Uniqueness of the Fock quantization of a free scalar field on S^1 with time dependent mass

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We analyze the quantum description of a free scalar field on the circle in the presence of an explicitly time-dependent potential, also interpretable as a time-dependent mass. Classically, the field satisfies a linear wave equation of the form $\ddot{\xi} - \xi'' + f(t)\xi = 0$. We prove that the representation of the canonical commutation relations corresponding to the particular case of a massless free field ($f = 0$) provides a unitary implementation of the dynamics for sufficiently general mass terms, $f(t)$. Furthermore, this representation is uniquely specified, among the class of representations determined by S^1-invariant complex structures, as the only one allowing a unitary dynamics. These conclusions can be extended in fact to fields on the two-sphere possessing axial symmetry. This generalizes a uniqueness result previously obtained in the context of the quantum field description of the Gowdy cosmologies, in the case of linear polarization and for any of the possible topologies of the spatial sections.

I. INTRODUCTION

In recent years, a Fock quantization allowing a unitary dynamics of the linearly polarized Gowdy T^3 cosmological model [1] has been constructed [2–4]. Moreover, this quantization has been shown to be unique under very reasonable conditions, namely, S^1-invariance and unitarity of the dynamics for the metric field that describes the local degrees of freedom of these spacetimes [5,6]. These results are ground breaking, since uniqueness in the quantization of cosmological models is rare, and previous attempts on the quantization of this Gowdy model [7,8]—the simplest nonhomogeneous cosmological model—had failed even in achieving a unitary implementation of the dynamical evolution [2,6,8,9].

Following the same methods and ideas, a Fock quantization with unitary dynamics was later achieved [10,11] for the linearly polarized Gowdy models with the other two allowed spatial topologies, i.e., the $S^1 \times S^2$ and S^3 models. Uniqueness was proved again for these alternate topologies [12] (see also the partial discussion in [11]).

For any of the three considered topologies, the local degrees of freedom of the model are parametrized by a single scalar field, effectively living on S^1 in the T^3 case, and on S^2 in the remaining cases. Moreover, in all cases the dynamics is governed by linear wave equations similar to those of the free fields, though with a time-dependent mass term. Alternatively, the mass term can be regarded as a time-dependent potential.

On the one hand, it is clear that the compactness of the effective space plays a role in allowing the unitary implementation of the dynamics, and in the uniqueness of the quantization [5,6] (see also [13] for a detailed account of the role played in this respect by the long range behavior in the noncompact case). On the other hand, it is not known how the above results depend on the details of the models, e.g., on the specific form of the mass term appearing in the field equations. Since the results are valid for the very different mass terms appearing in the Gowdy T^3 model and in the $S^1 \times S^2$ and S^3 cases, as well as for a constant mass term, of course, one may suspect that the unitarity and uniqueness results may not be too sensitive to the particular time dependence of the mass. However, nothing in the works mentioned above allows one to reach this conclusion, since methods specially adapted to the specific mass term of each of the models were used.

In the present work we will show that, for fields on S^1 (as well as for axisymmetric fields on S^2, see below), the commented results about the unitary implementation of the dynamics and the uniqueness of the representation are generic, i.e., they are valid essentially as long as the

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dynamics is of the specified type, regardless of the particular form of the mass term—provided only that this term is given by a sufficiently regular function.

From a more general perspective, it is important to emphasize that the quantum representations in which the dynamics is unitary coincide, in all cases of a generic time-dependent mass, with the representation that is naturally associated with the free massless field. So, the general belief that in quantum field theory different dynamics require different representations is not necessarily true in the compact case, and therefore in some important cosmological models. Somehow, the field with a time-dependent mass term possesses a dynamical behavior which—in the compact case—is sufficiently close to the free massless evolution, so that both dynamics are implemented as unitary transformations in the same representation.

This paper is organized as follows. First, we summarize some aspects of the two-Killing vectors reduction of general relativity in Sec. II, as a motivation for the family of scalar fields that we are going to consider. Then, we specify this family in Sec. III and discuss its Fock quantization in the representation that is naturally associated with the free massless scalar field. In Sec. IV we show that this Fock quantization provides a unitary implementation of our field dynamics, even if the system contains a time-dependent mass term. Section V proves that the considered Fock quantization is in fact unique, inasmuch as it is the only Fock representation that is invariant under the symmetry group of the field equations and allows a unitary quantum evolution. Finally, Sec. VI presents our conclusions.

II. MOTIVATION FROM THE TWO-KILLING VECTORS REDUCTION OF GENERAL RELATIVITY

To motivate the interest of the systems that we will study in this work, we will start by presenting a brief summary on the reduction of general relativity to spacetimes that possess two commuting spacelike Killing vectors. We consider only the case in which the isometry group generated by these Killing vectors is compact. Moreover, we restrict our discussion to spacetimes for which each of these Killing vectors is hypersurface orthogonal, a situation which is often called the linearly polarized case.

In addition, we assume that the spacetime is globally hyperbolic, so that it is possible to perform a $3 + 1$ decomposition in sections of constant time t. Since the isometry group generated by the Killing vectors is Abelian (with non-null orbits), one can introduce spatial coordinates $\{\theta, \phi, \rho\}$ such that ∂_ϕ and ∂_ρ are the Killing vector fields, and the spacetime metric is independent of ϕ and ρ [14]. As a consequence of this independence, the integral $\int d\phi d\rho = V_0$ (which is finite because the isometry group is compact) appears as a global factor in the gravitational action of General Relativity. We absorb its numerical value

in Newton's constant G by adopting units such that $8\pi G = V_0$.

On the other hand, one can fix the gauge freedom associated with the momentum constraints (also called diffeomorphism constraints) of the two directions ϕ and ρ by demanding the vanishing of the components $h_{\theta\phi}$ and $h_{\theta\rho}$ of the induced metric [15,16]. This gauge fixing [16], together with the assumption of hypersurface orthogonality, implies that the metric can be written globally in a diagonal form, except for the presence of a θ-component of the shift. The reduced metric can be parametrized in the following way [2]:

$$ds^2 = e^{\gamma - \psi}[-\tau^2 \underset{\sim}{N}^2 dt^2 + (d\theta + N^\theta dt)^2] + e^{-\psi}\tau^2(d\phi)^2 + e^{\psi}(d\rho)^2. \tag{1}$$

This reduced system still possesses two constraints: the densitized Hamiltonian constraint, $\tilde{H}$, and the momentum constraint of the θ-direction, H_θ. They take the expressions:

$$\tilde{H} = \tfrac{1}{2}(\psi'\tau)^2 + \tfrac{1}{2}p_\psi^2 + \tau(2\tau'' - \tau'\gamma' - p_\gamma p_\tau). \tag{2}$$

$$H_\theta = -2p_\gamma' + p_\tau\tau' + p_\gamma\gamma' + p_\psi\psi'. \tag{3}$$

In these formulas, the p's denote the momenta canonically conjugate to the metric fields, and the prime stands for the spatial derivative with respect to θ. In principle, one may introduce an additional gauge fixing to remove these constraints and further reduce the system. The particular details depend on those of the family of spacetimes that one considers, such as the isometry group and the spatial topology. Nonetheless, let us admit for the moment that one can adopt (globally) a gauge in which $\underset{\sim}{N}\tau = 1$. The generator of the time evolution is then the integrated Hamiltonian constraint $\int d\theta(\tilde{H}/\tau)$.

