PHYSICAL REVIEW D **79,** 085001 (2009)

Hard thermal loops in static external fields

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(Received 6 February 2009; published 1 April 2009)

We examine, in the imaginary-time formalism, the high temperature behavior of n-point thermal loops in static Yang-Mills and gravitational fields. We show that in this regime, any hard thermal loop gives the same leading contribution as the one obtained by evaluating the loop integral at zero external energies and momenta.

DOI: 10.1103/PhysRevD.79.085001 PACS numbers: 11.10.Wx

I. INTRODUCTION

In thermal field theory, much attention has been devoted to the study of the high temperature behavior of Green functions [1–4], when all their external energies and momenta are much smaller than the temperature T. These so-called hard thermal loops [5,6] are an important ingredient in a resummation procedure which is necessary to control infrared divergences and give meaningful results in perturbation theory [7]. These thermal amplitudes enjoy some simple gauge invariant and symmetry properties, being in general nonlocal functionals of the external fields. There are two special cases, namely, the static and the long wavelength limit, when these amplitudes become local functions of the external fields, which are independent of energies as well as of momenta. Nevertheless, these two limits yield in general distinct results for hard thermal self-energy functions [8–14]. Moreover, these limits also lead to different hard thermal loop effective actions [15,16].

Such a behavior may be more readily understood in the analytically continued imaginary-time formalism [17–19]. In this approach, the bosonic Green functions, for example, are defined at integral values of $k_{l0}/2\pi iT$, where k_{l0} is the energy of the lth external particle. Hence, any factors like $\exp(k_{l0}/T)$ can be set equal to unity. This suppresses any factors which could be exponentially increasing after analytic continuation to general values of k_{l0}. Then, the integrands of the thermal amplitudes become rational functions of k_{l0} and various limits can be taken.

Let us consider a typical term which appears in the integrand of any one-loop thermal diagram, namely:

$$f(k_0, \vec{k}, \vec{Q}) = \frac{N(k_0 + P) - N(Q)}{k_0 + P - Q}, \tag{1}$$

where N is the thermal distribution function for bosons or fermions

$$N(z) = \frac{1}{e^{z/T} \mp 1}. \tag{2}$$

$Q = |\vec{Q}|$ is the magnitude of the internal momentum, $P = |\vec{Q} + \vec{k}|$ and k_0, $\vec{k}$ are some linear combinations of external energies and momenta. As we have mentioned, before analytic continuation all k_{l0} are integer multiples of $2\pi iT$, so that one may use the relation:

$$N(k_0 + P) = N(P), \tag{3}$$

in which case (1) can be written in the form:

$$\tilde{f}(k_0, \vec{k}, \vec{Q}) = \frac{N(P) - N(Q)}{k_0 + P - Q}. \tag{4}$$

However, after analytic continuation, there is yet another important difference between (1) and (4): whereas f is an analytic function at $k_\mu = 0$, $\tilde{f}$ is no longer analytic at this point. For instance, in the static limit $k_0 = 0$, (4) may be approximated when $\vec{k} \to 0$ by $dN(Q)/dQ$. On the other hand, (4) would vanish in the long wavelength limit, when $\vec{k} \to 0$ first. Furthermore, it is easy to verify that if we put in the thermal loop, from the start, all $k_{l\mu} = 0$, the corresponding term in the integrand would be just:

$$f(0, 0, Q) = \frac{dN(Q)}{dQ} \equiv N'(Q), \tag{5}$$

in accordance with the result obtained in the static limit. This agreement occurs only in the static limit, because this is the single case when analytic continuation does not modify the original function. Thus, only the local form obtained in the static limit for a general hard thermal loop is equivalent, to leading order, to the result got by setting in the loop integrand all external energies and momenta equal to zero.

The above general result can be explicitly verified in thermal perturbation theory. We examine the 2-point functions in thermal Yang-Mills and gravity theories in Sec. II, where we establish this result in a way which clearly generalizes to higher-point functions. Then, in Sec. III, we exemplify some relevant aspects of the above argument in the context of gluon and graviton 3-point functions at finite temperature. We conclude this paper with a brief discussion in Sec. IV.

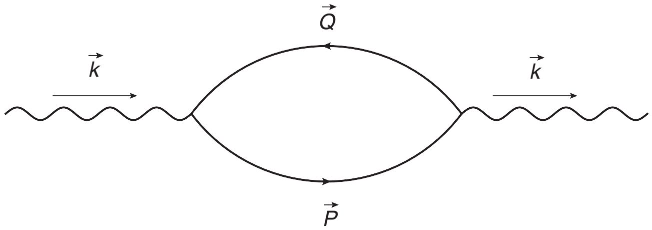

FIG. 1. A 2-point self-energy diagram. Wavy lines denote external gluons or gravitons and solid lines indicate internal thermal particles.

