Geometry and symmetry structures in two-time gravity

Itzhak Bars and Shih-Hung Chen
Department of Physics and Astronomy University of Southern California, Los Angeles, California 90089-2535, USA
(Received 19 November 2008; published 27 April 2009)

Two-time (2T) gravity in $d+2$ dimensions predicts 1T general relativity in d dimensions, augmented with a local scale symmetry known as the Weyl symmetry in 1T field theory. The emerging general relativity comes with a number of constraints, particularly on scalar fields and their interactions in 1T field theory. These constraints, detailed in this paper, are footprints of 2T gravity and could be a basis for testing 2T physics. Some of the conceptually interesting consequences of the ''accidental'' Weyl symmetry include that the gravitational constant emerges from vacuum values of the dilaton and other Higgs-type scalars and that it changes after every cosmic phase transition (inflation, grand unification, electroweak phase transition, etc.). We show that this consequential Weyl symmetry in d dimensions originates from coordinate reparametrization, not from scale transformations, in the $d+2$ spacetime of 2T gravity. To recognize this structure we develop in detail the geometrical structures, curvatures, symmetries, etc. of the $d+2$ spacetime which is restricted by a homothety condition derived from the action of 2T gravity. Observers that live in d dimensions perceive general relativity and all degrees of freedom as shadows of their counterparts in $d+2$ dimensions. Kaluza-Klein type modes are removed by gauge symmetries and constraints that follow from the 2T-gravity action. However some analogs to Kaluza-Klein modes, which we call ''prolongations'' of the shadows into the higher dimensions, remain but they are completely determined, up to gauge freedom, by the shadows in d dimensions.

DOI: 10.1103/PhysRevD.79.085021 PACS numbers: 12.60.−i, 11.30.Ly, 14.80.Bn, 14.80.Mz

I. INTRODUCTION

Two-time (2T) gravity [1] in $d+2$ dimensions has successfully reproduced the usual one time general relativity (GR) as a shadow in $(d-1)+1$ dimensions. Taken together with similar recent results for the standard model [2] and $\mathcal{N}=1, 2, 4$ supersymmetric 2T field theory [3,4], these 2T theories correctly describe $3+1$ dimensional nature directly in $4+2$ dimensions. The phenomenologically successful theories now have counterparts in $4+2$ dimensions, thus providing a new perspective on the significance of space and time and lending a new outlook on unification of 1T-physics theories.

Briefly, the relation between the $4+2$ and $3+1$ theory is as follows. After gauge fixing and solving some kinematic equations of motion, the $4+2$ field theory yields various ''shadows'' in $3+1$ dimensions. The ''conformal shadow'' of the $4+2$ theories coincides with the standard familiar theories, except for some additional new constraints. These new constraints on 1T field theory—in particular on scalar fields and their interactions—are consistent with everything we know so far. Potentially there are measurable phenomenological consequences of these new restrictions within the conformal shadow that could distinguish 2T physics from other approaches, as explained at the end of Sec. II.

In addition, a main novelty in 2T physics is that this formalism produces many 1T-physics shadows from the same parent theory. The conformal shadow mentioned in the previous paragraph is only one of many. The other shadows provide 1T field theories that are dual to the familiar ones and these may be turned into computational tools for extracting nonperturbative physics. The shadows give different perspectives of the $4+2$ theory as viewed by observers that are stuck in $3+1$ dimensions. The different embedding of $3+1$ dimensions into $4+2$ dimensions contains moduli that appear in $3+1$ dimensions as parameters of the 1T shadow theory, such as mass, curvature, or interaction with backgrounds, which offer different glimpses of the higher dimensions. Dualities transform shadows with different $3+1$ geometries or different values of the parameters. The shadows and dualities are most easily understood in the worldline formulation of 2T physics.[1] While the investigation of dualities in the 1T field theory formalism are ongoing [6], some of the simpler cases have been reported in [5] for scalar fields and in [7] for Dirac and Yang-Mills fields.

Through the dualities, and through hidden symmetries related to the higher spacetime, the parent theory in $d+2$ dimensions provides a new kind of unification of various 1T-physics field theories.

In this paper we will concentrate exclusively on the conformal shadow of 2T gravity in $d+2$ dimensions in order to clarify further its geometrical and symmetry properties. Specifically, we will investigate not only the shadow in $(d-1)+1$ dimensions but also its prolongation into $d+2$ dimensions. By this we mean that there are Riemann

[1]For examples of $(d-1)+1$ shadows that emerge from flat $d+2$ spacetime, see Tables I, II, and III in [5].

curvature components R^{M}_{NPQ} and other geometrical fields that are nonvanishing not only in the shadow in $(d-1)+1$ dimensions but also in $d+2$ dimensions. We will show that all such nonvanishing components of the prolongation of the shadow are actually fully determined, up to gauge freedom, by the fields within the shadow in $(d-1)+1$ dimensions.

Concentrating only on the shadow with an effective action principle in $(d-1)+1$ is self-consistent as shown in [1]. However, the extension of the shadow into the higher spacetime is likely to be important for discussing the dualities among shadows as well as for grasping the higher $d+2$ dimensional properties of the underlying theory.

Another important property of the conformal shadow for gravity in $(d-1)+1$ is that general relativity comes with a local rescaling Weyl symmetry [8–11], along with a dilaton that compensates for the local rescaling of the metric. This is one of the important restrictions imposed on 1T physics by 2T physics, as reported in [1]. The physical effect of this is that the gravitational constant is not a parameter but emerges in 1T physics from the vacuum value of the dilaton.[2]

A further property associated with the Weyl symmetry is that every scalar field in 1T physics beyond the dilaton (such as inflaton, Higgs, etc.) must be a conformal scalar that has a special fixed dimensionless coupling to the curvature scalar R. The physical effect of this is that the gravitational constant changes as a function of cosmic time after every phase transition in the universe (inflation, grand unification, etc.). In this paper we will clarify the origin of this important accidental local scale symmetry in the conformal shadow. It will be shown that it emerges as a remnant from symmetry under coordinate transformation (not scale transformations) of the higher dimensional 2T gravity.

In Sec. II we will briefly review the basic setup of 2T gravity, display its reduction to ordinary 1T general relativity augmented with the local Weyl symmetry, and explain the physically significant constraints that this structure puts on 1T field theory coupled to gravity. The rest of the paper develops the technical aspects of the geometry and symmetries to explain in detail how the reduced 1T theory of Sec. II is recovered from 2T gravity. In Sec. III we discuss the kinematics of 2T curved spacetime in $d+2$ dimensions. This involves solving the kinematical equations of motion that follow from the 2T-gravity action and working out the general consequences that the geometry of the 2T spacetime is restricted by a homothety condition on the metric. In Sec. IV the dynamical and kinematical equations are discussed. Their relation to an $\mathrm{Sp}(2,R)$ gauge symmetry of an underlying worldline particle theory is explained in Sec. V. Then in Sec. VI we show how spacetime in $(d-1)+1$ dimensions is embedded in spacetime in $(d+2)$ dimensions by making gauge choices and solving the kinematic equations. This leads to an explanation of the origin of the local scaling symmetry in general relativity in $(d-1)+1$ dimensions known as the Weyl symmetry. It will be shown that it originates from general coordinate transformations, not from local rescalings, in $d+2$ dimensions. In Sec. VII we calculate the components of the Riemann tensors and of the $\mathrm{SO}(d,2)$ "Lorentz" curvature in tangent space that describe the geometry of the prolongation of the conformal shadow into the higher dimensions. In Sec. VIII we discuss in more detail the emerging 1T dynamical equations of motion of both the shadow fields in d dimensions and their prolongations to higher dimensions and show that, up to gauge freedom, the prolongations are completely determined by the shadow fields in d dimensions. This leads to one of our main conclusions, that the shadow fields themselves are determined self-consistently by the action only within the shadow in d dimensions, *independently of the prolongations*, which was one of the goals in our investigation.

II. CONSTRAINTS IN 1T FIELD THEORY INDUCED BY 2T GRAVITY

In this section we briefly review 2T gravity to explain the constraints that it induces in 1T field theory, particularly involving scalar fields. We will see that the gravitational constant emerges from the vacuum values of the scalars and that it appears in several places in the action of 1T field theory. The structure of scalars that emerges in 1T field theory, shown in Eq. (2.15) and related discussion, is consistent with current observations but this structure could be one of the future tests for the predictions of 2T physics.

2T gravity, without any matter, includes three fields which we call the gravity triplet: the metric G_{MN}, the dilaton Ω, and another scalar field W, all in $d+2$ dimensions X^M. The action for pure 2T gravity is [1]

$$S^0 = \gamma \int d^{d+2}X\sqrt{G}\Big\{\delta(W)\Big[\Omega^2 R(G) + \frac{1}{2a}\partial\Omega\cdot\partial\Omega - V(\Omega)\Big] + \delta'(W)[\Omega^2(4-\nabla^2 W) + \partial W\cdot\partial\Omega^2]\Big\}. \tag{2.1}$$

Here $R(G)$ is the Riemann curvature scalar, a is the special constant

$$a \equiv \frac{d-2}{8(d-1)}, \tag{2.2}$$

[2]The massless Goldstone boson that emerges from the spontaneous breaking of scale symmetry [8–19], i.e. the fluctuation of the dilaton around its vacuum value, is eliminated by a Weyl gauge choice in our theory, so it does not generate any long range forces that could compete with the long range effects of gravity [20].

while the potential V can only have the form $V(\Omega) = \lambda\Omega^{2d/(d-2)}$ with a dimensionless coupling λ. The overall constant γ can be absorbed away by rescaling the fields, but is used for convenience to normalize the 1T shadow action that emerges in two lower dimensions. The action with this structure of kinetic terms, value of a, and form of V is unique under certain local gauge symmetries discussed in [1].

The unusual features of this action as a field theory include the delta function $\delta(W)$ and its derivative $\delta'(W)$. All fields are varied freely to derive equations of motion or to verify symmetries. The variations contain terms proportional to $\delta(W)$, $\delta'(W)$, and $\delta''(W)$ where more derivatives on $\delta(W)$ or $\delta'(W)$ emerge from integration by parts and the chain rule $\partial_M\delta(W) = \delta'(W)\partial_M W$, etc. The coefficients of $\delta(W)$, $\delta'(W)$, and $\delta''(W)$ for each general variation δG^{MN}, $\delta\Omega$, and δW give three equations of motion for each field. We will discuss some of the equations of motion later. There are remarkable consistencies between these equations all due to the noteworthy symmetries of this action. This symmetry in field theory captures the essentials of an underlying $\mathrm{Sp}(2, R)$ symmetry (see Sec. V) that makes position and momentum $X^M(\tau)$, $P_M(\tau)$ indistinguishable at every instant τ at the level of a worldline formulation of a particle in the presence of gravity [1].

Part of the gauge symmetry can be used to fix $W(X)$ to any function of X^M that can vanish in some region of spacetime X^M. To understand the role of W the reader is reminded that, in 2T field theory in *flat space*, W is replaced by a fixed function $W_{\text{flat}} = X^M X^N \eta_{MN}$ where η_{MN} is the $\mathrm{SO}(d, 2)$ invariant flat metric. When 2T field theory in flat $d + 2$ dimensions is reduced to shadows in $(d - 1) + 1$ dimensions, then, in the conformal shadow, the $\mathrm{SO}(d, 2)$ symmetry of W becomes the conformal symmetry of 1T field theory in Minkowski space.[3]

The symmetries of the action (2.1) do not allow a gravitational constant in $d+2$ dimensions; however Newton's constant emerges in the shadow in d dimensions from the vacuum value of the dilaton $\langle\Omega\rangle$ when the equations of motion are used to reduce this theory to the conformal shadow. The conformal shadow action derived from (2.1) in [1] (see Sec. VIII for justification) has the familiar form of a conformal scalar

$$S^0_{\text{shadow}} = \int d^d x\sqrt{-g}\bigg(\frac{1}{2a} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + R\phi^2 - V(\phi)\bigg), \tag{2.3}$$

where $g_{\mu\nu}(x)$ is the metric in d dimensions, $R(g)$ is its Riemann curvature scalar, and a has the special value in Eq. (2.2). The relation of $g_{\mu\nu}(x)$ to $G_{MN}(X)$ and of $\phi(x)$ to $\Omega(X)$ will be displayed below. Suffice it for now to say that $(g_{\mu\nu}, \phi)$ are the shadows of the higher dimensional fields as seen by observers living in d dimensions. There are no Kaluza-Klein (KK) type physical degrees of freedom, as those are removed by the gauge symmetries of 2T physics. But there are some analogs to KK modes, which we call "prolongations" of the shadow into the higher dimensions, determined by the shadow fields $(g_{\mu\nu}, \phi)$ as will be discussed later in this paper.

In the conformal shadow in Eq. (2.3) there is an accidental Weyl symmetry that plays multiple important roles. Because of the special value of a, Eq. (2.3) is the well-known action of a conformal scalar that has a Weyl symmetry $S^0_{\text{shadow}}(g', \phi') = S^0_{\text{shadow}}(g, \phi)$ under local rescalings [9] $g'_{\mu\nu}(x) = e^{2\lambda(x)} g_{\mu\nu}(x)$, $\phi'(x) = e^{-[(d-2)/2]\lambda(x)}\phi(x)$ with an arbitrary $\lambda(x)$. The original action in $d + 2$ dimensions (2.1) does not have a Weyl symmetry, so the symmetry in the conformal shadow appears to be "accidental." Later in this paper it will be explained how this symmetry originates in the coordinate transformations (not in scale transformations) in higher dimensions. Using this local symmetry, $\phi(x)$ can be gauge fixed to a constant ϕ_0, so that the action (2.3) becomes precisely pure general relativity in d dimensions

$$S^{0,\text{fixed}}_{\text{shadow}} = \int d^d x\sqrt{-g}\bigg(\frac{R(g)}{2\kappa_d^2} - \frac{\Lambda_d}{2\kappa_d^2}\bigg),$$
$$\text{with} \quad \frac{1}{2\kappa_d^2} \equiv \phi_0^2, \qquad \Lambda_d \equiv \lambda(\phi_0)^{4/(d-2)}. \tag{2.4}$$

Note that according to the sign of the kinetic term in Eq. (2.3), the field $\phi(x)$ has negative norm, but this sign is required in order to obtain a positive gravitational constant while being consistent with the Weyl symmetry. Of course, by having the Weyl symmetry, the negative norm ghost, which is also a Goldstone boson of scale transformations, is removed from the physical spectrum. This nice feature is a consequence of the symmetries of the higher dimensional 2T-gravity theory.

When matter is included the Weyl gauge can be chosen in various other ways (see below), and then one finds more physical effects of the dilaton beyond its footprints in the form of the gravitational constant in d dimensions $(2\kappa_d^2)^{-1}$ and an undetermined cosmological constant Λ_d (λ has any sign or magnitude).

We now outline the coupling of the gravity triplet (W, Ω, G^{MN}) to matter fields of the type Klein-Gordon $S_i(X)$, Dirac $\Psi(X)$, and Yang-Mills $A_M(X)$ [1]. In *flat* 2T field theory these fields must have the following engineering dimensions [2]:

$$\dim(X^M) = 1, \qquad \dim(S_i) = -\frac{d-2}{2},$$
$$\dim(\Psi) = -\frac{d}{2}, \qquad \dim(A_M) = -1. \tag{2.5}$$

The general 2T field theory of these fields in flat space in

[3]In other 1T shadows[1] this $\mathrm{SO}(d, 2)$ of the flat 2T theory is still a symmetry that is usually hidden and often not noticed in 1T physics before discovering it through a shadow of 2T physics.

TABLE I. Matter S_i, Ψ, A_M in interaction with the gravity triplet (W, Ω, G^{MN}).

