

Symplectic covariance of the $\mathcal{N} = 2$ hypermultiplets

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The main objective of this article is to recast the hypermultiplets sector of five-dimensional ungauged $\mathcal{N} = 2$ supergravity into a manifestly symplectic-covariant form. We propose that this facilitates the construction and analysis of hypermultiplet fields coupled to p -brane sources and discuss examples.

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I. INTRODUCTION

The study of $\mathcal{N} = 2$ supergravity (SUGRA) theories has gained interest in recent years for a variety of reasons. For example, $\mathcal{N} = 2$ branes are particularly relevant to the conjectured equivalence between string theory on anti-de Sitter (AdS) space and certain superconformal gauge theories living on the boundary of the space (the AdS/conformal field theory (CFT) duality) [1]. Also interesting is that many results were found to involve the so-called attractor mechanism (e.g. [2–4]); the study of which developed very rapidly with many intriguing outcomes (e.g. [5–7]). The subject is also important in the context of string theory compactifications, as it is known that the behavior of the lower dimensional fields is contingent upon the topology of the underlying submanifold. In addition, many $D = 4, 5$ results were shown to be related to higher dimensional ones via wrapping over specific cycles of manifolds with special holonomy. For example, M branes wrapping Kähler calibrated cycles of a Calabi-Yau (CY) threefold [8] dimensionally reduce to black holes and strings coupled to the vector multiplets of five-dimensional $\mathcal{N} = 2$ supergravity [9], while M branes wrapping special Lagrangian calibrated cycles reduce to configurations carrying charge under the hypermultiplet scalars [10–14]. Studying how higher dimensional results are related to lower dimensional ones may eventually provide clues to the explicit structure of the compact space and the choice of compactification mechanism, thereby contributing to more understanding of the string theory landscape. It becomes then an important issue indeed, as far as the string theoretic view of the universe is concerned, to study such compactifications by classifying lower dimensional solutions and analyzing how they relate to higher dimensional ones.

In reviewing the literature, one notices that most studies in $\mathcal{N} = 2$ SUGRA in any number of dimensions specifically address the vector multiplets sector; setting the hypermultiplets to zero. This is largely due to the fact that the standard representation of the hypermultiplet scalars as coordinates on a quaternionic manifold is somewhat hard

to deal with. It has been shown, however, that certain duality maps relate the target space of a given higher dimensional fields' sector to that of a lower dimensional one [15]. Particularly relevant to this work is the so-called c-map which relates the quaternionic structure of the $D = 5$ hypermultiplets to the more well-understood special geometric structure of the $D = 4$ vector multiplets. This means that one can recast the $D = 5$ hypermultiplet fields into a form that makes full use of the methods of special geometry. This was done in [16] and applied in the same reference as well as in [12] and others. Using this method, finding solutions representing the five-dimensional hypermultiplet fields often means coming up with ansätze that have special geometric form. This can be, and has been, done by building on the considerable $D = 4$ vector multiplets literature, and in most cases the solutions are remarkably similar. For example, $D = 5$ hypermultiplet couplings to 2-branes and instantons [12,16] lead to the same type of attractor equations found for the vector multiplets coupled to $D = 4$ black holes (e.g. [17–20]).

Despite the power of the c-map method, it is still a highly tedious process to find solutions representing the full set of hypermultiplet fields. This is particularly serious in view of the fact that the most general solutions necessarily depend on the structure of the underlying Calabi-Yau manifold. Since no explicit (nontrivial) compact CY threefolds are known, the best one can do is to derive constraints on the fields; for example, the aforementioned attractor equations. And even then, deriving these equations is a long and difficult process. One may then desire to find an approach to constructing $D = 5$ hypermultiplet solutions that is more systematic and hopefully easily generalizable to other types of fields in other dimensions. One way of doing this, which we propose in this article, is by exploiting the symplectic nature of the theory. It has long been known that quaternionic and special Kähler geometries contain symplectic isometries and that the hypermultiplets action (with or without gravity) is in fact symplectically invariant. Furthermore, direct examination of known constructions reveals that they are written in terms of symplectic invariants and that this seems to be a recurrent theme. So the question becomes, can one construct solutions based solely on symplectic invariance? If so, what is the simplest form

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of the theory's field/supersymmetry equations that reduces the amount of work needed to verify these ansätze? In this paper, this is exactly what we attempt to explore.

The paper is structured in the following way: Section II reviews the definition of the space of complex structure moduli of Calabi-Yau manifolds. In Sec. III we discuss special Kähler geometry with particular emphasis on its symplectic structure. In so doing, we set the notation needed for dealing with symplectic invariants, collect all the necessary equations from the literature, as well as derive new quantities. Section IV reviews the dimensional reduction of $D = 11$ SUGRA over a Calabi-Yau threefold with nontrivial complex structure moduli. Finally, in Sec. V we put everything together and reformulate the theory into a symplectically covariant form and write down the field and SUSY equations in the simplest way possible. It is our hope that the equations of this section can be used in future research to straightforwardly write down and study solution ansätze. We conclude by showing how this approach is applied to two known $D = 5$ results.

II. THE SPACE OF COMPLEX STRUCTURE MODULI OF CALABI-YAU MANIFOLDS

A Calabi-Yau manifold \mathcal{M} is defined as a Kähler manifold endowed with Ricci flat metrics. The fields of string/SUGRA theories dimensionally reduced over CY threefolds generally correspond to the parameters that describe possible deformations of \mathcal{M} . This parameters' space factorizes, at least locally, into a product manifold $\mathcal{M}_C \otimes \mathcal{M}_K$, with \mathcal{M}_C being the manifold of complex structure moduli and \mathcal{M}_K being a complexification of the parameters of the Kähler class. These so-called moduli spaces turn out to belong to the category of special Kähler manifolds (defined in the next section).

Calabi-Yau threefolds admit a single $(3, 0)$ cohomology form; i.e. they have Hodge number $h_{3,0} = 1$, which we will call Ω (the holomorphic volume form) and an arbitrary number of $(1, 1)$ and $(2, 1)$ forms determined by the corresponding h 's (whose values depend on the particular choice of CY manifold). The Hodge number $h_{2,1}$ determines the dimensions of \mathcal{M}_C , while $h_{1,1}$ determines the dimensions of \mathcal{M}_K . The pair (\mathcal{M}, K) , where K is the Kähler form of \mathcal{M} , can be deformed by either deforming the complex structure of \mathcal{M} or by deforming the Kähler form K (or both). In particular, \mathcal{M}_C corresponds to special Lagrangian cycles of the CY space \mathcal{M} that are completely specified by knowledge of the unique $(3, 0)$ form Ω and the arbitrary number of $(2, 1)$ forms.

The following basic properties of Ω can be found:

$$\begin{aligned} \int_{\mathcal{M}} \Omega \wedge \bar{\Omega} &= -ie^{-\mathcal{K}}, \\ \int_{\mathcal{M}} \Omega \wedge \nabla_i \Omega &= \int_{\mathcal{M}} \bar{\Omega} \wedge \nabla_{\bar{i}} \bar{\Omega} = 0, \\ \int_{\mathcal{M}} \nabla_i \Omega \wedge \nabla_{\bar{j}} \bar{\Omega} &= iG_{i\bar{j}} e^{-\mathcal{K}}, \quad (i = 1, \dots, h_{2,1}), \end{aligned} \quad (1)$$

where \mathcal{K} is the Kähler potential of \mathcal{M}_C , $G_{i\bar{j}}$ is a complex metric on \mathcal{M}_C , and ∇ is defined by

$$\nabla_i = \partial_i + \frac{1}{2}(\partial_i \mathcal{K}), \quad \nabla_{\bar{i}} = \partial_{\bar{i}} - \frac{1}{2}(\partial_{\bar{i}} \mathcal{K}), \quad (2)$$

based on the $U(1)$ Kähler connection

$$\mathcal{P} = -\frac{i}{2}[(\partial_i \mathcal{K})dz^i - (\partial_{\bar{i}} \mathcal{K})d\bar{z}^{\bar{i}}]. \quad (3)$$

The space \mathcal{M}_C can be described in terms of the periods of Ω . Let (A^I, B_J) , where $I, J, K = 0, \dots, h_{2,1}$, be a canonical H^3 homology basis such that

$$\begin{aligned} A^I \cap B_J &= \delta^I_J, & B_I \cap A^J &= -\delta^J_I, \\ A^I \cap A^J &= B_I \cap B_J = 0, \end{aligned} \quad (4)$$

and let (α_I, β^J) be the dual cohomology basis forms such that

$$\begin{aligned} \int_{\mathcal{M}} \alpha_I \wedge \beta^J &= \int_{A^I} \alpha_I = \delta^J_I, \\ \int_{\mathcal{M}} \beta^I \wedge \alpha_J &= \int_{B_J} \beta^I = -\delta^I_J, \\ \int_{\mathcal{M}} \alpha_I \wedge \alpha_J &= \int_{\mathcal{M}} \beta^I \wedge \beta^J = 0. \end{aligned} \quad (5)$$

The periods of Ω are then defined by

$$Z^I = \int_{A^I} \Omega, \quad F_I = \int_{B_I} \Omega, \quad (6)$$

such that

$$\Omega = Z^I \alpha_I - F_I \beta^I, \quad (7)$$

and the Kähler potential of \mathcal{M}_C becomes

$$\mathcal{K} = -\ln[i(\bar{Z}^I F_I - Z^I \bar{F}_I)]. \quad (8)$$

The so-called periods matrix is defined by

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} + 2i \frac{N_{IK} Z^K N_{JL} Z^L}{Z^P N_{PQ} Z^Q} = \theta_{IJ} - i\gamma_{IJ}, \quad (9)$$

where $F_{IJ} = \partial_I F_J$ (the derivative is with respect to Z^I), $N_{IJ} = \text{Im}(F_{IJ})$, and $\gamma^{IJ} \gamma_{JK} = \delta^K_I$.