It is not difficult to check that the resulting equation of motion for the field ψ is

$$\ddot{\psi} + \frac{\dot{\tau}}{\tau}\dot{\psi} - \frac{\tau'}{\tau}\psi' - \psi'' = 0. \tag{4}$$

The dot denotes the derivative with respect to t. Introducing the rescaling $\psi = (\xi/\sqrt{\tau})y(\theta)$, where $y(\theta)$ is a fixed function of the spatial coordinate θ only, the above field equation translates into the following equation for ξ:

$$\ddot{\xi} - \xi'' - 2\frac{y'}{y}\xi' + f(t, \theta)\xi = 0, \tag{5}$$

where

$$f(t, \theta) = \left(\frac{\dot{\tau}}{2\tau}\right)^2 - \frac{\ddot{\tau}}{2\tau} - \left(\frac{\tau'}{2\tau}\right)^2 + \frac{\tau''}{2\tau} - \frac{y''}{y}. \tag{6}$$

Obviously, Eq. (5) becomes a two-dimensional wave equation with a spatially constant potential $f = f(t)$ if τ is independent of θ and the function y is chosen equal to the unity. Explicitly,

$$\ddot{\xi} - \xi'' + f(t)\xi = 0, \qquad f(t) = \left(\frac{\dot{\tau}}{2\tau}\right)^2 - \frac{\ddot{\tau}}{2\tau}. \qquad (7)$$

In addition, if τ admits an expression of the form $\tau(t, \theta) = w(t)z(\theta)$, one gets a spatially constant potential with the choice $y(\theta) = \sqrt{z(\theta)}$. The field equation is then

$$\ddot{\xi} - \frac{1}{z(\theta)}[z(\theta)\xi']' + f(t)\xi = 0,$$
$$f(t) = \left(\frac{\dot{w}}{2w}\right)^2 - \frac{\ddot{w}}{2w}. \qquad (8)$$

At least locally, this equation can be understood as the wave equation of a ϕ-independent field ξ propagating in a 3-dimensional spacetime with the static metric

$$g_{ab} = -dt_a dt_b + d\theta_a d\theta_b + z^2(\theta)d\phi_a d\phi_b. \qquad (9)$$

The most interesting situation in which our previous discussion finds a straightforward application is in the case of the linearly polarized Gowdy cosmologies [1]. These cosmological solutions are (globally hyperbolic) spacetimes whose spatial sections are compact and which possess two Killing vector fields with the properties that we have assumed. Actually, the spatial sections must be homeomorphic to either the three-torus, T^3, the three-sphere, S^3, or the three-handle, $S^2 \times S^1$. For these spacetimes, the gauge $\underline{N}\tau = 1$ is indeed allowed [2,10]. In the case of the topology of the three-torus, this gauge is introduced by fixing the freedom associated with the densitized Hamiltonian constraint, what in turn is achieved by choosing the metric function τ (essentially) as the time coordinate, namely, $\tau = Ct$ where C is a constant of motion [2,3]. The corresponding field ξ is defined on the circle, while the potential of the corresponding wave equation is given by the function $1/(2t)^2$. The gauge fixing procedure can be paralleled in the case of the other two topologies, choosing $\tau(t, \theta) = C \sin t \sin\theta$, where C denotes again a constant of motion. In accordance to our above discussion, the choice $y = \sqrt{\sin\theta}$ leads then to a time-dependent potential, which turns out to be given by the function $f(t) = (1 + \csc^2 t)/4$. The corresponding field ξ is defined on the sphere S^2, with the coordinate θ being the zenith angle. This field is axisymmetric, since its only spatial dependence is on θ. Finally, in these circumstances, the term with spatial derivatives in the wave equation (8) can be interpreted just as the Laplacian on the two-sphere acting on the axisymmetric field ξ.

III. FOCK QUANTIZATION OF THE MODEL

A. The classical model

After the above motivation, we can now specify the models on which we will concentrate our discussion. The system which we want to study is a scalar field $\xi(t, \theta)$ propagating in a $(1 + 1)$-spacetime with the topology of $\mathbb{I} \times S^1$ and provided with the static metric $g_{ab} =$

$-dt_a dt_b + d\theta_a d\theta_b$. The time domain $\mathbb{I}$ is an interval of the real line and, in most of the practical situations, will be taken to coincide either with $\mathbb{R}^+$ or with $\mathbb{R}$. In addition, the field is subject to a time-dependent potential of the form $V(\xi) = f(t)\xi^2/2$, where $f(t)$ is a sufficiently regular function on the interval $\mathbb{I}$. As we have already pointed out, this potential can be thought of as a time-dependent mass term. Later in this work we will comment on the extension of our analysis to the case of an axisymmetric field on S^2, instead of a field on the circle.

In the canonical approach, in principle, the system can be described by the action

$$S(t_i, t_f) = \int_{t_i}^{t_f} dt \left[\left(\oint d\theta P\dot{\varphi}\right) - H\right],$$
$$H = \frac{1}{2}\oint d\theta[P^2 + (\varphi')^2 + f\varphi^2], \qquad (10)$$

where H is the Hamiltonian and φ and P are, respectively, the configuration and momentum of the field ξ.

The canonical phase space is the space of Cauchy data $\{(\varphi, P)\} = \{(\xi_{|t_0}, \dot{\xi}_{|t_0})\}$ at some fixed time t_0. The corresponding nonzero Poisson brackets are $\{\varphi(\theta), P(\theta')\} = \delta(\theta - \theta')$, where $\delta(\theta)$ is the Dirac delta on S^1. Varying the action (10) with respect to φ and P, we arrive to the field equations

$$\dot{\varphi} = P, \qquad \dot{P} = \varphi'' - f(t)\varphi, \qquad (11)$$

so that ξ satisfies the linear wave equation

$$\ddot{\xi} - \xi'' + f(t)\xi = 0. \qquad (12)$$

Alternatively to the space of Cauchy data $\Gamma = \{(\varphi, P)\}$, the phase space can be described as the space $S = \{\xi\}$ of solutions to Eq. (12). Both Γ and S are symplectic linear spaces, with the respective symplectic structures (independent of the choice of time section)

$$\sigma[(\varphi_1, P_1), (\varphi_2, P_2)] = \oint d\theta(\varphi_2 P_1 - \varphi_1 P_2), \qquad (13)$$

and

$$\Omega(\xi_1, \xi_2) = \oint d\theta(\xi_2 \partial_t \xi_1 - \xi_1 \partial_t \xi_2). \qquad (14)$$

Since the Hamiltonian (10) does not depend on the spatial variable θ, the field equations are invariant under S^1-translations:

$$T_\alpha: \theta \mapsto \theta + \alpha \quad \forall \ \alpha \in S^1. \qquad (15)$$

So, the translations T_α form a group of symmetries of the dynamics.[1]

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[1]Moreover, in situations such as the case of the linearly polarized Gowdy T^3 cosmologies, these symmetries are in fact gauge transformations of the reduced system obtained after an almost complete gauge fixing [5].