Quantity	Flat	Curved
Metric	η^{MN}	$G^{MN}(X)$
Volume element	$(d^{d+2}X)\delta(X^2)$	$(d^{d+2}X)\sqrt{G}\delta(W(X))$
Explicit X	X^M	$V^M(X) = \frac{1}{2}G^{MN}\partial_N W$
Gamma matrix, vielbein	γ_M	$E^a_M(X)\gamma_a$
Spin connection	$\gamma^M\partial_M\Psi$	$E^{Mc}\gamma_c(\partial_M + \frac{1}{4}\gamma_{ab}\omega^{ab}_M(X))\Psi$
Yang-Mills	Specialize η^{MN}	$\Omega^{2(d-4)/(d-2)}\,\mathrm{Tr}(-\frac{1}{4}F_{MN}F_{KL})G^{MK}G^{NL}$
Yukawa	Specialize $X^M\gamma_M$	$\Omega^{-(d-4)/(d-2)}V^M(g_i\bar{\Psi}^L\gamma_M\Psi^R S_i + \mathrm{H.c.})$
{Real scalar fields S_i, Ω extra $\frac{-1}{a}$ for dilaton}	{Complex $\varphi = \frac{S_1+iS_2}{\sqrt{2}}$}	$\{G^{MN}(\frac{1}{2a}\partial_M\Omega\partial_N\Omega - \frac{1}{2}\sum_i\partial_M S_i\partial_N S_i) + (\Omega^2 - a\sum_i S_i^2)R(G) - V(\Omega, S_i)\}$
$\delta'(W)$ term, scalars only	$W_{\mathrm{flat}} = X^2$	$\{(\Omega^2 - a\sum_i S_i^2)(4 - \nabla^2 W) + \partial W\cdot\partial(\Omega^2 - a\sum_i S_i^2)\}\delta'(W)$

$d+2$ dimensions was given in [2]. The matter part of the theory in curved space follows from the flat theory in [2] by making the substitutions indicated in Table I [1]. The dilaton Ω couples to Yang-Mills fields with factor $\Omega^{[2(d-4)/(d-2)]}$ and Yukawa terms with factor $\Omega^{-(d-4)/(d-2)}$ as in Table I. This coupling of Ω is dictated by the symmetries of the theory consistently with the dimensions in Eq. (2.5). When $d+2=6$, these factors become 1, so this coupling of the dilaton disappears for this special case. An important property of $V(\Omega, S_i)$ related again to the dimensions (2.5) and symmetries is that it must have the homogeneity property[4]

$$V(t\Omega, tS_i) = t^{2d/(d-2)}V(\Omega, S_i). \tag{2.6}$$

A general function with this property may be written in the form $V(\Omega, S_i) = \Omega^{2d/(d-2)}f(S_i/\Omega)$, where $f(\sigma_i)$ is an arbitrary function of the scale invariant variables $\sigma_i = S_i/\Omega$.

We emphasize an important property of the scalars S_i (including the Higgs field in the standard model). The symmetries require that, except for an overall normalization for each scalar, the *quadratic* part of the Lagrangian for any scalar $S_i(X)$ must have exactly the same structure as the one for the dilaton field Ω in the pure gravity action Eq. (2.1). This structure is included for scalars Ω, S_i in Table I, where the sign and structure of the curvature term relative to the kinetic term is fixed by the constant a. Furthermore, the symmetry requires also a $\delta'(W)$ term for the quadratic term in the scalars Ω, S_i as shown in the table.[5]

[4]Then $V(\Omega, S_i)$ has dimension $-d$ under the scaling of scalars $(\Omega, S_i) \to e^{-[(d-2)/2]\lambda}(\Omega, S_i)$, that is $V \to e^{-d\lambda}V$.

[5]In flat space the $\delta'(X^2)$ term can be rewritten as a $\delta(X^2)$ term that modifies the naive kinetic term. As indicated in the table, for 2T field theory in flat spacetime the function W is replaced by $W_{\mathrm{flat}} = X^2$. Then it can be verified that for each scalar Ω, S_i the kinetic terms $(\frac{1}{2a}\partial\Omega\cdot\partial\Omega - \frac{1}{2}\partial S_i\cdot\partial S_i)$ that are multiplied with $\delta(X^2)$ combine with the $\delta'(X^2)$ terms in Table I to become simply $\delta(X^2)(\frac{1}{2a}\Omega\partial^2\Omega - \frac{1}{2}S_i\partial^2 S_i)$ after dropping a total derivative, thus avoiding any $\delta'(X^2)$ terms, as in the general 2T-field theory in flat space [2].

The overall sign and magnitude of the normalization for the kinetic term $(\frac{-1}{2})$ of a real scalar $\frac{-1}{2}G^{MN}\partial_M S_i\partial_N S_i$ is fixed by the requirements of unitarity (no negative norms) and conventional definition of norm. For the dilaton the norm differs by an overall $\frac{-1}{a}$; the magnitude can be changed by rescaling the field Ω so it is not significant, but the sign is significant (negative norm) and is needed to produce a positive Newton constant from the vacuum values of the scalars in the shadow 1T theory as explained above. This negative norm ghost is harmless since it is removable in the shadow by using the leftover Weyl symmetry arising from coordinate transformations.

The form of the shadow action with only scalar matter fields was derived in [1] (see Sec. VIII for justification). It has the form of conformal scalars coupled to gravity

$$\begin{aligned} S_{\mathrm{shadow}}(g, \phi, s_i) = \int d^dx\sqrt{-g}\Big(&\frac{1}{2a}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \\ &- \frac{1}{2}g^{\mu\nu}\partial_\mu s_i\partial_\nu s_i + (\phi^2 - as_i^2)R \\ &- V(\phi, s_i)\Big), \end{aligned} \tag{2.7}$$

where a sum over i is implied. The equations of motion that follow from this action include

$$R_{\mu\nu}(g) - \tfrac{1}{2}g_{\mu\nu}R(g) = T_{\mu\nu}(\phi, s_i), \tag{2.8}$$

with the energy momentum tensor $T_{\mu\nu}$ given by

$$\begin{aligned} T_{\mu\nu} = \frac{1}{(\phi^2 - as_i^2)}\Big[&\Big(-\frac{1}{2a}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\partial_\mu s_i\partial_\nu s_i\Big) \\ &- (g_{\mu\nu}\nabla^2 - \nabla_\mu\partial_\nu)(\phi^2 - as_i^2) \\ &+ \frac{1}{2}g_{\mu\nu}\Big(\frac{1}{2a}\partial\phi\cdot\partial\phi - \frac{1}{2}\partial s_i\cdot\partial s_i - V(\phi, s_i)\Big)\Big]. \end{aligned} \tag{2.9}$$

The relation of the shadow ($g_{\mu\nu}$, ϕ, and s_i) to (G_{MN}, Ω, and S_i) and their prolongations will be given below. If there are N real scalars s_i in addition to the dilaton ϕ, then the kinetic and curvature terms have an automatic global symmetry SO$(N,1)$, with a SO$(N,1)$ diagonal metric $(-1/a, 1, 1, \ldots, 1)$ as seen in the expression $(\phi^2 - as_i^2)R$ and the kinetic term. This symmetry could be explicitly broken by the potential $V(\phi, s_i)$ which is arbitrary except for the homogeneity condition in Eq. (2.6).[6] The vacuum is determined by the properties of $V(\phi, s_i)$ and therefore the gravitational constant and the cosmological constants are now functions of the vacuum values of all the scalars

$$\frac{1}{2\kappa_d^2} = \langle \phi^2 - as_i^2 \rangle, \qquad \frac{\Lambda_d}{2\kappa_d^2} = V(\langle\phi\rangle, \langle s_i \rangle), \tag{2.10}$$

generalizing Eq. (2.4). These gravitational and cosmological "constants" (κ_d, Λ_d) induced by the various fundamental scalars (Ω, S_i) are not really constants since they must change after every cosmic phase transition of the Universe as a whole (inflation, grand unification, electroweak phase transition, etc.) as the various vacuum expectations values $\langle\phi\rangle$, $\langle s_i\rangle$ turn on at critical values of cosmic temperature or cosmic time. The cosmological implications of this are under study [21].

An important fact again is the presence of the accidental Weyl symmetry in the action (2.7), $S_{\text{shadow}}(g', \phi', s_i') = S_{\text{shadow}}(g, \phi, s_i)$, under local rescalings with an arbitrary *local* gauge parameter $\lambda(x)$

$$g'_{\mu\nu} = e^{2\lambda} g_{\mu\nu}, \qquad \phi' = e^{-[(d-2)/2]\lambda}\phi, \qquad s_i' = e^{-[(d-2)/2]\lambda}\phi. \tag{2.11}$$

This symmetry persists in the shadow action with additional matter fields when the fermions and gauge fields are included in the 2T action according to Table I. The Weyl symmetry can be used to remove the dilatonic Goldstone boson (now a mixture of many fields ϕ, s_i). The remaining physical scalar fields, *after* the phase transitions that produce the gravitational constant $(2\kappa_d^2)^{-1}$, can be neatly described by fixing the Weyl gauge so that the dilaton ϕ gets determined by the other scalars as follows:

$$\phi(x) = \pm\left(as_i^2(x) + \frac{1}{2\kappa_d^2}\right)^{1/2}. \tag{2.12}$$

This gauge choice reduces the curvature term in Eq. (2.7) to simply $R(g)/(2\kappa_d^2)$, thus conveniently describing gravity after the phase transition in the Einstein frame. However, while this gauge choice is convenient to describe gravity in the traditional setting, the gravitational constant enters in a few other places in the action as described below. In particular, the kinetic term of the scalars in the shadow action (2.7) turns into a nonlinear sigma model for the group SO$(N,1)$ [see Eq. (2.15)]. The scale of the nonlinearity in the sigma model is determined by the gravitational constant $(2\kappa_d^2)^{-1}$ as in Eq. (2.12).

Taking advantage of the homogeneity of the potential, and using $t = (\phi^2 - as_i^2)^{-1/2}$ in Eq. (2.6), V can be written in the form

$$\begin{aligned} V(\phi, s_i) &= (\phi^2 - as_i^2)^{d/(d-2)} \\ &\quad \times V\left(\frac{\phi}{\sqrt{\phi^2 - as_i^2}}, \frac{s_i}{\sqrt{\phi^2 - as_i^2}}\right) \\ &\equiv v(\sigma_i). \end{aligned} \tag{2.13}$$

In the Weyl gauge (2.12), after replacing the overall coefficient by the constant $(\phi^2 - as_i^2)^{d/(d-2)} = (2\kappa_d^2)^{-d/(d-2)}$, and absorbing it into the definition of $v(\sigma_i)$, we see that the leftover $v(\sigma_i)$ is an *arbitrary* function of the N variables $\sigma_i = s_i/\phi$ that are invariants under the local scale transformations of Eq. (2.11). Of course at this fixed gauge there are now some scales in the theory, namely, the gravitational scale κ_d and the other scales $\langle s_i/\phi\rangle = \langle\sigma_i\rangle$ generated by phase transitions that follow from the properties of $v(\sigma_i)$.

The kinetic term of the scalars in the nonlinear sigma model can also be written in terms of the σ_i or the s_i. For example, if we parametrize the N fields as $s_i = sn_i$ where $n_i(x)$ is a unit vector $\sum_i n_i n_i = 1$ and $s(x)$ is the magnitude of the SO(N) vector $s_i(x)$, then we can write $\sum_i s_i^2 = s^2$ so that ϕ in Eq. (2.12) becomes a function of a single field $s(x)$

$$\phi = \pm(2\kappa_d^2)^{-(1/2)}\sqrt{1 + 2a\kappa_d^2 s^2}. \tag{2.14}$$

Replacing these forms in the action (2.7) we obtain an ordinary looking general relativity (after the phase transitions) coupled to an arbitrary[6] potential $v(s, n_i)$ with a nonordinary kinetic energy term for scalars where the gravity scale κ_d appears nontrivially

$$\begin{aligned} S_{\text{shadow}}^{\text{fixed}}(g, s, n_i) = \int d^d x\sqrt{-g}\bigg\{&\frac{1}{2\kappa_d^2}R(g) - v(s, n_i) \\ &- \frac{1}{2}g^{\mu\nu}\left[\frac{\partial_\mu s\partial_\nu s}{1 + 2a\kappa_d^2 s^2} + s^2\sum_i \partial_\mu n_i \partial_\nu n_i\right]\bigg\}. \end{aligned} \tag{2.15}$$

The last term $\sum_i \partial n_i \cdot \partial n_i$, with $\vec{n}\cdot\vec{n} = 1$, is a nonlinear SO(N) sigma model, while taken together as a whole the scalar kinetic terms form a nonlinear SO$(N,1)$ sigma model coupled to gravity. The scale of nonlinearity of the SO$(N,1)$ sigma model is also *determined uniquely* by the gravity scale κ_d and the constant a given in Eq. (2.2). That the gravitational constant $(2\kappa_d^2)^{-1}$ should appear in this way in 1T field theory, in addition to the traditional term $R(g)/2\kappa_d^2$, is a prediction of the symmetries of 2T gravity.

Additional places in 1T field theory action where $(2\kappa_d^2)^{-1}$ appears as a consequence of 2T gravity include the dilaton factor $\phi^{2(d-4)/(d-2)}$ for Yang-Mills kinetic

[6] Of course, in a complete model of fundamental interactions, various Yang-Mills gauge symmetries also put constraints on the structure of the potential $V(\Omega, S_i)$ in addition to the homogeneity condition (2.6).

terms, and the dilaton factor $\phi^{-(d-4)/(d-2)}$ for Yukawa terms, which come from the similar terms in 2T field theory as shown in Table I. In these expressions $\phi = \pm(2\kappa_d^2)^{-(1/2)}\sqrt{1+2a\kappa_d^2 s^2}$ as above. Evidently when $d = 4$ these factors disappear, but they could play a physical role in a theory with d less than or larger than 4, thus providing additional signals of 2T physics.

The σ_i or s_i can also be parametrized in other convenient ways to take advantage of both the SO$(N, 1)$ symmetry of the kinetic term and of possible other symmetries[6] of the potential term $V(\Omega, S_i)$. Presumably the symmetries of the potential, indeed of the full theory, include at the very least the $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ symmetry of the standard model, which is possibly embedded in a larger grand unified symmetry group. In this context N is the total number of all the real scalars in the theory besides the dilaton. In the physical applications of these concepts in a complete theory, one would be advised to take advantage of the flavor/color or grand unified symmetry structures in choosing the most convenient parametrization of the SO$(N, 1)$ sigma model.

We see that, in addition to the possibility of a changing gravitational constant after each phase transition, some general constraints have emerged from 2T gravity on the structure of scalars in 1T field theory. The constraints described in the above paragraphs permeate to the shadow 1T field theory in d dimensions and show up in the couplings among scalars in the kinetic terms, potential energy $v(s, n_i)$, and gauge bosons and fermions through the factors $\phi^{2(d-4)/(d-2)}$ and $\phi^{-(d-4)/(d-2)}$, respectively, thus leaving footprints that observers in the conformal shadow in d dimensions can use to infer properties of the underlying 2T theory. These properties of scalars, including the inflaton and the Higgs, are currently under investigation in cosmological and LHC contexts [21].

Additional and deeper observable properties of the 2T theory can be obtained by studying the other shadows related to the conformal shadow by duality transformations as in the examples in [5,7] and their generalizations that are still under investigation [6].

III. KINEMATICS OF 2T CURVED SPACE IN $d + 2$ DIMENSIONS

A. Kinematic equations

The equations of motion that follow from the 2T-gravity action in Eq. (2.1) and Table I can be divided into two categories: dynamical equations and kinematical equations. The dynamical equations are those proportional to the delta function $\delta(W)$—these provide the dynamics including the field interactions in $d+2$ dimensions. The kinematical equations are those proportional to the derivatives of the delta function $\delta'(W)$ and $\delta''(W)$. We recall that such derivatives emerge from integration by parts in computing the variation of the action.

A remarkable property of the kinematical equations that will be emphasized below is that they are universal and have a geometrical character. They can be shown to be independent of interactions and they are the same for each field independent of which other fields are included in the action. Although these properties may not be immediately apparent when the kinematic equations are derived from the action, it follows after some rewriting of the equations as seen below. There is an important underlying symmetry for this result, namely, Sp$(2, R)$ which will be discussed in Sec. V. The kinematical equations provide the instructions for how to relate the fields in $(d + 2)$ dimensions to the shadows in d dimensions, so their solutions reduce the original 2T theory to various shadows in d dimensions, such as the conformal shadow in Eq. (2.4).