Finally, we note that one can choose a set of independent "special coordinates" z as follows:

$$z^I = \frac{Z^I}{Z^0}, \quad (10)$$

which are identified with the moduli of the complex structure z^i .

III. SPECIAL GEOMETRY AND SYMPLECTIC COVARIANCE

The space \mathcal{M}_C is described by special Kähler geometry, which we define in this section. The language we will use relies heavily on the symplectic structure of special manifolds. Some of the notation and equations used here are

original to this work. Our objective is to develop a working formulation of symplectic vector spaces that should facilitate the analysis of solutions in the hypermultiplets sector of $D = 5$ $\mathcal{N} = 2$ SUGRA, as well as any other theory with symplectic structure.

The symplectic group $Sp(2m, \mathbb{F}) \subset GL(2m, \mathbb{F})$ is the isometry group of a nondegenerate alternating bilinear form on a vector space of rank $2m$ over \mathbb{F} , where this last is usually either \mathbb{R} or \mathbb{C} , although other generalizations are possible. For our purposes, we take $\mathbb{F} = \mathbb{R}$ and $m = h_{2,1} + 1$. In other words, $Sp(2h_{2,1} + 2, \mathbb{R})$ is the group of the real bilinear matrices

$$\Lambda = \begin{bmatrix} {}^{11}\Lambda_J^I & {}^{12}\Lambda^{IJ} \\ {}^{21}\Lambda_{IJ} & {}^{22}\Lambda_J^I \end{bmatrix} \in Sp(2h_{2,1} + 2, \mathbb{R}) \quad (11)$$

that leave the totally antisymmetric symplectic matrix:

$$\mathbf{S} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \delta_I^J \\ -\delta_J^I & 0 \end{bmatrix} \quad (12)$$

invariant; i.e.

$$\Lambda^T \mathbf{S} \Lambda = \mathbf{S}, \quad \Lambda^T \mathbf{S}^T \Lambda = \mathbf{S}^T, \quad (13)$$

implying $|\Lambda| = 1$. The inverse of Λ is found to be

$$\Lambda^{-1} = \mathbf{S}^{-1} \Lambda^T \mathbf{S} = \begin{bmatrix} {}^{22}\Lambda_J^I & -{}^{12}\Lambda^{IJ} \\ -{}^{21}\Lambda_{IJ} & {}^{11}\Lambda_J^I \end{bmatrix}, \quad (14)$$

such that, using (13), $\Lambda^{-1} \Lambda = \mathbf{S}^{-1} \Lambda^T \mathbf{S} \Lambda = \mathbf{S}^{-1} \mathbf{S} = \mathbb{1}$ as needed. Also note that $\mathbf{S}^{-1} = \mathbf{S}^T = -\mathbf{S}$. We adopt the language that there exists a vector space \mathbf{Sp} such that the symplectic matrix \mathbf{S} acts as a metric on that space. Symplectic vectors in \mathbf{Sp} can be written in a ‘‘ket’’ notation as follows:

$$|A\rangle = \begin{pmatrix} a^I \\ \tilde{a}_I \end{pmatrix}, \quad |B\rangle = \begin{pmatrix} b^I \\ \tilde{b}_I \end{pmatrix}. \quad (15)$$

On the other hand, ‘‘bra’’ vectors defining a space dual to \mathbf{Sp} can be found by contraction with the metric in the usual way, yielding

$$\begin{aligned} \langle A| &= (\mathbf{S}\mathbf{A})^T = \mathbf{A}^T \mathbf{S}^T = (a^J \quad \tilde{a}_J) \begin{bmatrix} 0 & -\delta_J^I \\ \delta_J^I & 0 \end{bmatrix} \\ &= (\tilde{a}_I \quad -a^I), \end{aligned} \quad (16)$$

such that the inner product on \mathbf{Sp} is the ‘‘bra(c)ket’’:

$$\begin{aligned} \langle A|B\rangle &= \mathbf{A}^T \mathbf{S}^T \mathbf{B} = (\tilde{a}_I \quad -a^I) \begin{pmatrix} b^I \\ \tilde{b}_I \end{pmatrix} \\ &= \tilde{a}_I b^I - a^I \tilde{b}_I = -\langle B|A\rangle. \end{aligned} \quad (17)$$

In this language, the matrix Λ can simply be thought of as a rotation operator in \mathbf{Sp} . So a rotated vector is

$$|A'\rangle = \pm |\Lambda A\rangle = \pm \Lambda \mathbf{A}. \quad (18)$$

This is easily shown to preserve the inner product (17):

$$\langle A'|B'\rangle = (\pm)^2 \mathbf{A}^T \Lambda^T \Lambda^T \mathbf{S}^T \Lambda \mathbf{B} = \mathbf{A}^T \mathbf{S}^T \mathbf{B} = \langle A|B\rangle, \quad (19)$$

where (13) was used. In fact, one can *define* (13) based on the requirement that the inner product is preserved. To facilitate future calculations, we define the symplectic invariant

$$\begin{aligned} \langle A|\Lambda|B\rangle &\equiv \langle A|\Lambda B\rangle = \mathbf{A}^T \mathbf{S}^T \Lambda \mathbf{B} = \langle A\Lambda^{-1}|B\rangle \\ &= -\langle B\Lambda|A\rangle. \end{aligned} \quad (20)$$

The matrix Λ we will be using in the remainder of the paper has the property

$${}^{22}\Lambda_J^I = -{}^{11}\Lambda_J^I \rightarrow \Lambda^{-1} = -\Lambda, \quad (21)$$

which, via (20), leads to

$$\langle A|\Lambda|B\rangle = \langle A|\Lambda B\rangle = -\langle A\Lambda|B\rangle. \quad (22)$$

The choice (21) is not the only natural one. A consequence of it is that Λ is not symmetric, but $\mathbf{S}\Lambda$ is. On the other hand an equivalent choice would be a symmetric Λ , in which case it would be $\mathbf{S}\Lambda$ that satisfies (21). Within the context of special geometry, we have opted for a nonsymmetric Λ since it makes some later equations simpler.

Now consider the algebraic product of the two symplectic scalars

$$\langle A|B\rangle \langle C|D\rangle = (\mathbf{A}^T \mathbf{S}^T \mathbf{B})(\mathbf{C}^T \mathbf{S}^T \mathbf{D}). \quad (23)$$

The ordinary outer product of matrices is defined by

$$\mathbf{B} \otimes \mathbf{C}^T = \begin{pmatrix} b^I \\ \tilde{b}_I \end{pmatrix} \otimes (c^J \quad \tilde{c}_J) = \begin{bmatrix} b^I c^J & b^I \tilde{c}_J \\ \tilde{b}_I c^J & \tilde{b}_I \tilde{c}_J \end{bmatrix}, \quad (24)$$

which allows us to rewrite (23)

$$\langle A|B\rangle \langle C|D\rangle = \mathbf{A}^T \mathbf{S}^T (\mathbf{B} \otimes \mathbf{C}^T \mathbf{S}^T) \mathbf{D} = \langle A|\mathbf{B} \otimes \mathbf{C}^T \mathbf{S}^T|D\rangle. \quad (25)$$

Comparing the terms of (25), we conclude that one way a symplectic outer product can be defined is

$$|B\rangle \langle C| = \mathbf{B} \otimes \mathbf{C}^T \mathbf{S}^T = \begin{bmatrix} b^I \tilde{c}_J & -b^I c^J \\ \tilde{b}_I \tilde{c}_J & -\tilde{b}_I c^J \end{bmatrix}. \quad (26)$$

Note that the order of vectors in (26) is important, since generally

$$|B\rangle \langle C| = [\mathbf{S}|C\rangle \langle B|\mathbf{S}]^T. \quad (27)$$

However, if the outer product $|B\rangle \langle C|$ satisfies the property (21), i.e.