Given the periodicity in the spatial coordinate θ, one can equivalently use the Fourier coefficients of φ and P as coordinates of our phase space. Let us be more explicit: employing the Fourier series expansion of φ and P

$$\varphi = \frac{q_0}{\sqrt{2\pi}} + \sum_{n>0}\left(q_n\frac{\cos(n\theta)}{\sqrt{\pi}} + x_n\frac{\sin(n\theta)}{\sqrt{\pi}}\right), \quad (16)$$

$$P = \frac{p_0}{\sqrt{2\pi}} + \sum_{n>0}\left(p_n\frac{\cos(n\theta)}{\sqrt{\pi}} + y_n\frac{\sin(n\theta)}{\sqrt{\pi}}\right), \quad (17)$$

we see that the space of Cauchy data $\{(\varphi, P)\}$ is in a one-to-one correspondence with the space of real Fourier coefficients $\{(q_n, x_n, p_n, y_n); n \in \mathbb{N}^+\}\bigcup\{(q_0, p_0)\}$. Besides, from the basic Poisson bracket between φ and P, one can check that $\{q_0, p_0\} = 1$ and $\{q_n, p_{n'}\} = \{x_n, y_{n'}\} = \delta_{nn'}$, the rest of brackets being equal to zero. Thus, the canonical phase space can be coordinatized either by $\{(\varphi, P)\}$ or by the set of canonical pairs $\{(q_n, p_n), (x_n, y_n); n \in \mathbb{N}^+\}\bigcup\{(q_0, p_0)\}$.

By substituting the expansions (16) and (17) in Eq. (11) we obtain

$$\dot{q}_n = p_n, \qquad \dot{p}_n = -(n^2 + f)q_n, \quad (18)$$

and therefore

$$\ddot{q}_n + (n^2 + f)q_n = 0. \quad (19)$$

These formulas are also valid for q_0 and p_0, just by letting n vanish. In addition, by replacing q_n with x_n and p_n with y_n (with $n > 0$), one obtains the equations of motion corresponding to the pair (x_n, y_n).

Let us introduce now the complex phase space variables

$$a_n = \frac{1}{\sqrt{2n}}(nq_n + ip_n), \qquad \tilde{a}_n = \frac{1}{\sqrt{2n}}(nx_n + iy_n),$$

$$n \in \mathbb{N}^+. \quad (20)$$

These are just the usual annihilationlike variables for a system of harmonic oscillators with frequencies equal to n. Of course, that system corresponds to the particular case $f = 0$ in Eq. (12), i.e., to the free massless field case. We can use the above set of complex variables, together with the associated creationlike variables obtained by complex conjugation, in order to coordinatize the inhomogeneous sector ($n \neq 0$) of the canonical phase space. Besides, from now on, we ignore the zero mode ($n = 0$) in our analysis, since its dynamics is decoupled from that of the inhomogeneous sector, the mode can be quantized by standard methods, and it plays no role in the subsequent discussion of unitarity and uniqueness of the quantum representation.

For the nonzero modes, the dynamics is dictated by the inhomogeneous part $\bar{H}$ of the Hamiltonian (10), which in terms of our new set of variables adopts the expression:

$$\bar{H} = \sum_{n>0}\left[\left(n + \frac{f}{2n}\right)(a_n^*a_n + \tilde{a}_n^*\tilde{a}_n)\right.$$
$$\left. + \frac{f}{4n}(a_na_n + \tilde{a}_n\tilde{a}_n + a_n^*a_n^* + \tilde{a}_n^*\tilde{a}_n^*)\right]. \quad (21)$$

The symbol $*$ denotes complex conjugation.

The finite transformations generated by the Hamiltonian $\bar{H}$ are linear symplectic transformations which can be decomposed in 2×2 blocks, one for each fixed pair $\mathcal{A}_n = (a_n, a_n^*)$ and $\tilde{\mathcal{A}}_n = (\tilde{a}_n, \tilde{a}_n^*)$. Furthermore, the blocks for these two pairs (with the same mode number n) coincide. Thus, the classical evolution of the annihilation and creationlike variables from time t_0 to time t is totally determined by a sequence of 2×2 matrices $\mathcal{U}_n(t, t_0)$, with $n \in \mathbb{N}^+$, of the form:

$$\begin{pmatrix} a_n(t) \\ a_n^*(t) \end{pmatrix} = \mathcal{U}_n(t, t_0)\begin{pmatrix} a_n(t_0) \\ a_n^*(t_0) \end{pmatrix},$$
$$\begin{pmatrix} \tilde{a}_n(t) \\ \tilde{a}_n^*(t) \end{pmatrix} = \mathcal{U}_n(t, t_0)\begin{pmatrix} \tilde{a}_n(t_0) \\ \tilde{a}_n^*(t_0) \end{pmatrix}, \quad (22)$$

$$\mathcal{U}_n(t, t_0) = \begin{pmatrix} \alpha_n(t, t_0) & \beta_n(t, t_0) \\ \beta_n^*(t, t_0) & \alpha_n^*(t, t_0) \end{pmatrix}, \quad (23)$$

where $\alpha_n(t, t_0)$ and $\beta_n(t, t_0)$ are Bogoliubov coefficients which depend on the specific function $f(t)$ of the system.

At this stage of the discussion, a couple of remarks about the extension of our analysis are in order.

(1) In the field models of the considered type which arise from symmetry reductions of General Relativity to cosmological scenarios, the time interval $\mathbb{I}$ is contained, typically, in the positive semiaxis $\mathbb{R}^+$. The origin $t = 0$ corresponds to a big bang singularity. This singularity generically implies that the function f is no longer well behaved at that point. In cases like the Gowdy T^3 cosmologies, the time domain is unlimited in the evolution of the system apart from this singularity, and thus $\mathbb{I}$ coincides with $\mathbb{R}^+$. On the other hand, the function f may have more than one singularity, and in this case the time domain is further restricted to a bounded interval, as it happens to be the case for the Gowdy $S^1 \times S^2$ and S^3 cosmological models, discussed below when one allows the compact spatial (topological) manifold to differ from the circle. Obviously, one can consider more general settings than these models inspired in cosmology. In particular, for smooth functions f one can extend its domain of definition $\mathbb{I}$ to the whole real line, and the study applies then to scalar fields in $\mathbb{R} \times S^1$.

(2) As we have anticipated, our discussion can be extended to axisymmetric fields on S^2. With this aim, let us start by considering the action of a free scalar field on $\mathbb{I} \times S^2$ in the presence of a time-dependent potential. This action is of the form (10) with the spatial integration performed over S^2 instead of over

the circle, and with the quadratic term in spatial derivatives, $(\varphi')^2$, of the Hamiltonian replaced with $\eta^{ij}\partial_i\varphi\partial_j\varphi$, where the indices i and j denote spatial indices on the sphere, and η_{ij} is the round metric on S^2, namely, $\eta_{ij} = d\theta_i d\theta_j + \sin\theta^2 d\phi_i d\phi_j$. The resulting field equation is similar to Eq. (12), but with the second spatial derivative replaced with $\Delta\xi$, Δ being the Laplace-Beltrami operator on the two-sphere. The expansion of the Cauchy initial data is now performed in terms of spherical harmonics, $Y_{lm}(\theta, \phi)$, which are eigenfunctions of the operator Δ with eigenvalue equal to $l(l + 1)$. The requirement of axisymmetry restricts the harmonics in this expansion to the set $\{Y_{l0}, l \in \mathbb{N}\}$, the only spherical harmonics which are independent of ϕ. The coefficients of ξ in this expansion in harmonics satisfy an equation identical to (19), except for the substitution of n by $l + \frac{1}{2}$ and the redefinition of the function $f(t)$, whose role is played now by $\bar{f}(t) := f(t) - 1/4$. From this point on, the discussion is completely parallel to that presented for the field on the circle, with the only caveat that the mode numbers n correspond now to positive half-integers, $l + \frac{1}{2}$, rather than to positive integers, a fact which, nonetheless, does not affect the computations nor the rationale of our analysis.

