in the boundary conformal field theory, and contain in addition operator products involving states in continuous representations. A discussion of the relation between our results and some conclusions in [5] can be found in Appendix A 2.

Implementing the truncation in the procedure followed in Sec. III in order to directly obtain a consistent OPE does not seem possible because it would break analyticity. Therefore, an inevitable conclusion is that either the prescription must be modified in order to avoid inconsistencies with the spectral flow symmetry, i.e., the route we have followed to relate the OPE in the H_3^+ and the AdS_3 models is not self-consistent, or the structure constants must be further constrained. Nevertheless, although the physical process determining the truncation is not completely understood, several consistency checks have been performed in Sec. IV and the OPE displayed in items (1) – (3) can be taken to stand on solid foundations.

The full consistency of the fusion rules should follow from a proof of factorization and crossing symmetry of the four-point functions. A preliminary analysis of the factorization of amplitudes involving states in different sectors of the theory was presented in Sec. V. Based on the factorization *ansatz*, we proposed an expression for generic four-point functions and we showed that some terms are redundant in w-conserving amplitudes. We illustrated in one example that the amplitudes must factorize as expected in order to avoid inconsistencies, i.e., if the bootstrap approach holds, only states according to the fusion rules determined in Sec. IV must propagate in the intermediate channels. Analogously as the OPE, the factorization also agrees with the spectral flow selection rules. However, more work is necessary to put this *ansatz* on a firmer mathematical ground. In particular, additional information on the action of the spectral flow operation on descendant operators is required to verify crossing symmetry.

Given that scattering amplitudes of string theory on AdS_3 should be obtained from correlation functions in the $SL(2,\mathbb{R})$ WZNW model, our results constitute a step forward towards the construction of the S matrix in string theory on Lorentzian AdS_3 and to learn more about the dual conformal field theory on the boundary through AdS/CFT, in the spirit of [5]. Indeed an important application of our results would be to construct the S matrix of long strings in AdS_3 which describes scatterings in the CFT defined on the Lorentzian two-dimensional boundary. In particular, the OPE $\hat{\mathcal{C}}^{\alpha_1,w_1}_{j_1} \otimes \hat{\mathcal{C}}^{\alpha_2,w_2}_{j_2}$ obtained in Sec. IV sustains the expectations in [5] that short and long strings should appear as poles in the scattering of asymptotic states of long strings.

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APPENDIX A: APPENDICES

1. Analytic structure of W_1

The purpose of this appendix is to study the analytic structure of W_1. In particular, we are especially interested in possible zeros appearing in W_1 which are not evident in the expression (3.17), but are very important in our definition of the OPE.

Let us recall some useful identities relating different expressions for

$$G\begin{bmatrix} a, b, c \\ e, f \end{bmatrix}$$

[20],

$$G\begin{bmatrix} a, b, c \\ e, f \end{bmatrix} = \frac{\Gamma(b)\Gamma(c)}{\Gamma(e-a)\Gamma(f-a)} G\begin{bmatrix} e-a, f-a, u \\ u+b, u+c \end{bmatrix}, \tag{A1}$$

$$G\begin{bmatrix} a, b, c \\ e, f \end{bmatrix} = \frac{\Gamma(b)\Gamma(c)\Gamma(u)}{\Gamma(f-a)\Gamma(e-b)\Gamma(e-c)} G\begin{bmatrix} a, e-b, e-c \\ e, a+u \end{bmatrix}, \tag{A2}$$

where u is defined as $u = e + f - a - b - c$. Using the permutation symmetry among a, b, c and e, f, which is evident from the series representation of the hypergeometric function ${}_3F_2$, seven new identities may be generated. In what follows we use these identities in order to obtain the greatest possible amount of information on W_1.

Consider for instance C^{12} defined in (3.14). Using (A1), it can be rewritten for $j_1 = -m_1 + n_1$, with n_1 a non-negative integer, as

$$C^{12} = \frac{\Gamma(-N)\Gamma(-j_{13})\Gamma(-j_{12})\Gamma(1+j_2+m_2)}{\Gamma(-j_3-m_3)} \sum_{n=0}^{n_1} \binom{n_1}{n} \times \frac{(-)^n}{\Gamma(n-2j_1)} \frac{\Gamma(n-j_{12})}{\Gamma(-j_{12})} \frac{\Gamma(n+1+j_{23})}{\Gamma(1+j_{23})} \times \frac{\Gamma(1+j_3-m_3)}{\Gamma(1+j_3-m_3-n_1+n)}. \tag{A3}$$

Using (A2) instead of (A1), one finds an expression for C^{12} equal to (A3) with $j_3 \to -1 - j_3$.