II. THE 2-POINT FUNCTION

Although the high temperature limit of self-energy functions is well known, we briefly discuss them here in order to present the main points of the argument in a simple form which can be easily generalized. Thus, let us consider the diagram in Fig. 1 where, for our purpose, we need not specify the nature of the particles in the loop. Furthermore, since we treat in a unified way thermal loops in either external Yang-Mills or graviton fields, we suppress for simplicity the color and Lorentz indices. Then, the contribution of this diagram may be written in the form:

$$\Pi = \frac{\mathcal{C}_2}{(2\pi)^3}\int d^3Q I, \tag{6}$$

where $\mathcal{C}_2$ is a Casimir for the internal particles and

$$\begin{aligned} I &= T\sum_{Q_0}\frac{1}{\vec{Q}^2 - Q_0^2}\frac{1}{\vec{P}^2 - (Q_0+k_0)^2}t(Q_0) \\ &= \frac{1}{2\pi i}\int_C dQ_0 N(Q_0)\frac{1}{\vec{Q}^2 - Q_0^2}\frac{1}{\vec{P}^2 - (Q_0+k_0)^2}t(Q_0). \end{aligned} \tag{7}$$

Here the sum is over even (odd) integer values of $Q_0/\pi i T$ and C is a contour surrounding all poles of N in an anticlockwise sense. The numerator t is a tensor and we have indicated, for simplicity, only its dependence on the internal energy Q_0. In fact, in the Yang-Mills theory t is a quadratic function of the energies and momenta, whereas in gravity t becomes a function of fourth degree in the energies and momenta. Evaluating (7) in terms of the poles outside C, and writing $Q = |\vec{Q}|$, $P = |\vec{P}|$, we get:

$$\begin{aligned} I = -\frac{1}{4PQ}\Big\{ & N(Q)t(Q)\left[\frac{1}{Q-P+k_0} - \frac{1}{Q+P+k_0}\right] \\ & + (Q,P \to -Q,-P) + N(P)t(P-k_0) \\ & \times\left[\frac{1}{P-Q-k_0} - \frac{1}{P+Q-k_0}\right] \\ & + (Q,P\to -Q,-P)\Big\}, \end{aligned} \tag{8}$$

where we used the relation $N(P-k_0) = N(P)$ etc., since in the imaginary-time formalism $k_0/2\pi i T$ is an integer.

We now consider the leading high temperature contribution in the static case ($k_0 = 0$), which comes from the region $|\vec{k}| \ll P$, $Q \sim T$. To this end, we can make appropriate expansions like:

$$t(P) = t(Q) + (P-Q)t'(Q) + \cdots \tag{9}$$

and similar ones for $N(P)$. Then, it is easy to check that to leading order, which is obtained in the limit $|\vec{k}|/Q \to 0$, (8) reduces to:

$$\begin{aligned} I_S = -\frac{1}{4Q^2}\Big\{ & \left[N'(Q) - \frac{N(Q)}{Q}\right][t(Q) + t(-Q)] \\ & + N(Q)[t'(Q) - t'(-Q)]\Big\}, \end{aligned} \tag{10}$$

where we used the relation

$$N(Q) + N(-Q) \pm 1 = 0 \tag{11}$$

and omitted a T-independent term. At this point we note that in the integral (6), only those components of I_S with an even number of Q_i, and hence an even number of Q_0, do actually contribute. Thus $t(Q_0)$ becomes effectively an even function of Q_0 while $t'(Q_0)$ becomes an odd function, so that one may further simplify (10) as:

$$I_S = -\frac{1}{2Q^2}\left[N'(Q)t(Q) + N(Q)t'(Q) - \frac{N(Q)}{Q}t(Q)\right]. \tag{12}$$

Now, let us compare this result with the one obtained by setting, from the start, $k_\mu = 0$ in the diagram in Fig. 1. This gives (compare with (7)):

$$I_0 = \frac{1}{2\pi i}\int_C dQ_0 N(Q_0)\frac{1}{(Q_0^2 - Q^2)^2}t(Q_0). \tag{13}$$

We can evaluate the Q_0 integral in (13) by residues, in terms of the double poles outside C. Using Eq. (11) together with the properties mentioned afterwards, one can express the contributions from the double poles at $Q_0 = -Q$ in terms of those at $Q_0 = Q$. Then, omitting a T-independent term, one readily gets the result:

$$I_0 = -2\frac{d}{dQ_0}\left[\frac{N(Q_0)t(Q_0)}{(Q_0+Q)^2}\right]_{Q_0=Q}. \tag{14}$$

This is actually equal to the result given in (12), as expected from the argument given in the previous section.