B. Complex structure

Let us start our discussion about the quantization process by considering the choice of complex structure for the system, which is the mathematical structure that encodes all the ambiguity which is physically relevant in the Fock quantization. Recall that a complex structure J is a symplectic transformation in S (or in Γ), compatible with the symplectic structure [in the sense that their combination $\Omega(J\cdot, \cdot)$ provides a positive definite bilinear map], and such that its square equals minus the identity, $J^2 = -1$.

The time evolution can be viewed as a map that relates copies of Γ at different times, e.g., $\{(\mathcal{A}_n(t_0), \tilde{\mathcal{A}}_n(t_0)); n \in \mathbb{N}^+\}$ at t_0 with $\{(\mathcal{A}_n(t), \tilde{\mathcal{A}}_n(t)); n \in \mathbb{N}^+\}$ at t. For definiteness, we choose once and for all an initial reference time t_0. Associated with $\{(\mathcal{A}_n(t_0), \tilde{\mathcal{A}}_n(t_0)); n \in \mathbb{N}^+\}$ [which we will denote by $\{(\mathcal{A}_n(t_0), \tilde{\mathcal{A}}_n(t_0))\}$ in the following, to simplify the notation], there is a natural field decomposition $\xi = \xi^+ + \xi^-$, where

$$\xi^+(t, \theta) = \sum_{n>0} \frac{1}{\sqrt{2\pi n}} M_n(t)[\cos(n\theta)a_n(t_0)$$
$$+ \sin(n\theta)\tilde{a}_n(t_0)], \qquad (24)$$

ξ^- is the complex conjugate of ξ^+, and $M_n(t) = \alpha_n(t, t_0) + \beta_n^*(t, t_0)$. Expressing the cosine and sine functions in terms of exponentials, we can rewrite ξ^+ as

$$\xi^+(t, \theta) = \sum_{n\neq 0} \frac{1}{\sqrt{4\pi|n|}} M_{|n|}(t)e^{in\theta}b_n, \qquad (25)$$

where $b_n := \frac{1}{\sqrt{2}}[a_n(t_0) - i\tilde{a}_n(t_0)]$ and $b_{-n} := \frac{1}{\sqrt{2}}[a_n(t_0) + i\tilde{a}_n(t_0)]$. The explicit decomposition of the solutions in complex conjugate pairs defines the Ω-compatible complex structure J_0 on S:

$$J_0(M_{|n|}(t)e^{in\theta}) = iM_{|n|}(t)e^{in\theta},$$
$$J_0(M_{|n|}^*(t)e^{-in\theta}) = -iM_{|n|}^*(t)e^{-in\theta}. \qquad (26)$$

In the $\{(\mathcal{A}_n(t_0), \tilde{\mathcal{A}}_n(t_0))\}$ basis, the complex structure J_0 is given by a block diagonal matrix, with 4×4 blocks $(J_0)_n = \text{diag}(i, -i, i, -i)$ for each value of n.

Since the reference time t_0 has been chosen arbitrarily in $\mathbb{I}$, we can reproduce our analysis for any other time value in this interval, let us say $t = T$. In that case, the field $\xi(t, \theta)$ will be decomposed again in "positive" and "negative" frequency solutions analog to ξ^+ and ξ^- [see Eq. (25)], but now in terms of the coefficients $b_n(T)$ and the modes $M_{|n|}^T(t)$ which, in turn, are obtained by replacing t_0 with T in the expressions given above for b_n, b_{-n}, and $M_{|n|}(t)$. Thus, for each copy $\{(\mathcal{A}_n(T), \tilde{\mathcal{A}}_n(T))\}$ of Γ [which, alternatively, can be coordinatized by $\{\mathcal{B}_n(T)\} := \{(b_n(T), b_{-n}^*(T), b_{-n}(T), b_n^*(T))\}$], we obtain a natural field decomposition, and hence a natural complex structure J_T (note that we have called $J_{t_0} = J_0$ to simplify the notation). In this way, we arrive at a uniparametric family of solution spaces of positive [negative] frequency, $S_T^+ := \{\xi^+ = (\xi - iJ_T\xi)/2\}$ $[S_T^- = (S_T^+)^*]$, which is induced by the evolution.

Note also that, in the alternative $\{\mathcal{B}_n\}$ description of Γ, the time evolution is dictated precisely by the sequence of 2×2 matrices $\mathcal{U}_n(t, t_0)$, though now acting on the pairs (b_n, b_{-n}^*) and (b_{-n}, b_n^*). In the $\{\mathcal{B}_n\}$ basis, the complex structure J_0 is also given by a block diagonal matrix, with the 4×4 blocks $(J_0)_n = \text{diag}(i, -i, i, -i)$.

Finally, it is worth recalling that, in the free massless field case, one has $\alpha_n(t, t_0) = e^{-in(t-t_0)}$ and $\beta_n(t, t_0) = 0$ (with $n > 0$), so that the positive mode solutions $M_{|n|}(t)e^{in\theta}$ associated with the complex structure J_0 in the free case are simply the usual subfamily of plane waves $e^{-i|n|(t-t_0)+in\theta}$.

C. Quantum representation

Starting with (S, J_0) one can construct in a standard way the Hilbert space of the quantum theory. The first step is to complete the space of positive frequency solutions specified by J_0 with respect to the norm $\sqrt{\langle\xi^+, \xi^+\rangle}$, where $\langle\xi_1^+, \xi_2^+\rangle := -i\Omega[(\xi_1^+)^*, \xi_2^+]$. The result is the so-called one-particle Hilbert space, $\mathcal{H}$. Next, the Hilbert space is obtained by considering the symmetrized tensor product of n copies of $\mathcal{H}$, one for each $n \in \mathbb{N}$, and collecting the resulting spaces via the direct sum operation. In short, the

Hilbert space is the Fock space associated with $\mathcal{H}$:

$$\mathcal{F}(\mathcal{H}) = \oplus_{n=0}^{\infty}(\otimes_{(s)}^{n}\mathcal{H}).$$

In this prescription, the field operator $\hat{\xi}$ is written in terms of the annihilation and creation operators corresponding to the positive and negative parts defined by the complex structure J_0, namely,

$$\hat{\xi}(t; \theta) = \sum_{n \neq 0} \frac{1}{\sqrt{4\pi|n|}} M_{|n|}(t) e^{in\theta} \hat{b}_n + \text{H.c.}, \qquad (27)$$

where "H.c." stands for "Hermitian conjugate." One can also rewrite the field operator in terms of the annihilation $\{\hat{a}_n, \hat{\tilde{a}}_n\}$ and creation $\{\hat{a}_n^{\dagger}, \hat{\tilde{a}}_n^{\dagger}\}$ operators associated with the positive and negative frequency solutions corresponding to excitations of the "oscillators" q_n and x_n:

$$\hat{\xi}(t; \theta) = \sum_{n > 0} \frac{1}{\sqrt{2\pi n}} M_n(t) [\cos(n\theta)\hat{a}_n(t_0)$$
$$+ \sin(n\theta)\hat{\tilde{a}}_n(t_0)] + \text{H.c.} \qquad (28)$$

In the Heisenberg picture, time evolution is, in principle, provided by the Bogoliubov transformation (23), what means that one can define operators $\hat{a}_n(t)$, $\hat{a}_n^{\dagger}(t)$ at time t, related with the operators $\hat{a}_n(t_0)$, $\hat{a}_n^{\dagger}(t_0)$ at time t_0 according to

$$\begin{pmatrix} \hat{a}_n(t) \\ \hat{a}_n^{\dagger}(t) \end{pmatrix} = \begin{pmatrix} \alpha_n(t, t_0) & \beta_n(t, t_0) \\ \beta_n^*(t, t_0) & \alpha_n^*(t, t_0) \end{pmatrix} \begin{pmatrix} \hat{a}_n(t_0) \\ \hat{a}_n^{\dagger}(t_0) \end{pmatrix}, \qquad (29)$$

and a completely similar expression for $(\hat{\tilde{a}}_n, \hat{\tilde{a}}_n^{\dagger})$.