$$[|B\rangle \langle C|]^{-1} = -|B\rangle \langle C|, \quad (28)$$

then it is invariant under the interchange $B \leftrightarrow C$:

$$|B\rangle \langle C| = |C\rangle \langle B|. \quad (29)$$

The definition of a special Kähler manifold goes like this: Let \mathcal{L} denote a complex $U(1)$ line bundle whose first Chern class equals the Kähler form \mathcal{K} of a Hodge-Kähler

manifold \mathcal{M} . Now consider an additional holomorphic flat vector bundle of rank $(2h_{2,1} + 2)$ with structural group $Sp(2h_{2,1} + 2, \mathbb{R})$ on $\mathcal{M} : \mathcal{S}\mathcal{V} \rightarrow \mathcal{M}$. Construct a tensor bundle $\mathcal{S}\mathcal{V} \otimes \mathcal{L}$. This then is a special Kähler manifold if for some holomorphic section $|\Psi\rangle$ of such a bundle the Kähler 2-form is given by

$$K = -\frac{i}{2\pi} \partial \bar{\partial} \ln(i\langle \Psi | \bar{\Psi} \rangle), \quad (30)$$

or in terms of the Kähler potential

$$\mathcal{K} = -\ln(i\langle \Psi | \bar{\Psi} \rangle) \rightarrow \langle \bar{\Psi} | \Psi \rangle = ie^{-\mathcal{K}}. \quad (31)$$

Now, this exactly describes the space of complex structure moduli \mathcal{M}_C if one chooses

$$|\Psi\rangle = \begin{pmatrix} Z^I \\ F_I \end{pmatrix}, \quad (32)$$

which, via (31), leads directly to Eq. (8) defining the Kähler potential of \mathcal{M}_C . We then identify \mathcal{M}_C as a special Kähler manifold with metric $G_{i\bar{j}}$.

It can be easily demonstrated that the matrix

$$\Lambda = \begin{bmatrix} \gamma^{IK} \theta_{KJ} & -\gamma^{IJ} \\ (\gamma_{IJ} + \gamma^{KL} \theta_{IK} \theta_{JL}) & -\gamma^{JK} \theta_{KI} \end{bmatrix} \quad (33)$$

satisfies the symplectic condition (13), where γ and θ are defined by (9). Its inverse is then

$$\Lambda^{-1} = -\Lambda = \begin{bmatrix} -\gamma^{JK} \theta_{KI} & \gamma^{IJ} \\ -(\gamma_{IJ} + \gamma^{KL} \theta_{IK} \theta_{JL}) & \gamma^{IK} \theta_{KJ} \end{bmatrix}. \quad (34)$$

The symplectic structure manifest here is a consequence of the topology of the Calabi-Yau manifold \mathcal{M} , the origins of which can be traced to the completeness relations (5), clearly

$$\int_{\mathcal{M}} \begin{bmatrix} \alpha_I \wedge \alpha_J & \alpha_I \wedge \beta^J \\ \beta^I \wedge \alpha_J & \beta^I \wedge \beta^J \end{bmatrix} = \begin{bmatrix} 0 & \delta_I^J \\ -\delta_I^J & 0 \end{bmatrix} = \mathbf{S}. \quad (35)$$

In fact, if one defines the symplectic vector

$$|\Theta\rangle = \begin{pmatrix} \beta^I \\ \alpha_I \end{pmatrix}, \quad (36)$$

then it is easy to check that

$$\int_{\mathcal{M}} \Theta \otimes \Theta^T = \mathbf{S}^T \rightarrow \int_{\mathcal{M}} |\Theta\rangle \wedge \langle \Theta| = -\mathbb{1}. \quad (37)$$

Next, we construct a basis in \mathbf{Sp} . Properly normalized, the periods vector (32) provides such a basis

$$|V\rangle = e^{\mathcal{K}/2} |\Psi\rangle = \begin{pmatrix} L^I \\ M_I \end{pmatrix}, \quad (38)$$

such that, using (31)

$$\langle \bar{V} | V \rangle = (L^I \bar{M}_I - \bar{L}^I M_I) = i. \quad (39)$$

Since $|V\rangle$ is a scalar in the (i, j, k) indices, it couples only to the $U(1)$ bundle via the Kähler covariant derivative

$$\begin{aligned} |\nabla_i V\rangle &= \left[\partial_i + \frac{1}{2} (\partial_i \mathcal{K}) \right] V, \\ |\nabla_{\bar{i}} V\rangle &= \left[\partial_{\bar{i}} - \frac{1}{2} (\partial_{\bar{i}} \mathcal{K}) \right] V, \\ |\nabla_i \bar{V}\rangle &= \left[\partial_i - \frac{1}{2} (\partial_i \mathcal{K}) \right] \bar{V}, \\ |\nabla_{\bar{i}} \bar{V}\rangle &= \left[\partial_{\bar{i}} + \frac{1}{2} (\partial_{\bar{i}} \mathcal{K}) \right] \bar{V}. \end{aligned} \quad (40)$$

Using this, one can construct the orthogonal \mathbf{Sp} vectors

$$|U_i\rangle = |\nabla_i V\rangle = \begin{pmatrix} \nabla_i L^I \\ \nabla_i M_I \end{pmatrix} = \begin{pmatrix} f_i^I \\ h_{iI} \end{pmatrix}, \quad (41)$$

$$|U_{\bar{i}}\rangle = |\nabla_{\bar{i}} \bar{V}\rangle = \begin{pmatrix} \nabla_{\bar{i}} \bar{L}^I \\ \nabla_{\bar{i}} \bar{M}_I \end{pmatrix} = \begin{pmatrix} \bar{f}_{\bar{i}}^I \\ \bar{h}_{\bar{i}I} \end{pmatrix}, \quad (42)$$

with

$$\begin{aligned} |\nabla_i U_j\rangle &= \left[\partial_i + \frac{1}{2} (\partial_i \mathcal{K}) \right] U_j, \\ |\nabla_{\bar{i}} U_j\rangle &= \left[\partial_{\bar{i}} - \frac{1}{2} (\partial_{\bar{i}} \mathcal{K}) \right] U_j, \\ |\nabla_i U_{\bar{j}}\rangle &= \left[\partial_i - \frac{1}{2} (\partial_i \mathcal{K}) \right] U_{\bar{j}}, \\ |\nabla_{\bar{i}} U_{\bar{j}}\rangle &= \left[\partial_{\bar{i}} + \frac{1}{2} (\partial_{\bar{i}} \mathcal{K}) \right] U_{\bar{j}}. \end{aligned} \quad (43)$$

Note that $|U_i\rangle$ also couples to the metric $G_{i\bar{j}}$ via the Levi-Civita connection. So its full covariant derivative is defined by

$$\begin{aligned} |\mathcal{D}_i U_j\rangle &= |\nabla_i U_j\rangle - \Gamma_{ij}^k |U_k\rangle, & |\mathcal{D}_{\bar{i}} U_j\rangle &= |\nabla_{\bar{i}} U_j\rangle, \\ |\mathcal{D}_i U_{\bar{j}}\rangle &= |\nabla_i U_{\bar{j}}\rangle, & |\mathcal{D}_{\bar{i}} U_{\bar{j}}\rangle &= |\nabla_{\bar{i}} U_{\bar{j}}\rangle - \Gamma_{\bar{i}\bar{j}}^{\bar{k}} |U_{\bar{k}}\rangle. \end{aligned} \quad (44)$$

It can be demonstrated that these quantities satisfy the properties

$$|\nabla_i \bar{V}\rangle = |\nabla_{\bar{i}} V\rangle = 0, \quad (45)$$

$$\langle U_i | U_j \rangle = \langle U_{\bar{i}} | U_{\bar{j}} \rangle = 0, \quad (46)$$

$$\langle \bar{V} | U_i \rangle = \langle V | U_{\bar{i}} \rangle = \langle V | U_i \rangle = \langle \bar{V} | U_{\bar{i}} \rangle = 0, \quad (47)$$

$$|\nabla_{\bar{j}} U_i\rangle = G_{i\bar{j}} |V\rangle, \quad |\nabla_i U_{\bar{j}}\rangle = G_{i\bar{j}} |\bar{V}\rangle, \quad (48)$$

$$G_{i\bar{j}} = (\partial_i \partial_{\bar{j}} \mathcal{K}) = -i \langle U_i | U_{\bar{j}} \rangle. \quad (49)$$