Both the kinematic and dynamical equations for 2T gravity were derived from the action in [1]. In this section we deal mainly with the kinematics. For the pure gravity triplet (G_{MN}, Ω, W) the kinematic equations have the following form:

$$G^{MN}\partial_M W \partial_N W = 4W, \tag{3.1}$$

$$G^{MN}\partial_M W \partial_N \Omega = 4a\Omega(6 - \nabla^2 W), \tag{3.2}$$

$$\nabla_M \partial_N W = G_{MN}[-6 + \nabla^2 W + 8a(6 - \nabla^2 W)], \tag{3.3}$$

where ∇ is the covariant derivative in the curved space with metric G_{MN}. After contracting the third equation with G^{MN} and taking account of $\nabla^2 W = G^{MN}\nabla_M\partial_N W$, one can solve for $\nabla^2 W$ and find

$$\nabla^2 W = \frac{6(d+2)(8a-1)}{(d+2)(8a-1)+1} = 2(d+2), \tag{3.4}$$

where the special value of a in Eq. (2.2) is used. Note that the result is independent of the metric, and, in particular, it is true in flat space for $W_{\mathrm{flat}} = X^2$ as listed in Table I, namely, $(\nabla^2 W)_{\mathrm{flat}} = \eta^{MN}\partial_M\partial_N(X^2) = 2\partial_M X^M = 2(d+2)$.

With this result, the kinematic equations for the gravity triplet (3.1)–(3.3) simplify to

$$W = V\cdot V, \qquad V\cdot\partial\Omega = -\frac{d-2}{2}\Omega, \qquad G_{MN} = \nabla_M V_N, \tag{3.5}$$

where the dot products are constructed with G^{MN} and the vector V_M or V^M is defined as the derivative of $W(X)$

$$V_M \equiv \tfrac{1}{2}\partial_M W, \qquad V^M = \tfrac{1}{2}G^{MN}\partial_N W. \tag{3.6}$$

For this form of V_M the expression for $G_{MN} = \frac{1}{2}\nabla_M\partial_N W = \frac{1}{2}\partial_M\partial_N W - \frac{1}{2}\Gamma^K_{MN}\partial_K W$ is symmetric $G_{MN} = \nabla_M V_N = \nabla_N V_M$ since the Christoffel symbol [22]

$$\Gamma^K_{MN} = \tfrac{1}{2}G^{KQ}(\partial_M G_{NQ} + \partial_N G_{MQ} - \partial_Q G_{MN}) \tag{3.7}$$

is symmetric. In particular, in flat space all the kinematic equations above are satisfied by

$$W_{\rm flat} = X^2, \qquad V_M^{\rm flat} = X_M, \qquad G_{MN}^{\rm flat} = \eta_{MN}, \qquad (\Gamma_{MN}^K)_{\rm flat} = 0 \tag{3.8}$$

as listed in Table I.

The form of the metric in (3.5) that emerged from the 2T gravity action satisfies a special geometric property. By using the definition of the Lie derivative $\mathcal{L}_V$ for the vector V, which on a tensor is given by $\mathcal{L}_V G_{MN} \equiv \nabla_M V_N + \nabla_N V_M$, we can recognize that the form G_{MN} given in (3.5) and (3.6) is equivalent to writing the following *homothety* equations for the metric and its inverse:

$$\mathcal{L}_V G_{MN} = 2G_{MN}, \qquad \mathcal{L}_V G^{MN} = -2G^{MN}. \tag{3.9}$$

The Lie derivative amounts to a general coordinate transformation of the metric using the vector $V_M(X)$ as the parameter of transformation, therefore we can say that under such a transformation the metric yields a factor of 2

$$\begin{aligned} 2G_{MN} &= \mathcal{L}_V G_{MN} = \nabla_M V_N + \nabla_N V_M \\ &= V^K \partial_K G^{MN} + \partial_M V^K G_{KN} + \partial_N V^K G_{MK}. \end{aligned} \tag{3.10}$$

The equivalence of the homothety conditions above to the kinematic equations of motion (3.5) that emerged from the action is shown by inserting the Christoffel symbol (3.7) into $\nabla_M V_N = \partial_M V_N - \Gamma_{MN}^K V_K$.

After coupling the gravity triplet to any matter as in Table I, the kinematic equations initially derived from the action appear to have couplings between the gravity triplet (G_{MN}, Ω, W) and matter fields [1]. However, after using the kinematic equations for matter as well, one finds that they simplify to the form above (3.5) and (3.6) regardless of the type of matter they couple to [1]. Furthermore, the kinematic equations for matter fields $(S_i,\ \Psi,\ A_M)$ also simplify to the following form [1]:

$$V \cdot DS_i = -\frac{d-2}{2} S_i, \qquad V \cdot D\Psi = -\frac{d}{2}\Psi, \qquad V^M F_{MN} = 0, \tag{3.11}$$

where $F_{MN} = \partial_M A_N - \partial_N A_M - ig[A_M, A_N]$ is the Yang-Mills field strength, and D_M is the Yang-Mills covariant derivative. These equations can also be rewritten as the response to the Lie derivative $\mathcal{L}_V$ applied on the corresponding fields of various spins.

It is now evident that all kinematic equations (3.5) and (3.11) derived from the action (2.1) have a geometrical meaning and they are the same for each field irrespective of the interactions and irrespective of which other fields are included in the action. This is why we call these "kinematic" equations. The deeper significance of this structure is an underlying Sp(2, R) symmetry explained in Sec. V.

B. Kinematics of the metric, vielbein and Dirac gamma matrices

The kinematic equations described above required the peculiar homothety condition (3.9) that the metric must satisfy $\mathcal{L}_V G_{MN} = 2G_{MN}$, which in turn requires that the metric must also be constructed from the potential $W(X)$

$$G_{MN} = \nabla_M V_N, \quad \text{with} \quad V_N = \tfrac{1}{2}\partial_N W \quad \text{and} \quad W = V \cdot V. \tag{3.12}$$

This is a nonlinear equation since Γ_{MN}^P is constructed from the metric as in (3.7). These equations are solved by choosing gauges and convenient coordinates. Then the solution is expressed in terms of the shadow field $g_{\mu\nu}(x)$ in two lower dimensions, and its prolongations, all of which remain arbitrary as far as the homothety condition (3.12) is concerned, as will be discussed below. In this section we develop properties of the curved space described by a metric that satisfies (3.12) without choosing any gauges.

Before imposing the homothety condition (3.12), we recall the well-known usual formulation of curved space, with any signature in any number of dimensions. We define a base space and a tangent space. The vielbein that connects the two spaces $E_M{}^a(X)$ is labeled by an index M in base space and an index a in tangent space. The metric in flat tangent space is the SO(d, 2) invariant flat metric η_{ab} while the curved space metric $G_{MN}(X)$ is constructed from the vielbein as a SO(d, 2) invariant

$$G_{MN}(X) = E_M{}^a(X) E_N{}^b(X) \eta_{ab}. \tag{3.13}$$

We introduce the usual affine connection $\Gamma_{MN}^P(X)$ of Eq. (3.7) which is symmetric in base space $\Gamma_{MN}^P = \Gamma_{NM}^P$, and the SO($d$, 2) Yang-Mills field known as the "spin connection" $\omega_M{}^{ab}(X)$ which is antisymmetric in tangent space $\omega_M{}^{ab} = -\omega_M{}^{ba}$. We will use the following notation for various covariant derivatives:

$$\nabla_M = \partial_M - \Gamma_M; \qquad D_M = \partial_M + \omega_M; \qquad \hat{D}_M = \partial_M - \Gamma_M + \omega_M. \tag{3.14}$$

The first one ∇_M is covariant when applied on a field with only base space indices, the second one D_M is covariant when applied on a field with only tangent space indices, and the third one $\hat{D}_M$ is covariant when applied on a field with both base and tangent space indices. In many expressions we will write $\hat{D}_M$ and let it be understood that sometimes Γ_M or ω_M would drop out automatically depending on the field. However, when it becomes useful we will specialize $\hat{D}_M$ to ∇_M or D_M or even ∂_M.

Using these definitions, the covariant derivative of $E_M{}^a$ that is gauge invariant under general coordinate transformations as well as under the tangent space local SO(d, 2) transformations is

$$\hat{D}_M E_N{}^a = \partial_M E_N{}^a - \Gamma_{MN}{}^P E_P{}^a + \omega_M{}^{ab} E_{Nb}. \tag{3.15}$$

A symmetric connection $\Gamma^P_{MN} = \Gamma^P_{NM}$ demands vanishing torsion $T_{MN}{}^a$

$$T_{MN}{}^a \equiv \hat{D}_M E_N{}^a - \hat{D}_N E_M{}^a = \partial_{[M} E_{N]}{}^a + \omega^{ab}_{[M} E_{N]b} = D_{[M} E_{N]}{}^a = 0, \tag{3.16}$$

where Γ^P_{MN} dropped out due to antisymmetrization. $T_{MN}{}^a = 0$ is an equation from which the spin connection $\omega_M{}^{ab}$ is solved as a function of E^a_M as

$$\omega^{ab}_M = E^{Na} E^{Pb} (C_{MNP} - C_{NPM} - C_{PMN}), \quad C_{MNP} \equiv -\tfrac{1}{2} E_{Mc} (\partial_N E^c_P - \partial_P E^c_N). \tag{3.17}$$

Furthermore, the two connections Γ^P_{MN} and $\omega_M{}^{ab}$ are related to each other by requiring that the covariant derivative of E^a_M in Eq. (3.15) vanishes

$$\hat{D}_M E_N{}^a = 0 \rightarrow \omega_M{}^{ab} = \tfrac{1}{2} E^{N[a} (\nabla_M E^{b]}_N). \tag{3.18}$$

Equation (3.18) insures that the covariant derivative of the metric vanishes $\hat{D}_P G_{MN} = \nabla_P G_{MN} = 0$, and since ω_M drops out it can be written as

$$\nabla_P G_{MN} = \partial_P G_{MN} - \Gamma_{PM}{}^Q G_{QN} - \Gamma_{PN}{}^Q G_{MQ} = 0. \tag{3.19}$$

Then one can show that the Γ^P_{MN} which solves both equations (3.18) and (3.19) is nothing but the usual Christoffel connection constructed from the metric G_{MN} given in Eq. (3.7).

The curvature tensor is constructed just as in Yang-Mills theory from the spin connection $\omega_M{}^{ab}$ which is nothing but the Yang-Mills gauge field for the SO$(d, 2)$ local symmetry in tangent space

$$R_{MN}{}^{ab} = \partial_M \omega_N{}^{ab} - \partial_N \omega_M{}^{ab} + \omega_M{}^{ak} \omega_{Nk}{}^b - \omega_N{}^{ak} \omega_{Mk}{}^b. \tag{3.20}$$

This Yang-Mills field strength coincides with the standard curvature tensor R^P_{QMN} constructed from the affine connection

$$R^P_{QMN} = \partial_M \Gamma^P_{NQ} - \partial_N \Gamma^M_{MQ} + \Gamma^P_{MS} \Gamma^S_{NQ} - \Gamma^P_{NS} \Gamma^S_{MQ} \tag{3.21}$$

after converting the base indices to tangent indices

$$R_{MN}{}^{ab} = -R^P_{QMN} E^a_P E^{Qb}. \tag{3.22}$$

As is well known, the curvature with all lower indices $G_{PK} R^K_{QMN} = R_{PQMN}$ is antisymmetric in $M \leftrightarrow N$ and separately under $P \leftrightarrow Q$, but is symmetric under the interchange of the pairs $MN \leftrightarrow PQ$, and satisfies the cyclic identity

$$R_{MNPQ} = -R_{NMPQ} = -R_{MNQP} = R_{PQMN}, \quad R^P_{QMN} + R^P_{MNQ} + R^P_{NQM} = 0. \tag{3.23}$$

We now turn to the special kinematics of 2T physics. The specialty in 2T physics is that the metric is constructed from the covariant derivative of the vector V_M as in Eq. (3.12). Applying the standard formalism above, and imposing the homothety condition (3.12), we obtain the following seven lemmas that describe certain general properties of this special gravitational system that are useful in our work:

(1) The vielbein E^a_M is constructed from a vector V^a in tangent space

$$E_M{}^a = D_M V^a = \partial_M V^a + \omega^{ab}_M V_b, \tag{3.24}$$

where $W = V^a V_a$ and

$$V^a = V_M E^{Ma} = \tfrac{1}{2} (\partial_M W) E^{Ma} \quad \text{or} \quad V_M = E^a_M V_a = \tfrac{1}{2} (\partial_M W). \tag{3.25}$$

This is shown by reconstructing the metric and using the following series of steps to prove that it agrees with Eq. (3.12) as follows:

$$\begin{aligned} G_{MN}(X) &= E_M{}^a(X) E_N{}^b(X) \eta_{ab} \\ &= D_M V^a E_N{}^b(X) \eta_{ab} = \hat{D}_M (V_b E_N{}^b(X)) \\ &= \hat{D}_M [D_N (\tfrac{1}{2} V^a V^b \eta_{ab})] = \tfrac{1}{2} \nabla_M \partial_N W \\ &= \nabla_M V_N. \end{aligned} \tag{3.26}$$

In going from the first to the second line we used $\hat{D}_M E_N{}^b = 0$ of Eq. (3.18); the rest of the steps are evident. Hence the structure of the vielbein in (3.24) is equivalent to the homothety condition (3.12) on G_{MN}.

(2) The vanishing of torsion $T^a_{MN} = 0$ requires the following kinematic conditions on the curvature:

$$R_{MN}{}^{ab} V_b = 0, \qquad V^P R_{PQMN} = 0, \qquad R_{MNPQ} V^P = 0, \qquad V^M R_{MN}{}^{ab} = 0. \tag{3.27}$$

The first form is shown by inserting $E_N{}^a = \hat{D}_N V^a$ in the vanishing torsion $0 = T_{MN}{}^a \equiv \hat{D}_{[M} E_{N]}{}^a = [\hat{D}_M, \hat{D}_N] V^a = R^{ab}_{MN} V_b$. The second form follows from the first by replacing tangent indices by base indices, or directly by writing the vanishing torsion in the form $T^P_{MN} = [\nabla_M, \nabla_N] V^P = V^Q R^P_{QMN} = 0$. The third form follows from the second by using the identity $R_{PQMN} = R_{MNPQ}$. The last form follows from converting the P, Q indices to ab indices in the third form. It should be noted that the last form $V^M R_{MN}{}^{ab} = 0$ is the standard kinematic equation required by Sp$(2, R)$ constraints on any Yang-Mills field strength for any gauge group $V^M F_{MN}{}^a$ as given in Eq. (3.11).

(3) The SO$(d, 2)$ Dirac gamma matrices with base space indices $\gamma_M \equiv E^a_M \gamma_a$ are covariantly constant

$$\hat{D}_M\gamma_N = \partial_M\gamma_N - \Gamma^P_{MN}\gamma_P + \tfrac{1}{4}\omega^{ab}_M(\gamma_{ab}\gamma_N - \gamma_N\gamma_{ab}) = 0. \quad (3.28)$$

Here the covariant derivative $\hat{D}_M$ includes ω^{ab}_M because $(\gamma_M)_{A\dot{B}}$ has spinor indices $A\dot{B}$ in tangent space. To show this result, consider first the tangent space gamma matrices γ_a, which are pure constants that satisfy $\partial_M\gamma_a = 0$. For these the covariant derivative also gives $\hat{D}_M\gamma_a = 0$ because it reduces to the ordinary derivative

$$\hat{D}_M\gamma_a = D_M\gamma_a = \partial_M\gamma_a = 0, \quad (3.29)$$

because the ω^{ab}_M contributions for all tangent space indices a, A, and $\dot{B}$ in $(\gamma_a)_{A\dot{B}}$ cancel each other. Then the result in Eq. (3.28) is shown by rewriting $\hat{D}_M\gamma_N = \hat{D}_M(E^a_N\gamma_a) = (\hat{D}_M E^a_N)\gamma_a = 0$ which follows from (3.18).

(4) The covariant derivative of the gamma matrix $V = V^N\gamma_N = V^a\gamma_a$, which appears in the Yukawa couplings in Table I, gives γ_M

$$\hat{D}_M V = D_M V = \gamma_M. \quad (3.30)$$

This is shown by writing $D_M V = (D_M V^a)\gamma_a = E^a_M\gamma_a = \gamma_M$.

(5) The ordinary derivative of $V^2 = V^a V_a = V^M V_M$ gives $2V_M$

$$\partial_M V^2 = 2V_M. \quad (3.31)$$

This is shown by writing $\partial_M V^2 = D_M V^2 = 2(D_M V^a)V_a = 2E_M{}^a V_a = 2V_M$. Of course, this is in agreement with the fact that $W = V^2$ and the definition $V_M = \frac{1}{2}\partial_M W$.