A key question is to elucidate whether the above transformations correspond to unitary transformations in $\mathcal{F}(\mathcal{H})$, i.e., whether or not the dynamics is implementable in a unitary way on the Fock representation determined by J_0 (in the following, we will call it the J_0-Fock representation). Let us recall that a symplectic transformation R can be unitarily implemented on a Fock representation, constructed from a complex structure J, if and only if $(R + JRJ)$ is an operator of the Hilbert-Schmidt type on the corresponding one-particle Hilbert space [17,18]. Equivalently, R is implementable as a unitary transformation if and only if the representations defined by J and RJR^{-1} are unitarily equivalent, i.e., if and only if $(J - RJR^{-1})$ is Hilbert-Schmidt.

In the case of the family of symplectic transformations $\mathcal{U}(t, t_0)$ defined by the classical dynamics [and specified by the matrices $\mathcal{U}_n(t, t_0)$], the Hilbert-Schmidt condition for a unitary implementation in the J_0-Fock representation becomes a square summability condition on the coefficients β_n, namely, $\sum_{n=1}^{\infty} |\beta_n(t, t_0)|^2 < \infty$ $\forall t \in \mathbb{I}$, given a fixed reference time t_0. Before proving that this condition is indeed satisfied, let us conclude the subsection with some additional comments.

(1) In order to construct the Fock representation, we could have considered the space S_T^+, determined by the complex structure J_T, rather than the space of positive frequency solutions specified by J_0. Since the time T can take any value in $\mathbb{I}$, we would have obtained in this way a uniparametric family of J_T-Fock representations. Clearly, the J_0-Fock representation belongs to this family and corresponds to $T = t_0$. Note that unitary implementability of the dynamics on the J_0-Fock representation (and actually on any representation within the family) amounts to the unitary equivalence of all the J_T-Fock representations.

(2) By considering the counterpart of J_0 on the canonical phase space Γ (rather than on S), one can construct the functional representation which is unitarily equivalent to the J_0-Fock description (see [19] for a detailed treatment in complex variables and [20] for the Gelfand-Naimark-Segal relationship between Schrödinger and Fock representations). The result is a Schrödinger representation of the canonical commutation relations on the Hilbert space $\mathcal{H} = L^2(\mathcal{Q}, \mu)$ of square integrable functions on the infinite dimensional linear space $\mathcal{Q} = \{(q_n, x_n); n \in \mathbb{N}^+\} \cong (\mathbb{R}^2)^{\mathbb{N}^+}$, with respect to the Gaussian measure

$$d\mu = \prod_{n > 0}\left(e^{-n(q_n^2 + x_n^2)} \frac{n}{\pi} dq_n dx_n\right). \qquad (30)$$

The basic operators of configuration ($\hat{q}_n$ and $\hat{x}_n$) and momentum ($\hat{p}_n$ and $\hat{y}_n$) act as multiplicative and derivative operators, respectively:

$$\hat{q}_n \Psi = q_n \Psi, \qquad \hat{x}_n \Psi = x_n \Psi, \qquad (31)$$

$$\hat{p}_n \Psi = -i \frac{\partial}{\partial q_n} \Psi + inq_n \Psi,$$
$$\hat{y}_n \Psi = -i \frac{\partial}{\partial x_n} \Psi + inx_n \Psi. \qquad (32)$$

Here, $\Psi \in \mathcal{H}$ is an arbitrary "wave function". If one employs relation (20) to introduce operators $(\hat{a}_n, \hat{\tilde{a}}_n)$ and $(\hat{a}_n^{\dagger}, \hat{\tilde{a}}_n^{\dagger})$, it is easy to check that these provide the annihilation and creation operators of the representation. Thus, the constructed representation is just the one which is naturally associated with the free massless field case. A different way to see this fact is by computing the counterpart of the complex structure J_0 on the canonical phase space, j_0, which in terms of the variables (φ, P) takes the familiar form

$$j_0 = \begin{pmatrix} 0 & -(-\Delta)^{-1/2} \\ (-\Delta)^{1/2} & 0 \end{pmatrix}. \qquad (33)$$

Obviously, the massless free field dynamics is implemented as a unitary transformation in this representation; actually, the corresponding coefficients

β_n vanish identically, since the complex structure is invariant under the free field dynamics. Nevertheless, in the next section we will prove a nontrivial result, namely, that the dynamics of the field ξ that we are studying admits also a unitary implementation in the considered representation tailored to the free massless field.

IV. UNITARY DYNAMICS

In this section, we want to address the question of whether the sequences $\{\beta_n\}$ are square summable or not. Thus, we will be interested in the large n limit of the coefficients β_n, and therefore in the behavior of the equations of motion (19) for large n.

Let us start by writing the general solution to those equations of motion in the form

$$q_n(t) = A_n \exp[n\Theta_n(t)] + A_n^* \exp[n\Theta_n^*(t)], \qquad (34)$$

where, for each n, A_n is a complex constant and Θ_n is a particular complex solution of the characteristic equation

$$n\ddot{\Theta}_n + n^2\dot{\Theta}_n^2 + n^2 + f = 0, \qquad (35)$$

arising from Eqs. (19) and (34).

A simple calculation shows the relation between $\Theta_n(t)$ and the modes $M_n(t)$ associated with the complex structure J_0:

$$M_n(t) = - \exp\{n[\Theta_n(t) - \Theta_n(t_0)]\}\frac{1 - i\dot{\Theta}_n^*(t_0)}{2\,\mathrm{Im}\dot{\Theta}_n(t_0)}$$
$$+ \exp\{n[\Theta_n^*(t) - \Theta_n^*(t_0)]\}\frac{1 - i\dot{\Theta}_n^*(t_0)}{2\,\mathrm{Im}\dot{\Theta}_n(t_0)}. \qquad (36)$$

It is worth pointing out that Eq. (35) involves only the function $\dot{\Theta}_n$ and its derivative. Actually, it is just a first-order differential equation of the Riccati type for $\dot{\Theta}_n$. Hence, the functions Θ_n are determined only up to additive constants. We use this freedom to set $\Theta_n(t_0) = 0$. Let us consider now the freedom in choosing a particular solution to Eq. (35) for each $n \in \mathbb{N}^+$. By computing the relation between the initial data $(q_n(t_0), p_n(t_0))$, on the one hand, and $\dot{\Theta}_n(t_0)$ and the complex arbitrary constants A_n appearing in Eq. (34), on the other hand, one can check that it is possible to reach any value of the initial data while setting $\dot{\Theta}_n(t_0) = -i$. This condition fixes then the solution to Eq. (35). The choice is motivated by our knowledge of the free massless scalar field, case in which $\dot{\Theta}_n = -i$ is satisfied not only initially, but at all times. Substituting the resulting relation between $(q_n(t_0), p_n(t_0))$ and A_n in Eq. (34), it is easy to obtain the evolution matrices in terms of the original variables (q_n, p_n). Changing from those variables to the annihilation and creationlike variables (a_n, a_n^*), one can deduce the expression of the Bogoliubov coefficients $\alpha_n(t, t_0)$ and $\beta_n(t, t_0)$ as functions of the real and imaginary parts of $\Theta_n(t)$, which we call $r_n(t)$ and $s_n(t)$, respectively:

$$\alpha_n(t, t_0) = \tfrac{1}{2}e^{nr_n(t)}e^{ins_n(t)}[1 + i\dot{r}_n(t) - \dot{s}_n(t)], \qquad (37)$$

$$\beta_n(t, t_0) = \tfrac{1}{2}e^{nr_n(t)}e^{-ins_n(t)}[1 + i\dot{r}_n(t) + \dot{s}_n(t)]. \qquad (38)$$