Special Kähler manifolds admit a completely symmetric and covariantly holomorphic tensor C_{ijk} and its antiholomorphic conjugate $C_{\bar{i}\bar{j}\bar{k}}$ such that the following restriction

on the curvature is true:

$$R_{\bar{i}j\bar{k}l} = G_{j\bar{k}}G_{l\bar{i}} + G_{l\bar{k}}G_{j\bar{i}} - C_{rlj}C_{\bar{s}\bar{i}\bar{k}}G^{r\bar{s}}, \quad (50)$$

generally referred to in the literature as the special Kähler geometry constraint. It can be shown that

$$|\mathcal{D}_i U_j\rangle = G^{k\bar{i}}C_{ijk}|U_{\bar{j}}\rangle, \quad (51)$$

which leads to

$$C_{ijk} = -i\langle \mathcal{D}_i U_j | U_k \rangle. \quad (52)$$

The following identities may now be derived:

$$\mathcal{N}_{IJ}L^J = M_I, \quad \bar{\mathcal{N}}_{IJ}f_i^J = h_{i|I}, \quad (53)$$

$$\bar{\mathcal{N}}_{IJ}\bar{L}^J = \bar{M}_I, \quad \mathcal{N}_{IJ}f_{\bar{i}}^J = h_{\bar{i}|I},$$

$$\gamma_{IJ}L^I\bar{L}^J = \frac{1}{2}, \quad G_{i\bar{j}} = 2\gamma_{IJ}f_i^J f_{\bar{j}}^I, \quad (54)$$

as well as the very useful (and quite essential for our purposes)

$$\gamma^{IJ} = 2(L^I\bar{L}^J + G^{i\bar{j}}f_i^I f_{\bar{j}}^J),$$

$$(\gamma_{IJ} + \gamma^{KL}\theta_{IK}\theta_{JL}) = 2(M_I\bar{M}_J + G^{i\bar{j}}h_{i|I}h_{\bar{j}|J}),$$

$$\gamma^{IK}\theta_{KJ} = 2(\bar{L}^I M_J + G^{i\bar{j}}f_i^I h_{\bar{j}|J}) + i\delta_J^I$$

$$= 2(L^I\bar{M}_J + G^{i\bar{j}}h_{i|I}f_{\bar{j}}^J) - i\delta_J^I$$

$$= (L^I\bar{M}_J + \bar{L}^I M_J)$$

$$+ G^{i\bar{j}}(f_i^I h_{\bar{j}|J} + h_{i|I} f_{\bar{j}}^J). \quad (55)$$

Equations (55) lead to a second form for the symplectic matrix (33)

$$\Lambda = \begin{bmatrix} (L^I\bar{M}_J + \bar{L}^I M_J) + G^{i\bar{j}}(f_i^I h_{\bar{j}|J} + h_{i|I} f_{\bar{j}}^J) & -2(L^I\bar{L}^J + G^{i\bar{j}}f_i^I f_{\bar{j}}^J) \\ 2(M_I\bar{M}_J + G^{i\bar{j}}h_{i|I}h_{\bar{j}|J}) & -(L^J\bar{M}_I + \bar{L}^J M_I) - G^{i\bar{j}}(f_i^J h_{\bar{j}|I} + h_{i|I} f_{\bar{j}}^J) \end{bmatrix} \quad (56)$$

with inverse

$$\Lambda^{-1} = -\Lambda = \begin{bmatrix} -(L^J\bar{M}_I + \bar{L}^J M_I) - G^{i\bar{j}}(f_i^J h_{\bar{j}|I} + h_{i|I} f_{\bar{j}}^J) & 2(L^I\bar{L}^J + G^{i\bar{j}}f_i^I f_{\bar{j}}^J) \\ -2(M_I\bar{M}_J + G^{i\bar{j}}h_{i|I}h_{\bar{j}|J}) & (L^I\bar{M}_J + \bar{L}^I M_J) + G^{i\bar{j}}(f_i^I h_{\bar{j}|J} + h_{i|I} f_{\bar{j}}^J) \end{bmatrix}. \quad (57)$$

By inspection, one can write down the following important result:

$$\Lambda = |V\rangle\langle\bar{V}| + |\bar{V}\rangle\langle V| + G^{i\bar{j}}|U_i\rangle\langle U_{\bar{j}}| + G^{i\bar{j}}|U_{\bar{j}}\rangle\langle U_i|, \\ \Lambda^{-1} = -|V\rangle\langle\bar{V}| - |\bar{V}\rangle\langle V| - G^{i\bar{j}}|U_i\rangle\langle U_{\bar{j}}| - G^{i\bar{j}}|U_{\bar{j}}\rangle\langle U_i|. \quad (58)$$

In other words, the rotation matrix in \mathbf{Sp} is expressible as the outer product of the basis vectors; a result which, in retrospect, seems obvious. Note that since Λ satisfies the property (21), it is invariant under the interchange $V \leftrightarrow \bar{V}$ and/or $U_i \leftrightarrow U_{\bar{j}}$. This makes manifest the fact that Λ is a real matrix; $\Lambda = \bar{\Lambda}$. Now, applying $\Lambda^{-1}\Lambda = \mathbb{1}$, we end up with the condition

$$|\bar{V}\rangle\langle V| + G^{i\bar{j}}|U_i\rangle\langle U_{\bar{j}}| = |V\rangle\langle\bar{V}| + G^{i\bar{j}}|U_{\bar{j}}\rangle\langle U_i| - i, \quad (59)$$

which can be checked explicitly using (55). This can be used to write Λ in an even simpler form

$$\Lambda = 2|V\rangle\langle\bar{V}| + 2G^{i\bar{j}}|U_{\bar{j}}\rangle\langle U_i| - i, \\ \Lambda^{-1} = -2|V\rangle\langle\bar{V}| - 2G^{i\bar{j}}|U_{\bar{j}}\rangle\langle U_i| + i. \quad (60)$$

For future convenience we also compute

$$\mathcal{D}_i \Lambda = \nabla_i \Lambda = \partial_i \Lambda \\ = 2|U_i\rangle\langle\bar{V}| + 2|\bar{V}\rangle\langle U_i| + 2G^{j\bar{p}}G^{k\bar{p}}C_{ijk}|U_{\bar{r}}\rangle\langle U_{\bar{p}}|. \quad (61)$$

It is clearly easier, and possibly more intuitive, to work with an expression such as (60) over something like (56), or even (33). It is indeed this very fact that has motivated this work in its entirety. Finally, we note that our discussion here is based on a definition of special manifolds that is not the only one in existence. See, for instance, [21] for details. Explicit examples of special manifolds in various dimensions are given in, for example, [22]. More detail on this obviously vast topic may be found in [23–31].

IV. $D = 5$ $\mathcal{N} = 2$ SUPERGRAVITY WITH HYPERMULTIPLETS

The dimensional reduction of $D = 11$ supergravity over a Calabi-Yau manifold \mathcal{M} yields ungauged $D = 5$ $\mathcal{N} = 2$ SUGRA. We look at the case where only the complex structure of \mathcal{M} is deformed. We will follow, and slightly extend, the notation of [16].

The unique supersymmetric gravity theory in 11 dimensions has the following bosonic action:

$$S_{11} = \int_{11} \left(\mathcal{R} \star 1 - \frac{1}{2} \mathcal{F} \wedge \star \mathcal{F} - \frac{1}{6} \mathcal{A} \wedge \mathcal{F} \wedge \mathcal{F} \right), \quad (62)$$

where \mathcal{R} is the $D = 11$ Ricci scalar, \mathcal{A} is the 3-form gauge potential, $\mathcal{F} = d\mathcal{A}$, and \star is the Hodge star operator. The dimensional reduction is traditionally done using the metric

$$ds^2 = e^{(2/3)\sigma} g_{\mu\nu} dx^\mu dx^\nu + e^{-(\sigma/3)} ds_{\text{CY}}^2, \quad (63)$$

$\mu, \nu = 0, \dots, 4,$

where $g_{\mu\nu}$ is the target five-dimensional metric, ds_{CY}^2 is a metric on the six-dimensional compact subspace \mathcal{M} , the dilaton σ is a function in x^μ only, and the warp factors are chosen to give the conventional numerical coefficients in five dimensions.

The flux compactification of the gauge field is done by expanding \mathcal{A} into two forms, one is the five-dimensional gauge field A while the other contains the components of \mathcal{A} on \mathcal{M} written in terms of the cohomology forms (α_I, β^I) as follows:

$$\begin{aligned} \mathcal{A} &= A + \sqrt{2}(\zeta^I \alpha_I + \tilde{\zeta}_I \beta^I), \\ \mathcal{F} = d\mathcal{A} &= F + \sqrt{2}[(\partial_\mu \zeta^I) \alpha_I + (\partial_\mu \tilde{\zeta}_I) \beta^I] \wedge dx^\mu. \end{aligned} \quad (64)$$

Because of the 11-dimensional Chern-Simons term, the coefficients ζ^I and $\tilde{\zeta}_I$ appear as pseudoscalar axion fields in the lower dimensional theory. We also note that A in five dimensions is dual to a scalar field which we will call a (known as the universal axion). The set $(a, \sigma, \zeta^0, \tilde{\zeta}_0)$ is known as the universal hypermultiplet.¹ The rest of the hypermultiplets are $(z^i, \bar{z}^{\bar{i}}, \zeta^i, \tilde{\zeta}_i)$, where we recognize the z 's as the CY's complex structure moduli. Note that the total number of scalar fields in the hypermultiplets sector is $4(h_{2,1} + 1)$ (each hypermultiplet has four real scalar fields) which comprises a quaternionic manifold as noted earlier. Also included in the hypermultiplets are the fermionic partners of the hypermultiplet scalars known as the hyperini (singular: hyperino).

