Fermionic Casimir densities in toroidally compactified spacetimes with applications to nanotubes

S. Bellucci[1,*] and A. A. Saharian[2,†]

[1]*INFN, Laboratori Nazionali di Frascati, Via Enrico Fermi 40, 00044 Frascati, Italy*

[2]*Department of Physics, Yerevan State University, 1 Alex Manoogian Street, 0025 Yerevan, Armenia*

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Fermionic condensate and the vacuum expectation values of the energy-momentum tensor are investigated for a massive spinor field in higher-dimensional spacetimes with an arbitrary number of toroidally compactified spatial dimensions. By using the Abel-Plana summation formula and the zeta function technique we present the vacuum expectation values in two different forms. Applications of the general formulas to cylindrical and toroidal carbon nanotubes are given. We show that the topological Casimir energy is positive for metallic cylindrical nanotubes and is negative for semiconducting ones. The toroidal compactification of a cylindrical nanotube along its axis increases the Casimir energy for metallic-type (periodic) boundary conditions along its axis and decreases the Casimir energy for the semiconducting-type compactifications.

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I. INTRODUCTION

Many of high energy theories of fundamental physics, including supergravity and superstring theories, are formulated in spacetimes having compact spatial dimensions. From an inflationary point of view, universes with compact dimensions, under certain conditions, should be considered a rule rather than an exception [1]. The models of a compact universe with nontrivial topology may play an important role by providing proper initial conditions for inflation. There are many reasons to expect that in string theory the most natural topology for the universe is that of a flat compact three-manifold [2]. The quantum creation of the universe having toroidal spatial topology is discussed in [3] and in Ref. [4] within the framework of various supergravity theories. An interesting application of the quantum field theoretical models with nontrivial topology of spatial dimensions recently appeared in nanophysics [5]. In a sheet of hexagons from the graphite structure, known as graphene, the long-wavelength description of the electronic states can be formulated in terms of the Dirac-like theory of massless spinors in three-dimensional spacetime with the Fermi velocity playing the role of speed of light (see, e.g., Ref. [6]). Single-walled carbon nanotubes are generated by rolling up a graphene sheet to form a cylinder and the background spacetime for the corresponding Dirac-like theory has topology $R^2 \times S^1$. Compactifying the direction along the cylinder axis we obtain another class of graphene-made structures called toroidal carbon nanotubes with the background topology $R^1 \times (S^1)^2$.

The compactification of spatial dimensions leads to a number of interesting quantum field theoretical effects which include instabilities in interacting field theories [7], topological mass generation [8], and symmetry breaking [9]. In the case of nontrivial topology, the boundary conditions imposed on fields give rise to the modification of the spectrum for vacuum fluctuations and, as a result, to the Casimir-type contributions in the vacuum expectation values of physical observables (for the topological Casimir effect and its role in cosmology, see [10–14] and references therein). The Casimir effect is common to all systems characterized by fluctuating quantities and has important implications on all scales, from cosmological to subnuclear. In the Kaluza-Klein-type models this effect has been used as a stabilization mechanism for moduli fields which parametrize the size and the shape of the extra dimensions. The Casimir energy can also serve as a model for dark energy needed for the explanation of the present accelerated expansion of the universe (see [15] and references therein). In addition to its fundamental interest, the Casimir effect also plays an important role in the fabrication and operation of nano- and micro-scale mechanical systems (see, for instance, [16]) and has become an increasingly popular topic in quantum field theory.

The effects of the toroidal compactification of spatial dimensions on the properties of quantum vacuum for various spin fields have been discussed by several authors (see, for instance, [4,10–14,17,18] and references therein). In the present paper, we investigate one-loop quantum effects arising from vacuum fluctuations of a massive fermionic field on background of higher-dimensional spacetimes with an arbitrary number of toroidally compactified spatial dimensions. We will assume generalized periodicity conditions along compactified dimensions with arbitrary phases. Important quantities that characterize the quantum fluctuations are the fermionic condensate and the expectation value of the energy-momentum tensor. In the next section, by using the Abel-Plana summation formula, we derive a recurrence formula relating the fermionic condensates in topologies $R^p \times (S^1)^q$ and $R^{p+1} \times (S^1)^{q-1}$. An alternative expression for the topological part in the fermi-

*bellucci@lnf.infn.it
†saharian@ictp.it

onic condensate is obtained by using the zeta function technique. In Sec. III, we consider the corresponding formulas for the vacuum expectation values of the energy-momentum tensor. In Sec. IV, we give applications of general formulas to the Casimir effect for electrons in a carbon nanotube within the framework of a three-dimensional Dirac-like model. The main results of the paper are summarized in Sec. V. In the appendix, we show the equivalence of two representations for the vacuum expectation values obtained by the Abel-Plana summation formula and by the zeta function method.

II. FERMIONIC CONDENSATE

We consider a quantum fermionic field on background of $(D+1)$-dimensional flat spacetime with spatial topology $R^p \times (S^1)^q$, $p+q=D$. The corresponding line element has the form

$$ds^2 = dt^2 - \sum_{l=1}^{D}(dz^l)^2, \tag{1}$$

where $-\infty < z^l < \infty$, $l=1,\ldots,p$, and $0 \leq z^l \leq L_l$ for $l=p+1,\ldots,D$. The dynamics of the field is governed by the Dirac equation

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0. \tag{2}$$

In the $(D+1)$-dimensional spacetime the Dirac matrices are $N \times N$ matrices with $N = 2^{[(D+1)/2]}$, where the square brackets mean the integer part of the enclosed expression. We will assume that these matrices are given in the chiral representation:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^\mu = \begin{pmatrix} 0 & \sigma_\mu \\ -\sigma_\mu^+ & 0 \end{pmatrix}, \qquad \mu = 1,2,\ldots,D, \tag{3}$$

with the relation $\sigma_\mu \sigma_\nu^+ + \sigma_\nu \sigma_\mu^+ = 2\delta_{\mu\nu}$. For example, in $D=4$ the first four matrices γ^μ, $\mu = 0, 1, 2, 3$, can be taken the same as the corresponding matrices in four-dimensional spacetime and $\gamma^4 = \gamma^0\gamma^1\gamma^2\gamma^3$. In this case, σ_1, σ_2, σ_3 are the standard Pauli matrices and

$$\sigma_4 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}. \tag{4}$$

Note that, unlike to the Pauli matrices, σ_4 is anti-Hermitian.

In this paper, we are interested in the effects of nontrivial topology on the vacuum expectation values (VEVs) of the energy-momentum tensor and the fermionic condensate, assuming that along the compactified dimensions the field obeys the boundary conditions (no summation over $l = p+1,\ldots,D$)

$$\psi(t, \mathbf{z}_p, \mathbf{z}_q + L_l \mathbf{e}_l) = e^{2\pi i \alpha_l}\psi(t, \mathbf{z}_p, \mathbf{z}_q), \tag{5}$$

with constant phases α_l. In (5), $\mathbf{z}_p = (z^1,\ldots,z^p)$ and $\mathbf{z}_q = (z^{p+1},\ldots,z^D)$ denote the coordinates along uncompactified and compactified dimensions, respectively; $\mathbf{e}_l$ is the unit vector along the direction of the coordinate z^l. First we consider the fermionic condensate.