In the equations of motion (19), the n^2 term dominates over the mass term $f(t)$ in the limit of large n modes, and we thus expect that the solutions $q_n(t)$ converge to those corresponding to the massless case for the same initial conditions, at least for sufficiently regular functions $f(t)$ on $\mathbb{I}$. Hence, the exponential $e^{nr_n(t)}$ tends to 1, and thus $\beta_n(t, t_0)$ is square summable if and only if so is $2e^{-nr_n(t)}\beta_n(t, t_0)$. Therefore, in the following we will focus our analysis on the behavior of $\dot{r}_n$ and $\dot{s}_n$ for large n.

Let us write the functions $\dot{\Theta}_n$ in the form

$$\dot{\Theta}_n = -i + \frac{W_n}{n}. \qquad (39)$$

The initial condition on $\dot{\Theta}_n$ translates then into the vanishing of W_n at t_0. Besides, from Eq. (35), it follows that the functions W_n satisfy the first-order differential equations

$$\dot{W}_n = 2inW_n - W_n^2 - f, \qquad (40)$$

also of the Riccati type. We want to show now that, in the large n-limit, the desired solutions to Eq. (40) admit "ultraviolet modes" of order $1/n$. Thus, in particular, the sequences $W_n(t)$ tend to zero, and the sequences $W_n(t)/n$ are square summable $\forall t$. The argument is the following. In the asymptotic limit of large n, the quadratic term W_n^2 in Eq. (40) is expected to be dominated by the linear term in W_n, whose coefficient is proportional to n and therefore grows in the asymptotic regime under consideration. We will hence neglect that quadratic term, show that the resulting linear equation admits solutions $\bar{W}_n$ of order $1/n$, and check that, in the asymptotic limit $n \to \infty$, the contribution of the quadratic term for such solutions is in fact negligible in our original differential equation.

Thus, let us consider the linear equation obtained from Eq. (40) after removing the quadratic term W_n^2:

$$\dot{\bar{W}}_n = 2in\bar{W}_n - f. \qquad (41)$$

The solution to Eq. (41) satisfying the initial condition $W_n(t_0) = 0$ is given by

$$\bar{W}_n(t) = - \exp(2int) \int_{t_0}^{t} d\bar{t} f(\bar{t}) \exp(-2in\bar{t}). \qquad (42)$$

A simple integration by parts leads then to

$$\bar{W}_n(t) = - \frac{if(t)}{2n} + \frac{if(t_0)e^{2in(t-t_0)}}{2n} - \frac{\exp(2int)}{2in} \int_{t_0}^{t} d\bar{t}\,\dot{f}(\bar{t})$$
$$\times \exp(-2in\bar{t}), \qquad (43)$$

and one can easily check that the absolute value of the last term is bounded by $\frac{1}{2n}\int_{t_0}^{t} dt|\dot{f}|$. It is therefore clear that, for

sufficiently regular $f(t)$, there is a function $C(t)$, independent of n, such that the absolute value of the solutions (42) is bounded by $C(t)/n$. To reach this conclusion, one only needs that the function $f(t)$ is differentiable (so that $\dot{f}$ exists) and its derivative is integrable in every interval $[t_0, t] \subset \mathbb{I}$ [for instance, these conditions are satisfied if $f(t)$ is a C^1 function in $\mathbb{I}$]. Then, in particular, the sequence $\bar{W}_n(t)$ tends to zero in the asymptotic limit of infinite n, $\forall\, t \in \mathbb{I}$.

We now return to the original differential equation (40). Given the behavior of the functions (42), the quadratic term $\bar{W}_n^2$ is bounded in absolute value by $C(t)^2/n^2$, and is hence negligible, in particular, compared with the linear term in that equation. Therefore, the functions $\bar{W}_n(t)$ defined in formula (42) can be taken as asymptotic solutions, in the limit of large n, to Eq. (40) up to subdominant terms, terms which in any case do not affect the square summability of the sequence $W_n(t)/n$.

Considering again the real and imaginary contributions of the solutions $\dot{\Theta}_n$, and splitting also W_n in real and imaginary parts, $W_n = R_n + iI_n$, one gets

$$\dot{r}_n(t) = \frac{R_n(t)}{n}, \qquad \dot{s}_n(t) = -1 + \frac{I_n(t)}{n}. \qquad (44)$$

According to our analysis, the real functions $R_n(t)$ and $I_n(t)$ tend to zero in the asymptotic limit of large n, for all allowed values of the time t.[2]

It is now a simple exercise to check that the coefficients $\beta_n(t, t_0)$, given in Eq. (38), are square summable for all t, since the leading term of $|\beta_n|$ in the asymptotic limit of large n is just $(R_n + iI_n)/(2n)$. This proves that, for a free real scalar field ξ on S^1, or an axisymmetric one on S^2, which is subject to a time-dependent potential $V(\xi) = f(t)\xi^2/2$ (or, equivalently, with a time-dependent mass), there exists at least one Fock representation in which the dynamics is implemented as a unitary transformation. This Fock representation is the one naturally associated with the massless scalar field with vanishing potential. In particular, this representation is S^1-invariant, i.e., it is defined by a complex structure J_0 which is invariant under the action of the group of S^1-translations (15)—if we consider the case of a field on S^2 instead, the invariance is under the group $SO(3)$. This means that the representation also provides us with a unitary implementation of the symmetry group of the field equations.

A natural question is whether or not the above result holds for other nonequivalent Fock representations which satisfy as well the requirement of invariance under the symmetries of the field equations. An answer in the positive would imply that one cannot pick out a preferred

S^1-invariant Fock representation [or an $SO(3)$-invariant one for axisymmetric fields on the sphere] by demanding the unitary implementation of the dynamics. In contrast, if the answer is in the negative, the J_0-Fock representation would be confirmed as the unique (up to unitary equivalence) Fock representation which is invariant under the symmetry group of the field equations and fulfills the demand of allowing a unitary quantum evolution. In the next section we will show that this is indeed the case.

V. UNIQUENESS OF THE QUANTIZATION

Since, for the analyzed case of a free scalar field on the circle with a time-dependent mass or potential, we are interested just in S^1-invariant Fock representations, we will consider only complex structures J that are invariant under the group of translations (15): $J = T_\alpha^{-1} J T_\alpha \; \forall \alpha \in S^1$. We will refer to such complex structures simply as invariant ones. Now, given a Ω-compatible invariant complex structure J, it can be shown [5] that it is related to J_0 via $J = K_J J_0 K_J^{-1}$, where K_J is a block diagonal symplectic transformation, with 4×4 blocks of the form

$$(K_J)_n = \begin{pmatrix} (\mathcal{K}_J)_n & 0 \\ 0 & (\mathcal{K}_J)_n \end{pmatrix}, \qquad (\mathcal{K}_J)_n = \begin{pmatrix} \kappa_n & \lambda_n \\ \lambda_n^* & \kappa_n^* \end{pmatrix}$$

$$|\kappa_n|^2 - |\lambda_n|^2 = 1, \quad \forall\, n \in \mathbb{N}^+. \qquad (45)$$

The fact that $|\kappa_n|^2 - |\lambda_n|^2 = 1$, implies that $|\kappa_n| \geq 1$ and $|\kappa_n| \geq |\lambda_n|$. Consequently, in particular, the sequence $\{\lambda_n/\kappa_n^*\}$ is bounded.