There is a third expression in which C^{12} can be written as a finite sum for generic j_2, j_3. This follows from (3.14), using the identity obtained from (A2) with $(e \leftrightarrow f)$. This expression is explicitly invariant under $j_3 \to -1 - j_3$.

Consider for instance (A3). All quotients inside the sum are such that the arguments in the Γ functions of the denominator equal those in the numerator up to a positive integer, except for the one with $\Gamma(n - 2j_1)$ which is regular and nonvanishing for $\mathrm{Re} j_1 < -\frac{1}{2}$. Then, each quotient is separately regular. Eventually, some of them may vanish, but not for all values of n. In particular, for $n = 0$ the first two quotients equal one. The last factor may vanish for $n = 0$, but for $n = n_1$ it equals one. However, particular configurations of j_i, m_i may occur such that one of the first two quotients vanishes for certain values of n, namely, $n = n_{\min}, n_{\min} + 1, \ldots, n_1$, and the last one vanishes for other special values, namely, $n = 0, 1, \ldots, n_{\max}$. Thus, if $n_{\max} \geq n_{\min}$, all terms in the sum cancel and C^{12} vanishes as a simple zero. In fact, let us consider for instance both $1 + j_{23} = -p_3$ and $1 + j_3 - m_3 = 1 + n_3$, with p_3, n_3 non-negative integers. This requires $\Phi^{j_2,w_2}_{m_2,\bar{m}_2} \in \mathcal{D}^{-,w_2}_{j_2}$ and $j_3 = j_1 - j_2 - 1 - p_3 = m_3 + n_1 - n_2 - 1 - p_3$, which impose $p_3 < n_1$ and allow one to rewrite the sum in (A3) as

$$\sum_{n=0}^{p_3} \frac{1}{n!} \frac{n_1!}{(n_1-n)!} \frac{p_3!}{(p_3-n)!} \frac{\Gamma(n-j_{12})}{\Gamma(-j_{12})} \frac{1}{\Gamma(n-2j_1)} \times \frac{n_3!}{\Gamma(1+n_3-n_1+n)}. \tag{A4}$$

Finally, taking into account that $1 + n_3 - n_1 + n = -n_2 - (p_3 - n) \leq 0$, for $n = 0, 1, \ldots, p_3$, the sum vanishes as a simple zero. A similar analysis for $j_{12} = p_3 \geq 0$ and $1 + j_3 - m_3 = 1 + n_3 \geq 1$ shows that no zeros appear in this case when $\Phi^{j_2,w_2}_{m_2,\bar{m}_2}$ is the spectral flow image of a primary field.

From the expression obtained for C^{12} by changing $j_3 \to -1 - j_3$, one finds zeros again for $\Phi^{j_2,w_2}_{m_2,\bar{m}_2} \in \mathcal{D}^{-,w_2}_{j_2}$. These appear when both $j_3 = j_2 - j_1 + p_3$ and $j_3 = -m_3 - 1 - n_3$ hold simultaneously.

Finally, repeating the analysis for the sum in the third expression for C^{12}, i.e., that explicitly symmetric under $j_3 \to -1 - j_3$, one finds the same zeros as in the previous cases.

Let us now consider the analytic structure of $W_1 =: D_1 C^{12}\bar{C}^{12}$. Expression (3.17) together with the discussions above allow one to rewrite W_1 as

$$W_1(j_i; m_i, \bar{m}_i) = \frac{(-)^{m_3-\bar{m}_3+\bar{n}_1}\pi^2\gamma(-N)}{\gamma(-2j_1)\gamma(1+j_{12})\gamma(1+j_{13})} \times \frac{\Gamma(1+j_2+m_2)}{\Gamma(-j_2-\bar{m}_2)} \times \frac{\Gamma(1+j_3+m_3)}{\Gamma(-j_3-\bar{m}_3)} E_{12}\bar{E}_{12}, \tag{A5}$$

where E_{12} is given by $\Gamma(-2j_1)$ times (A4). E_{12} has no poles but it may vanish for certain special configurations if $\Phi^{j_2,w_2}_{m_2,\bar{m}_2} \in \mathcal{D}^{-,w_2}_{j_2}$, namely, $n_2 < n_1 - p_3$ and $j_3 = m_3 + n_3$ or $j_3 = -m_3 - 1 - n_3$, with $n_3 = 0, 1, 2, \ldots$, where $p_3 = -1 - j_{23}$ in the former and $p_3 = j_{13}$ in the latter. The same result applies to $\bar{E}_{12}$, changing n_i by $\bar{n}_i$. Obviously one might find, using other identities, new zeros for special configurations. This could be a difficult task, because the series does not reduce to a finite sum in general. Fortunately, it is not necessary for our purposes.