The bosonic action of the ungauged five-dimensional $\mathcal{N} = 2$ supergravity theory with vanishing vector multiplets is

¹So-called because it appears in all Calabi-Yau compactifications, irrespective of the detailed structure of the CY manifold. We recall that the dilaton σ is proportional to the natural logarithm of the volume of \mathcal{M} .

$$\begin{aligned} S_5 &= \int_5 \left\{ R \star 1 - \frac{1}{2} d\sigma \wedge \star d\sigma - G_{ij} dz^i \wedge \star dz^{\bar{j}} \right. \\ &\quad - F \wedge (\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I) - \frac{1}{2} e^{-2\sigma} F \wedge \star F \\ &\quad - e^\sigma [(\gamma_{IJ} + \gamma^{KL} \theta_{IK} \theta_{JL}) d\zeta^I \wedge \star d\zeta^J \\ &\quad \left. + \gamma^{IJ} d\tilde{\zeta}_I \wedge \star d\tilde{\zeta}_J + 2\gamma^{IK} \theta_{JK} d\zeta^J \wedge \star d\tilde{\zeta}_I \right\}. \end{aligned} \quad (65)$$

Variation of the action gives the following field equations for σ , $(z^i, \bar{z}^{\bar{i}})$, A , and $(\zeta^I, \tilde{\zeta}_I)$:

$$(\Delta\sigma) \star 1 - e^\sigma X + e^{-2\sigma} F \wedge \star F = 0, \quad (66)$$

$$\begin{aligned} (\Delta z^i) \star 1 + \Gamma_{jk}^i dz^j \wedge \star dz^k - \frac{1}{2} e^\sigma G^{i\bar{j}} (\partial_{\bar{j}} X) \star 1 &= 0, \\ (\Delta \bar{z}^{\bar{i}}) \star 1 + \Gamma_{\bar{j}\bar{k}}^{\bar{i}} d\bar{z}^{\bar{j}} \wedge \star d\bar{z}^{\bar{k}} - \frac{1}{2} e^\sigma G^{\bar{i}j} (\partial_j X) \star 1 &= 0, \end{aligned} \quad (67)$$

$$d^\dagger [e^{-2\sigma} F + \star(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I)] = 0, \quad (68)$$

$$\begin{aligned} d^\dagger [e^\sigma \gamma^{IK} \theta_{JK} d\zeta^J + e^\sigma \gamma^{IJ} d\tilde{\zeta}_J + \zeta^I \star F] &= 0, \\ d^\dagger [e^\sigma (\gamma_{IJ} + \gamma^{KL} \theta_{IK} \theta_{JL}) d\zeta^J + e^\sigma \gamma^{JK} \theta_{IK} d\tilde{\zeta}_J - \tilde{\zeta}_I \star F] &= 0, \end{aligned} \quad (69)$$

where d^\dagger is the adjoint exterior derivative and Δ is the Laplace de-Rahm operator. For compactness we have defined

$$\begin{aligned} X &= (\gamma_{IJ} + \gamma^{KL} \theta_{IK} \theta_{JL}) d\zeta^I \wedge \star d\zeta^J \\ &\quad + \gamma^{IJ} d\tilde{\zeta}_I \wedge \star d\tilde{\zeta}_J + 2\gamma^{IK} \theta_{JK} d\zeta^J \wedge \star d\tilde{\zeta}_I, \end{aligned} \quad (70)$$

as well as used the Bianchi identity $dF = 0$ to get the given form of (69). From a five-dimensional perspective, the moduli $(z^i, \bar{z}^{\bar{i}})$ behave as scalar fields. We recall, however, that the behavior of the other fields is dependent on the moduli, i.e. they are functions in them. Hence it is possible to treat (67) as constraints that can be used to reduce the degrees of freedom of the other field equations. Certain assumptions, however, are needed to perform this, so we will not do so here since our objective is to discuss the field equations in their most general form. This is more properly done in the context of specific solution ansätze.

Equations (68) and (69) are clearly the statements that the forms

$$\begin{aligned} \mathcal{J}_2 &= e^{-2\sigma} F + \star(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I), \\ \mathcal{J}_3^I &= e^\sigma \gamma^{IK} \theta_{JK} d\zeta^J + e^\sigma \gamma^{IJ} d\tilde{\zeta}_J + \zeta^I \star F, \\ \tilde{\mathcal{J}}_{5|I} &= e^\sigma (\gamma_{IJ} + \gamma^{KL} \theta_{IK} \theta_{JL}) d\zeta^J \\ &\quad + e^\sigma \gamma^{JK} \theta_{IK} d\tilde{\zeta}_J - \tilde{\zeta}_I \star F \end{aligned} \quad (71)$$

are conserved. These are, in fact, Noether currents corresponding to certain isometries of the quaternionic manifold defined by the hypermultiplets as discussed in various

sources [15,32]. From a five-dimensional perspective, they can be thought of as the result of the invariance of the action under particular infinitesimal shifts of A and $(\zeta, \tilde{\zeta})$ [16,33]. The charge densities corresponding to them can then be found in the usual way by

$$\mathcal{Q}_2 = \int \mathcal{J}_2, \quad \mathcal{Q}_5^I = \int \mathcal{J}_5^I, \quad \tilde{\mathcal{Q}}_{5|I} = \int \tilde{\mathcal{J}}_{5|I}. \quad (72)$$

The geometric way of understanding these charges is noting that they descend from the 11-dimensional electric and magnetic M-brane charges, hence the (2, 5) labels.² M2 branes wrapping special Lagrangian cycles of \mathcal{M} generate \mathcal{Q}_2 while the wrapping of M5 branes excite $(\mathcal{Q}_5^I, \tilde{\mathcal{Q}}_{5|I})$.

Finally, for completeness sake we also give da , where a is the universal axion dual to A . Since (68) is equivalent to $d^2a = 0$, we conclude that

$$da = e^{-2\sigma} \star F - (\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I), \quad (73)$$

where a is governed by the field equation

$$d^\dagger[e^{2\sigma} da + e^{2\sigma}(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I)] = 0; \quad (74)$$

as a consequence of $dF = 0$. Both terms involving F in (65) could then be replaced by the single expression³

$$S_a = \frac{1}{2} \int e^{2\sigma} [da + (\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I)] \wedge \star [da + (\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I)]. \quad (75)$$

The full supersymmetric action is invariant under the following SUSY variations. For the gravitini:

$$\delta_\epsilon \psi^A = \tilde{\nabla} \epsilon^A + [\mathcal{G}]^A_B \epsilon^B, \quad (76)$$

$$[\mathcal{G}] = \begin{bmatrix} \frac{1}{4}(v - \bar{v} - Y) & -\bar{u} \\ u & -\frac{1}{4}(v - \bar{v} - Y) \end{bmatrix},$$

where the indices A and B run over (1, 2), $\tilde{\nabla}$ is given by

$$\tilde{\nabla} = dx^\mu \left(\partial_\mu + \frac{1}{4} \omega_\mu^{\hat{\mu}\hat{\nu}} \Gamma_{\hat{\mu}\hat{\nu}} \right), \quad (77)$$

where the ω 's are the usual spin connections, hatted indices denote dimensions in a flat tangent space, and the ϵ 's are the SUSY parameters. The other quantities in (76) are

$$u = e^{\sigma/2} (M_I d\zeta^I + L^I d\tilde{\zeta}_I), \quad \bar{u} = e^{\sigma/2} (\bar{M}_I d\zeta^I + \bar{L}^I d\tilde{\zeta}_I),$$

$$v = \frac{1}{2} d\sigma + \frac{i}{2} e^{-\sigma} \star F, \quad \bar{v} = \frac{1}{2} d\sigma - \frac{i}{2} e^{-\sigma} \star F, \quad (78)$$

and

²This is the reverse situation to that of [16], where the (dual) Euclidean theory was studied.

³Alternatively, one may dualize the action by introducing a as a Lagrange multiplier and modifying the action accordingly [16].

$$Y = \frac{\bar{Z}^I N_{IJ} dZ^J - Z^I N_{IJ} d\bar{Z}^J}{\bar{Z}^I N_{IJ} Z^J} \quad (79)$$

which is proportional to the $U(1)$ Kähler connection defined by (3).