(6) The various fields V_a, V_M, and V automatically satisfy the following kinematic equations:

$$(V^M D_M - 1)V_a = 0, \qquad (V^M\nabla_M - 1)V_N = 0, \qquad (V^M D_M - 1)V = 0. \quad (3.32)$$

These follow from $D_M V_a = E_{Ma}$, $\nabla_M V_N = G_{MN}$, and $D_M V = \gamma_M$ derived above.

(7) The following kinematic property in $d+2$ dimensions is automatically satisfied

$$D_M(\sqrt{G}\delta(V^2)\gamma^M V) = d\sqrt{G}\delta(V^2). \quad (3.33)$$

To show this, first recall that the divergence of any vector $\nabla_M v^M = \partial_M v^M + \Gamma_{MP}{}^M v^P$ can be rewritten as $\nabla_M v^M = G^{-1/2}\partial_M(\sqrt{G}v^M)$. Applying this to the vector $v^M \equiv \delta(V^2)\gamma^M V$, gives $D_M(\sqrt{G}v^M) = \sqrt{G}\hat{D}_M v^M$ where $\hat{D}_M$ appears. Then use the properties derived in the lemmas above as follows:

$$\begin{aligned} &D_M[\sqrt{G}\delta(V^2)\gamma^M V] \\ &= \sqrt{G}\hat{D}_M[\delta(V^2)\gamma^M V] \\ &= \sqrt{G}(\hat{D}_M\delta(V^2))\gamma^M V + \sqrt{G}\delta(V^2)(\hat{D}_M\gamma^M)V \\ &\quad + \sqrt{G}\delta(V^2)\gamma^M\hat{D}_M V \\ &= \sqrt{G}\delta'(V^2)(2V_M)\gamma^M V + \sqrt{G}\delta(V^2)(0)V \\ &\quad + \sqrt{G}\delta(V^2)\gamma^M\gamma_M \\ &= \sqrt{G}\delta'(V^2)2V^2 + \sqrt{G}\delta(V^2)(d+2) \\ &= d\sqrt{G}\delta(V^2). \end{aligned}$$

To get to the last step we have used the property of the delta function $V^2\delta'(V^2) = -\delta(V^2)$.

IV. DYNAMICAL AND KINEMATIC EQUATIONS OF MOTION

The dynamical equations of motion derived from the action (2.1), and its generalization from Table I, are those proportional to $\delta(W)$ for the general variation of every field. The dynamical equations that follow from varying the metric, dilaton, and scalars are [1]

$$\delta\Omega\colon [\nabla^2\Omega - 2aR\Omega + a\partial_\Omega V(\Omega, S_i)]_{W=0} = 0, \quad (4.1)$$

$$\delta S_i\colon [\nabla^2 S_i - 2aRS_i - \partial_{S_i}V(\Omega, S_i)]_{W=0} = 0, \quad (4.2)$$

$$\delta G^{MN}\colon [R_{MN}(G) - \tfrac{1}{2}G_{MN}R(G) - T_{MN}]_{W=0} = 0, \quad (4.3)$$

where the stress tensor T_{MN} is

$$\begin{aligned} T_{MN} = \frac{1}{\Omega^2 - aS_i^2}\bigg[&\frac{-1}{2a}\partial_M\Omega\partial_N\Omega + \frac{1}{2}\partial_M S_i\partial_N S_i \\ &+ G_{MN}\bigg(\frac{1}{4a}(\partial\Omega)^2 - \frac{1}{4}(\partial S_i)^2 - \frac{1}{2}V(\Omega, S_i)\bigg) \\ &- (G_{MN}\nabla^2 - \nabla_M\partial_N)(\Omega^2 - aS_i^2)\bigg] \end{aligned} \quad (4.4)$$

and as usual $R_{MN}(G) \equiv R^P_{MPN}$ and $R(G) \equiv G^{MN}R_{MN}$. These equations are to be solved at $W = 0$ because of the delta function $\delta(W)$ that multiplies them, but we will at first manipulate them for any W.

We now simplify these equations as follows. Contracting Eq. (4.3) with G^{MN}, we can solve for $R(G) = \frac{-2}{d}G^{MN}T_{MN}$ and get

$$\begin{aligned} (\Omega^2 - aS_i^2)R(G) = \bigg[&-\frac{1}{2a}\partial\Omega\cdot\partial\Omega + \frac{1}{2}\partial S_i\cdot\partial S_i \\ &+ \frac{2(d+1)}{d}\nabla^2(\Omega^2 - aS_i^2) \\ &+ \frac{d+2}{d}V(\Omega, S_i)\bigg]. \end{aligned} \quad (4.5)$$

Multiply Eqs. (4.1) and (4.2) by $(-\Omega/a)$, S_i respectively, sum over i and add them, to get

$$0 = 2R(\Omega^2 - aS_i^2) - \frac{1}{a}(\Omega\nabla^2\Omega - aS_i\nabla^2 S_i) - (\Omega\partial_\Omega + S_i\partial_{S_i})V(\Omega, S). \tag{4.6}$$

In this equation we insert the expression in (4.5) and use the homogeneity of the potential (2.6) to write $(\Omega\partial_\Omega + S_i\partial_{S_i})V(\Omega, S) = \frac{2d}{d-2}V(\Omega, S)$, and after some simplifications we obtain

$$\nabla^2(\Omega^2 - aS_i^2) = -V(\Omega, S_i). \tag{4.7}$$

Inserting this back into (4.5) yields

$$R(G) = \frac{-\frac{1}{2a}\partial\Omega\cdot\partial\Omega + \frac{1}{2}\partial S_i\cdot\partial S_i - V(\Omega, S_i)}{\Omega^2 - aS_i^2}. \tag{4.8}$$

Using both Eqs. (4.7) and (4.8), the energy momentum tensor in Eq. (4.4) simplifies to

$$T_{MN} = \frac{1}{(\Omega^2 - aS_i^2)}\bigg[-\frac{1}{2a}\partial_M\Omega\partial_N\Omega + \frac{1}{2}\partial_M S_i\partial_N S_i - \frac{1}{2}G_{MN}(\Omega^2 - aS_i^2)R + \nabla_M\partial_N(\Omega^2 - aS_i^2)\bigg]. \tag{4.9}$$

Inserting (4.8) and (4.9) into (4.3) yields

$$R_{MN}(G) = S_{MN}(\Omega, S_i), \tag{4.10}$$

where $S_{MN}(\Omega, S_i)$ is given by

$$S_{MN}(\Omega, S_i) \equiv \frac{1}{(\Omega^2 - aS_i^2)}\bigg[-\frac{1}{2a}\partial_M\Omega\partial_N\Omega + \frac{1}{2}\partial_M S_i\partial_N S_i + \nabla_M\partial_N(\Omega^2 - aS_i^2)\bigg]. \tag{4.11}$$

Of course, T_{MN} and S_{MN} are related by $T_{MN} = S_{MN} - \frac{1}{2}G_{MN}G^{PQ}S_{PQ}$. This is as much as we can simplify the dynamical equations before choosing gauges and imposing $W = 0$.

We also gather the kinematic equations satisfied by these fields and W as discussed in the previous section, with $V_M \equiv \frac{1}{2}\partial_M W$.

$$W = V\cdot V, \qquad G_{MN} = \nabla_M V_N, \qquad V^P R_{PQMN} = 0, \qquad V^M R_{MN} = 0, \tag{4.12}$$

$$V\cdot\partial\Omega = -\frac{d-2}{2}\Omega, \qquad V\cdot\partial S_i = -\frac{d-2}{2}S_i, \qquad V^M S_{MN} = 0. \tag{4.13}$$

A remarkable property is that the variation of the action with respect to W does not give a new equation besides those kinematic or dynamical equations that are obtained from the variation of the other fields. This was explained [1] as being due to a local symmetry that allows $W(X)$ to be set to any desired function of X^M. Although W is set to zero eventually in the dynamical equations (4.1)–(4.3), its first and second derivatives that are related to V_M and G_{MN} do not vanish [see e.g. the flat case in Eq. (3.8)]. Exercising the freedom in choosing some $W(X)$ is one of the steps that defines the shadow in lower dimensions. The selection that leads to the conformal shadow will be described in the next section.

V. THE UNDERLYING Sp(2, R)

In the previous section we showed that the kinematic equations have a geometrical significance. Now we emphasize that both the kinematic and dynamical equations are intimately related to the fundamental $\mathrm{Sp}(2,R)$ gauge symmetry that is at the root of 2T physics. The significance of the kinematic equations is that they impose part of the *gauge invariant physical state* conditions under $\mathrm{Sp}(2,R)$ which is explained as follows. It was shown in [1] that the three generators Q_{ij} of $\mathrm{Sp}(2,R)$ in the presence of gravity are given by the following three functions of phase space (X^M, P_M):

$$Q_{11} = W(X), \qquad Q_{12} = Q_{21} = V^M(X)P_M, \qquad Q_{22} = G^{MN}(X)P_M P_N. \tag{5.1}$$

These Q_{ij} form the $\mathrm{Sp}(2,R)$ Lie algebra under Poisson brackets provided the fields $W(X)$, $V^M(X)$, and $G^{MN}(X)$ satisfy the kinematic equations in Eqs. (3.5), (3.6), and (3.9). The reader can check that in flat space $W_{\rm flat} = X^2$, $V^M_{\rm flat} = X_M$, and $G^{MN}_{\rm flat} = \eta^{MN}$ satisfy the $\mathrm{Sp}(2,R)$ closure property under Poisson brackets. These Q_{ij} generate a *local gauge symmetry on the worldline* for a particle interacting with gravity, thus making its position and momentum $X^M(\tau)$, $P_M(\tau)$ indistinguishable at every worldline instance [1]. In the quantum theory of such a particle, its physical states must be $\mathrm{Sp}(2,R)$ gauge invariant, and hence these Q_{ij} must vanish on the first quantized wave functions. In position space the first quantized wave functions are the fields in 2T field theory. Therefore these fields must satisfy $Q_{ij} \sim 0$ after a proper quantum ordering of X, P, and replacing the momentum by a derivative $P_M = -i\partial_M$. The kinematic equations in (3.5), (3.6), and (3.9) imposed by the action are the precise expressions of the vanishing of the generator $Q_{12} = (-iV^M\partial_M + \cdots) \sim 0$ after appropriate quantum ordering for matter or gravitational fields of various spins. The vanishing of $Q_{11} = W(X)$ is imposed through the delta function $\delta(W)$ and its derivatives, and finally the vanishing of $Q_{22} = (-G^{MN}\partial_M\partial_N + \cdots)$ amounts to the *dynamical* equations of motion.[7] Thus we see that all the equations of motion generated by the

[7] The dots $\cdots$ in the expressions of Q_{ij} are the corrections due to interactions. This general property is explained in Refs. [1,2,23]. These corrections, in the case of gravity, are precisely supplied directly by the action in Eq. (2.1) and Table I, so they are determined and written out fully in the kinematic and dynamical equations discussed in this paper as well as Ref. [1].

2T field theory have the significance of imposing the physical state condition under the Sp(2, R) gauge symmetry, or more precisely, its extension that includes particles with spin as well as interactions, as explained in [2,23].

One additional point of clarification about the role of the underlying Sp(2, R), as reflected in the kinematics, is in order. The Becchi-Rouet-Stora-Tyutin (BRST) field theory formulation in [23] is technically a fuller approach for imposing Sp(2, R), but the extra baggage of the BRST formalism, in the form of ghosts and redundant gauge degrees of freedom, can be avoided by appreciating a few simple aspects related to Sp(2, R) as just outlined in the previous paragraph. A related point is that the underlying Sp(2, R) provides the key for the resolution of an ambiguity about the kinematic equations as derived from the action (2.1) and Table I. This ambiguity is avoided through the BRST approach, but is more easily resolved directly as follows. The variation of the action for each field yields a linear superposition of the delta function and its derivatives of the form $A\delta(W) + B\delta'(W) + C\delta''(W) = 0$. These imply three equations that are satisfied at $W = 0$, but there is ambiguity in identifying the proper forms of A, B, and C that should vanish at $W = 0$. This is because these distributions satisfy $W\delta''(W) = -2\delta'(W)$ and $W\delta'(W) = -\delta(W)$. Therefore, if we add to B a term that is proportional to W, that term feeds into a term added to A. Similarly any terms proportional to W and W^2 in C feed into B and A, respectively. In the BRST approach the ambiguities of adding such terms to B or C are just gauge degrees of freedom which in any case drop out automatically in the physical sector. When the BRST approach is short-circuited as explained in [2], this ambiguity is resolved by recognizing that the $B = C = 0$ kinematic equations amount to demanding the closure of the underlying Sp(2, R) Lie algebra, as made clear by the Q_{ij} in Eq. (5.1) for the corresponding worldline particle model [1]. This closure demands that the equations $B = C = 0$ must be valid for all W, not only $W = 0$, so that Sp(2, R) is defined and its Lie algebra is satisfied without restrictions on the phase space degrees of freedom. This is necessary for it to be a gauge symmetry of the particle model. The upshot is that the particle model can be used as a guide to identify the correct forms of B, C and then demand $B = C = 0$ not only at $W = 0$ but at all W, which means that if B, C are expanded in powers of W, the coefficient of each power of W should vanish. This is a shortcut to insure self-consistency of all the equations of motion, including the dynamical equations, derived from the action (i.e. consistency of having first class constraints Q_{11}, Q_{12}, and Q_{22}, which then are set to zero). By satisfying Sp(2, R) in this way, the ambiguities in A, B, and C are resolved at any W. This insures the validity of the underlying Sp(2, R) gauge symmetry and turns the ambiguities into gauge freedom, consistent with the BRST approach [2]. Thus, the physical sector that is gauge invariant under Sp(2, R), namely, $B = C = 0$ at any W, and $A|_{W=0} = 0$, are the consistent field equations of motion.

Accordingly, it should be emphasized that the kinematic equations above (4.12) and (4.13), which are consistent with the particle model [1], are to be solved at any W, not only at $W = 0$, while the dynamical equations (4.1)–(4.11) need to be satisfied only at $W = 0$. This is the procedure followed in the following sections to obtain the conformal shadow and its prolongation.

The same result is also obtained without using the guidance of the particle model discussed in the two previous paragraphs, but only using the gauge symmetry in the equations of motion $A\delta(W) + B\delta'(W) + C\delta''(W) = 0$ that follow from the 2T field theory action. To explain this gauge symmetry we will make a coordinate transformation, $X^M \to (w, u, x^\mu)$, such that $W(X) = w$ is one of the coordinates, as in the next section. Furthermore, to simplify the discussion we will concentrate only on a single scalar field, say the dilaton $\Omega(w, u, x^\mu)$, and suppress the coordinates u, x^μ since they are irrelevant to the discussion. A similar discussion will hold for each field in the theory.

We want to show that the action has a gauge symmetry under the gauge transformation $\delta_\Lambda \Omega = \Lambda_\Omega(w, u, x)$ for off-shell arbitrary Ω as well as off-shell other fields. The variation of the action with respect to the field Ω takes the form

$$\delta_\Lambda S = \int dwdud^dx \delta_\Lambda \Omega(w)[A_\Omega(w)\delta(w) + B_\Omega(w)\delta'(w) + C_\Omega(w)\delta''(w)]. \tag{5.2}$$

Of course, A_Ω, B_Ω, and C_Ω depend on w through Ω and other fields as well. Because of the delta functions we need to analyze the expansion of each term in powers of w and then do the integral over w. Hence we have

$$\Omega(w) = \Omega_0 + w\Omega_1 + \tfrac{1}{2}w^2\Omega_2 + \cdots, \tag{5.3}$$

$$A_\Omega(w) = A_0 + wA_1 + \tfrac{1}{2}w^2A_2 + \cdots, \tag{5.4}$$

$$B_\Omega(w) = B_0 + wB_1 + \tfrac{1}{2}w^2B_2 + \cdots, \tag{5.5}$$

$$C_\Omega(w) = C_0 + wC_1 + \tfrac{1}{2}w^2C_2 + \cdots, \tag{5.6}$$

$$\Lambda_\Omega(w) = \Lambda_0 + w\Lambda_1 + \tfrac{1}{2}w^2\Lambda_2 + \cdots. \tag{5.7}$$

Then the integral gives

$$\delta_\Lambda S = \int dud^dx[\Lambda_0(A_0 - B_1 + C_2) + \Lambda_1(-B_0 + 2C_1) + \Lambda_2 C_0]. \tag{5.8}$$

It is possible to make $\delta_\Lambda S = 0$ with a choice of gauge parameters Λ_0, Λ_1, and Λ_2 that are related to each other, when all the fields are off shell. There are three local parameters but only one condition; hence two of the parameters among Λ_0, Λ_1, and Λ_2 can be chosen arbitrarily such that the action is gauge invariant $\delta_\Lambda S = 0$ off shell. This 2-parameter gauge symmetry is a remnant of the Sp(2, R) BRST gauge symmetry discussed in [23]. A similar local symmetry is valid separately for each field in any 2T-field theory. This was called the 2T-gauge symmetry in [2].