2. Relation to [5]

This appendix contains some comments about the relation between our work and [5]. For simplicity, we use the conventions of the latter, related to ours by $j \to -j$ in the x basis, up to normalizations. The range of j for discrete representations is now $\frac{1}{2} < j < \frac{k-1}{2}$ and for continuous representations, $j = \frac{1}{2} + i\mathbb{R}$.

One of the aims of [5] was to study the factorization of four-point functions involving $w = 0$ short strings in the boundary conformal field theory. The x basis seems appropriate for this purpose since $x_i, \bar{x}_i$ can be interpreted as the coordinates of the boundary. Naturally, both the OPE and the factorization look very different in the m and x basis. For instance, it is not obvious how discrete series would appear in the OPE or factorization of fields in continuous representations if they are to be obtained from the analogous expressions in the H_3^+ model in the x basis. However, when discrete representations are involved, there are certain similarities. Actually, in accord with the fusion rules $\hat{\mathcal{D}}^{+,w_1}_{j_1} \otimes \hat{\mathcal{D}}^{+,w_2}_{j_2}$ obtained in Sec. IV, $w = 1$ long strings and $w = 0$ short strings were found in the factorization studied in [5]. Conversely, it was interpreted that $w = 1$ short strings do not propagate in the intermediate channels, while we found spectral flow nonpreserving contributions of discrete representations in the OPE. In this Appendix we analyze this issue. We reexamine the three-point functions involving two $w = 0$ strings and one $w = 1$ short string and certain divergences in the four-point functions of $w = 0$ short strings, namely, the so-called Poles$_2$, which seem to break the factorization.

a. Three-point functions involving one $w = 1$ short string and two $w = 0$ strings

The w-conserving two-point functions of short strings in the target space ($w \geq 0$) are given by

$$\langle \Phi^{w,j}_{J,\bar{J}}(x_1,\bar{x}_1)\Phi^{w,j}_{J,\bar{J}}(x_2,\bar{x}_2)\rangle \sim |2j - 1 \pm (k-2)w| \times \frac{\Gamma(2j+p)\Gamma(2j+\bar{p})}{\Gamma(2j)^2 p!\bar{p}!} \frac{\mathcal{B}(j)}{x_{12}^{2J}\bar{x}_{12}^{2\bar{J}}}, \tag{A6}$$

where $\mathcal{B}(j) = B(-j)$ and the upper (lower) sign holds for $J = j + p + \frac{k}{2}w$ ($J = -j - p + \frac{k}{2}w$), p, $\bar{p}$ being nonnegative integers. Three-point functions of $w = 0$ string states are

$$\langle \Phi_{j_1}(x_1,\bar{x}_1)\Phi_{j_2}(x_2,\bar{x}_2)\Phi_{j_3}(x_3,\bar{x}_3)\rangle = C(j_1,j_2,j_3)\prod_{i>j}|x_{ij}|^{-2j_{ij}}, \tag{A7}$$

and for one $w = 1$ short string and two $w = 0$ strings they are given by (we omit the x, $\bar{x}$ dependence)

$$\langle \Phi^{j_1,w=1}_{J_1,\bar{J}_1}(x_1,\bar{x}_1)\Phi_{j_2}(x_2,\bar{x}_2)\Phi_{j_3}(x_3,\bar{x}_3)\rangle \sim \frac{1}{\Gamma(0)}\mathcal{B}(j_1)C\left(\frac{k}{2} - j_1, j_2, j_3\right)\frac{\Gamma(j_2+j_3-J_1)}{\Gamma(1-j_2-j_3+\bar{J}_1)} \times \frac{\Gamma(j_1+J_1-\frac{k}{2})}{\Gamma(1-j_1-\bar{J}_1+\frac{k}{2})}\frac{1}{\gamma(j_1+j_2+j_3-\frac{k}{2})}. \tag{A8}$$

The $\Gamma(0)^{-1}$ factor is absent when the $w = 1$ operator is a long string state. This three-point function was obtained in [5] from an equivalent expression in the m basis. J_1, $\bar{J}_1$ label the global $SL(2,\mathbb{R})$ representations and can be written in terms of parameters m_1, $\bar{m}_1$ as $J_1 = \mp m_1 + \frac{k}{2}$, $\bar{J}_1 = \mp\bar{m}_1 + \frac{k}{2}$, depending if the correlator involved the field $\Phi^{j_1,w_1=\pm 1}_{m_1,\bar{m}_1}$.