Finally, the hyperini equations are

$$\delta_\epsilon \xi_1^I = e^{1I}{}_\mu \Gamma^\mu \epsilon_1 - \bar{e}^{2I}{}_\mu \Gamma^\mu \epsilon_2, \quad (80)$$

$$\delta_\epsilon \xi_2^I = e^{2I}{}_\mu \Gamma^\mu \epsilon_1 + \bar{e}^{1I}{}_\mu \Gamma^\mu \epsilon_2,$$

written in terms of the quantities

$$e^{1I} = e^{1I}{}_\mu dx^\mu = \begin{pmatrix} u \\ E^{\hat{I}} \end{pmatrix},$$

$$e^{2I} = e^{2I}{}_\mu dx^\mu = \begin{pmatrix} v \\ E^{\hat{I}} \end{pmatrix},$$

$$E^{\hat{I}} = e^{\sigma/2} e^{\hat{I}j} (h_{jI} d\zeta^I + f_j^I d\tilde{\zeta}_I),$$

$$\bar{E}^{\hat{I}} = e^{\sigma/2} e^{\hat{I}\bar{j}} (h_{\bar{j}I} d\zeta^I + f_{\bar{j}}^I d\tilde{\zeta}_I) \quad (81)$$

and the beins of the special Kähler metric

$$e^{\hat{I}} = e^{\hat{I}j} dz^j, \quad \bar{e}^{\hat{I}} = e^{\hat{I}\bar{j}} d\bar{z}^{\bar{j}}, \quad G_{i\bar{j}} = e^{\hat{k}}_i e^{\hat{l}}_{\bar{j}} \delta_{\hat{k}\hat{l}}. \quad (82)$$

V. THE THEORY IN SYMPLECTIC FORM

In this section we arrive at our main objective: recasting the action (65) and its associated field and SUSY equations into a manifestly symplectic form based on the language defined in Sec. III. The reader should be convinced by now that this is a straightforward matter and can be achieved by direct examination of the equations involved. We give as much detail as possible for the sake of future reference. Finally, we show how a calculation based on the symplectic formulation may be carried out by direct application to the results of [12,16].

A. Reformulation

The action (65) is invariant under rotations in \mathcal{Sp} , so by inspection it is clear that R , $d\sigma$, dz , and F are themselves symplectic invariants, whose explicit form will depend on the specific ansätze used. The axion fields $(\zeta, \tilde{\zeta})$, however, can be thought of as components of an \mathcal{Sp} ‘‘axions vector.’’ If we define

$$|\Xi\rangle = \begin{pmatrix} \zeta^I \\ -\tilde{\zeta}_I \end{pmatrix}, \quad |d\Xi\rangle = \begin{pmatrix} d\zeta^I \\ -d\tilde{\zeta}_I \end{pmatrix}, \quad (83)$$

then clearly

$$\langle \Xi | d\Xi \rangle = \zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I, \quad (84)$$

as well as

$$\begin{aligned} \langle \partial_\mu \Xi | \Lambda | \partial^\mu \Xi \rangle &= -(\gamma_{IJ} + \gamma^{KL} \theta_{IK} \theta_{JL}) (\partial_\mu \zeta^I) (\partial^\mu \zeta^J) \\ &\quad - \gamma^{IJ} (\partial_\mu \tilde{\zeta}_I) (\partial^\mu \tilde{\zeta}_J) \\ &\quad - 2\gamma^{IK} \theta_{JK} (\partial_\mu \zeta^J) (\partial^\mu \tilde{\zeta}_I), \end{aligned} \quad (85)$$

such that (70) becomes

$$\begin{aligned} X &= (\gamma_{IJ} + \gamma^{KL} \theta_{IK} \theta_{JL}) d\zeta^I \wedge \star d\zeta^J \\ &\quad + \gamma^{IJ} d\tilde{\zeta}_I \wedge \star d\tilde{\zeta}_J + 2\gamma^{IK} \theta_{JK} d\zeta^J \wedge \star d\tilde{\zeta}_I \\ &= -\langle \partial_\mu \Xi | \Lambda | \partial^\mu \Xi \rangle \star 1. \end{aligned} \quad (86)$$

Also note that we chose the minus sign in the definition (83) such that the resulting equations agree with the form of the theory used in previous work, particularly [12,13,16]. Replacing the minus sign with a positive sign would result in the appearance of minus signs in various locations in the action, field, and SUSY equations.

As a consequence of this language, the field expansion (64) could be rewritten

$$\begin{aligned} \mathcal{A} &= A + \sqrt{2} \langle \Theta | \Xi \rangle, \\ \mathcal{F} &= d\mathcal{A} = F + \sqrt{2} \langle \Theta | d\Xi \rangle. \end{aligned} \quad (87)$$

The bosonic action in manifest symplectic covariance is hence

$$\begin{aligned} S_5 &= \int_5 \left[R \star 1 - \frac{1}{2} d\sigma \wedge \star d\sigma - G_{ij} dz^i \wedge \star dz^j \right. \\ &\quad \left. - F \wedge \langle \Xi | d\Xi \rangle - \frac{1}{2} e^{-2\sigma} F \wedge \star F \right. \\ &\quad \left. + e^\sigma \langle \partial_\mu \Xi | \Lambda | \partial^\mu \Xi \rangle \star 1 \right]. \end{aligned} \quad (88)$$

The equations of motion are now

$$(\Delta\sigma) \star 1 + e^\sigma \langle \partial_\mu \Xi | \Lambda | \partial^\mu \Xi \rangle \star 1 + e^{-2\sigma} F \wedge \star F = 0, \quad (89)$$

$$(\Delta z^i) \star 1 + \Gamma_{jk}^i dz^j \wedge \star dz^k + \frac{1}{2} e^\sigma G^{ij} \partial_j \langle \partial_\mu \Xi | \Lambda | \partial^\mu \Xi \rangle \star 1 = 0,$$

$$(\Delta z^{\bar{i}}) \star 1 + \Gamma_{\bar{j}\bar{k}}^{\bar{i}} dz^{\bar{j}} \wedge \star dz^{\bar{k}} + \frac{1}{2} e^\sigma G^{\bar{i}\bar{j}} \partial_{\bar{j}} \langle \partial_\mu \Xi | \Lambda | \partial^\mu \Xi \rangle \star 1 = 0, \quad (90)$$

$$d^\dagger [e^{-2\sigma} F + \star \langle \Xi | d\Xi \rangle] = 0, \quad (91)$$

$$d^\dagger [e^\sigma \langle \Lambda d\Xi \rangle + \star F | \Xi \rangle] = 0. \quad (92)$$

Note that, as is usual for Chern-Simons actions, the explicit appearance of the gauge potential $|\Xi\rangle$ in (91) and (92) does not have an effect on the physics since

$$\begin{aligned} d^\dagger \star \langle \Xi | d\Xi \rangle &\rightarrow d \langle \Xi | d\Xi \rangle = \left\langle d\Xi \middle| d\Xi \right\rangle, \\ d^\dagger \star F | \Xi \rangle &\rightarrow d[F | \Xi \rangle] = F \wedge | d\Xi \rangle, \end{aligned} \quad (93)$$

where the Bianchi identities on A and $|\Xi\rangle$ were used. Now,

if $|\Xi\rangle$ is taken to be independent of the moduli, then we can write

$$\partial_j \langle \partial_\mu \Xi | \Lambda | \partial^\mu \Xi \rangle = \langle \partial_\mu \Xi | \partial_j \Lambda | \partial^\mu \Xi \rangle. \quad (94)$$

Furthermore, since the exponents of both the \mathcal{M}_C Kähler potential \mathcal{K} and the dilaton σ are proportional to the volume of the CY submanifold, then they can be taken to be proportional to each other, i.e. following [18]:

$$\sigma = c\mathcal{K}, \quad (95)$$

where c is some arbitrary constant. The Noether currents and charges become

$$\begin{aligned} \mathcal{J}_2 &= e^{-2\sigma} F + \star \langle \Xi | d\Xi \rangle, \\ |\mathcal{J}_5\rangle &= e^\sigma \langle \Lambda d\Xi \rangle + \star F | \Xi \rangle, \\ \mathcal{Q}_2 &= \int \mathcal{J}_2, \quad |\mathcal{Q}_5\rangle = \int |\mathcal{J}_5\rangle. \end{aligned} \quad (96)$$

The equations of the universal axion (73)–(75) are now

$$da = e^{-2\sigma} \star F - \langle \Xi | d\Xi \rangle, \quad (97)$$

$$d^\dagger [e^{2\sigma} da + e^{2\sigma} \langle \Xi | d\Xi \rangle] = 0, \quad (98)$$

and

$$S_a = \frac{1}{2} \int e^{2\sigma} [da + \langle \Xi | d\Xi \rangle] \wedge \star [da + \langle \Xi | d\Xi \rangle]. \quad (99)$$