Using this gauge symmetry we can choose arbitrarily the prolongations $\Omega_1(u, x)$ and $\Omega_2(u, x)$ in the expansion of Eq. (5.3). It is convenient to make the choice of Ω_1, Ω_2 such that $B_1 = C_2$ and $C_1 = 0$. These gauge choices hold when all the fields are off shell.

Now we investigate the on-shell equations of motion which are obtained from the above procedure by taking $\delta\Omega_0$, $\delta\Omega_1$, and $\delta\Omega_2$ arbitrary and independent of each other. So the equations of motion for the on-shell $\Omega_{0,1,2}$ are

$$A_0 - B_1 + C_2 = 0, \qquad -B_0 + 2C_1 = 0, \qquad C_0 = 0. \tag{5.9}$$

In the gauge we have chosen they become $A_0 = 0$, $B_0 = 0$, $B_1 = C_2$, $C_0 = 0$, and $C_1 = 0$.

Now we investigate what C_2 is in more detail. It was shown in [1] that in the variation of the action with respect to every field the term $C\delta''(W)$ is always of the form $C \sim (G^{MN}\partial_M W \partial_N W - 4W)$ up to a field dependent proportionality factor. In the next section we show that in the coordinate system $W(X) = w$, this expression becomes zero automatically by constraining only the G^{ww} component of the metric $G^{MN}(X)$. Therefore, we automatically obtain $C_2 = 0$.

With this result for $C_2 = 0$ taken into account, we now see that, in our chosen gauge, the on-shell dynamics must satisfy

$$\begin{aligned} &A_0 = 0, \qquad B_0 = 0, \qquad B_1 = 0, \\ &C_0 = 0, \qquad C_1 = 0, \qquad C_2 = 0. \end{aligned} \tag{5.10}$$

The coefficients of the higher powers of w in the expansion of $A_\Omega(w)$, $B_\Omega(w)$, and $C_\Omega(w)$, such as $A_{n\geq1}$, $B_{n\geq2}$, and $C_{n\geq3}$ are arbitrary because they never enter in the equations. So they could be chosen arbitrarily without any consequence for the dynamics of the fields Ω_0, Ω_1, and Ω_2 which do appear in the equations. In particular, imposing $B_{n\geq2} = 0$ and $C_{n\geq3} = 0$ has no consequences for the field components Ω_0, Ω_1, and Ω_2 since they only restrict $\Omega_{n\geq3}$. The latter are pure gauge freedom which never appear in the equations or even in the off-shell action. Similar statements apply to the other fields.

This is in agreement with the procedure we discussed above, of solving the equations $A\delta(W) + B\delta'(W) + C\delta''(W) = 0$ for all the fields by imposing $A|_{W=0}$ while taking $B = C = 0$ at all W. As we have shown, this is the consequence of a gauge choice, consistent with the gauge symmetries of the action in Eq. (2.1), as well as with the Sp(2, R) gauge symmetry properties of the worldline formulation of particle dynamics in the presence of gravity.

VI. GENERAL RELATIVITY AS A SHADOW WITH WEYL SYMMETRY

In this section we determine the shadow fields and their prolongations. For scalar fields Ω, $S_i(X)$, these are defined by expanding the field in powers of $W(X)$, as done below. The zeroth order term is the shadow. The coefficients of all higher powers are Kaluza-Klein–type degrees of freedom, which we call prolongations of the shadow. For fields that have spin indices, such as $G_{MN}(X)$, $R_{MNPQ}(X)$, the zeroth order term has components that point in two lower dimensions, such as $g_{\mu\nu}$, $R_{\mu\nu\lambda\sigma}$, as well as components that point in the additional two dimensions. In traditional Kaluza-Klein terminology the extra components are additional KK degrees of freedom. In our case all such KK-type degrees of freedom, as well as the coefficients of the higher powers in W, are called prolongations.

We will take advantage of gauge symmetries to eliminate some of the redundant gauge degrees of freedom to clearly identify the physical degrees of freedom recognized in 1T field theory in d dimensions. The result will be that the fields in $d+2$ dimensions $G_{MN}(X)$, $\Omega(X)$, and $S_i(X)$ will be reduced to the fields in d dimensions $g_{\mu\nu}(x)$, $\phi(x)$, and $s_i(x)$ by a series of steps that involve gauge fixing as well as solving the kinematic equations. The prolongations of the shadows $g_{\mu\nu}(x)$, $\phi(x)$, and $s_i(x)$ from the "wall" x^μ into the higher dimensional space X^M, namely, the full $G_{MN}(X)$, $\Omega(X)$, and $S_i(X)$, will also be discussed. In this section the shadows and their prolongations will be allowed to be arbitrary fields in d dimensions, restricted only by the kinematic conditions, but in the following section, by using the dynamical equations of the full theory it will be shown that all prolongations become functions of only the shadow fields $g_{\mu\nu}(x)$, $\phi(x)$, and $s_i(x)$. One of the goals in this section is to show that the accidental Weyl symmetry of Eq. (2.11) acting on $g_{\mu\nu}(x)$, $\phi(x)$, and $s_i(x)$ in general relativity in the shadow action (2.7) emerges from the local coordinate reparametrization symmetry in the higher spacetime X^M. It will be clarified how a generalization of the Weyl symmetry acts also on the prolongations.

Among the local symmetries in 2T gravity there are obviously general coordinate transformations and the local symmetry that allows arbitrary transformation of W [1] as emphasized above. Exercising the freedom of making gauge choices for these local symmetries defines the properties of the emergent spacetimes for the shadows in the lower dimensions.

To begin this process we parametrize the spacetime X^M in terms of $d+2$ coordinates (w, u, and x^μ) and define the tangent basis in base space $\partial_M = (\partial_w, \partial_u, \partial_\mu)$ relative to these coordinates. In this basis we use the general coordinate transformations to gauge fix $d+2$ components of the metric, $G^{w\nu} = 0$, $G^{wu} = -1$, and $G^{uu} = 0$, leading to the following gauge fixed form of $G^{MN}(X)$:

$$G^{MN} = \begin{array}{c} M\backslash N \\ w \\ u \\ \mu \end{array} \begin{array}{c} \begin{array}{ccc} w & u & v \end{array} \\ \begin{pmatrix} G^{ww} & -1 & 0 \\ -1 & 0 & G^{u\nu} \\ 0 & G^{\mu u} & G^{\mu\nu} \end{pmatrix} \end{array}. \tag{6.1}$$

Next we select $W(w, u, x^\mu) = w$ to be simply one of the coordinates, which immediately gives $V_M = \frac{1}{2}\partial_M W = (\frac{1}{2}, 0, 0)_M$. Inserting this in the kinematic equation $W = G^{MN}V_M V_N$ gives $w = \frac{1}{4}G^{ww}$ which fixes another component of the metric. The result of these steps is then

$$W(X) = w, \qquad G^{ww}(X) = 4w, \qquad V_M(X) = (\tfrac{1}{2}, 0, 0)_M, \qquad V^M(X) = (2w, -\tfrac{1}{2}, 0)^M. \tag{6.2}$$

This choice of W gives $V^M\partial_M = 2w\partial_w - \frac{1}{2}\partial_u$, and the kinematic conditions for the scalars Ω, S_i in (4.13) become $(2w\partial_w - \frac{1}{2}\partial_u + \frac{d-2}{2})\Omega(w, u, x^\mu) = 0$, and similarly for S_i. Their general solution for any w is

$$\Omega(X) = e^{(d-2)u}\phi(x, we^{4u}), \qquad S_i(X) = e^{(d-2)u}s_i(x, we^{4u}), \tag{6.3}$$

where, except for the overall factors of $e^{(d-2)u}$, the fields $\phi(x, we^{4u})$, $s_i(x, we^{4u})$ are general functions of the variables x^μ and the combination we^{4u}.

Now we impose the kinematic equation $G_{MN} = \nabla_M V_N$ in the form of the homothety condition $\mathcal{L}_V G^{MN} = -2G^{MN}$ as explained in (3.9)

$$V^K\partial_K G^{MN} - \partial_K V^M G^{KN} - \partial_K V^N G^{MK} = -2G^{MN}. \tag{6.4}$$

This is already satisfied for the fixed metric components $G^{ww} = 4w$, $G^{wu} = -1$, and $G^{uu} = G^{w\mu} = 0$, while it gives the following conditions on the remaining metric components:

$$(2w\partial_w - \tfrac{1}{2}\partial_u)G^{\mu u} = -2G^{\mu u}, \qquad (2w\partial_w - \tfrac{1}{2}\partial_u)G^{\mu\nu} = -2G^{\mu\nu}. \tag{6.5}$$

Their general solutions are

$$G^{\mu u}(X) = e^{4u}\gamma^\mu(x, we^{4u}), \qquad G^{\mu\nu}(X) = e^{4u}\tilde{g}^{\mu\nu}(x, we^{4u}), \tag{6.6}$$

where γ^μ and $\tilde{g}^{\mu\nu}$ are general functions of x^μ and we^{4u}.

We see that the solutions to the kinematic conditions are given in terms of functions of fewer than $d + 2$ variables. We find that there are remaining coordinate transformation symmetries in $d + 1$ variables that can remove the $\gamma^\mu(x, we^{4u})$, thus reducing further the degrees of freedom. To explain this we first examine the coordinate transformations that maintain the restricted form of G^{MN} that emerged above. The infinitesimal general coordinate transformation of the scalars W, Ω, and S_i and the metric G^{MN} are

$$\delta_\varepsilon X^M = \varepsilon^M(X), \qquad \delta_\varepsilon W = \varepsilon^K\partial_K W, \qquad \delta_\varepsilon\Omega = \varepsilon^K\partial_K\Omega, \qquad \delta_\varepsilon S_i = \varepsilon^K\partial_K S_i, \tag{6.7}$$

$$\delta_\varepsilon G^{MN} = \varepsilon^K\partial_K G^{MN} - (\partial_K\varepsilon^M)G^{KN} - (\partial_K\varepsilon^N)G^{KM}. \tag{6.8}$$

The remaining symmetry should not change the form of $W = w$ and the fixed metric components G^{ww}, G^{wu}, $G^{w\mu}$, and G^{uu} given above. This requirement is satisfied by the following form of infinitesimal coordinate transformations $\varepsilon^M(X)$:

$$\varepsilon^w(X) = 0, \qquad \varepsilon^u(X) = \Lambda(x, we^{4u}), \qquad \varepsilon^\mu(X) = \varepsilon^\mu(x, we^{4u}), \tag{6.9}$$

which give $\delta_\varepsilon W = \delta_\varepsilon G^{ww} = \delta_\varepsilon G^{wu} = \delta_\varepsilon G^{uu} = \delta_\varepsilon G^{w\mu} = 0$. In what follows we will show that $\varepsilon^\mu(x, we^{4u})$ at $w = 0$ will be related to coordinate transformations in the d dimensional shadow, while $\Lambda(x, we^{4u})$ at $w = 0$, which comes from coordinate transformations of u, will be related to local scale transformations in the d dimensional shadow.

The coordinate transformations of (u, x^μ) with parameters $\Lambda(x, we^{4u})$, $\varepsilon^\mu(x, we^{4u})$ give nonzero $\delta_\varepsilon G^{\mu u}$, $\delta_\varepsilon G^{\mu\nu}$, $\delta_\varepsilon\Omega$, and $\delta_\varepsilon S_i$. We focus on $\delta_\varepsilon\Omega$ and $\delta_\varepsilon G^{\mu u}$ which follow from (6.7) and (6.8)

$$\delta_\varepsilon\Omega = \Lambda\partial_u\Omega + \varepsilon^\lambda\partial_\lambda\Omega, \tag{6.10}$$

$$\delta_\varepsilon G^{\mu u} = \{\Lambda\partial_u G^{\mu u} + \varepsilon^\lambda\partial_\lambda G^{\mu u} + (\partial_w\varepsilon^\mu) - (\partial_\lambda\varepsilon^\mu)G^{\lambda u} - (\partial_\lambda\Lambda)G^{\mu\lambda} - (\partial_u\Lambda)G^{\mu u}\}. \tag{6.11}$$

Evidently there is enough gauge freedom in $\Lambda(x, we^{4u})$ to gauge fix $\Omega = e^{(d-2)u}\phi(x, we^{4u})$ completely to any desired form as a function of (x, we^{4u}). We will take advantage of this freedom later.[8]

Similarly, there is enough gauge freedom in $\varepsilon^\mu(x, we^{4u})$ to gauge fix $G^{\mu u} = e^{4u}\gamma^\mu(x, we^{4u}) = 0$. Then the gauge fixed form of G^{MN} for any w becomes

[8]Convenient gauges will be mentioned in discussing Eqs. (8.8)–(8.10). We mention here other possibilities that may serve different purposes. One possible partial gauge choice is to make Ω independent of w, as $\Omega = e^{(d-2)u}\phi(x)$, while S_i remains as given in (6.3). With this there still remains the gauge freedom of making $\phi(x)$ a constant. Another gauge of interest is to fix Ω such that $\Omega^2 - aS_i^2 = e^{2(d-2)u}[\phi^2(x) - s_i^2(x)]$ is independent of w, where $\phi(x)$, $s_i(x)$ are the shadows defined by the expansions in Eqs. (6.20) and (6.21), and again the x dependence can be further gauge fixed to a constant. This is similar to Eq. (2.12), but includes the u, w dependence, thus providing the prolongation of the shadow for Eq. (2.12). The expansion in powers of w in the second gauge gives the details of how the prolongations are gauge fixed, namely, $\phi\phi_1 - as_i s_{1i} = 0$ and $\phi\phi_2 - as_i s_{2i} + \phi_1^2 - as_{1i}^2 = 0$, etc. (rather than $\phi_1 = \phi_2 = 0$, etc., in the first gauge) where ϕ_1, ϕ_2, s_{1i}, and s_{2i} are defined in Eqs. (6.20) and (6.21).

$$G^{MN}=\begin{array}{c|c} M\backslash N & \begin{array}{ccc} w & u & \nu \end{array} \\ \begin{array}{c} w\\ u\\ \mu \end{array} & \begin{pmatrix} 4w & -1 & 0\\ -1 & 0 & 0\\ 0 & 0 & e^{4u}\tilde{g}^{\mu\nu}(x,we^{4u}) \end{pmatrix} \end{array}. \tag{6.12}$$

The metric with lower indices is then

$$G_{MN}=\begin{array}{c|c} M\backslash N & \begin{array}{ccc} w & u & \nu \end{array} \\ \begin{array}{c} w\\ u\\ \mu \end{array} & \begin{pmatrix} 0 & -1 & 0\\ -1 & -4w & 0\\ 0 & 0 & e^{-4u}\tilde{g}_{\mu\nu}(x,we^{4u}) \end{pmatrix} \end{array}. \tag{6.13}$$

We may now ask if there is any more remaining symmetry that does not change the gauge fixed forms of G_{MN}? For keeping the form of G_{MN} we need $\delta_\varepsilon G^{\mu u}=0$ for the expressions in Eq. (6.11) after setting $G^{\mu u}=0$. This is satisfied by parameters that obey the condition $\partial_w\varepsilon^\mu = G^{\mu\nu}\partial_\nu\Lambda$, with an arbitrary $\Lambda(x,we^{4u})$. To analyze further the meaning of the remaining symmetry we expand in powers of w

$$\delta x^\mu=\varepsilon^\mu(x,we^{4u})=\varepsilon_0^\mu(x)+we^{4u}\varepsilon_1^\mu(x)+\cdots, \tag{6.14}$$

$$\delta u=\Lambda(x,we^{4u})=\Lambda_0(x)+we^{4u}\Lambda_1(x)+\cdots. \tag{6.15}$$

The remaining symmetry has as independent parameters only the lowest component $\varepsilon_0^\mu(x)$, and all $\Lambda(x,we^{4u})$

$$\begin{aligned} &\text{independent: } \varepsilon_0^\mu(x), \text{ and } \Lambda_0(x), \Lambda_1(x), \Lambda_2(x), \ldots,\\ &\text{dependent: } \varepsilon_1^\mu(x)=g^{\mu\nu}\partial_\nu\Lambda_0,\\ &\qquad\varepsilon_2^\mu(x)=g^{\mu\nu}\partial_\nu\Lambda_1-g_1^{\mu\nu}\partial_\nu\Lambda_0, \text{ etc.,} \end{aligned} \tag{6.16}$$

where $g^{\mu\nu}$, $g_1^{\mu\nu}$ are defined by the expansion of the metric in powers of we^{4u} given in Eq. (6.24). Among these, $\varepsilon_0^\mu(x)$ corresponds to general coordinate transformations of x^μ while $\Lambda_0(x)$ is the gauge parameter of local scale transformations on the remaining local fields, known as the Weyl transformations in 1T field theory, as explained below.