On the other hand, for the alternate case of an axisymmetric field on S^2, the complex structures that descend from $SO(3)$-invariant ones were discussed in Ref. [11]. Besides, it was shown in Ref. [12] that these invariant complex structures can be parametrized in the same way as the S^1-invariant ones. Using this common parametrization, all of the following discussion applies as well for this other family of field models.

Let us return to the mainstream of our argumentation. Given a symplectic transformation R, it is not difficult to see that it admits a unitary implementation with respect to the complex structure $J = K_J J_0 K_J^{-1}$ if and only if $K_J^{-1} R K_J$ is unitarily implementable with respect to J_0. Thus, the time evolution $\mathcal{U}$ (specified by the sequence of matrices $\{\mathcal{U}_n\}$) will be unitarily implementable with respect to the Fock representation determined by $J = K_J J_0 K_J^{-1}$ if and only if the J_0-Fock representation admits a unitary implementation of the symplectic map $K_J^{-1} \mathcal{U} K_J$. This last condition amounts to the square summability of the sequences

$$\beta_n^J(t, t_0) = (\kappa_n^*)^2 \beta_n(t, t_0) - \lambda_n^2 \beta_n^*(t, t_0)$$
$$+ 2i\kappa_n^* \lambda_n \, \mathrm{Im}[\alpha_n(t, t_0)], \quad \forall\, t \in \mathbb{I}, \qquad (46)$$

where α_n and β_n are the Bogoliubov coefficients corresponding to J_0, given in Eqs. (37) and (38). Summarizing, a different Fock representation, defined by a different invari-

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[2]Of course, a more detailed analysis of the asymptotic behavior of the solutions $\dot{\Theta}_n$ can be performed, as Eq. (43) already suggests. However, for our purposes the current estimate is sufficient.

ant complex structure $J = K_J J_0 K_J^{-1}$, allows a unitary implementation of the ξ field dynamics if and only if the sequence (46) is square summable at every instant of time t in the domain $\mathbb{I}$.

On the other hand, we recall that the Fock representation specified by $J = K_J J_0 K_J^{-1}$ and the J_0-Fock representation are unitarily equivalent if and only if the sequence $\{\lambda_n\}$ is square summable (details on this point can be found in Ref. [5]). In the rest of this section, we will demonstrate that, if the sequences $\{\beta_n^J(t, t_0)\}$ $\forall t \in \mathbb{I}$ are square summable, the same must necessarily happen to the sequence $\{\lambda_n\}$. This will prove that the J_0-Fock representation is in fact unique, up to unitary equivalence.

Let us suppose that the dynamics, for some function $f(t)$, is unitarily implemented in the invariant Fock representation determined by the complex structure J; that is, let us suppose that $\{\beta_n^J(t, t_0)\}$ is square summable $\forall t \in \mathbb{I}$. Since $|\kappa_n| > 1$, it then follows that the sequence provided by

$$\frac{\beta_n^J(t, t_0)}{(\kappa_n^*)^2} = \beta_n(t, t_0) - \left(\frac{\lambda_n}{\kappa_n^*}\right)^2 \beta_n^*(t, t_0)$$

$$+ 2i\left(\frac{\lambda_n}{\kappa_n^*}\right)\text{Im}[\alpha_n(t, t_0)] \qquad (47)$$

is also square summable. Moreover, we already know that the sequence $\{\lambda_n/\kappa_n^*\}$ is bounded and that the sequence $\{\beta_n(t, t_0)\}$ is square summable (this was shown in the previous section). These facts guarantee then that $\{\beta_n(t, t_0) - (\lambda_n/\kappa_n^*)^2 \beta_n^*(t, t_0)\}$ is square summable. Since the space of square summable sequences is a linear space, one is led to conclude that the sequence $\{(\lambda_n/\kappa_n^*) \text{Im}[\alpha_n(t, t_0)]\}$ has to be square summable as well.

The analysis that we presented in Sec. IV to show the square summability of $\{\beta_n(t, t_0)\}$ at all instants of time t can now be applied to check that the sequence $\{\text{Im}[\alpha_n(t, t_0)] + \sin(n(t - t_0))\}$ is also square summable. Thus, from the bound on λ_n/κ_n^* and using linearity, we conclude that $\{(\lambda_n/\kappa_n^*) \sin(nT)\}$ is also a square summable sequence. Here, we have called $T = t - t_0$ in order to simplify the notation. Therefore, the function

$$g(T) := \lim_{M \to \infty} \sum_{n=1}^{M} \frac{|\lambda_n|^2}{|\kappa_n|^2} \sin^2(nT) \qquad (48)$$

exists for all T in the interval $\bar{\mathbb{I}}$, obtained from $\mathbb{I}$ after a negative shift by t_0. In particular, the function $g(T)$ is well defined at least on some closed subinterval of the form $\bar{\mathbb{I}}_L = [a, a + L] \subseteq \bar{\mathbb{I}}$ (for a suitable choice of the time a), where L is some finite number strictly smaller than the length of $\mathbb{I}$. Related to this number L, let us introduce also, for later use, a fixed positive integer n_0 such that the product $n_0 L$ is larger than the unity, a condition that can always be fulfilled.

We can now apply Luzin's theorem [21], which ensures that, for every $\delta > 0$, there exist: (i) a measurable set $E_\delta \subset$

$\bar{\mathbb{I}}_L$ such that its complement $\bar{E}_\delta$ with respect to $\bar{\mathbb{I}}_L$ satisfies $\int_{\bar{E}_\delta} dt < \delta$, and (ii) a function $F_\delta(T)$, continuous on $\bar{\mathbb{I}}_L$, which coincides with $g(T)$ in E_δ. We then get

$$\sum_{n=1}^{M} \frac{|\lambda_n|^2}{|\kappa_n|^2} \int_{E_\delta} \sin^2(nT)dT \leq \int_{E_\delta} g(T)dT =: I_\delta,$$

$$\forall\ M \in \mathbb{N}^+, \qquad (49)$$

where $I_\delta = \int_{E_\delta} F_\delta(T)dT$ is some finite number. Since

$$\int_{E_\delta} \sin^2(nT)dT = \int_{\bar{\mathbb{I}}_L} \sin^2(nT)dT - \int_{\bar{E}_\delta} \sin^2(nT)dT$$

$$\geq \frac{L}{2} - \frac{1}{2n_0} - \delta, \quad \forall\ n \geq n_0, \qquad (50)$$

we have that

$$\sum_{n=n_0}^{M} \frac{|\lambda_n|^2}{|\kappa_n|^2} \leq \frac{2n_0 I_\delta}{n_0 L - 1 - 2n_0\delta} \quad \forall\ M > n_0. \qquad (51)$$

Here, we have used that it is possible to choose $2\delta < (L - 1/n_0)$, that

$$\int_{\bar{\mathbb{I}}_L} \sin^2(nT)dT = \frac{L}{2} - \frac{\sin[2n(a + L)]}{4n} + \frac{\sin(2na)}{4n}$$

$$\geq \frac{L}{2} - \frac{1}{2n} \quad \forall\ n \in \mathbb{N}^+, \qquad (52)$$

and that $(L - 1/n) \geq (L - 1/n_0)\ \forall n \geq n_0$. Let us emphasize that Eq. (51) is valid for arbitrary large M. Then, the right-hand side of that equation, which does not depend on M, provides a bound to the increasing sequence of partial sums $\{\sum_{n=n_0}^{M}(|\lambda_n|^2/|\kappa_n|^2)\}$, where n_0 is fixed. As a result, the sequence $\{\lambda_n/\kappa_n\}$ is necessarily square summable.