For the topology under consideration, we will denote the fermionic condensate $\langle 0|\bar{\psi}\psi|0\rangle$ (with $|0\rangle$ being the amplitude for the vacuum state) by $\langle\bar{\psi}\psi\rangle_{p,q}$. We expand the field operator in terms of the complete set of positive and negative frequency eigenfunctions $\{\psi_\beta^{(+)}, \psi_\beta^{(-)}\}$:

$$\hat{\psi} = \sum_\beta [\hat{a}_\beta \psi_\beta^{(+)} + \hat{b}_\beta^+ \psi_\beta^{(-)}], \tag{6}$$

where $\hat{a}_\beta$ is the annihilation operator for particles, and $\hat{b}_\beta^+$ is the creation operator for antiparticles. By using the commutation relations for these operators, the condensate is presented in the form of the mode sum

$$\langle\bar{\psi}\psi\rangle_{p,q} = \sum_\beta \bar{\psi}_\beta^{(-)}(x)\psi_\beta^{(-)}(x). \tag{7}$$

In order to evaluate the condensate with this formula, we need the explicit form of the eigenfunctions satisfying the boundary conditions (5).

In accordance with the problem symmetry the dependence of these functions on the spacetime coordinates can be taken in the plane-wave form $e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$, $\omega = \sqrt{k^2+m^2}$, with the wave vector $\mathbf{k}$. From the Dirac equation we find

$$\psi_\beta^{(+)} = \frac{e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}}{(2^{p+1}\pi^p V_q \omega)^{1/2}} \begin{pmatrix} w_\sigma^{(+)}\sqrt{\omega+m} \\ (\mathbf{n}\cdot\boldsymbol{\sigma})w_\sigma^{(+)}\sqrt{\omega-m} \end{pmatrix}, \tag{8}$$

$$\psi_\beta^{(-)} = \frac{e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega t}}{(2^{p+1}\pi^p V_q \omega)^{1/2}} \begin{pmatrix} (\mathbf{n}\cdot\boldsymbol{\sigma})w_\sigma^{(-)}\sqrt{\omega-m} \\ w_\sigma^{(-)}\sqrt{\omega+m} \end{pmatrix}, \tag{9}$$

where $\beta = (\mathbf{k}, \sigma)$, $\mathbf{n} = \mathbf{k}/k$, and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_D)$; $V_q = L_{p+1}\cdots L_D$ is the volume of the compactified subspace. In these expressions $w_\sigma^{(+)}$, $\sigma = 1,\ldots,N/2$, are one-column matrices having $N/2$ rows with the elements $w_l^{(\sigma)} = \delta_{l\sigma}$ and $w_\sigma^{(-)} = iw_\sigma^{(+)}$. The eigenfunctions (8) and (9) are normalized in accordance with the condition

$$\int d^D x \psi_\beta^{(\pm)+}\psi_{\beta'}^{(\pm)} = \delta_{\beta\beta'}. \tag{10}$$

In the discussion below we will decompose the wave vector into components along the uncompactified and compactified dimensions: $\mathbf{k} = (\mathbf{k}_p, \mathbf{k}_q)$, $k = \sqrt{\mathbf{k}_p^2 + \mathbf{k}_q^2}$. The eigenvalues for the components along the compactified dimensions are determined from the boundary conditions (5):

$$\mathbf{k}_q = (2\pi(n_{p+1}+\alpha_{p+1})/L_{p+1},\ldots,2\pi(n_D+\alpha_D)/L_D),$$
$$n_{p+1},\ldots,n_D = 0, \pm 1, \pm 2, \ldots. \tag{11}$$

For the components along the uncompactified dimensions one has $-\infty < k_l < \infty$, $l = 1,\ldots,p$.

Substituting the eigenfunctions (9) into formula (7), for the fermionic condensate we find the expression

$$\langle\bar{\psi}\psi\rangle_{p,q} = -\frac{mN}{2^{p+1}\pi^p V_q}\int d\mathbf{k}_p \sum_{\mathbf{n}_q\in\mathbf{Z}^q}\frac{1}{\omega}, \quad (12)$$

with $\mathbf{n}_q = (n_{p+1}, \ldots, n_D)$ and

$$\omega^2 = \mathbf{k}_p^2 + \sum_{l=p+1}^{D}[2\pi(n_l+\alpha_l)/L_l]^2 + m^2. \quad (13)$$

We implicitly assume the presence of a cutoff function in (12) which makes the integrosum finite.

For the further evaluation of formula (12) we apply to the sum over n_{p+1} the Abel-Plana summation formula in the form [19]

$$\sum_{n_{p+1}=-\infty}^{+\infty} f(|n_{p+1}+\alpha_{p+1}|) = 2\int_0^\infty dx f(x) + i\int_0^\infty dx\sum_{\lambda=\pm 1}\frac{f(ix)-f(-ix)}{e^{2\pi(x+i\lambda\alpha_{p+1})}-1}. \quad (14)$$

As a result, the fermionic condensate is presented in the decomposed form

$$\langle\bar{\psi}\psi\rangle_{p,q} = \langle\bar{\psi}\psi\rangle_{p+1,q-1} + \Delta_{p+1}\langle\bar{\psi}\psi\rangle_{p,q}, \quad (15)$$

where $\langle\bar{\psi}\psi\rangle_{p+1,q-1}$ corresponds to the first term on the right-hand side of (14) and is the fermionic condensate for the topology $R^{p+1}\times(S^1)^{q-1}$. The second term on the right-hand side of formula (15) is induced by the compactness of the z^{p+1} direction and is given by the formula

$$\Delta_{p+1}\langle\bar{\psi}\psi\rangle_{p,q} = -\frac{2^{-1-p}mNL_{p+1}}{\pi^{(p+1)/2}\Gamma((p+1)/2)V_q}\sum_{\mathbf{n}_{q-1}\in\mathbf{Z}^{q-1}}\sum_{\lambda=\pm1}\int_{\omega_{\mathbf{n}_{q-1}}}^\infty du \times\frac{(u^2-\omega_{\mathbf{n}_{q-1}}^2)^{(p-1)/2}}{e^{L_{p+1}u+2\pi i\lambda\alpha_{p+1}}-1}, \quad (16)$$

where $\mathbf{n}_{q-1} = (n_{p+2}, \ldots, n_D)$ and

$$\omega_{\mathbf{n}_{q-1}}^2 = \sum_{l=p+2}^{D}[2\pi(n_l+\alpha_l)/L_l]^2 + m^2. \quad (17)$$

Note that the expression on the right-hand side of (16) is finite and the introduction of the cutoff function is necessary in the first term on the right-hand side of (15) only.