Finally, after performing the d^3Q integration in (6), one obtains the well known leading T^2 contribution for the static gluon 2-point function, while the static graviton 2-point function gives a leading contribution of order T^4.

III. THE 3-POINT FUNCTION

Consider the triangle diagram shown in Fig. 2. Its contribution in the imaginary-time formalism may be written in the form:

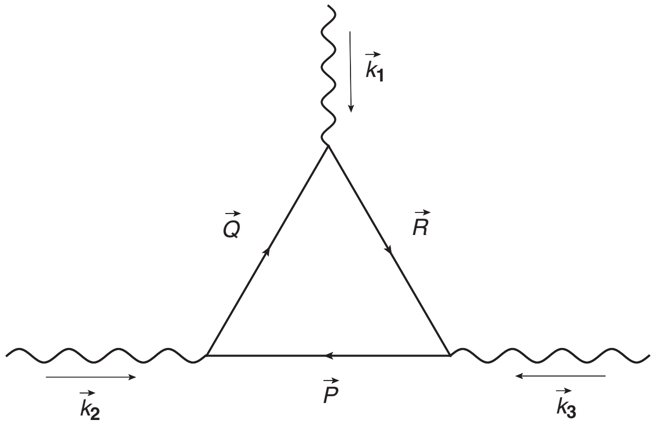

FIG. 2. One-loop 3-point vertex diagram. Color and Lorentz indices are suppressed.

$$\Gamma = \frac{\mathcal{C}_3}{(2\pi)^4 i} \int d^3Q \int_C dQ_0 N(Q_0) \frac{1}{\vec{Q}^2 - Q_0^2} \times \frac{1}{\vec{R}^2 - (Q_0 + k_{10})^2} \frac{1}{\vec{P}^2 - (Q_0 - k_{30})^2} t(Q_0). \quad (15)$$

Here the Casimir $\mathcal{C}_3$ gives the number of internal degrees of freedom. The tensor t, whose color and Lorentz indices have been suppressed for conciseness, is a function of the energies and momenta of degree three or six in the Yang-Mills or gravity theory, respectively. Evaluating the Q_0 integral by residues and using $N(P + k_{30}) = N(P)$ etc., we obtain

$$\Gamma = -\frac{1}{(2\pi)^3} \int d^3Q[I + I' - (I_1 + I_1' + I_2 + I_2' + I_3 + I_3')], \quad (16)$$

where

$$I = \frac{1}{8PQR}\left[\frac{t(Q)N(Q)}{D_3 D_1} + \frac{t(P + k_{30})N(P)}{D_2 D_3} + \frac{t(R - k_{10})N(R)}{D_1 D_2}\right], \quad (17)$$

$$I_1 = \frac{1}{8PQR}\left[\frac{t(Q)N(Q)}{D_3 E_1} + \frac{t(P + k_{30})N(P)}{D_3 F_2} + \frac{t(-R - k_{10})N(-R)}{E_1 F_2}\right]. \quad (18)$$

Here we have used the notation

$$D_1 = k_{10} + Q - R, \qquad E_1 = k_{10} + Q + R, \qquad F_2 = k_{20} - R - P \quad (19)$$

and cyclic permutations ($k_1 \rightarrow k_2 \rightarrow k_3$ and $Q \rightarrow P \rightarrow R$), so that

$$D_1 + D_2 + D_3 = 0. \quad (20)$$

I_2 and I_3 are obtained by cyclic permutations on (18) and:

$$I'(P, Q, R) = I(-P, -Q, -R), \qquad I_1'(P, Q, R) = I_1(-P, -Q, -R). \quad (21)$$

In the static limit $k_{i0} = 0$, the leading contributions at high T come from the region $|\vec{k}_i| \ll P, Q, R \sim T$. One can thus make appropriate expansions like

$$t(P) = t(Q) + (P - Q)t'(Q) + \frac{1}{2}(P - Q)^2 t''(Q) + \cdots \quad (22)$$

Let us first consider the contributions to (16) associated with the $t(Q)$ term in the above expansions. Then, using similar expansions for $N(P)$, etc., as well as the relation (20), one finds that the corresponding terms in (17), (18), and (21), give:

$$I_a = I_a' = -\frac{1}{16} \frac{N''(Q)}{Q^3} t(Q), \quad (23)$$

$$I_{1a} = I_{1a}' = \frac{1}{16} \frac{N(Q) - QN'(Q)}{Q^5} t(Q). \quad (24)$$

Inserting (23) and (24), etc., into (16), we get the contribution

$$\Gamma_a = \frac{\mathcal{C}_3}{8} \int \frac{d^3Q}{(2\pi)^3} \frac{1}{Q^5}[Q^2 N''(Q) - 3QN'(Q) + 3N(Q)]t(Q). \quad (25)$$