Employing the square summability of $\{\lambda_n/\kappa_n\}$ and the identity $|\kappa_n|^2 - |\lambda_n|^2 = 1$, it is straightforward to see that the sequence $\{\kappa_n\}$ is bounded. Thus, the sequence $\{\lambda_n = \kappa_n(\lambda_n/\kappa_n)\}$ is square summable, as we wanted to show. Therefore, we conclude that the J_0-Fock representation is the *unique* (up to unitary equivalence) invariant quantum description in which the dynamics is implemented as a unitary transformation and, consequently, the unique invariant quantum theory where the Schrödinger picture can be consistently defined.

VI. DISCUSSION AND CONCLUSIONS

It is well known that, in contrast to systems with a finite-dimensional linear phase space, there are inequivalent representations of the canonical commutation relations in quantum field theory.[3] Clearly, this raises the issue of which choice of representation, if any, is the adequate one for a given classical field theory. Since, in addition to

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[3]We mean representations of the regular type, i.e., giving rise to irreducible representations of the Weyl relations satisfying the standard criteria of weak continuity.

its kinematics, a field theory is characterized by its dynamics and its group of symmetries, it is most natural to take into account these ingredients and let them play a fundamental role when elucidating the appropriate quantum representation. In fact, from a physical point of view, it is highly questionable that one could accept as a satisfactory quantization of the system a representation of the canonical commutation relations which fails to produce a unitary implementation of the dynamics, or of the symmetries.

Therefore, given a classical field theory, the first fundamental issue that arises regarding its quantization is whether there exists or not a quantum representation with a proper unitary dynamics and symmetry group. Note that, for Poincaré invariant theories, dynamics and symmetries are unified under the Poincaré group, and one looks in fact for unitary implementations precisely of that group. In the particular case of free scalar fields on Minkowski spacetime, representations with the desired properties are known to exist, of course. These are the familiar free field representations, defined by Poincaré invariant complex structures, and are distinguished by the value of the mass. This means that, for each mass m, there exists a different, unitarily inequivalent, representation of the canonical commutation relations. The dynamics of the massive field with mass equal to m is unitarily implemented on the corresponding m-representation, but not on any of the distinct representations defined by $m' \neq m$.

The situation described above is a neat example of the necessity of invoking the dynamics, and the symmetries, in the quantization process. It illustrates the general belief that, in quantum field theory, the representation depends on the dynamics. One can, furthermore, expect that dynamical (or energy related) considerations might fix the representation uniquely[4] (up to unitary equivalence). This is indeed the case for free fields.

In the present work, we have carried out the analysis of the canonical quantization of a free scalar field $\xi(t, \theta)$ on the circle in the presence of an explicitly time-dependent potential $V(\xi) = f(t)\xi^2/2$, interpretable as a time-dependent mass. With respect to the set of field theories commented above, there are two major differences. On the one hand, the effective space where the field lives is now the compact space S^1, which is an important simplification. On the other hand, the field is not truly free, as the "mass" term depends on time. Time translation invariance is therefore lost and, though linear, the dynamics is nontrivial. Thus, the existence of a representation with unitary dynamics is not granted *a priori*.

The first result that we have demonstrated is that, for the considered type of field theories, there exists indeed a representation which allows a unitary implementation of the dynamics, namely, the representation which is naturally associated with the massless free field on S^1. Moreover,

this result holds for all sufficiently regular functions $f(t)$ [it suffices that $f(t)$ is differentiable and its derivative is integrable in every compact subinterval of the domain of definition].

This result is better understood by reminding the reader that the above mentioned inequivalence between the free field representations in Minkowski spacetime is due to the long range behavior (see Ref. [13]), which is absent in the S^1 case. Actually, the representations of the canonical commutation relations associated with free fields on S^1 *are* all unitarily equivalent [5], for any value of the mass. We have shown that, remarkably, the zero mass representation (and therefore the free field representation for any other value of the mass) also supports the dynamics of our field for every choice of the (regular) function $f(t)$. In this sense, the free field representation emerges in the compact case as a fixed stage where (at least some) different representations of interest are simultaneously realized. Moreover, the free field representation is defined by a complex structure which is invariant under S^1-translations, and therefore carries as well a unitary implementation of that symmetry group.

The second result that we have proved is the uniqueness of the quantum representation. In addition to a unitary implementation of the dynamics, we require that the representation is defined by a S^1-invariant complex structure. Under these conditions, we have shown that any representation which supports a unitary dynamics for the field ξ, for a given function $f(t)$, is unitarily equivalent to the massless free field representation. Thus, our conditions provide a successful uniqueness criterion.

It may be worth commenting on a couple of points to clarify the significance of this result. On the one hand, the set of representations defined by S^1-invariant complex structures is quite large and definitely contains different unitary equivalence classes. Therefore, the fact that unitary implementation of the dynamics selects precisely one equivalence class has to be considered nontrivial. Of course, it would be desirable to extend our results in order to include representations defined by general S^1-invariant algebraic states, rather than just states induced by complex structures. However, since our symmetry group of (constant) S^1-translations is relatively small, that would leave us with a huge set of states with, at least to our knowledge, no manageable characterization. On the other hand, we note that the condition of S^1-invariance, though not strictly necessary for the unitary implementation of S^1-translations, is certainly natural, and follows the general procedure for the implementation of symmetries (provided that invariant states exist, of course).

Although we have centered our attention on the S^1 case, mainly for simplicity, we have seen that all our results extend to the case of axisymmetric fields on the two-sphere, with the proper adaptations [for instance, the symmetry group would be $SO(3)$ rather than S^1]. In this regard,

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[4]Leaving aside situations of spontaneous symmetry breaking.

let us stress that the general analysis here presented is motivated by the current interest in the quantization of symmetry reductions of general relativity in the presence of two commuting spacelike Killing vectors. Actually, the two considered cases of the field on the circle and the axisymmetric field on the sphere cover all symmetry reductions of this kind found in cosmology, when one restricts to compact spatial sections, provided that the Killing vectors are hypersurface orthogonal (linear polarization). Such symmetry reductions correspond to the families of Gowdy spacetimes. Thus, our analysis includes the recent treatments of the quantum Gowdy cosmologies: specifically, the linearly polarized T^3 model [2–6] is described by a scalar field ξ on S^1 with $f(t) = 1/(2t)^2$, whereas the Gowdy $S^1 \times S^2$ and S^3 models [10–12] admit a description in terms of an axisymmetric field ξ on S^2 with $f(t) = (1 +$ $\csc^2 t)/4$. With the present unified treatment, we expect to have contributed both to a better understanding of the previously obtained results, and to an extension of them which can find applications in other symmetry reductions of general relativity or, more generally, in quantum field theory on curved backgrounds.

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