Expanding the function $1/(e^y-1)$ in the integrand of formula (16), we find an alternative form:

$$\Delta_{p+1}\langle\bar{\psi}\psi\rangle_{p,q} = -\frac{2NmL_{p+1}}{(2\pi)^{p/2+1}V_q}\sum_{n=1}^\infty\cos(2\pi n\alpha_{p+1}) \times\sum_{\mathbf{n}_{q-1}\in\mathbf{Z}^{q-1}}\omega_{\mathbf{n}_{q-1}}^p f_{p/2}(nL_{p+1}\omega_{\mathbf{n}_{q-1}}), \quad (18)$$

with the notation $f_\nu(x) = K_\nu(x)/x^\nu$. From here it follows that in the case of the periodic boundary condition along the direction z^{p+1} ($\alpha_{p+1}=0$) the contribution to the fermionic condensate due to the compactness of the corresponding direction is always negative independently of the boundary conditions along the other directions. In the limit when the length of one of the compactified dimensions, say z^l, $l\geq p+2$, is large, the main contribution into the sum over n_l in (18) comes from large values of n_l, and in the leading order we can replace the summation by the integration in accordance with

$$\frac{1}{L_l}\sum_{n_l=-\infty}^{+\infty} f(2\pi|n_l+\alpha_l|/L_l)\to\frac{1}{\pi}\int_0^\infty dy f(y).$$

The integral over y is evaluated by using the formula

$$\frac{1}{\pi}\int_0^\infty dy(y^2+b^2)^{p/2}f_{p/2}(c\sqrt{y^2+b^2}) = \frac{b^{p+1}}{\sqrt{2\pi}}f_{(p+1)/2}(cb), \quad (19)$$

and from (18) the corresponding formula is obtained for the topology $R^{p+1}\times(S^1)^{q-1}$. In the limit $L_l \ll L_{p+1}$, $l = p+2, \ldots, D$, the main contribution into the topological part (18) comes from the term with $\mathbf{n}_{q-1}=0$, and in the leading order we have

$$\Delta_{p+1}\langle\bar{\psi}\psi\rangle_{p,q} \approx -\frac{2Nm^{p+1}L_{p+1}}{(2\pi)^{p/2+1}V_q}\times\sum_{n=1}^\infty\cos(2\pi n\alpha_{p+1})f_{p/2}(nL_{p+1}m). \quad (20)$$

As we could expect, for large masses, $mL_{p+1}\gg 1$, the fermionic condensate given by formula (18) is exponentially suppressed.

After the recurring application of formula (18), the topological part of the fermionic condensate for spatial topology $R^p\times(S^1)^q$ is presented in the form

$$\langle\bar{\psi}\psi\rangle_{p,q} = \sum_{j=p}^{D-1}\Delta_{j+1}\langle\bar{\psi}\psi\rangle_{j,D-j}. \quad (21)$$

For a massless field the fermionic condensate vanishes.

An alternative form for the topological part in the fermionic condensate is obtained by making use of the zeta function technique [11,20]. We introduce the zeta function density

$$\zeta(s) = \frac{1}{V_q}\int\frac{d\mathbf{k}_p}{(2\pi)^p}\sum_{\mathbf{n}_q\in\mathbf{Z}^q}\frac{1}{\omega^{2s}}, \quad (22)$$

with ω defined by relation (13). In the case $\alpha_l=0$, $m=0$, the point $\mathbf{n}_q=0$ is to be excluded from the sum. After the integration over $\mathbf{k}_p$, this function is presented in the form

$$\zeta(s) = \frac{\Gamma(s-p/2)}{(4\pi)^{p/2}\Gamma(s)V_q}\sum_{\mathbf{n}_q\in\mathbf{Z}^q} \times\left\{\sum_{l=p+1}^{D}[2\pi(n_l+\alpha_l)/L_l]^2+m^2\right\}^{p/2-s}. \quad (23)$$

An exponentially convergent expression for the analytic continuation of the function (23) is given by the generalized Chowla-Selberg formula [21]. The application of this formula to Eq. (23) gives

$$\zeta(s) = \zeta_{\rm M}(s) + \zeta_{p,q}(s), \quad (24)$$

where

$$\zeta_{\rm M}(s) = \int\frac{d\mathbf{k}_D}{(2\pi)^D}\frac{1}{(k_D^2+m^2)^s} = \frac{m^{D-2s}}{(4\pi)^{D/2}}\frac{\Gamma(s-D/2)}{\Gamma(s)}, \quad (25)$$

is the corresponding zeta function in the usual Minkowski spacetime and the part

$$\zeta_{p,q}(s) = \frac{2^{1-s}m^{D-2s}}{(2\pi)^{D/2}\Gamma(s)}\sideset{}{'}\sum_{\mathbf{m}_q\in\mathbf{Z}^q}\cos(2\pi\mathbf{m}_q\cdot\boldsymbol{\alpha}_q) \times f_{D/2-s}(mg(\mathbf{L}_q,\mathbf{m}_q)), \quad (26)$$

with $\mathbf{L}_q = (L_{p+1},\ldots,L_D)$ and $\boldsymbol{\alpha}_q = (\alpha_{p+1},\ldots,\alpha_D)$, is induced by the nontrivial topology. The prime on the summation sign in (26) means that the term $\mathbf{m}_q = 0$ should be excluded from the sum and we have used the notation

$$g(\mathbf{L}_q,\mathbf{m}_q) = \left(\sum_{i=p+1}^{D}L_i^2m_i^2\right)^{1/2}. \quad (27)$$

The topological part in (24) is an analytic function at the physical point $s = 1/2$ and for the fermionic condensate one directly finds

$$\langle\bar\psi\psi\rangle_{p,q} = -\frac{mN}{2}\zeta_{p,q}(1/2) = -\frac{Nm^D}{(2\pi)^{(D+1)/2}}\sideset{}{'}\sum_{\mathbf{m}_q\in\mathbf{Z}^q}\cos(2\pi\mathbf{m}_q\cdot\boldsymbol{\alpha}_q) \times f_{(D-1)/2}(mg(\mathbf{L}_q,\mathbf{m}_q)). \quad (28)$$

In the case $p = D-1$, $q = 1$ this formula coincides with (18). In the appendix we prove the equivalence of two representations (21) and (28) for the topological part in the fermionic condensate for general case. Note that in (28) we can write the function $\cos(2\pi\mathbf{m}_q\cdot\boldsymbol{\alpha}_q)$ in the form of the product $\prod_{i=p+1}^{D}\cos(2\pi m_i\alpha_i)$.

III. ENERGY-MOMENTUM TENSOR

In order to find the VEV for the operator of the energy-momentum tensor, we substitute the expansion (6) and the analog expansion for the operator $\hat{\bar\psi}$ into the corresponding expression for spinor fields,

$$T_{\mu\nu}\{\hat{\bar\psi},\hat\psi\} = \frac{i}{2}[\hat{\bar\psi}\gamma_{(\mu}\partial_{\nu)}\hat\psi - (\partial_{(\mu}\hat{\bar\psi})\gamma_{\nu)}\hat\psi]. \quad (29)$$

Similar to the case of the fermionic condensate, by making use of the commutation relations for the annihilation and creation operators, one finds the following mode-sum formula:

$$\langle 0|T_{\mu\nu}|0\rangle = \langle T_{\mu\nu}\rangle_{p,q} = \sum_\beta T_{\mu\nu}\{\bar\psi_\beta^{(-)}(x),\psi_\beta^{(-)}(x)\}. \quad (30)$$

Substituting the eigenfunctions (9) into this mode-sum formula, for the energy density and vacuum stresses one finds (no summation over $l = 1,\ldots,D$)

$$\langle T_0^0\rangle_{p,q} = -\frac{N}{2(2\pi)^pV_q}\int d\mathbf{k}_p\sum_{\mathbf{n}_q\in\mathbf{Z}^q}\omega, \quad (31)$$

$$\langle T_l^l\rangle_{p,q} = \frac{N}{2(2\pi)^pV_q}\int d\mathbf{k}_p\sum_{\mathbf{n}_q\in\mathbf{Z}^q}\frac{k_l^2}{\omega}. \quad (32)$$

As in the case of the fermionic condensate, we will assume that some cutoff function is present, without writing it explicitly.