As observed in [5], when $J_1 = \frac{k}{2} - j_1 - p$, $\bar{J}_1 = \frac{k}{2} - j_1 - \bar{p}$, the factor $\frac{\Gamma(j_1+J_1-\frac{k}{2})}{\Gamma(1-j_1-\bar{J}_1+\frac{k}{2})}$ cancels the $\Gamma(0)$ and the three-point function is finite and can be interpreted as a w-conserving amplitude. To see this, recall that if it was obtained from a $w = -1$ three-point function in the m basis and $m_1 = j_1 + p$, then

$$\langle \Phi^{j_1,w=1}_{J_1,\bar{J}_1}(x_1,\bar{x}_1)\Phi_{j_2}(x_2,\bar{x}_2)\Phi_{j_3}(x_3,\bar{x}_3)\rangle \sim (-)^{p+\bar{p}}\mathcal{B}(j_1)C\left(\frac{k}{2} - j_1, j_2, j_3\right) \times \frac{\Gamma(j_2+j_3+j_1-\frac{k}{2}+p)}{p!\Gamma(j_2+j_3+j_1-\frac{k}{2})}\frac{\Gamma(j_2+j_3+j_1-\frac{k}{2}+\bar{p})}{\bar{p}!\Gamma(j_2+j_3+j_1-\frac{k}{2})} \tag{A9}$$

reduces to (A7) when $p = \bar{p} = 0$ and $j_1 \to \frac{k}{2} - j_1$, as expected from spectral flow symmetry. Similarly, if $w = +1$ and $m_1 = -j_1 - p$, the same interpretation holds.

On the contrary, for $w=-1$ ($w=+1$) and $m_1=-j_1-p$ ($m_1=j_1+p$), the $\Gamma(j_1+J_1-\frac{k}{2})$ does not cancel the factor $\Gamma(0)^{-1}$ and then, it was concluded in [5] that the three-point function vanishes in this case.

However, notice that if $J_1=\frac{k}{2}+j_1+n=j_2+j_3+p$, $\bar{J}_1=\frac{k}{2}+j_1+\bar{n}=j_2+j_3+\bar{p}$, $n, \bar{n}\in\mathbb{Z}_{\geq 0}$, the r.h.s. of (A8) can also be rewritten as the r.h.s. of (A9), but now this nonvanishing amplitude corresponds to a $w=1$ three-point function which is not equivalent to a w-conserving one. Indeed, (A9) is regular as long as $n<p$ ($\bar{n}<\bar{p}$) and when $n\geq p$ ($\bar{n}\geq\bar{p}$) there are divergences in $C(\frac{k}{2}-j_1, j_2, j_3)$ at $j_1=j_2+j_3-\frac{k}{2}-q$ with $q=0,1,2,\cdots$. Using the spectral flow symmetry, the $w=1$ short string can be identified with a $w=2$ short string with $\tilde{j}_1=\frac{k}{2}-j_1=k-j_2-j_3+q$, which correspond to the Poles$_2$ in [5].

b. Factorization of four-point functions of $w=0$ short strings

The four-point amplitude of $w=0$ short strings was extensively studied in [5]. The conformal blocks were rearranged as sums of products of positive powers of x times functions of $u=z/x$. In order to perform the integral over the world sheet before the j integral, it was necessary to change the j-integration contour from $\frac{1}{2}+i\mathbb{R}$ to $\frac{k-1}{2}+i\mathbb{R}$, and in this process two types of sequences of poles were picked up, namely,

$$\text{Poles}_1\colon\ j_3=j_1+j_2+n,$$
$$\text{Poles}_2\colon\ j_3=k-j_1-j_2+n,$$

where $n=0,1,2,\ldots$. Only values of n for which $j_3<\frac{k-1}{2}$ contribute to the factorization, so Poles$_1$ appear when $j_1+j_2<\frac{k-1}{2}$ and Poles$_2$ when $j_1+j_2>\frac{k+1}{2}$. The contributions from Poles$_1$ were identified as two particle states of short strings in the boundary conformal field theory, but no interpretation was found for Poles$_2$ as s-channel exchange.