The remaining gauge parameters $\Lambda_{n\geq 1}(x)$ are generalizations of the Weyl symmetry $\Lambda_0(x)$. They can be used to make convenient gauge choices.[8]

The transformation of the scalars in (6.3) and metric components in (6.13) under the remaining symmetry (6.14) and (6.15) can be extracted from the general coordinate transformation rules (6.7) and (6.8) in the form

$$\begin{aligned}\delta\phi(x,we^{4u})=&[\Lambda(x,we^{4u})(4w\partial_w+d-2)\\ &+\varepsilon^\mu(x,we^{4u})\partial_\mu]\phi(x,we^{4u}),\end{aligned} \tag{6.17}$$

$$\begin{aligned}\delta s_i(x,we^{4u})=&[\Lambda(x,we^{4u})(4w\partial_w+d-2)\\ &+\varepsilon^\mu(x,we^{4u})\partial_\mu]s_i(x,we^{4u}),\end{aligned} \tag{6.18}$$

$$\begin{aligned}\delta\tilde{g}_{\mu\nu}(x,we^{4u})=&\Lambda(x,we^{4u})(4w\partial_w-4)\tilde{g}_{\mu\nu}(x,we^{4u})\\ &+\mathcal{L}_\varepsilon\tilde{g}_{\mu\nu}(x,we^{4u}),\end{aligned} \tag{6.19}$$

where $\mathcal{L}_\varepsilon\tilde{g}^{\mu\nu}(x,we^{4u})$ is the Lie derivative using the vector $\varepsilon^\mu(x,we^{4u})$. After inserting in these expressions the field configurations (6.3)–(6.13) and the form of the remaining parameters (6.14) and (6.15), the result can be expanded in powers of w to extract term by term the transformation properties of the shadows in x^μ and their prolongations into the u and w dimensions. To do this we expand every field in powers of w to define the shadow fields in d dimensions $\phi(x)$, $s_i(x)$, and $g_{\mu\nu}(x)$ as the zeroth order terms, while their prolongations $\phi_n(x)$, $s_{ni}(x)$, and $g_{n\mu\nu}(x)$ are defined as the coefficients of the higher powers of we^{4u} as follows:

$$\phi(x,we^{4u})=\phi(x)+we^{4u}\phi_1(x)+\tfrac{1}{2}(we^{4u})^2\phi_2(x)+\cdots, \tag{6.20}$$

$$s_i(x,we^{4u})=s_i(x)+we^{4u}s_{1i}(x)+\tfrac{1}{2}(we^{4u})^2s_{2i}(x)+\cdots. \tag{6.21}$$

Similarly we have for the metric

$$\begin{aligned}\tilde{g}_{\mu\nu}(x,we^{4u})=&g_{\mu\nu}(x)+we^{4u}g_{1\mu\nu}(x)\\ &+\tfrac{1}{2}(we^{4u})^2g_{2\mu\nu}(x)+\cdots.\end{aligned} \tag{6.22}$$

For the determinant we get

$$\begin{aligned}\sqrt{G}&=e^{-2du}\sqrt{-\tilde{g}(x,we^{4u})}\\ &=e^{-2du}\sqrt{-g}\bigg[1+\frac{we^{4u}}{2}g_{1\lambda}^\lambda+\frac{(we^{4u})^2}{4}(g_{2\lambda}^\lambda+(g_{1\lambda}^\lambda)^2)\\ &\quad+\cdots\bigg].\end{aligned} \tag{6.23}$$

The inverse metric is also computed in terms of $g_{\mu\nu}, g_{1\mu\nu}, g_{2\mu\nu},\ldots$ as

$$\begin{aligned}\tilde{g}^{\mu\nu}(x,we^{4u})=&g^{\mu\nu}(x)-we^{4u}g_1^{\mu\nu}(x)\\ &-\tfrac{1}{2}(we^{4u})^2(g_2^{\mu\nu}-2g_{1\sigma}^{\nu}g_1^{\sigma\mu})(x)+\cdots.\end{aligned} \tag{6.24}$$

Here the upper indices on $g_1^{\mu\nu}$, $g_2^{\mu\nu}$, etc. are raised or lowered by using the lowest component of the metric $g_{\mu\nu}$; so $g_1^{\mu\nu}$, $g_2^{\mu\nu}$ do not mean the inverses of $g_{1\mu\nu}$, $g_{2\mu\nu}$. Inserting these expressions allows us to extract the following transformation rules for the shadow fields $\phi(x)$, $s_i(x)$, and $g_{\mu\nu}(x)$ by setting $w=0$ in Eqs. (6.17)–(6.19):

$$\delta\phi(x)=(d-2)\Lambda_0(x)\phi(x)+\varepsilon_0^\mu(x)\partial_\mu\phi(x), \tag{6.25}$$

$$\delta s_i(x)=(d-2)\Lambda_0(x)s_i(x)+\varepsilon_0^\mu(x)\partial_\mu s_i(x), \tag{6.26}$$

$$\delta g_{\mu\nu}(x)=-4\Lambda_0(x)g_{\mu\nu}(x)+\mathcal{L}_{\varepsilon_0}g_{\mu\nu}(x). \tag{6.27}$$

In these expressions it is clear that $\Lambda_0(x)$ is the infinitesimal parameter of the Weyl transformations, which is seen by comparing to Eq. (2.11) and setting $\Lambda_0(x) = -\lambda(x)/2$. This shows that the local scale symmetry in 1T field theory comes from the coordinate reparametrization symmetry $\delta u = [\Lambda(x, we^{4u})]_{w=0}$ of the 2T field theory. This was one of the points we wanted to prove in this section.

The higher powers in w of Eqs. (6.17)–(6.19) give the nontrivial transformation rules for the prolongations under coordinate, Weyl, and generalized Weyl transformations $\varepsilon_0^\mu(x)$, $\Lambda_0(x)$, and $\Lambda_{n\geq 1}(x)$, as follows:

$$\delta\phi_1(x) = \{(d+2)\Lambda_0\phi_1(x) + (d-2)\Lambda_1\phi + \varepsilon_0^\mu\partial_\mu\phi_1 + \varepsilon_1^\mu(x)\partial_\mu\phi\}, \tag{6.28}$$

$$\delta\phi_2(x) = \{(d+6)\Lambda_0\phi_2 + 2(d+2)\Lambda_1\phi_1 + (d-2)\Lambda_2\phi + \varepsilon_0^\mu\partial_\mu\phi_2 + 2\varepsilon_1^\mu\partial_\mu\phi_1 + \varepsilon_2^\mu\partial_\mu\phi\}, \tag{6.29}$$

and similarly for δs_{ni}. Evidently the terms containing $\Lambda_0(x)$ and ε_0^μ are the local scale transformations and local coordinate transformations on these fields. Recall that $\varepsilon_{n\geq 1}^\mu$ are functions of the Λ_n as given in (6.16). Similarly, for the metric prolongations we get the following transformation laws under the coordinate, Weyl, and generalized Weyl transformations:

$$\delta g_{1\mu\nu}(x) = \{0\times\Lambda_0(x)g_{1\mu\nu}(x) + 4\Lambda_1(x)g_{\mu\nu} + \mathcal{L}_{\varepsilon_0}g_{1\mu\nu}(x) + \mathcal{L}_{\varepsilon_1}g_{\mu\nu}(x)\}, \tag{6.30}$$

$$\delta g_{2\mu\nu}(x) = \{4\Lambda_0(x)g_{2\mu\nu}(x) + 8\Lambda_1(x)g_{1\mu\nu} + 4\Lambda_2 g_{\mu\nu} + \mathcal{L}_{\varepsilon_0}g_{2\mu\nu}(x) + 2\mathcal{L}_{\varepsilon_1}g_{1\mu\nu}(x) + \mathcal{L}_{\varepsilon_2}g_{\mu\nu}(x)\}. \tag{6.31}$$

VII. RIEMANN AND LORENTZ CURVATURES

A. Christoffel connection and curvature

We are now ready to use the gauge fixed metric in Eqs. (6.12), (6.13), (6.22), and (6.24) to compute the curvatures at any w. For the Christoffel connection $\Gamma_{MN}^P \equiv \frac{1}{2}G^{PQ}(\partial_M G_{NQ} + \partial_N G_{MQ} - \partial_Q G_{MN})$ we obtain

$$\Gamma_{MN}^w = \begin{array}{c|ccc} M\backslash N & w & u & \nu \\ \hline w & 0 & 2 & 0 \\ u & 2 & 8w & 0 \\ \mu & 0 & 0 & -2e^{-4u}\tilde{g}_{\mu\nu} \end{array}, \qquad \Gamma_{MN}^u = \begin{array}{c|ccc} M\backslash N & w & u & \nu \\ \hline w & 0 & 0 & 0 \\ u & 0 & -2 & 0 \\ \mu & 0 & 0 & \frac{e^{-4u}}{2}\partial_w\tilde{g}_{\mu\nu} \end{array}, \tag{7.1}$$

$$\Gamma_{MN}^\lambda = \begin{array}{c|ccc} M\backslash N & w & u & \nu \\ \hline w & 0 & 0 & \frac{1}{2}\tilde{g}^{\lambda\sigma}\partial_w\tilde{g}_{\sigma\nu} \\ u & 0 & 0 & -2\delta_\nu^\lambda + 2w\tilde{g}^{\lambda\sigma}\partial_w\tilde{g}_{\sigma\nu} \\ \mu & \frac{1}{2}\tilde{g}^{\lambda\sigma}\partial_w\tilde{g}_{\sigma\mu} & -2\delta_\mu^\lambda + 2w\tilde{g}^{\lambda\sigma}\partial_w\tilde{g}_{\sigma\mu} & \Gamma_{\rho\sigma}^\mu(\tilde{g}) \end{array}. \tag{7.2}$$

Expanding $\Gamma_{\rho\sigma}^\mu(\tilde{g})$ in the last line in powers of w gives $\Gamma_{\rho\sigma}^\mu(\tilde{g}) = \Gamma_{\rho\sigma}^\mu(g) + we^{4u}\Gamma_{1\rho\sigma}^\mu + \cdots$, where the zeroth order term is the usual $\Gamma_{\rho\sigma}^\mu(g)$ in d dimensions and the first order term is

$$\Gamma_{1\rho\sigma}^\mu = \{-\tfrac{1}{2}g_1^{\mu\nu}(\partial_\rho g_{\sigma\nu} + \partial_\sigma g_{\rho\nu} - \partial_\nu g_{\rho\sigma}) + \tfrac{1}{2}g^{\mu\nu}(\partial_\rho g_{1\sigma\nu} + \partial_\sigma g_{1\rho\nu} - \partial_\nu g_{1\rho\sigma})\}. \tag{7.3}$$

Even though w is set to zero eventually, one must first take derivatives of Γ_{MN}^P with respect to w in computing various components of the curvature $R_{MNPQ}(G)$. Therefore w dependent terms in Γ_{MN}^P (i.e. prolongations of its shadow) will contribute to the curvature in zeroth order in powers of w because of derivatives with respect to w.

To calculate the Riemann tensor $R_{PMN}^Q \equiv \partial_M\Gamma_{NP}^Q - \partial_N\Gamma_{MP}^Q + \Gamma_{MS}^Q\Gamma_{NP}^S - \Gamma_{NS}^Q\Gamma_{MP}^S$, we recall that the zero torsion condition imposes the following kinematical constraint (3.23) on the curvature:

$$V_Q R_{PMN}^Q = V^Q R_{QPMN} = V^Q R_{MNPQ} = V^Q R_{NPQ}^M = 0. \tag{7.4}$$

With the gauge choice of Eq. (6.2) these conditions become

$$R_{PMN}^w = 0, \qquad R_{uPMN} = 4wR_{wPMN}, \qquad R_{MNu}^P = 4wR_{MNw}^P. \tag{7.5}$$

From the form of the gauge fixed metric in Eq. (6.12) we also obtain

$$R_{PMN}^u = -R_{wPMN}. \tag{7.6}$$

From these it is easy to see consequences such as

$$R_{wMN}^u = R_{uMN}^u = R_{uwMN} = R_{MNuw} = 0, \tag{7.7}$$

$$R_{MNw}^u = R_{NMw}^u, \qquad R_{MNu}^\lambda = 4wR_{MNw}^\lambda, \text{ etc.} \tag{7.8}$$

Using the antisymmetry and cyclic properties in (3.23), these kinematic relations explain many of the results in the following lists for the Riemann tensor computed by using the Christoffel connection in (7.1) and (7.2) at any w

$$R^{w}_{MPN} = 0, \tag{7.9}$$

$$R^{u}_{PMN} = \begin{cases} R^{u}_{wMN} = R^{u}_{uMN} = 0, & R^{u}_{\rho\mu\nu} = \frac{1}{2}\nabla_{[\mu}g_{1\nu]\rho} + \cdots, \\ R^{u}_{\rho\mu u} = R^{u}_{\mu\rho u} = 4wR^{u}_{\rho\mu w}, & R^{u}_{\rho\mu w} = \frac{e^{4u}}{4}(g^{\sigma}_{1\mu}g_{1\sigma\rho} - 2g_{2\rho\mu}) + \cdots, \end{cases} \tag{7.10}$$

where the covariant derivative ∇_μ is with respect to the metric $g_{\mu\nu}(x)$. The curvatures on the first column are either identically zero or vanish when $w = 0$, while those in the second column $R^{u}_{\rho\mu w}$, $R^{u}_{\rho\mu\nu}$ do not vanish even at $w = 0$. The $+\cdots$ means there are terms proportional to higher powers of w but are of no interest in our analysis. Similarly we obtain R^{λ}_{PMN} with analogous properties for the first and second columns

$$R^{\lambda}_{PMN}\colon \begin{cases} R^{\lambda}_{P\mu u} = 4wR^{\lambda}_{P\mu w}, & R^{\lambda}_{Pwu} = 0, \\ R^{\lambda}_{w\mu u} = R^{\lambda}_{u\mu w} = 4wR^{\lambda}_{w\mu w}, & R^{\lambda}_{w\mu w} = \frac{e^{8u}}{4}(g^{\lambda\nu}_{1}g_{1\mu\nu} - 2g^{\lambda}_{2\mu}) + \cdots, \\ R^{\lambda}_{u\mu u} = 16w^2R^{\lambda}_{w\mu w}, & R^{\lambda}_{w\mu\nu} = \frac{e^{4u}}{2}\nabla_{[\mu}g^{\lambda}_{1\nu]} + \cdots, \\ R^{\lambda}_{\rho\mu u} = -R^{\lambda}_{\rho u\mu} = 4wR^{\lambda}_{\rho\mu w}, & R^{\lambda}_{\rho\mu w} = -R^{\lambda}_{\rho w\mu} = \frac{e^{4u}}{2}g^{\lambda\sigma}\nabla_{[\sigma}g_{1\rho]\mu} + \cdots, \\ R^{\lambda}_{u\mu\nu} = 4wR^{\lambda}_{w\mu\nu}, & R^{\lambda}_{\rho\mu\nu} = R^{\lambda}_{\rho\mu\nu}(g) - g^{\lambda}_{1[\mu}g_{\nu]\rho} - \delta^{\lambda}_{[\mu}g_{1\nu]\rho} + \cdots. \end{cases} \tag{7.11}$$

At $w = 0$ the nonvanishing components of R_{QPMN} with all lower indices are $R_{w\mu w\nu}$, $R_{\mu\nu\lambda w}$, and $R_{\mu\nu\lambda\sigma}$

$$w \to 0\colon \begin{cases} R_{w\mu w\nu}(G) = \frac{e^{4u}}{4}(g_{1\mu\sigma}g^{\sigma}_{1\nu} - 2g_{2\mu\nu}) + \cdots, \\ R_{\mu\nu\lambda w}(G) = \frac{1}{2}(\nabla_\mu g_{1\nu\lambda} - \nabla_\nu g_{1\mu\lambda}) + \cdots, \\ R_{\mu\nu\lambda\sigma}(G) = e^{-4u}[R_{\mu\nu\lambda\sigma}(g) + g_{1\sigma[\mu}g_{\nu]\lambda} - g_{1\lambda[\mu}g_{\nu]\sigma}] + \cdots. \end{cases} \tag{7.12}$$

In the last expression it should be noted that $R^{\lambda}_{\rho\mu\nu}(G)$ differs from $R^{\lambda}_{\rho\mu\nu}(g)$, the latter being the standard Riemann tensor constructed from the metric $g_{\mu\nu}$. The difference is accounted by the contributions of the prolongations of the metric which contribute to $R^{\lambda}_{\rho\mu\nu}(G)$ even when $w = 0$.