Next, we look at the SUSY variations. The gravitini equations can be explicitly written as follows:

$$\delta_\epsilon \psi^1 = \tilde{\nabla} \epsilon_1 + \frac{1}{4} (ie^{-\sigma} \star F - Y) \epsilon_1 - e^{\sigma/2} \langle \bar{V} | d\Xi \rangle \epsilon_2, \quad (100)$$

$$\delta_\epsilon \psi^2 = \tilde{\nabla} \epsilon_2 - \frac{1}{4} (ie^{-\sigma} \star F - Y) \epsilon_2 + e^{\sigma/2} \langle V | d\Xi \rangle \epsilon_1, \quad (101)$$

while the hyperini variations are

$$\begin{aligned} \delta_\epsilon \xi_1^0 &= e^{\sigma/2} \langle V | \partial_\mu \Xi \rangle \Gamma^\mu \epsilon_1 \\ &\quad - \left[\frac{1}{2} (\partial_\mu \sigma) - \frac{i}{2} e^{-\sigma} (\star F)_\mu \right] \Gamma^\mu \epsilon_2, \\ \delta_\epsilon \xi_2^0 &= e^{\sigma/2} \langle \bar{V} | \partial_\mu \Xi \rangle \Gamma^\mu \epsilon_2 \\ &\quad + \left[\frac{1}{2} (\partial_\mu \sigma) + \frac{i}{2} e^{-\sigma} (\star F)_\mu \right] \Gamma^\mu \epsilon_1, \end{aligned} \quad (102)$$

$$\begin{aligned} \delta_\epsilon \xi_1^{\hat{i}} &= e^{\sigma/2} e^{\hat{i}j} \langle U_j | \partial_\mu \Xi \rangle \Gamma^\mu \epsilon_1 - e^{\hat{i}j} (\partial_\mu z^{\bar{j}}) \Gamma^\mu \epsilon_2, \\ \delta_\epsilon \xi_2^{\hat{i}} &= e^{\sigma/2} e^{\hat{i}\bar{j}} \langle U_{\bar{j}} | \partial_\mu \Xi \rangle \Gamma^\mu \epsilon_2 + e^{\hat{i}j} (\partial_\mu z^j) \Gamma^\mu \epsilon_1. \end{aligned} \quad (103)$$

For easy reference, we also compute

$$\begin{aligned}
 dG_{i\bar{j}} &= G_{k\bar{j}}\Gamma_{ri}^k dz^r + G_{i\bar{k}}\Gamma_{\bar{r}\bar{j}}^{\bar{k}} dz^{\bar{r}}, \\
 dG^{i\bar{j}} &= -G^{p\bar{j}}\Gamma_{rp}^i dz^r - G^{i\bar{p}}\Gamma_{\bar{r}\bar{p}}^{\bar{j}} dz^{\bar{r}}, \\
 |dV\rangle &= dz^i|U_i\rangle - i\mathcal{P}|V\rangle, \\
 |d\bar{V}\rangle &= dz^{\bar{i}}|U_{\bar{i}}\rangle + i\mathcal{P}|\bar{V}\rangle, \\
 |dU_i\rangle &= G_{i\bar{j}}dz^{\bar{j}}|V\rangle + \Gamma_{ik}^r dz^k|U_r\rangle \\
 &\quad + G^{j\bar{l}}C_{ijk}dz^k|U_{\bar{l}}\rangle - i\mathcal{P}|U_i\rangle, \\
 |dU_{\bar{i}}\rangle &= G_{\bar{j}\bar{i}}dz^{\bar{j}}|\bar{V}\rangle + \Gamma_{\bar{i}\bar{k}}^{\bar{r}} dz^{\bar{k}}|U_{\bar{r}}\rangle + G^{l\bar{j}}C_{\bar{i}\bar{j}\bar{k}}dz^{\bar{k}}|U_{\bar{l}}\rangle \\
 &\quad + i\mathcal{P}|U_{\bar{i}}\rangle, \\
 d\Lambda &= (\partial_i\Lambda)dz^i + (\partial_{\bar{i}}\Lambda)dz^{\bar{i}}, \tag{104}
 \end{aligned}$$

where \mathcal{P} is the $U(1)$ connection defined by (3) and $(\partial_i\Lambda, \partial_{\bar{i}}\Lambda)$ are given by (61).

B. Examples

The analysis of solution ansätze representing hypermultiplet fields should now reduce to the problem of constructing and manipulating symplectic quantities. Using the language developed in this paper, we now demonstrate how this can be done by applying the symplectic method to two known results.

In [12,13] we studied the dimensional reduction of M5 branes wrapping special Lagrangian cycles of a Calabi-Yau threefold and showed explicitly that it led to Bogomol'nyi-Prasad-Sommerfield (BPS) 2-branes coupled to the five-dimensional $\mathcal{N} = 2$ hypermultiplets with constant universal axion ($F = da = 0$). The case with nontrivial complex structure moduli led to constraint equations on the solution that turned out to be of the attractor type. We will not reproduce the entire calculation here, but rather only show enough to demonstrate how the symplectic method greatly reduces the effort involved.

The $D = 5$ spacetime metric due to the presence of the 2-brane was found to be of the form

$$ds^2 = (-dt^2 + dx_1^2 + dx_2^2) + e^{-2\sigma}(dx_3^2 + dx_4^2), \tag{105}$$

where (x^1, x^2) define the spatial directions tangent to the brane and (x^3, x^4) define those transverse to it. The constraint equations on the dilaton and moduli are

$$\begin{aligned}
 d(e^{-(\sigma/2)}) &= \langle d\mathcal{H}|V\rangle = \langle d\mathcal{H}|\bar{V}\rangle, \\
 dz^i &= -e^{\sigma/2}G^{i\bar{j}}\langle d\mathcal{H}|U_{\bar{j}}\rangle, \\
 dz^{\bar{i}} &= -e^{\sigma/2}G^{\bar{i}j}\langle d\mathcal{H}|U_j\rangle, \tag{106}
 \end{aligned}$$

where

$$|\mathcal{H}\rangle = \begin{pmatrix} H^I \\ \tilde{H}_I \end{pmatrix} \tag{107}$$

is taken to be dependent only on the (x^3, x^4) coordinates, such that the moduli dependence is carried exclusively by

$|V\rangle$ and $|U\rangle$. The field equations are straightforwardly satisfied if $|\mathcal{H}\rangle$ is taken to be radial and harmonic in the transverse plane, i.e.

$$|\Delta\mathcal{H}\rangle = 0, \tag{108}$$

which is generally solved by

$$|\mathcal{H}\rangle = |\lambda\rangle + \ln r|\varpi\rangle, \tag{109}$$

where r is the radial coordinate in the (x^3, x^4) plane, $|\lambda\rangle$ is an arbitrary constant, and

$$|\varpi\rangle = \begin{pmatrix} q^I \\ \tilde{q}_I \end{pmatrix}, \tag{110}$$

defines constant ‘‘electric’’ and ‘‘magnetic’’ charges excited by the wrapping of the M5 brane over each homology cycle on the submanifold \mathcal{M} . It follows then that

$$|d\mathcal{H}\rangle = \frac{dr}{r}|\varpi\rangle \quad \text{and} \quad |\star d\mathcal{H}\rangle = d\varphi|\varpi\rangle, \tag{111}$$

where φ is the angular coordinate in the (x^3, x^4) plane. We take the axions vector to be of the simple form

$$|d\Xi\rangle = \pm|\star d\mathcal{H}\rangle = \pm d\varphi|\varpi\rangle. \tag{112}$$

The dilaton equation (89) is now

$$(\Delta\sigma)\star 1 + e^\sigma\left\langle d\Xi\Big|_{\wedge}\star d\Xi\right\rangle = 0. \tag{113}$$

The first term of (113) gives

$$\begin{aligned}
 (\Delta\sigma)\star 1 &= -2e^\sigma\langle\star d\mathcal{H}|V\rangle \wedge \langle\bar{V}|d\mathcal{H}\rangle \\
 &\quad - 2e^\sigma G^{i\bar{j}}\langle\star d\mathcal{H}|U_{\bar{j}}\rangle \wedge \langle U_i|d\mathcal{H}\rangle. \tag{114}
 \end{aligned}$$

Now, with the knowledge that

$$\left\langle\star d\mathcal{H}\Big|_{\wedge}d\mathcal{H}\right\rangle \propto \langle\varpi|\varpi\rangle = 0, \tag{115}$$

as well as

$$\begin{aligned}
 \left\langle d\Xi\Big|_{\wedge}\star d\Xi\right\rangle &= 2\langle\star d\mathcal{H}|V\rangle \wedge \langle\bar{V}|d\mathcal{H}\rangle \\
 &\quad + 2G^{i\bar{j}}\langle\star d\mathcal{H}|U_{\bar{j}}\rangle \wedge \langle U_i|d\mathcal{H}\rangle, \tag{116}
 \end{aligned}$$

it is clear that the second term of (113) exactly cancels the first.