Recall that we found Poles$_1$ among the w-conserving discrete contributions to the OPE $\mathcal{D}^{+,w_i}_{j_i}\times\mathcal{D}^{+,w_i}_{j_i}$ [see (3.28)] and Poles$_2$ in the w-violating terms with $\tilde{j}_3=\frac{k}{2}-j_3=j_1+j_2-\frac{k}{2}-n$. Therefore, it seems tempting to consider Poles$_2$ as two particle states of $w=1$ short strings in the boundary conformal field theory. However, neither the powers of $x, \bar{x}$ nor the residues of the poles in the four-point function studied in [5] allow this interpretation and thus the Poles$_2$ had to be truncated. Clearly, more work is necessary to determine the four-point function and understand the factorization.

3. KZ equations in the m basis and the factorization *ansatz*

We studied some features of the factorization of four-point functions in Sec. V. The purpose of this Appendix is to show some consistency conditions of the expressions used in that section.

Let us start by considering the KZ equation for w-conserving n-point functions in the m basis, namely, [10]

$$\mathcal{E}_i\kappa^{-1}\left\langle\prod_{\ell=1}^{n}\Phi^{j_\ell,w_\ell}_{m_\ell,\bar{m}_\ell}(z_\ell,\bar{z}_\ell)\right\rangle=0, \tag{A10}$$

where

$$\mathcal{E}_i\equiv(k-2)\frac{\partial}{\partial z_i}+\sum_{j\neq i}\frac{Q_{ij}}{z_{ji}},\qquad Q_{ij}=-2t_i^3t_j^3+t_i^-t_j^++t_i^+t_j^-, \tag{A11}$$

t^a are defined by $\tilde{J}^a_0|j,m,\bar{m},w\rangle=-t^a|j,m,\bar{m},w\rangle$, $|j,m,\bar{m},w\rangle$ being the state corresponding to the field $\Phi^{j,w}_{m,\bar{m}}$, and κ was introduced in Sec. V.

Since a generic w-conserving four-point function can be obtained from the expression involving four $w=0$ fields, we concentrate on

$$\left\langle\prod_{i=1}^{4}\Phi^{j_i,w_i=0}_{m_i,\bar{m}_i}(z_i,\bar{z}_i)\right\rangle=|z_{34}|^{2(\tilde{\Delta}_2+\tilde{\Delta}_1-\tilde{\Delta}_4-\tilde{\Delta}_3)}\times|z_{14}|^{2(\tilde{\Delta}_2+\tilde{\Delta}_3-\tilde{\Delta}_4-\tilde{\Delta}_1)}\times|z_{13}|^{2(\tilde{\Delta}_4-\tilde{\Delta}_1-\tilde{\Delta}_2-\tilde{\Delta}_3)}\times|z_{24}|^{-4\tilde{\Delta}_2}\mathcal{F}_j(z,\bar{z}),$$

$\mathcal{F}_j(z,\bar{z})$ being a function of the cross ratios $z, \bar{z}$, not determined by conformal symmetry. The KZ equation (A10) implies the following constraint:

$$\frac{\partial\mathcal{F}_j(z,\bar{z})}{\partial z}=\frac{1}{k-2}\left[\frac{Q_{21}}{z}+\frac{Q_{23}}{z-1}\right]\mathcal{F}_j(z,\bar{z}). \tag{A12}$$

Assuming that $\mathcal{F}_j(z,\bar{z})$ has the following form

$$\mathcal{F}_j(z,\bar{z})=\sum_{N,\bar{N}=0}^{\infty}\int dj\left\{A_j^{(N,\bar{N})}\begin{bmatrix}j_1,j_2,j_3,j_4\\ m_1,m_2,\ldots,\bar{m}_4\end{bmatrix}\times z^{\Delta_j-\tilde{\Delta}_1-\tilde{\Delta}_2+N}\bar{z}^{\Delta_j-\tilde{\Delta}_1-\tilde{\Delta}_2+\bar{N}}\right\}, \tag{A13}$$

inserting it into (A12) with $\Delta_j=\tilde{\Delta}_j\equiv-\frac{j(1+j)}{k-2}$, then