We must now include the contributions to (16) associated with the terms $t'(Q)$ and $t''(Q)$ in expansions like (22). The calculation is straightforward and the corresponding expressions are:

$$\Gamma_b = \frac{\mathcal{C}_3}{8} \int \frac{d^3Q}{(2\pi)^3} \frac{1}{Q^4}[2QN'(Q) - 3N(Q)]t'(Q), \quad (26)$$

$$\Gamma_c = \frac{\mathcal{C}_3}{8} \int \frac{d^3Q}{(2\pi)^3} \frac{1}{Q^3} N(Q)t''(Q). \quad (27)$$

Thus, in the static case, the three terms (25)–(27), contribute to (16) as

$$\Gamma_S = \frac{\mathcal{C}_3}{8} \int \frac{d^3Q}{(2\pi)^3} \frac{1}{Q^5}\{Q^2[N''(Q)t(Q) + 2N'(Q)t'(Q) + N(Q)t''(Q)] - 3Q[N'(Q)t(Q) + N(Q)t'(Q)] + 3N(Q)t(Q)\}. \quad (28)$$

Let us next compare this result with the one obtained from the diagram in Fig. 2, when all external energies and momenta are vanishing. We then get:

$$\Gamma_0 = \frac{\mathcal{C}_3}{(2\pi)^4 i} \int d^3Q \int_C dQ_0 N(Q_0) \frac{1}{(Q^2 - Q_0^2)^3} t(Q_0). \quad (29)$$

One can evaluate the Q_0 integral by residues, in terms of the triple poles outside the contour C. Then, using the relation (11) and the properties mentioned afterward, we can write Γ_0 in the simple form:

$$\Gamma_0 = \mathcal{C}_3 \int \frac{d^3Q}{(2\pi)^3} \frac{d^2}{dQ_0^2} \left[\frac{N(Q_0)t(Q_0)}{(Q_0 + Q)^3} \right]_{Q_0=Q}. \tag{30}$$

This result is in fact equivalent to the one in (28), as expected from the general argument given in Sec. I. Clearly, this feature will also hold for higher-point functions.

IV. DISCUSSION

We have studied, in the analytically continued imaginary-time formalism, the behavior of hard thermal loops in static external Yang-Mills and gravitational fields. Amplitudes calculated in this formalism naturally give rise to retarded (advanced) hard thermal functions [20], whose imaginary parts vanish in the static limit. We have shown that for any hard thermal loop, the leading contributions in the static limit are the same as those obtained by evaluating the loop integral at zero external energy-momentum. This is consistent with the behavior of thermal self-energy loops noticed in [8–14]. This result may be useful to simplify the calculation of static limits in thermal field theories, which are relevant to study some physical properties of systems in thermal equilibrium, like plasma frequencies and screening lengths.

Although the above relationship holds both in the Yang-Mills as well as in the gravity theory, there is an important difference between these cases. One can show [15] that in the Yang-Mills theory, all higher order point functions vanish to leading order in the static limit. This fact can also be simply understood from our previous argument since, after setting in the thermal loop all external energies and momenta equal to zero, one can see by power counting that higher-point functions can no longer yield quadratic T^2 contributions. This implies, in particular, that (30) should vanish in Yang-Mills theory, a fact that also follows from Bose symmetry. Indeed, there appears in this case an antisymmetric color factor which requires another factor with an antisymmetric dependence on external momenta and energy. But such a factor will necessarily vanish in the zero energy-momenta limit.

On the other hand, in gravity, such an antisymmetric dependence cannot be present and then, since in (30) t is a function of sixth degree in Q, it will yield a leading contribution of order T^4. The fact that there are, for all n-point functions, static T^4 terms in gravity is of course connected with the quartic ultraviolet divergence of the zero-temperature loops. These static thermal functions are related by Ward identities [21], which are associated with the invariance of this system under time-independent coordinate transformations.

ACKNOWLEDGMENTS

J. F. is grateful to J. C. Taylor for a very helpful correspondence. He would also like to thank F. T. Brandt for useful discussions. SHP is supported by CNPq, proc. num. 150920/2007-5. This work was supported in part by CNPq and FAPESP, Brazil.

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