After the application of summation formula (14) to the series over n_{p+1}, we receive the following recurrence relation:

$$\langle T_\mu^\nu\rangle_{p,q} = \langle T_\mu^\nu\rangle_{p+1,q-1} + \Delta_{p+1}\langle T_\mu^\nu\rangle_{p,q}, \quad (33)$$

where $\langle T_\mu^\nu\rangle_{p+1,q-1}$ is the VEV of the energy-momentum tensor for the topology $R^{p+1}\times(S^1)^{q-1}$. The part $\Delta_{p+1}\langle T_\mu^\nu\rangle_{p,q}$ is induced by the compactness of the z^{p+1} direction and is given by the expression (no summation over l)

$$\Delta_{p+1}\langle T_l^l\rangle_{p,q} = \frac{(4\pi)^{-(p+1)/2}NL_{p+1}}{\Gamma((p+1)/2)V_q}\sum_{\mathbf{n}_{q-1}\in\mathbf{Z}^{q-1}}\sum_{\lambda=\pm1}\int_{\omega_{\mathbf{n}_{q-1}}}^\infty du \times\frac{f^{(l)}(u)(u^2-\omega_{\mathbf{n}_{q-1}}^2)^{(p-1)/2}}{e^{L_{p+1}u+2\pi i\lambda\alpha_{p+1}}-1}, \quad (34)$$

with the notations

$$f^{(l)}(u) = \frac{4(u^2-\omega_{\mathbf{n}_{q-1}}^2)}{p+1}, \quad l = 0,1,\ldots,p, \quad f^{(p+1)}(u) = -2u^2, \quad f^{(l)}(u) = k_l^2, \quad l = p+2,\ldots D. \quad (35)$$

Expanding the integrand, this expression can also be presented in the form (no summation over l)

$$\Delta_{p+1}\langle T_l^l\rangle_{p,q} = \frac{2NL_{p+1}}{(2\pi)^{p/2+1}V_q}\sum_{\mathbf{n}_{q-1}\in\mathbf{Z}^{q-1}}\sum_{n=1}^\infty\cos(2\pi n\alpha_{p+1}) \times\omega_{\mathbf{n}_{q-1}}^{p+2}F^{(l)}(nL_{p+1}\omega_{\mathbf{n}_{q-1}}), \quad (36)$$

with the notations

$$F^{(0)}(z) = F^{(l)}(z) = f_{p/2+1}(z), \qquad l = 1, \ldots, p,$$
$$F^{(p+1)}(z) = -f_{p/2}(z) - (p+1)f_{p/2+1}(z),$$
$$F^{(l)}(z) = (k_l/\omega_{\mathbf{n}_{q-1}})^2 f_{p/2}(z), \qquad l = p+2, \ldots, D. \tag{37}$$

It is easy to check that for a massless field the topological part (36) is traceless. As we see the vacuum stresses along the uncompactified dimensions are equal to the energy density. Of course, this property is a direct consequence of the boost invariance along the corresponding directions. In particular, from (36) it follows that in the case of periodic boundary conditions along the coordinate z^{p+1} ($\alpha_{p+1} = 0$), the compactification along this coordinate increases the vacuum energy density independently of the boundary conditions along the other directions. The limiting cases of general formulas for the VEV of the energy-momentum tensor are investigated in a way similar to that described before for the condensate.

From (33), for the VEV of the energy-momentum tensor in the topology $R^p \times (S^1)^q$ one finds

$$\langle T_\mu^\nu \rangle_{p,q} = \sum_{j=p}^{D-1} \Delta_{j+1} \langle T_\mu^\nu \rangle_{j,D-j}. \tag{38}$$

Now, by using the standard relations for the Mac-Donald function, it can be seen that the vacuum energy density and stresses along the compactified dimensions are related by the formula (no summation over l)

$$\partial_{L_l}(V_q \langle T_0^0 \rangle_{p,q}) = \frac{V_q}{L_l} \langle T_l^l \rangle_{p,q}, \qquad l = p+1, \ldots, D. \tag{39}$$

For the simplest Kaluza-Klein-type model with spatial topology $R^3 \times S^1$, from (36) for the energy density one finds ($L_{p+1} = L$, $\alpha_{p+1} = \alpha$)

$$\langle T_0^0 \rangle_{3,1} = \frac{1}{\pi^2 L^5} \sum_{n=1}^{\infty} \frac{\cos(2\pi n\alpha)}{n^5 e^{nmL}} [(nmL)^2 + 3nmL + 3]. \tag{40}$$

This quantity is positive for an untwisted field ($\alpha = 0$) and is negative for a twisted field ($\alpha = 1/2$). In the general case, the Casimir energy density is not a monotonic function of the size of the compactified dimension. This is seen from the left panel of Fig. 1 where we have plotted the quantity (40) as a function of the parameter mL for different values of the phase α (numbers near the curves). The values of the phase are chosen in a way to show the transition from the positive energies to negative ones. In the right panel of Fig. 1 we have presented the Casimir energy density (40) for a massless field as a function of the parameter α.

An alternative expression for the VEV of the energy density is obtained by using the integral representation of the corresponding zeta function given by (26):

$$\langle T_0^0 \rangle_{p,q} = -\frac{N}{2} \zeta_{p,q}(-1/2)$$
$$= \frac{N m^{D+1}}{(2\pi)^{(D+1)/2}} {\sum_{\mathbf{m}_q \in \mathbf{Z}^q}}' \cos(2\pi \mathbf{m}_q \cdot \boldsymbol{\alpha}_q)$$
$$\times f_{(D+1)/2}(mg(\mathbf{L}_q, \mathbf{m}_q)). \tag{41}$$

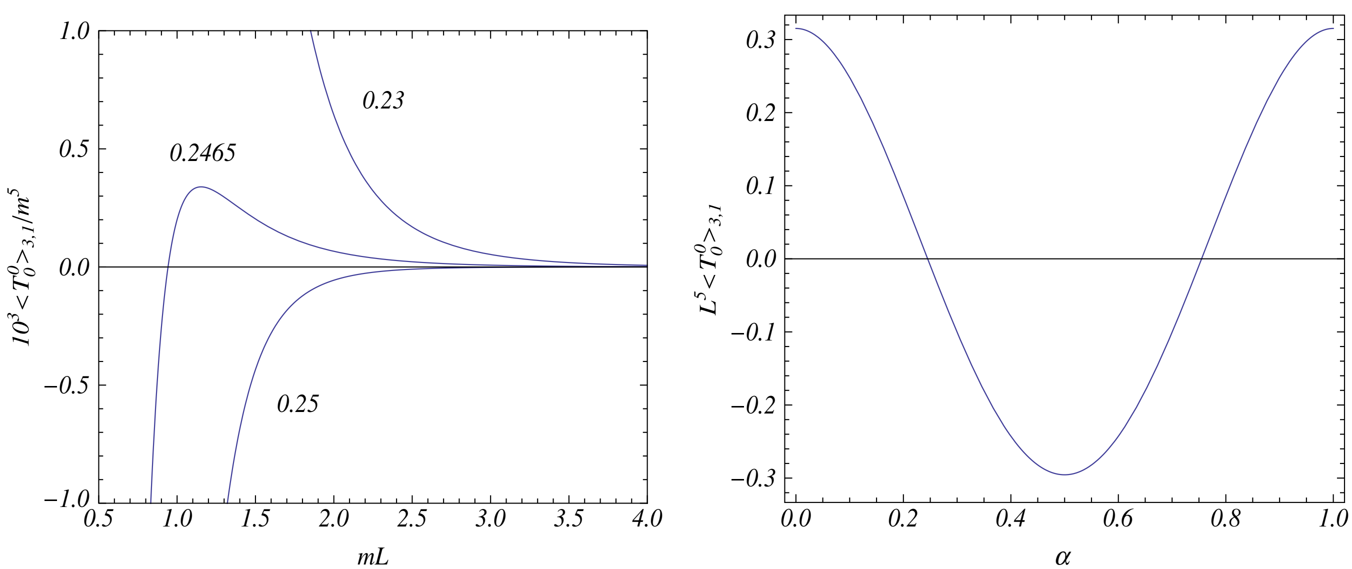

FIG. 1 (color online). The Casimir energy density in the Kaluza-Klein-type model with spatial topology $R^3 \times S^1$ as a function of the parameter mL for different values of α (left panel). The right panel presents the corresponding quantity for a massless field as a function of α.