$$A_j^{(0,0)}\begin{bmatrix}j_1,j_2,j_3,j_4\\ m_1,m_2,\ldots,\bar{m}_4\end{bmatrix}$$

satisfies

$$\{2m_1m_2 - j(1+j) + j_1(1+j_1) + j_2(1+j_2)\}A_j^{(0,0)}\begin{bmatrix} j_1, j_2, j_3, j_4 \\ m_1, m_2, \ldots, \bar{m}_4 \end{bmatrix} = (m_1 - j_1)(m_2 + j_2)A_j^{(0,0)}\begin{bmatrix} j_1, j_2, j_3, j_4 \\ m_1+1, m_2-1, \ldots, \bar{m}_4 \end{bmatrix} + (m_1 + j_1)(m_2 - j_2)A_j^{(0,0)}\begin{bmatrix} j_1, j_2, j_3, j_4 \\ m_1-1, m_2+1, \ldots, \bar{m}_4 \end{bmatrix}. \tag{A14}$$

The equations relating coefficients $A_j^{(N,\bar{N})}$ with N, $\bar{N} \neq 0$, are much more complicated because they mix terms with different values of m_i, $\bar{m}_i$ with terms at different levels N, $\bar{N}$.

This equation does not have enough information to determine $A_j^{(0,0)}$ completely. So we just check that the expression found in (5.5) is consistent with an analysis performed directly in the m basis. Inserting

$$A_j^{(0,0)}\begin{bmatrix} j_1, j_2, j_3, j_4 \\ m_1, m_2, \ldots, \bar{m}_4 \end{bmatrix} = \mathbb{A}_j^{w=0}(j_1, \ldots, j_4; m_1, \ldots, \bar{m}_4)$$

into (A14) reproduces the same equation with $A_j^{(0,0)}$ replaced by $W(j_1, j_2, j; m_1, m_2, m)$. Because of the complicated expressions known for W, we focus on the case in which one of the fields in the four-point function is a discrete primary, namely, $\Phi^{j_1,w_1=0}_{m_1,\bar{m}_1} \in \mathcal{D}^{+,w=0}_{j_1}$. In this case, using (3.17) one can show that (A14) is equivalent to

$$0 = \sum_{n=0}^{n_1-1}(-)^n\binom{n_1}{n}\left[j - m + \frac{(m_1-j_1)(1+j_1+m_1)}{n_1+1-n} + \frac{(m_2-j_2)(1+j_2+m_2)(n_1-n)}{n+1+j+m-n_1}\right]\frac{\Gamma(n-j_1-j_2+j)}{\Gamma(-j_1-j_2+j)} \times \frac{\Gamma(n+1+j+j_2-j_1)}{\Gamma(1+j+j_2-j_1)}\frac{\Gamma(-2j_1)}{\Gamma(n-2j_1)}\frac{\Gamma(1+j+m)}{\Gamma(n-n_1+1+j+m)} - (-)^{n_1}[m_1(1-m_1) + j_1(1+j_1)] \times \frac{\Gamma(n_1-j_1-j_2+j)}{\Gamma(-j_1-j_2+j)}\frac{\Gamma(n_1+1+j+j_2-j_1)}{\Gamma(1+j+j_2-j_1)}\frac{\Gamma(-2j_1)}{\Gamma(n_1-2j_1)},$$

where $n_1 = m_1 + j_1$ and $m = m_1 + m_2$. Using m conservation this can be rewritten as

$$0 = \sum_{n=0}^{n_1-1}(-)^n\binom{n_1}{n}\left[-n\frac{1-n+2j_1}{n_1+1-n} + \frac{(n-j_1-j_2+j)(n+1+j_2+j-j_1)}{n+1+j+m-n_1}\right]\frac{\Gamma(n-j_1-j_2+j)}{\Gamma(-j_1-j_2+j)} \times \frac{\Gamma(n+1+j_2+j-j_1)}{\Gamma(1+j_2+j-j_1)}\frac{\Gamma(-2j_1)}{\Gamma(n-2j_1)}\frac{\Gamma(1+j+m)}{\Gamma(n-n_1+1+j+m)} - (-)^{n_1}[m_1(1-m_1) + j_1(1+j_1)] \times \frac{\Gamma(n_1-j_1-j_2+j)}{\Gamma(-j_1-j_2+j)}\frac{\Gamma(n_1+1+j_2+j-j_1)}{\Gamma(1+j_2+j-j_1)}\frac{\Gamma(-2j_1)}{\Gamma(n_1-2j_1)}.$$