We can now compute the Ricci tensor $R_{MN} \equiv R^{P}_{MPN} = R^{w}_{MwN} + R^{u}_{MuN} + R^{\lambda}_{M\lambda N}$. The kinematic constraints $V^M R_{MN} = 0$, imply

$$R_{uN} = 4wR_{wN}. \tag{7.13}$$

Hence R_{uw}, $R_{u\nu}$ are related to R_{ww}, $R_{w\nu}$, respectively, by a factor of $4w$ while $R_{uu} = (4w)^2R_{ww}$, hence we have

$$R_{MN}(G) = \begin{array}{c|c} M\backslash N & \begin{matrix} w & u & \nu \end{matrix} \\ \begin{matrix} w \\ u \\ \mu \end{matrix} & \begin{pmatrix} R_{ww} & 4wR_{ww} & R_{w\nu} \\ 4wR_{ww} & (4w)^2R_{ww} & 4wR_{w\nu} \\ R_{w\mu} & 4wR_{w\mu} & R_{\mu\nu}(G) \end{pmatrix} \end{array}, \tag{7.14}$$

where

$$R_{ww}(G) = \frac{e^{8u}}{4}\,\mathrm{Tr}(g_1g_1 - 2g_2) + \cdots,$$
$$R_{w\mu}(G) = \frac{e^{4u}}{2}(\nabla_\lambda g^{\lambda}_{1\mu} - \nabla_\mu \mathrm{Tr}g_1) + \cdots, \tag{7.15}$$
$$R_{\mu\nu}(G) = R_{\mu\nu}(g) - (d-2)g_{1\mu\nu} - (\mathrm{Tr}g_1)g_{\mu\nu} + \cdots.$$

The trace notation Tr means that indices are contracted by using the lowest mode $g_{\mu\nu}$. The $+\cdots$ indicates that there are additional higher order terms in powers of w that are not of interest in our analysis. For $w = 0$ only R_{ww}, $R_{w\mu}$, and $R_{\mu\nu}$ have nonvanishing contributions while the other components of R_{MN} vanish. In the last expression we see that $R_{\mu\nu}(G)$ differs from $R_{\mu\nu}(g)$ which is the standard Ricci tensor constructed from the metric $g_{\mu\nu}$.

Finally the Ricci scalar, $R(G) = G^{MN}R_{MN} = 4wR_{ww} - 2R_{wu} + e^{4u}\tilde{g}^{\mu\nu}R_{\mu\nu}(G)$, is

$$R(G) = e^{4u}[R(g) - 2(d-1)e^{4u}\,\mathrm{Tr}g_1] + \cdots. \tag{7.16}$$

Again in the last expression $R(G)$ differs from $R(g)$ which is the standard curvature scalar.

As seen explicitly in all the expressions above, the prolongations of the shadow of the metric, namely, $g_{1\mu\nu}$, $g_{2\mu\nu}$ contribute nontrivially to the prolongations of the curvatures. Even when $w = 0$, there are nonvanishing curvature components, such as $R^{\lambda}_{w\mu w}$, $R^{\lambda}_{w\mu\nu}$, $R^{\lambda}_{\rho\mu w}$, and $R^{\lambda}_{\rho\mu\nu}$ that point not only in the x^μ directions but also in the w, u directions. The notation $R^{\lambda}_{\rho\mu\nu}(G)$, $R_{\mu\nu}(G)$, and $R(G)$ is used to distinguish them from $R^{\lambda}_{\rho\mu\nu}(g)$, $R_{\mu\nu}(g)$, and $R(g)$ where the latter depend only on the lowest mode $g_{\mu\nu}(x)$ while the former depend on $G_{\mu\nu}$ including the higher modes $g_{1\mu\nu}$, $g_{2\mu\nu}$. We will see however, that after taking into account the dynamical equations of motion, all extra curvature pieces get determined only in terms of the shadow fields $g_{\mu\nu}(x)$, $\phi(x)$, and $s_i(x)$, while the dynamics of these lowest modes interacting with $R^{\lambda}_{\rho\mu\nu}(g)$, $R_{\mu\nu}(g)$, and $R(g)$ will be given by standard general relativity (with the Weyl symmetry) as determined self-consistently only by the shadow action in Eq. (2.7).

B. Gauge fixed vielbein, spin connection, and SO$(d, 2)$ curvature

For completeness we record here also the gauge fixed forms of the vielbein, spin connection, and SO$(d, 2)$ curvature $E_M^a(w, u, x)$, $\omega_M^{ab}(w, u, x)$, and $R_{MN}^{ab}(w, u, x)$ that are compatible with the gauge fixed metric and its curvatures above.

We take the following form of the gauge fixed vielbein that satisfies $G_{MN} = E_M^a E_N^b \eta_{ab}$ up to a local SO$(d, 2)$ transformation in tangent space:

$$E_M{}^a = \begin{array}{c} M\backslash a \\ w \\ u \\ \mu \end{array} \begin{array}{c} \begin{array}{ccc} -' & +' & i \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 2w & 1 & 0 \\ 0 & 0 & e^{-2u}\tilde{e}_\mu^i(x, we^{4u}) \end{pmatrix} \end{array}. \tag{7.17}$$

Its inverse that satisfies $E_a^M E_M^b = \delta_a^b$ or $E_a^M E_N^a = \delta_N^M$ is

$$E_a^M = \begin{array}{c} a\backslash M \\ -' \\ +' \\ i \end{array} \begin{array}{c} \begin{array}{ccc} w & u & \mu \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ -2w & 1 & 0 \\ 0 & 0 & e^{2u}\tilde{e}_i^\mu(x, we^{4u}) \end{pmatrix} \end{array}, \tag{7.18}$$

where $\tilde{e}_\mu{}^i$ and $\tilde{e}_i{}^\mu$ are inverses of each other. These may be expanded in powers of we^{4u}

$$\tilde{e}_\mu{}^i(x, we^{4u}) = e_\mu{}^i + we^{4u}e_{1\mu}^i + \tfrac{1}{2}(we^{4u})^2 e_{2\mu}^i + \cdots, \tag{7.19}$$

$$\tilde{e}_i{}^\mu(x, we^{4u}) = e_i{}^\nu - we^{4u}e_{1i}^\nu + \frac{(we^{4u})^2}{2}(2e_{1\rho}^\nu e_{1i}^\rho - e_{2i}^\nu) + \cdots. \tag{7.20}$$

Here $e_i{}^\nu$ is the inverse of $e_\mu{}^i$ as usual, but e_{1i}^μ is not the inverse of $e_{1\mu}^i$, rather it is $e_{1\mu}^i$ with indices raised or lowered by using the appropriate tangent space or base space metrics, $e_{1i}^\mu = \eta_{ij}e_{1\nu}^j g^{\nu\mu}$, and similarly for e_{2i}^ν. From $\tilde{g}_{\mu\nu} = \tilde{e}_\mu^i\tilde{e}_\nu^i\eta_{ij}$ we can obtain relations between the expansion of the vielbein and the expansion of the metric given in (6.22)

$$g_{\mu\nu} = e_\mu{}^i e_\nu{}^j\eta_{ij}, \qquad g_{1\mu\nu} = (e_{1\mu}^i e_\nu{}^j + e_\mu{}^i e_{1\nu}^j)\eta_{ij},$$
$$g_{2\mu\nu} = (e_{2\mu}^i e_\nu{}^j + e_\mu{}^i e_{2\nu}^j + 2e_{1\mu}^i e_{1\nu}^j)\eta_{ij}. \tag{7.21}$$

Recall the gauge fixed versions of the vectors $V_M = \frac{1}{2}\partial_M W = (\frac{1}{2}, 0, 0)_M$ and $V^M = \frac{1}{2}\partial_N W G^{MN} = (2w, -\frac{1}{2}, 0)^M$ in Eq. (6.2). Their tangent space counterparts become $V_a = V_M E_a^M = \frac{1}{2}E_a^w$ and $V^a = V^M E_M^a = 2wE_w^a - \frac{1}{2}E_u^a$. Explicitly these are

$$\begin{aligned} &V_a = (\tfrac{1}{2}, -w, 0)_a, \\ &V^a = (w, -\tfrac{1}{2}, 0)^a, \text{ in the basis } a = (-', +', i), \end{aligned} \tag{7.22}$$

and have the dot product $V^aV_a = w$.

The spin connection is constructed by using the standard relation $\omega_M^{ab} = E^{Na}E^{Pb}(C_{MNP} - C_{NPM} - C_{PMN})$ given in Eqs. (3.17), with $C_{PMN} \equiv -\frac{1}{2}E_{Pa}(\partial_M E_N^a - \partial_N E_M^a)$. With the above gauge fixed form of E_M^a we obtain

$$\omega_w^{ab} = \begin{array}{c} a\backslash b \\ -' \\ +' \\ i \end{array} \begin{array}{c} \begin{array}{ccc} -' & +' & j \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\tilde{e}^{\sigma[i}\partial_w\tilde{e}_\sigma^{j]} \end{pmatrix} \end{array}, \qquad \omega_u^{ab} = \begin{array}{c} a\backslash b \\ -' \\ +' \\ i \end{array} \begin{array}{c} \begin{array}{ccc} -' & +' & j \end{array} \\ \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2w\tilde{e}^{\sigma[i}\partial_w\tilde{e}_\sigma^{j]} \end{pmatrix} \end{array}, \tag{7.23}$$

and

$$\omega_\lambda^{ab} = \begin{array}{c} a\backslash b \\ -' \\ +' \\ i \end{array} \begin{array}{c} \begin{array}{ccc} -' & +' & j \end{array} \\ \begin{pmatrix} 0 & 0 & e^{-2u}(-2e_\lambda^j + w\tilde{e}^{j\sigma}\partial_w\tilde{g}_{\lambda\sigma}) \\ 0 & 0 & \frac{e^{-2u}}{2}\tilde{e}^{j\sigma}\partial_w\tilde{g}_{\lambda\sigma} \\ e^{-2u}(2\tilde{e}_\lambda^i - w\tilde{e}^{i\sigma}\partial_w\tilde{g}_{\lambda\sigma}) & -\frac{e^{-2u}}{2}\tilde{e}^{i\sigma}\partial_w\tilde{g}_{\lambda\sigma} & \omega_\lambda^{ij}(\tilde{e}) \end{pmatrix} \end{array}, \tag{7.24}$$

where $\omega_\lambda^{ij}(\tilde{e})$ is the standard spin connection in d dimensions as constructed from $\tilde{e}_\lambda^i(x, we^{4u})$ including the prolongations of the shadow $\tilde{e}_\lambda^i(x)$.

With these explicit forms, it can be verified that the spin connection ω_M^{ab}, the vielbein E_M^a, and the vector V^a satisfy the kinematic relation

$$E_M^a = D_MV^a = \partial_MV^a + \omega_M^{ab}V_b, \tag{7.25}$$

that is required by 2T gravity as expected from Eq. (3.24). The kinematic equations have completely fixed all components of $\omega_M^{ab}(X)$ in terms of $e_\lambda^j(x, we^{4u})$ and explicit functions of the extra coordinates w, u. When $w = 0$ we recognize that the vielbein in d dimensions $e^j_\lambda(x)$ is basically the shadow component $\omega_\lambda^{-'i}$ of the spin connection that remains unrestricted as a function of x^μ as far as the kinematic equations are concerned.

The SO(d, 2) curvature is

$$R^{ab}_{MN} = \partial_M \omega^{ab}_N - \partial_N \omega^{ab}_M + \omega^{ak}_M \omega_{Nk}{}^b - \omega_N{}^{ak}\omega_{Mk}{}^b = -R^P_{QMN} E^a_P E^{Qb}. \tag{7.26}$$

With the help of the antisymmetry $R^{ab}_{MN} = -R^{ab}_{NM}$, $R^{ab}_{MN} = -R^{ba}_{MN}$, and the kinematic relations in Eq. (3.27), $R^{ab}_{uN} = 4wR^{ab}_{wN}$, $R^{a-'}_{MN} = 2wR^{a+'}_{MN}$, all the nonzero components of the curvature are determined as follows:

$$\begin{aligned}
R^{+'i}_{w\mu} &= \frac{e^{6u}}{2}\left(\frac{1}{2}g_1^{\lambda\sigma}g_{1\mu\sigma} - g^\lambda_{2\mu}\right)e^i_\lambda + \cdots,\\
R^{-'i}_{w\mu} &= 2wR^{+'i}_{w\mu},\\
R^{ij}_{w\mu} &= \tilde{e}^{\nu i}\tilde{e}^j_\rho R^\rho_{\nu w\mu}(G),\\
R^{+'i}_{u\mu} &= 4wR^{+'a}_{w\mu},\\
R^{-'i}_{u\mu} &= 2wR^{+'i}_{u\mu},\\
R^{ij}_{u\mu} &= 4wR^{ij}_{w\mu},\\
R^{+'i}_{\mu\nu} &= \frac{e^{2u}}{2}(\nabla_\mu g_{1\nu\lambda} - \nabla_\nu g_{1\mu\lambda})e^{i\lambda} + \cdots,\\
R^{-'i}_{\mu\nu} &= 2wR^{+'i}_{\mu\nu},\\
R^{ij}_{\mu\nu} &= \tilde{e}^{\sigma i}\tilde{e}^i_\rho R^\rho_{\sigma\mu\nu}(G),
\end{aligned} \tag{7.27}$$

where $R^\rho_{\nu w\mu}(G)$ and $R^\rho_{\sigma\mu\nu}(G)$ are given in Eq. (7.12). These are the curvatures at any w which include all the prolongations of the shadow into the higher dimensions. When $w = 0$, the nonzero terms are just $R^{+'i}_{w\mu}$, $R^{+'i}_{\mu\nu}$, $R^{ij}_{w\mu}$, and $R^{ij}_{\mu\nu}$ while all others vanish.

It should be noted that even at $w = 0$ there are nontrivial components of curvature pointing in the w direction in base space and in the $+'$ direction in tangent space. This is part of the information about the prolongation of the shadow. In the next section it will be shown that, after taking the dynamical equations into account, only the shadow fields $e^i_\mu(x)$, together with matter fields such as $\phi(x)$, $s_i(x)$, determine all curvature components including the prolongations, while the shadow fields satisfy among themselves the familiar general relativity equations (with a Weyl symmetry) which follows self-consistently from the 1T-physics shadow action in Eq. (2.7).