The equivalence of the representations (38) and (41) for the energy density is seen in a way similar to that used in the appendix for the fermionic condensate. The corresponding formulas for the vacuum stresses along compactified dimensions are obtained from relations (39) (no summation over l):

$$\langle T_l^l\rangle_{p,q} = \langle T_0^0\rangle_{p,q} - \frac{Nm^{D+3}L_l^2}{(2\pi)^{(D+1)/2}} \sum_{\mathbf{m}_q\in\mathbf{Z}^q}{}' m_l^2 \cos(2\pi\mathbf{m}_q\cdot\boldsymbol{\alpha}_q) \times f_{(D+3)/2}(mg(\mathbf{L}_q,\mathbf{m}_q)), \tag{42}$$

with $l = p+1, \ldots, D$. A number of special cases of formula (41) for the Casimir energy can be found in literature (see [4,10–14,17]). For a massless fermionic field from (41) we find (no summation over l)

$$\langle T_0^0\rangle_{p,q} = N\frac{\Gamma((D+1)/2)}{2\pi^{(D+1)/2}} \sum_{\mathbf{m}_q\in\mathbf{Z}^q}{}' \frac{\cos(2\pi\mathbf{m}_q\cdot\boldsymbol{\alpha}_q)}{g^{D+1}(\mathbf{L}_q,\mathbf{m}_q)}, \tag{43}$$

$$\langle T_l^l\rangle_{p,q} = \langle T_0^0\rangle_{p,q} - N(D+1) \times \frac{\Gamma((D+1)/2)}{2\pi^{(D+1)/2}} \sum_{\mathbf{m}_q\in\mathbf{Z}^q}{}' L_l^2 m_l^2 \frac{\cos(2\pi\mathbf{m}_q\cdot\boldsymbol{\alpha}_q)}{g^{D+3}(\mathbf{L}_q,\mathbf{m}_q)}, \tag{44}$$

where $l = p+1, \ldots, D$. Note that for a massless field the representation (36) has stronger convergence than the one given by (43) and (44): the summand in (36) decays exponentially instead of the power-law decay in (43) and (44).

IV. APPLICATIONS TO NANOTUBES

In this section, we specify the general results given above for the electrons on a carbon sheet rolled into a cylinder or torus making use of the description of the electronic states in terms of Dirac fermion fields. In this case, $D = 2$ and we consider the geometries of cylindrical and toroidal nanotubes separately. Note that the Dirac-like model for electrons in a carbon nanotube is valid provided that the cylinder circumference is much larger than the interatomic spacing. For typical nanotubes the corresponding ratio can be between 10 and 20 and this approximation is adequate [5,6].

A. Cylindrical nanotubes

A single wall cylindrical nanotube is a graphene sheet rolled into a cylindrical shape. For this case we have spatial topology $R^1\times S^1$ with the compactified dimension of the length L. Note that the carbon nanotube is characterized by its chiral vector $\mathbf{C}_h = n_w\mathbf{a}_1 + m_w\mathbf{a}_2$, with n_w, m_w being integers, and $L = |\mathbf{C}_h| = a\sqrt{n_w^2 + m_w^2 + n_w m_w}$. In the expression for the chiral vector, $\mathbf{a}_1$ and $\mathbf{a}_2$ are the basis vectors of the hexagonal lattice of graphene and $a = |\mathbf{a}_1| = |\mathbf{a}_2| = 2.46$ Å is the lattice constant. A zigzag nanotube corresponds to the special case $\mathbf{C}_h = (n_w, 0)$, and an armchair nanotube corresponds to the case $\mathbf{C}_h = (n_w, n_w)$. All other cases correspond to chiral nanotubes. The electron properties of carbon nanotubes can be either metallic or semiconductor-like depending on the manner the cylinder is obtained from the graphene sheet. In the case $n_w - m_w = 3q_w$, $q_w\in Z$, the nanotube will be metallic and in the case $n_w - m_w \neq 3q_w$ the nanotube will be a semiconductor with an energy gap inversely proportional to the diameter. In particular, the armchair nanotube is metallic and the $(n_w, 0)$ zigzag nanotube is metallic if and only if n_w is an integer multiple of 3.

In order to see the boundary conditions along the compactified dimension, we note that for the (n_w, m_w) nanotube the phase factor in the wave function is in the form $e^{i[m_1+(n_w-m_w)/3]\varphi}$, $m_1\in Z$, where φ is the angular variable along the compact dimension. From here it follows that for metallic nanotubes we have periodic boundary conditions ($\alpha_l = 0$) and for semiconductor nanotubes, depending on the chiral vector, we have two classes of inequivalent boundary conditions corresponding to $\alpha_l = \pi/3$ ($n_w - m_w = 3q_w + 2$) and $\alpha_l = 2\pi/3$ ($n_w - m_w = 3q_w + 1$). In the expression for the Casimir densities the phases α_l appear in the form $\cos(2\pi n\alpha_l)$ and, hence, the Casimir energy density and stresses are the same for these two cases.

Using the tight-binding approximation it can be seen that the electronic band structure close to the Dirac points shows a conical dispersion $E(\mathbf{k}) = v_F|\mathbf{k}|$, where $\mathbf{k}$ is the momentum measured relatively to the Dirac points and v_F represents the Fermi velocity which plays the role of speed of light. The corresponding low-energy excitations can be described by a pair of two-component Weyl spinors, which are composed of the Bloch states residing on the two different sublattices of the honeycomb lattice of the graphene sheet. The corresponding Fermi velocity is given by $v_F = 3ta/2$ ($v_F \approx 10^8$ cm/s in graphene), where t is the nearest neighbor hopping energy. Below, in specifying the formulas from previous section for the case $D = 2$, we consider a massive spinor field to keep the discussion general. The formulas for a massless case, appropriate for carbon nanotubes, will be given separately.