The moduli equations involve a slightly longer calculation. The first term of (90) gives

$$\begin{aligned}
 (\Delta z^i)\star 1 &= e^\sigma G^{i\bar{j}}G^{l\bar{m}}G^{k\bar{n}}C_{\bar{j}\bar{m}\bar{n}}\langle\star d\mathcal{H}|U_l\rangle \wedge \langle d\mathcal{H}|U_k\rangle \\
 &\quad + e^\sigma G^{i\bar{j}}\langle\star d\mathcal{H}|\bar{V}\rangle \wedge \langle d\mathcal{H}|U_{\bar{j}}\rangle \\
 &\quad + e^\sigma G^{i\bar{j}}\langle d\mathcal{H}|V\rangle \wedge \langle\star d\mathcal{H}|U_{\bar{j}}\rangle \\
 &\quad - e^\sigma G^{p\bar{j}}G^{r\bar{k}}\Gamma_{rp}^i\langle d\mathcal{H}|U_{\bar{k}}\rangle \wedge \langle\star d\mathcal{H}|U_{\bar{j}}\rangle. \tag{117}
 \end{aligned}$$

The second term is

$$\Gamma_{rp}^i dz^r \wedge \star dz^p = e^\sigma G^{p\bar{j}} G^{r\bar{k}} \Gamma_{rp}^i \langle d\mathcal{H}|U_{\bar{k}} \rangle \wedge \langle \star d\mathcal{H}|U_{\bar{j}} \rangle, \quad (118)$$

which cancels the last term of (117). Using (61), the last term of the moduli equation becomes

$$\begin{aligned} & \frac{1}{2} e^\sigma G^{i\bar{j}} \left\langle d\Xi | \partial_{\bar{j}} \Lambda | \star d\Xi \right\rangle \\ &= -e^\sigma G^{i\bar{j}} G^{l\bar{m}} G^{k\bar{n}} C_{\bar{j}\bar{m}\bar{n}} \langle \star d\mathcal{H}|U_l \rangle \wedge \langle d\mathcal{H}|U_k \rangle \\ & \quad - e^\sigma G^{i\bar{j}} \langle \star d\mathcal{H}|\bar{V} \rangle \wedge \langle d\mathcal{H}|U_{\bar{j}} \rangle \\ & \quad - e^\sigma G^{i\bar{j}} \langle d\mathcal{H}|V \rangle \wedge \langle \star d\mathcal{H}|U_{\bar{j}} \rangle, \end{aligned} \quad (119)$$

exactly canceling the remaining terms of (117).

The second example we wish to consider is that of [16]. The result discussed therein was that of instanton couplings to the hypermultiplets. Instantons are of course Euclidean solutions of the theory and may be thought of as being magnetically dual to the 2-branes discussed above (in $D = 5$). In order to consider this result, we analytically continue the action of the theory from a Minkowski background to a Euclidean metric. This is achieved by an ordinary Wick rotation which has the effect of changing $|\Xi\rangle \rightarrow i|\Xi\rangle$ in the field and SUSY equations. Furthermore, the vector $|\mathcal{H}\rangle$ satisfying the harmonic condition in Euclidean $D = 5$ space now becomes

$$|\mathcal{H}\rangle = |\lambda\rangle + \frac{1}{3r^3} |\varpi\rangle, \quad (120)$$

instead of (109), with (110) still valid. Note that the coordinate r is now radial in all the five flat dimensions. Hence

$$|d\mathcal{H}\rangle = -\frac{dr}{r^4} |\varpi\rangle. \quad (121)$$

Rewriting the constraint equations on the dilaton and moduli in our language we get

$$\begin{aligned} d(e^{\sigma/2}) &= -\langle d\mathcal{H}|V \rangle = -\langle d\mathcal{H}|\bar{V} \rangle, \\ dz^i &= e^{-(\sigma/2)} G^{i\bar{j}} \langle d\mathcal{H}|U_{\bar{j}} \rangle, \\ dz^{\bar{i}} &= e^{-(\sigma/2)} G^{\bar{i}j} \langle d\mathcal{H}|U_j \rangle \end{aligned} \quad (122)$$

while the axions can be written in the form

$$|d\Xi\rangle = -ie^{-\sigma} |\Lambda d\mathcal{H}\rangle. \quad (123)$$

Now the dilaton and moduli equations can be shown to be satisfied in a very similar manner as that of the first example and the $|\Xi\rangle$ field equation reduces to the harmonic condition on $|\mathcal{H}\rangle$:

$$d^\dagger[e^\sigma |\Lambda d\Xi\rangle] = -id^\dagger |\Lambda \Lambda d\mathcal{H}\rangle = i|\Delta\mathcal{H}\rangle = 0, \quad (124)$$

where the fact that $\Lambda^{-1} = -\Lambda$ was used. The hyperini variations (102) and (103) vanish for $\epsilon_1 = \pm\epsilon_2$ as follows:

$$\begin{aligned} \delta_\epsilon \xi_1^0 &= -ie^{-(\sigma/2)} \langle V|\Lambda|d\mathcal{H}\rangle + e^{-(\sigma/2)} \langle V|d\mathcal{H}\rangle \\ &= -i2e^{-(\sigma/2)} \langle V|V\rangle \langle \bar{V}|d\mathcal{H}\rangle \\ & \quad - i2e^{-(\sigma/2)} G^{i\bar{j}} \langle V|U_{\bar{j}} \rangle \langle U_i|d\mathcal{H}\rangle \\ & \quad - e^{-(\sigma/2)} \langle V|d\mathcal{H}\rangle + e^{-(\sigma/2)} \langle V|d\mathcal{H}\rangle = 0, \end{aligned} \quad (125)$$

where (47) was used. Also

$$\begin{aligned} \delta_\epsilon \xi_1^{\hat{i}} &= -ie^{-(\sigma/2)} e^{\hat{i}j} \langle U_j|\Lambda|d\mathcal{H}\rangle \\ & \quad - e^{-(\sigma/2)} e^{\hat{i}k} G^{\bar{k}j} \langle U_j|d\mathcal{H}\rangle \\ &= -i2e^{-(\sigma/2)} e^{\hat{i}j} \langle U_j|V\rangle \langle \bar{V}|d\mathcal{H}\rangle \\ & \quad - i2e^{-(\sigma/2)} e^{\hat{i}j} G^{m\bar{n}} \langle U_j|U_{\bar{n}} \rangle \langle U_m|d\mathcal{H}\rangle \\ & \quad - e^{-(\sigma/2)} e^{\hat{i}j} \langle U_j|d\mathcal{H}\rangle - e^{-(\sigma/2)} e^{\hat{i}j} \langle U_j|d\mathcal{H}\rangle \\ &= 2e^{-(\sigma/2)} e^{\hat{i}j} \langle U_j|d\mathcal{H}\rangle - 2e^{-(\sigma/2)} e^{\hat{i}j} \langle U_j|d\mathcal{H}\rangle = 0, \end{aligned} \quad (126)$$

where (49) was used. Similarly $\delta_\epsilon \xi_2^0 = 0$ and $\delta_\epsilon \xi_2^{\hat{i}} = 0$ are satisfied.

This is as far as we will go in demonstrating the use of the symplectic method in analyzing the hypermultiplets. We note that the calculations shown here are considerably shorter than their counterparts performed without using the symplectic language. In fact, the original details would indeed be too long to reasonably reproduce in print.

VI. CONCLUSION

In this work, we took a close look at the geometries responsible for the behavior of the hypermultiplet fields of five-dimensional $\mathcal{N} = 2$ supergravity with particular emphasis on the symplectic structure arising from the underlying topology of the Calabi-Yau subspace. We proposed the use of the mathematics of symplectic vector spaces to recast the theory in explicit symplectic covariance. We argued that this greatly simplifies the effort involved in analyzing the hypermultiplet fields, with or without gravitational coupling, and demonstrated this by partially applying it to two known results.

The five-dimensional hypermultiplets sector is hardly the only one exhibiting symplectic symmetry. In fact, the structures reviewed here are almost always discussed in the literature in the context of the four-dimensional vector multiplets where very similar analytical difficulties arise. In fact, it is because the special Kähler geometry of the $D = 4$ vector multiplets is so well researched that it became possible to apply similar techniques to the (c-mapped) $D = 5$ hypermultiplets. It is then natural to attempt to extend the symplectic formulation to the $D = 4$ theory as well as to any other theory, supersymmetric or not, exhibiting hidden or explicit Sp covariance. One hopes that this will help simplify tedious calculations as well as contribute to further understanding the behavior of such theories.

Finally, an immediate application of the symplectic formulation to analyzing solution ansätze for various possible situations seems to be the next natural thing to do. For example, an analysis of branes coupled to the full set of hypermultiplet fields can now be greatly simplified, even if

one is interested in a general understanding, rather than a detailed solution. Further classification of such solutions becomes a more manageable task. In the future, we plan to explore at least some of the above possibilities.

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