VIII. DYNAMICS OF SHADOWS AND PROLONGATIONS

Having chosen gauges and solved the kinematic equations in the previous sections, we are now ready to discuss the matching of geometry to matter through the dynamical equations derived in Sec. IV from the 2T-gravity action (2.1) and Table I[9]

$$\begin{aligned}
&[R_{MN}(G) - S_{MN}(\Omega, S_i)]_{W=0} = 0,\\
&\left[\frac{1}{\sqrt{G}}\partial_M(\sqrt{G}G^{MN}\partial_N\Omega) - 2a\Omega R(G) + a\partial_\Omega V(\Omega, S_i)\right]_{W=0} = 0,\\
&\left[\frac{1}{\sqrt{G}}\partial_M(\sqrt{G}G^{MN}\partial_N S_i) - 2aS_i R(G) - \partial_{S_i} V(\Omega, S_i)\right]_{W=0} = 0,
\end{aligned} \tag{8.1}$$

where S_{MN} was obtained in Eq. (4.11)

$$S_{MN}(\Omega, S_i) \equiv \frac{1}{(\Omega^2 - aS_i^2)}\left[-\frac{1}{2a}\partial_M\Omega\partial_N\Omega + \frac{1}{2}\partial_M S_i\partial_N S_i + \nabla_M\partial_N(\Omega^2 - aS_i^2)\right]. \tag{8.2}$$

Note that these 2T-gravity equations are imposed only at $w = 0$, unlike the kinematic equations that were solved at all w (see the explanation in Secs. IV and V). We want to compare these equations in $(d+2)$ dimensions to the equations of motion of general relativity in d dimensions

$$R_{\mu\nu}(g) = \frac{1}{(\phi^2 - as_i^2)}\left[-\frac{1}{2a}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\partial_\mu s_i\partial_\nu s_i + \nabla_\mu\partial_\nu(\phi^2 - as_i^2) + \frac{g_{\mu\nu}}{d-2}(V(\phi, s_i) + \nabla^2(\phi^2 - as_i^2))\right], \tag{8.3}$$

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) = 2a\phi R(g) - a\partial_\phi V(\phi, s_i),$$

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu s_i) = 2as_i R(g) + \partial_{s_i}V(\phi, s_i),$$

that follow directly from varying the conformal shadow action (2.7) and using (2.8) and (2.9).

In comparing the original and the shadow equations, we note that we lose two dimensions not only in the spacetime $X^M \to x^\mu$ but also in the components of the metric $G_{MN}(X) \to g_{\mu\nu}(x)$, and similarly for curvature, gauge fields, spinors, etc. Recall also that $R_{\mu\nu}(G)$, $R(G)$ are different than the $R_{\mu\nu}(g)$, $R(g)$ that appear in (8.3), as seen in Eqs. (4.8) and (7.12). The differences depend on the prolongations of the metric and the scalars given in Eqs. (6.20)–(6.24). Moreover, additional components of the tensor $R_{MN}(G)$ are restricted by the original Eqs. (8.1). So, going from (8.1) to (8.3) is not a naive dimensional reduction. The questions we need to investigate include the following.

[9]We have neglected gauge fields and spinor fields to keep our analysis simple. The same general conclusions about the shadows are obtained if all of the fields described in Table I, that would be required for the standard model coupled to gravity, are included in the present analysis.

(i) We recall that the conformal shadow action (2.7) was derived in [1] from the 2T-gravity action (2.1) by inserting directly the solution of the kinematic equations and the gauge fixing discussed above. Can the shadow equations (8.3) be derived from the original equations of motion (8.1) rather than from varying the shadow action? Sometimes these two procedures do not agree, so it is important to verify that they give the same result.

(ii) More importantly, are the prolongations additional Kaluza-Klein type degrees of freedom? What is the dynamics of the prolongations of the metric G_{MN}, curvature $R_{PQMN}(G)$, and scalars Ω, S_i that survived the gauge fixing and kinematic constraints of the previous sections, and do their dynamics restrict the dynamics of the shadow fields (ϕ, s_i, and $g_{\mu\nu}$) beyond the equations of motion in (8.3)? If additional restrictions on (ϕ, s_i, and $g_{\mu\nu}$) arise it would imply that the shadow action (2.7) misses information that influences the shadow fields.

As explained below, the answers are that there are nontrivial prolongations of the metric, curvature, and the scalars, which are however determined only by the shadows (ϕ, s_i, and $g_{\mu\nu}$). Meanwhile, the shadows themselves are determined self-consistently precisely as dictated by the shadow action (2.7) which yielded the general relativity equations (8.3).

To investigate these questions we insert the expansions in powers of w for the fields (6.20)–(6.23) and for the curvatures (4.8) and (7.15) into the original Eqs. (8.1). The derivatives ∂_w, ∂_u in the scalar equations give no new information at $w=0$ because such terms combine to expressions that are proportional to the kinematic conditions, which are already satisfied for the scalars. This is a nontrivial result that is true in curved space only for the special value of $a=(d-2)/8(d-1)$. Hence, for the scalar equations, even though the prolongations ϕ_1, ϕ_2, s_{1i}, and s_{2i}, etc. are nonzero, we obtain directly the naive reduction of the $d+2$ dimensional equations to d dimensions, in agreement with the shadow equations (8.3). The prolongations of the scalars ϕ_1, ϕ_2, s_{1i}, and s_{2i}, etc. are not fixed by the scalar equations in (8.1).

Turning to the curvature equation, $R_{MN}=S_{MN}$ at $w=0$, we begin by computing $[S_{\mu\nu}(\Omega,S_i)]_{w=0}$ from (8.2) as follows:

$$[S_{\mu\nu}(\Omega,S_i)]_{w=0}=\frac{1}{(\phi^2-as_i^2)}\bigg[-\frac{1}{2a}\partial_\mu\phi\partial_\nu\phi+\frac{1}{2}\partial_\mu s_i\partial_\nu s_i+\nabla_\mu\partial_\nu(\phi^2-as_i^2)-\{(\Gamma^w_{\mu\nu}\partial_w+\Gamma^u_{\mu\nu}\partial_u)(\Omega^2-aS_i^2)\}_{w=0}\bigg]. \tag{8.4}$$

After inserting the explicit Christoffel symbols $\Gamma^w_{\mu\nu}$, $\Gamma^u_{\mu\nu}$ in Eqs. (7.1–(7.3) and setting $w=0$, we obtain

$$\{(\Gamma^w_{\mu\nu}\partial_w+\Gamma^u_{\mu\nu}\partial_u)(\Omega^2-aS_i^2)\}_{w=0}=-4(\phi\phi_1-as_is_{1i})g_{\mu\nu}+(d-2)(\phi^2-as_i^2)g_{1\mu\nu}. \tag{8.5}$$

Now matching geometry with matter $[R_{\mu\nu}(G)-S_{\mu\nu}(\Omega,S_i)]_{w=0}=0$, where the curvature

$$R_{\mu\nu}(G)=R_{\mu\nu}(g)-(d-2)g_{1\mu\nu}-(Trg_1)g_{\mu\nu}+\cdots \tag{8.6}$$

was given in (7.15), we find

$$R_{\mu\nu}(g)=S_{\mu\nu}(\phi,s)+\bigg[4\frac{\phi\phi_1-as_is_{1i}}{\phi^2-as_i^2}+\mathrm{Tr}(g_1)\bigg]g_{\mu\nu}. \tag{8.7}$$

This agrees with the shadow equations (8.3) only if the term in brackets satisfies

$$4\frac{\phi\phi_1-as_is_{1i}}{\phi^2-as_i^2}+\mathrm{Tr}(g_1)=\frac{V(\phi,s_i)+\nabla^2(\phi^2-as_i^2)}{(d-2)(\phi^2-as_i^2)}. \tag{8.8}$$

In fact, this relation is exactly correct and can be derived directly from Eq. (4.7), which was obtained as a consequence of the original equations of 2T gravity (4.1)–(4.3).

We have thus shown that all the shadow equations (8.3) derived directly from the shadow action (2.7) are in exact agreement with solving directly the original equations of motion (8.1) in $d+2$ dimensions. This answers the concerns raised above in (i).

There remains to examine the rest of the original equations of motion (8.1) $R_{MN}=S_{MN}$ at $w=0$, to determine whether any additional constraints emerge on the shadow fields or their prolongations. On the geometry side we see from (7.14) that $[R_{uw}=R_{uu}=R_{u\mu}]_{w=0}=0$, and also on the matter side we find $[S_{uw}=S_{uu}=S_{u\mu}]_{w=0}=0$ for the special value of $a=(d-1)/8(d-2)$. Therefore the corresponding equations are identically satisfied without any conditions on the shadows or the prolongations. Proceeding further, from the remaining two cases $[R_{ww}(G)-S_{ww}(\Omega,S_i)]_{w=0}=0$ and $[R_{w\mu}(G)-S_{w\mu}(\Omega,S_i)]_{w=0}=0$, we get nontrivial equations that restrict the prolongations

$$\mathrm{Tr}(g_1g_1-2g_2)=\frac{8}{\phi^2-as_i^2}\bigg[-\frac{d}{d-2}(\phi_1^2-as_{1i}^2)+(\phi\phi_2-as_is_{2i})\bigg], \tag{8.9}$$

$$\nabla_\lambda g^\lambda_{1\mu}-\partial_\mu g^\lambda_{1\lambda}=\frac{2}{\phi^2-as_i^2}\bigg[-\frac{1}{2a}(\phi_1\partial_\mu\phi-as_{1i}\partial_\mu s_i)+2\partial_\mu(\phi\phi_1-as_is_{1i})-g^\lambda_{1\mu}\partial_\lambda(\phi^2-as_i^2)\bigg]. \tag{8.10}$$

From the first of these we may solve algebraically for $\mathrm{Tr}(g_2)$, and consider the second equation, along with (8.8), as equations of motion that restrict $g^{\rho}_{1\mu}$.

To show that there are solutions to the three prolongation equations (8.8)–(8.10), we provide an example with the following special form, which of course is not the general case:

$$g^{\nu}_{1\mu} = A_1(x)\delta^{\nu}_{\mu}, \qquad g^{\nu}_{2\mu} = A_2(x)\delta^{\nu}_{\mu}, \qquad \phi_1 = B_1(x)\phi, \qquad s_{1i} = B_1(x)s_i, \qquad \phi_2 = B_2(x)\phi, \qquad s_{2i} = B_2(x)s_i. \tag{8.11}$$

Furthermore, we use the Weyl gauge $(\phi^2 - as_i^2) = (2\kappa_d^2)^{-1}$ in Eq. (2.12) to simplify these equations. The three Eqs. (8.8)–(8.10) are then solved by

$$A_1(x) = \frac{2\kappa_d^2 V(\phi, s_i)}{(d-2)} + c, \tag{8.12}$$

$$B_1(x) = -\frac{\kappa_d^2(d-1)}{2(d-2)}V(\phi, s_i) - \frac{1}{4}cd, \tag{8.13}$$

$$8B_2 + 2dA_2 = d\left(A_1^2 + \frac{8B_1^2}{d-2}\right), \tag{8.14}$$

where c is an arbitrary constant. Hence the prolongations are determined by the shadow fields; however one combination of B_2, A_2 remains arbitrary.

Thus, we find that there are not sufficient equations to determine all of the degrees of freedom $g^{\nu}_{1\mu}$, $g^{\nu}_{2\mu}$, ϕ_1, ϕ_2, s_{1i}, and s_{2i} that participated in the dynamics at $w = 0$. This is a sign that there are gauge symmetries, so what cannot be determined by the equations of motion must be a gauge degree of freedom, at least on shell. We did identify an off-shell gauge symmetry, namely, the $\Lambda_n(x)$ in Eqs. (6.28)–(6.31) which is sufficient to explain why one function is gauge freedom in the example above, but the evidence is that there is more gauge freedom. In fact more gauge symmetry should be expected as in flat 2T-field theory [2], where in the expansion in powers of w of *matter fields* each coefficient except the zeroth order (i.e. each prolongation) is a gauge degree of freedom. In flat 2T field theory the prolongations decoupled completely from the shadow fields in flat space [2] consistent with being gauge freedom. However, what we have learned in this paper is that there are also some that, rather than being gauge freedom, are actually determined by the shadow fields via the geometry in curved space $g^{\nu}_{1\mu}$, $g^{\nu}_{2\mu}$ as seen in Eqs. (8.8)–(8.10).

In any case, an outcome of our analysis is that there are nontrivial prolongations which are determined by the shadow fields ϕ, s_i, and $g_{\mu\nu}$ up to gauge freedom. However, the shadow fields themselves ϕ, s_i, and $g_{\mu\nu}$ are determined self-consistently by the action (2.7) only within the shadow, as in Eqs. (8.3), independently of the prolongations.

IX. CONCLUSIONS

The decoupling of the dynamics of the shadow proven in the previous section is significant because it shows that general relativity in d dimensions, augmented with the Weyl symmetry, as expressed by the action (2.7), is the prediction of 2T gravity for observers asking questions only in d dimensions. Establishing this effective action principle, by analyzing the equations of motion in detail as we did above, was one of the aims of our analysis.

This shows that the full physical (gauge invariant) information in $(d + 2)$ dimensions is captured by the conformal shadow, so this is a "holographic" shadow. Turning this around, we can also claim that usual general relativity in d dimensions, augmented with the Weyl symmetry, is described directly in $d + 2$ dimensions in the form of 2T gravity.

We have shown quite generally that the Weyl symmetry in 1T field theory is directly related to higher spacetime general coordinate transformations that include an extra time dimension. Therefore local Weyl symmetry is a strong footprint of 2T physics. Just like other gauge symmetries, there are observable effects of the structure that this symmetry imposes on interactions.

As we have shown, as a consequence of 2T gravity, the graviton and the scalars must satisfy certain structures in 1T field theory. Dirac and Yang-Mills fields can be included in a straightforward way except for inserting the dilaton factors of $\phi^{2(d-4)/(d-2)}$ in Yang-Mills kinetic terms and $\phi^{-(d-4)/(d-2)}$ in Yukawa terms (as in Table I). With these dilaton factors the crucial Weyl symmetry is intact in every dimension d. These are some of the footprints of 2T gravity.

Some of the consequences of the emergent structures imposed by 2T gravity were outlined in the Introduction and Sec. II. Investigations of physical effects in the context of cosmology and LHC physics are currently in progress and will appear in future publications [21].

ACKNOWLEDGMENTS

We thank Yueh-Cheng Kuo, Guillaume Quelin, and Bora Orcal at USC for helpful discussions. This work was partially supported by the U.S. Department of Energy under Grant No. DE-FG03-84ER40168.

[1] I. Bars, Phys. Rev. D **77**, 125027 (2008).
[2] I. Bars, Phys. Rev. D **74**, 085019 (2006). For a summary see AIP Conf. Proc. **903**, 550 (2007).
[3] I. Bars and Y. C. Kuo, Phys. Rev. Lett. **99**, 041801 (2007); Phys. Rev. D **76**, 105028 (2007).
[4] I. Bars and Y. C. Kuo, Phys. Rev. D **79**, 025001 (2009).
[5] I. Bars, S. H. Chen, and G. Quelin, Phys. Rev. D **76**, 065016 (2007).
[6] I. Bars and G. Quelin (to be published).
[7] I. Bars, and G. Quelin, Phys. Rev. D **77**, 125019 (2008).
[8] B. Zumino, in *Lectures on Elementary Particles and Quantum Field Theory*, 1970 Brandeis Summer School, edited by S. Deser *et al.* (MIT Press, Cambridge, MA, 1970).
[9] C. Callan, S. Coleman, and R. Jackiw, Ann. Phys. (N.Y.) **59**, 42 (1970).
[10] A. Iorio, L. O'Raifeartaigh, I. Sachs, and C. Wiesendanger, Nucl. Phys. **B495**, 433 (1997).
[11] R. Jackiw, Teor. Mat. Fiz. **148**, 80 (2006) [Theor. Math. Phys. **148**, 941 (2006)].
[12] A. Salam and J. Strathdee, Phys. Rev. **184**, 1760 (1969).
[13] C. J. Isham, A. Salam, and J. Strathdee, Phys. Lett. **31B**, 300 (1970).
[14] C. J. Isham, A. Salam, and J. Strathdee, Ann. Phys. (N.Y.) **62**, 98 (1971).
[15] J. R. Ellis, Nucl. Phys. **B22**, 478 (1970); **B26**, 536 (1971).
[16] R. H. Dicke, Phys. Rev. **126**, 1875 (1962).
[17] S. Deser, Gen. Relativ. Gravit. **1**, 131 (1970).
[18] C. G. Callan, S. R. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2247 (1969).
[19] S. R. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2239 (1969).
[20] R. Sundrum, arXiv:hep-th/0312212.
[21] I. Bars and S. H. Chen (to be published).
[22] We use standard conventions for the Christoffel connections and Riemann curvatures, see. e.g., S. M. Carroll, *Spacetime and Geometry: An Introduction to General Relativity* (Addison-Wesley, New York, 2004).
[23] I. Bars and Y.-C. Kuo, Phys. Rev. D **74**, 085020 (2006).