In the case $D = 2$, the general formula for the fermionic condensate from Sec. II takes the form ($N = 2$, $p = 1$, $q = 1$, $V_q = L$, $L_{p+1}\equiv L$, $\alpha_{p+1}\equiv\alpha$)

$$\langle\bar\psi\psi\rangle_{1,1} = -\frac{m}{\pi L}S_\alpha(mL), \tag{45}$$

where we have defined

$$S_\alpha(x) = \sum_{n=1}^{+\infty}\cos(2\pi n\alpha)\frac{e^{-xn}}{n} = -\frac{1}{2}\ln[1 - 2e^{-x}\cos(2\pi\alpha) + e^{-2x}]. \tag{46}$$

In a similar way, for the VEV of the energy-momentum tensor from (36) we find (no summation over l)

$$\langle T_l^l \rangle_{1,1} = \frac{1}{\pi L^3} \sum_{n=1}^{\infty} \cos(2\pi n\alpha) G^{(l)}(nmL) \frac{e^{-nmL}}{n^3}, \tag{47}$$

with the notations

$$\begin{aligned} G^{(0)}(z) &= G^{(1)}(z) = 1 + z, \\ G^{(2)}(z) &= -(2 + 2z + z^2). \end{aligned} \tag{48}$$

In particular, for the energy density we have

$$\langle T_0^0 \rangle_{1,1} = \frac{1}{\pi L^3} S_\alpha^{(0)}(mL), \tag{49}$$

where the notation

$$S_\alpha^{(0)}(x) = \sum_{n=1}^{\infty} \cos(2\pi n\alpha) e^{-nx} \frac{1 + nx}{n^3} \tag{50}$$

is introduced. In Fig. 2, we have plotted the function $S_\alpha^{(0)}(x)$ for different values of α (numbers near the curves). In particular, the Casimir energy density is positive for armchair nanotubes (periodic boundary conditions).

In the case $m = 0$, we have

$$\langle T_0^0 \rangle_{1,1} = \langle T_1^1 \rangle_{1,1} = -\frac{1}{2} \langle T_2^2 \rangle_{1,1} = \frac{S_\alpha^{(0)}(0)}{\pi L^3}. \tag{51}$$

In particular, $S_0^{(0)}(0) = 1.202$, $S_{1/2}^{(0)}(0) = -0.902$, and $S_{1/3}^{(0)}(0) = -0.534$. Note that the corresponding fermionic condensate vanishes. In carbon nanotubes we have two sublattices and each of them gives the contribution to the Casimir densities given by (51). So, for the Casimir energy density on a carbon nanotube with radius L one has

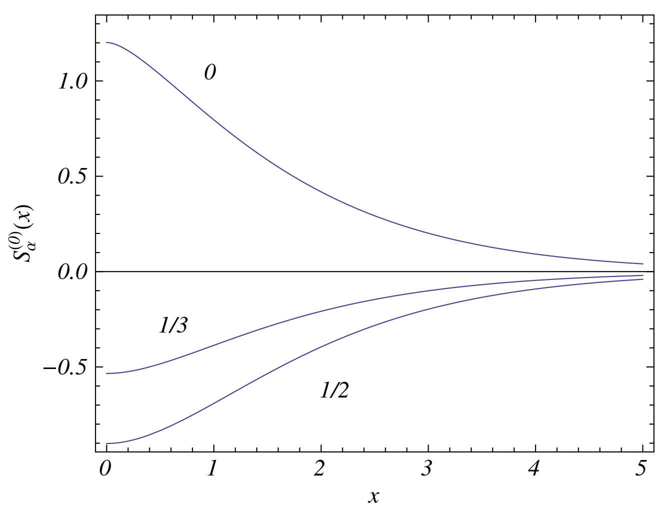

FIG. 2 (color online). The function $S_\alpha^{(0)}(x)$ from (50) for different values of the parameter α (numbers near the curves).

$$\langle T_0^0 \rangle_{1,1}^{(\mathrm{cn})} = \frac{2\hbar v_{\mathrm{F}}}{\pi L^3} S_\alpha^{(0)}(0), \tag{52}$$

where the standard units are restored. Hence, we see that the topological Casimir energy is positive for metallic nanotubes and is negative for semiconducting ones.

B. Toroidal nanotubes

For the geometry of a toroidal nanotube we have the spatial topology $(S^1)^2$ with $p = 0$ and $q = 2$. In this case, from the general formulas for the fermionic condensate we find

$$\begin{aligned} \langle \bar{\psi} \psi \rangle_{0,2} = &-\frac{m}{\pi} \sum_{j=1,2} \frac{S_{\alpha_j}(mL_j)}{L_j} \\ &- \frac{2m}{\pi} \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \cos(2\pi m_1 \alpha_1) \cos(2\pi m_2 \alpha_2) \\ &\times \frac{e^{-m\sqrt{m_1^2 L_1^2 + m_2^2 L_2^2}}}{\sqrt{m_1^2 L_1^2 + m_2^2 L_2^2}}, \end{aligned} \tag{53}$$

where the function $S_\alpha(x)$ is defined by (46).

The corresponding formulas for the energy density and the vacuum stresses have the form (no summation over l)

$$\begin{aligned} \langle T_0^0 \rangle_{0,2} = &\sum_{j=1,2} \frac{S_{\alpha_j}^{(0)}(mL_j)}{\pi L_j^3} + \frac{2}{\pi} \\ &\times \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{+\infty} \frac{\cos(2\pi m_1 \alpha_1) \cos(2\pi m_2 \alpha_2)}{\exp(mg(\mathbf{L}_2, \mathbf{m}_2))} \\ &\times \frac{1 + mg(\mathbf{L}_2, \mathbf{m}_2)}{g^3(\mathbf{L}_2, \mathbf{m}_2)}, \end{aligned} \tag{54}$$

$$\begin{aligned} \langle T_l^l \rangle_{0,2} = &\langle T_0^0 \rangle_{0,2} - \frac{m^5}{\pi} \sum_{j=1,2} \sum_{m_j=1}^{+\infty} \cos(2\pi m_j \alpha_j) L_l^2 m_l^2 \\ &\times \frac{3 + 3x + x^2}{x^5 e^x} \bigg|_{x = mL_j m_j} \\ &- \frac{2m^5}{\pi} \sum_{m_1=1}^{+\infty} \sum_{m_2=1}^{+\infty} \cos(2\pi m_1 \alpha_1) \cos(2\pi m_2 \alpha_2) \\ &\times L_l^2 m_l^2 \frac{3 + 3x + x^2}{x^5 e^x} \bigg|_{x = mg(\mathbf{L}_2, \mathbf{m}_2)}, \end{aligned} \tag{55}$$

with $l = 1$, 2 and $g(\mathbf{L}_2, \mathbf{m}_2) = \sqrt{m_1^2 L_1^2 + m_2^2 L_2^2}$. Alternative expressions for the topological parts are obtained from formulas (36) and (38). For a massless field we find

$$\langle T_0^0\rangle_{0,2} = \sum_{j=1,2} \frac{S_{\alpha_j}^{(0)}(0)}{\pi L_j^3} + \frac{2}{\pi} \times \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{+\infty} \frac{\cos(2\pi m_1\alpha_1)\cos(2\pi m_2\alpha_2)}{(m_1^2L_1^2 + m_2^2L_2^2)^{3/2}}, \quad (56)$$

$$\langle T_l^l\rangle_{0,2} = \langle T_0^0\rangle_{0,2} - \frac{3}{\pi}\sum_{j=1,2}\sum_{m_j=1}^{+\infty}\cos(2\pi m_j\alpha_j)\frac{L_l^2m_l^2}{L_j^5m_j^5} - \frac{6}{\pi} \times \sum_{m_1=1}^{+\infty}\sum_{m_2=1}^{+\infty} L_l^2m_l^2\frac{\cos(2\pi m_1\alpha_1)\cos(2\pi m_2\alpha_2)}{(m_1^2L_1^2 + m_2^2L_2^2)^{5/2}}. \quad (57)$$