To see that this vanishes, it is sufficient to note that

$$\sum_{n=0}^{n_1-1}(-)^n\binom{n_1}{n}\left[-n\frac{1-n+2j_1}{n_1+1-n}\right]\frac{\Gamma(n-j_1-j_2+j)}{\Gamma(-j_1-j_2+j)}\frac{\Gamma(n+1+j_2+j-j_1)}{\Gamma(1+j_2+j-j_1)}\frac{\Gamma(-2j_1)}{\Gamma(n-2j_1)}\frac{\Gamma(1+j+m)}{\Gamma(n-n_1+1+j+m)} - (-)^{n_1}[m_1(1-m_1) + j_1(1+j_1)]\frac{\Gamma(n_1-j_1-j_2+j)}{\Gamma(-j_1-j_2+j)}\frac{\Gamma(n_1+1+j_2+j-j_1)}{\Gamma(1+j_2+j-j_1)}\frac{\Gamma(-2j_1)}{\Gamma(n_1-2j_1)} = -\sum_{\tilde{n}=0}^{n_1-1}(-)^{\tilde{n}}\binom{n_1}{\tilde{n}}\left[\frac{(\tilde{n}-j_1-j_2+j)(\tilde{n}+1+j_2+j-j_1)}{\tilde{n}+1+j+m-n_1}\right]\frac{\Gamma(\tilde{n}-j_1-j_2+j)}{\Gamma(-j_1-j_2+j)}\frac{\Gamma(\tilde{n}+1+j_2+j-j_1)}{\Gamma(1+j_2+j-j_1)}\frac{\Gamma(-2j_1)}{\Gamma(\tilde{n}-2j_1)} \times \frac{\Gamma(1+j+m)}{\Gamma(\tilde{n}-n_1+1+j+m)},$$

where $\tilde{n} = n - 1$.

Let us now discuss the other possible *ansatz*, namely (5.11). To see that $\mathbb{A}_j^{w=1}$ also verifies the KZ equation, consider $\Delta_j = -\frac{j(1+j)}{k-2} - m - \frac{k}{4}$ and $m = m_1 + m_2 - \frac{k}{2}$ in (A12). In this case, the equation to be satisfied by $A_j^{(0,0)}$, obtained by replacing (A13) into (A12), is the following:

$$\{2m_1m_2 - j(1+j) + j_1(1+j_1) + j_2(1+j_2) - (k-2)\Big(m_1 + m_2 - \frac{k}{4}\Big)\Big\}A_j^{(0,0)}\begin{bmatrix} j_1, j_2, j_3, j_4 \\ m_1, m_2, \ldots, \bar{m}_4 \end{bmatrix}$$
$$= (m_1 - j_1)(m_2 + j_2)A_j^{(0,0)}\begin{bmatrix} j_1, j_2, j_3, j_4 \\ m_1 + 1, m_2 - 1, \ldots, \bar{m}_4 \end{bmatrix} + (m_1 + j_1)(m_2 - j_2)A_j^{(0,0)}\begin{bmatrix} j_1, j_2, j_3, j_4 \\ m_1 - 1, m_2 + 1, \ldots, \bar{m}_4 \end{bmatrix}$$
$$- (m_2 - j_2)(m_3 + j_3)A_j^{(0,0)}\begin{bmatrix} j_1, j_2, j_3, j_4 \\ m_1, m_2 + 1, m_3 - 1, \ldots, \bar{m}_4 \end{bmatrix}. \tag{A15}$$

It is not difficult to check that

$$A_j^{(0,0)}\begin{bmatrix} j_1, j_2, j_3, j_4 \\ m_1, m_2, \ldots, \bar{m}_4 \end{bmatrix} = \mathbb{A}_j^{w=1}(j_1, \ldots, j_4; m_1, \ldots, \bar{m}_4)$$

is a solution of this equation.

Obviously, $A_j^{w=-1}$ is also a solution of (A12) when $\Delta_j = -\frac{j(1+j)}{k-2} + m - \frac{k}{4}$ and $m = m_1 + m_2 + \frac{k}{2}$.

Here, we have considered the simple case of four $w = 0$ fields. However, these results can be generalized for arbitrary w-conserving correlators using the identity (5.9).

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