In particular, it is of interest to see the difference of the Casimir densities between the toroidal (with radii L_1 and L_2) and cylindrical (with radius L_2) geometries of the carbon nanotube. For the condensate this difference is directly given by formula (18) and one has

$$\langle\bar\psi\psi\rangle_{0,2} = \langle\bar\psi\psi\rangle_{1,1} - \frac{2m}{\pi L_2}\sum_{n=1}^{\infty}\cos(2\pi n\alpha_1) \times \sum_{n_2=-\infty}^{+\infty} K_0(n(L_1/L_2)\sqrt{4\pi^2(n_2+\alpha_2)^2 + m^2L_2^2}). \quad (58)$$

The first term on the right-hand side of this formula is the condensate for the topology $R^1 \times S^1$ with the length of the compactified dimension L_2. A similar formula for the VEV of the energy-momentum tensor follows from (36) (no summation over l):

$$\langle T_l^l\rangle_{0,2} = \langle T_l^l\rangle_{1,1} + \frac{2}{\pi L_2^3}\sum_{n=1}^{\infty}\cos(2\pi n\alpha_1) \times \sum_{n_2=-\infty}^{+\infty}[4\pi^2(n_2+\alpha_2)^2 + m^2L_2^2] \times F^{(l)}(n(L_1/L_2)\sqrt{4\pi^2(n_2+\alpha_2)^2 + m^2L_2^2}), \quad (59)$$

where the functions $F^{(l)}(z)$ are given by expressions (37) with $p = 0$. The second terms on the right-hand sides of formulas (58) and (59) are induced by the compactification of the cylinder (with radius L_2) along its axis. In Fig. 3, we have plotted these terms for the energy density, $\Delta_1\langle T_0^0\rangle_{0,2}$ (left panel), and for the stress along the axis of the cylinder, $\Delta_1\langle T_1^1\rangle_{0,2}$ (right panel), for a massless fermionic field as functions of the ratio L_1/L_2. The numbers near the curves correspond to the values of (α_1, α_2). As we have mentioned before the values of the phase $\alpha_l = 0$, $1/3$ are realized in carbon nanotubes. The vacuum stress $\Delta_1\langle T_2^2\rangle_{0,2}$ is related to the quantities plotted in Fig. 3 by the zero trace condition for the energy-momentum tensor of a massless field.

The corresponding formulas for the Casimir densities in toroidal nanotubes, which we denote by $\langle T_l^l\rangle_{0,2}^{(\mathrm{tn})}$, are obtained from (56), (57), and (59) in the massless limit with an additional factor of 2 which takes into account the presence of two sublattices: $\langle T_l^l\rangle_{0,2}^{(\mathrm{tn})} = 2\langle T_l^l\rangle_{0,2}|_{m=0}$. In standard units, the factor $\hbar v_F$ appears as well. Note that if the chiral vector $\mathbf{C}_h$ is directed along the axis z^2 then one has $L_2 = |\mathbf{C}_h|$. The translational vector defining the unit cell, $\mathbf{T}$, is perpendicular to $\mathbf{C}_h$ and its components are related to the components of the chiral vector by the formula

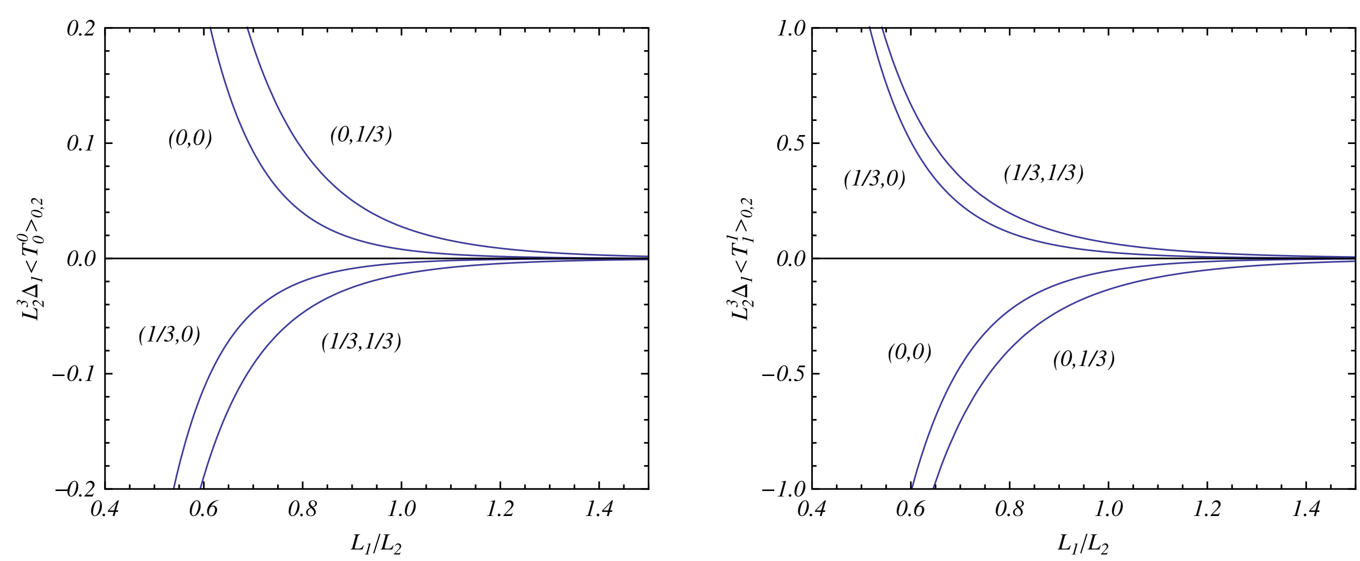

FIG. 3 (color online). The difference between the vacuum energy densities (left panel) and stresses (right panel) between the cylindrical (with radius L_2) and toroidal (with radii L_1 and L_2) geometries for a massless fermionic field. The numbers near the curves are the corresponding values for (α_1, α_2).

$$\mathbf{T} = \frac{n_w + 2m_w}{d_r}\mathbf{a}_1 - \frac{2n_w + m_w}{d_r}\mathbf{a}_2, \tag{60}$$

where $d_r = \gcd(n_w, m_w)$ if $(m_w - n_w)$ is not a multiple of $3 \times \gcd(n_w, m_w)$ and $d_r = 3 \times \gcd(n_w, m_w)$ if $(m_w - n_w)$ is a multiple of $3 \times \gcd(n_w, m_w)$. Here gcd means the greatest common divisor. Now for the length of the second toroidal dimension we have $L_1 = N_w|\mathbf{T}|$, where N_w is the number of unit cells along the corresponding direction. By taking into account that $|\mathbf{T}| = \sqrt{3}L_1/d_r$, for the ratio of the lengths of the torus in (59) one finds $L_1/L_2 = \sqrt{3}N_w/d_r$. From the graphs in Fig. 3 it follows that the toroidal compactification of a cylindrical nanotube along its axis increases the Casimir energy for periodic boundary conditions ($\alpha_1 = 0$) and decreases the Casimir energy for the semiconducting-type compactifications. In particular, the Casimir energy of the armchair cylindrical nanotube increases by the compactification if N_w is an integer multiple of 3 and decreases otherwise.

V. CONCLUSION

In the present paper we have investigated the topological Casimir effect for a massive spinor field on background of spacetime with an arbitrary number of toroidally compactified spatial dimensions. The boundary conditions along compactified dimensions are taken in general form with arbitrary phases. For the evaluation of the Casimir densities we have used the direct mode-summation method. By applying the Abel-Plana formula to the corresponding mode sums, we have derived recurrence formulas which relate the VEVs for the topologies $R^p \times (S^1)^q$ and $R^{p+1} \times (S^1)^{q-1}$. The part induced by the compactness of the $(p+1)$-th direction is given by expression (18) for the fermionic condensate and by expression (36) for the VEV of the energy-momentum tensor. The total topological VEVs are obtained after the summation over all compactified dimensions, formulas (21) and (38). Alternative expressions are obtained by using the generalized Chowla-Selberg formula for the analytic continuation of the corresponding zeta function. These expressions are given by formula (28) for the condensate and by formulas (41) and (42) for the energy density and vacuum stresses along compactified dimensions. Note that the stresses along the uncompactified dimensions coincide with the energy density. This property is a direct consequence of the boost invariance along the corresponding directions. For a massless fermionic field the condensate vanishes and the expressions for the VEVs of the energy density and vacuum stresses take the form (43) and (44). Note that, unlike in the case of a massive field, the convergence of the multiseries in the latter case is power law. In the representation based on the application of the Abel-Plana summation formula we have exponentially convergent multiseries in both cases of massive and massless fields. On the example of the simplest Kaluza-Klein-type model with spatial topology $R^3 \times S^1$ we have demonstrated that, unlike to the special cases of twisted and untwisted fields, in general, the Casimir energy density is not a monotonic function of the size of the internal space.

In Sec. IV, we specify the general formulas for the model with $D = 2$. This model may be used for the evaluation of the Casimir densities within the framework of the Dirac-like theory for the description of the electronic states in carbon nanotubes, where the role of speed of light is played by the Fermi velocity. Though the corresponding spinor field is massless, to keep the discussion general, we present the formulas for the cylindrical and toroidal geometries in the massive case and specify the results for the nanotubes separately. For carbon nanotubes the fermionic condensate vanishes and the VEV of the energy-momentum tensor is given by formula (52) for cylindrical nanotubes and by (56) and (57) (with an additional factor of 2 which takes into account the presence of two sublattices) for toroidal nanotubes. In the case of toroidal nanotubes, an alternative representation with the stronger convergence of the series is given by formula (59) with $m = 0$. The topological Casimir energy is positive for metallic cylindrical nanotubes and is negative for semiconducting ones. We have shown that the toroidal compactification of a cylindrical nanotube along its axis increases the Casimir energy for periodic boundary conditions and decreases the Casimir energy for the semiconducting-type compactifications. In particular, the Casimir energy of the armchair cylindrical nanotube increases by the compactification if the number of unit cells along the axis of cylinder is an integer multiple of 3 and decreases otherwise.

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APPENDIX: EQUIVALENCE OF TWO APPROACHES

In this section, we show that the formulas (21) and (28) for the topological part in the fermionic condensate are equivalent. First of all we note that from formula (28) one has

$$\langle \bar{\psi}\psi \rangle_{p,q} = \langle \bar{\psi}\psi \rangle_{p+1,q-1} - \frac{2Nm^D}{(2\pi)^{(D+1)/2}} \times \sum_{m_{p+1}=1}^{\infty} \cos(2\pi m_{p+1}\alpha_{p+1}) \times \sum_{\mathbf{m}_{q-1}\in\mathbf{Z}^{q-1}} \cos(2\pi \mathbf{m}_{q-1}\cdot\boldsymbol{\alpha}_{q-1}) \times f_{(D-1)/2}(mg(\mathbf{L}_q, \mathbf{m}_q)). \tag{A1}$$

Hence, we should prove the relation

$$\sum_{\mathbf{m}_{q-1}\in\mathbf{Z}^{q-1}} \cos(2\pi\mathbf{m}_{q-1}\cdot\boldsymbol{\alpha}_{q-1}) f_{(D-1)/2}(mg(\mathbf{L}_q,\mathbf{m}_q)) = \frac{(2\pi)^{(q-1)/2}L_{p+1}}{V_q m^{D-1}} \sum_{\mathbf{n}_{q-1}\in\mathbf{Z}^{q-1}} \omega^p_{\mathbf{n}_{q-1}} f_{p/2}(nL_{p+1}\omega_{\mathbf{n}_{q-1}}). \tag{A2}$$

For this we will use the Poisson's resummation formula

$$\sum_{\mathbf{m}_{q-1}\in\mathbf{Z}^{q-1}} F(\mathbf{x})\delta(\mathbf{x}-\mathbf{m}_{q-1}) = \sum_{\mathbf{n}_{q-1}\in\mathbf{Z}^{q-1}} F(\mathbf{x})e^{2i\pi\mathbf{n}_{q-1}\cdot\mathbf{x}}, \tag{A3}$$

for the function

$$F(\mathbf{x}) = \cos(2\pi\mathbf{x}\cdot\boldsymbol{\alpha}_{q-1}) \times f_{(D-1)/2}(m\sqrt{g^2(\mathbf{L}_{q-1},\mathbf{x}) + L^2_{p+1}m^2_{p+1}}). \tag{A4}$$

After the integration over $\mathbf{x}$ we find

$$\sum_{\mathbf{m}_{q-1}\in\mathbf{Z}^{q-1}} \cos(2\pi\mathbf{m}_{q-1}\cdot\boldsymbol{\alpha}_{q-1}) f_{(D-1)/2}(mg(\mathbf{L}_q,\mathbf{m}_q)) = \sum_{\mathbf{n}_{q-1}\in\mathbf{Z}^{q-1}} \int d\mathbf{x}\cos[2\pi\mathbf{x}\cdot(\boldsymbol{\alpha}_{q-1}+\mathbf{n}_{q-1})] \times f_{(D-1)/2}(m\sqrt{g^2(\mathbf{L}_{q-1},\mathbf{x}) + L^2_{p+1}m^2_{p+1}}). \tag{A5}$$

For the evaluation of the integral on the right-hand side we first introduce a new integration variable in accordance with $y_i = x_iL_i$ and then introduce spherical coordinates. The integration over the angular coordinates is expressed in terms of the Bessel function. At the final step the integral is evaluated by using the formula [22]

$$\int_0^\infty dyy^{\mu+1}J_\mu(by)f_\nu(c\sqrt{y^2+a^2}) = \frac{b^\mu}{c^{2\nu}}(b^2+c^2)^{\nu-\mu-1}f_{\nu-\mu-1}(a\sqrt{b^2+c^2}). \tag{A6}$$

This leads to the following result

$$\int d\mathbf{x}\cos(2\pi\mathbf{x}\cdot(\boldsymbol{\alpha}_{q-1}+\mathbf{n}_{q-1})) \times f_{(D-1)/2}(m\sqrt{g^2(\mathbf{L}_{q-1},\mathbf{x}) + L^2_{p+1}m^2_{p+1}}) = \frac{(2\pi)^{(q-1)/2}L_{p+1}}{m^{D-1}V_q}\omega^p_{\mathbf{n}_{q-1}} f_{p/2}(m_{p+1}L_{p+1}\omega_{\mathbf{n}_{q-1}}), \tag{A7}$$

where $\omega_{\mathbf{n}_{q-1}}$ is defined by relation (13). Substituting this relation into (A5) leads to the result (A2) which proves the equivalence of two expressions for the topological part.

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