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Post-Newtonian metric of general relativity including all the c^{-4} terms in the continuity of the IAU2000 resolutions

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The Solar System metric recommended by the International Astronomical Union (IAU) resolution B1.3, during its 24th general assembly, allows light propagation calculations until order c^{-3} only. However, an increasing number of forthcoming spatial experiments will require a modelization of the gravitational field including all the c^{-4} terms in the metric to describe light propagation at the required precision. This will be the case for missions planned or in project, like TIPO, ASTROD, LATOR. Hence, it is necessary to extend the IAU framework to include all the relevant contributing terms. This paper proposes such an extension.

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I. INTRODUCTION

Space missions such as ASTROD [1], LATOR [2], ODYSSEY [3], SAGAS [4], or TIPO [5] project to link clocks placed in the Solar System via a laser link at interplanetary distances (near Earth and Mars in the case of TIPO, for instance). The goal of such experiments is to measure accurately distances between two stations in order to get information on gravitation in the Solar System, or on the specific gravitational field of the Sun or of a planet. But light's trajectories depend on the space-time curvature, described in general relativity (GR) by a pseudo-Riemmanian 4-metric. Hence, considering high accurate clocks that should be available in a close future, it is necessary to take all the c^{-4} terms into account in the metric, as it is shown in Sec. V.

Various approximation methods have been developed to solve the GR equation. Two of them are the so-called post-Newtonian (PN) and post-Minkowskian (PM) approximations. In the PN scheme, the metric tensor is developed in powers of $1/c$, while it is developed in powers of G in the PM scheme.

The PN approach is relevant when considering a weak gravitational field generated by sources with weak velocities, i.e. velocities of the order $\sqrt{GM/r}$, M being some characteristic mass. The generally so-called nPN order terms in the metric are the terms of orders c^{-2n-2} in g_{00}, c^{-2n-1} in g_{0i}, and c^{-2n} in g_{ij}, i.e. the metric terms leading to c^{-2n} terms in the motion of a test body having a weak velocity (for instance, a test body describing a bounded orbit). On the other hand, in a PN metric, one can be interested in the motion of a test particle having a velocity of the order of c (as it is the case in light transfer problems, for instance). In this case, the c^{-2n} terms in all the components $g_{\alpha\beta}$ of the metric contribute at the same level, leading to c^{-2n} terms in the involved equations of motion. Since the IAU2000 [6] and Damour-Soffel-Xu (DSX) [7] metrics do not include c^{-4} terms in g_{ij}, it is useful to extend these frameworks by determining the c^{-4} term in g_{ij} for applications to the forthcoming space missions.

A lot of work has been done on the subject with different goals. In [8], authors are interested in hydrodynamical equations of GR in the slow motion approximation while in [9,10] the author is interested in gravitational radiation generation. They both use the same approach to get second order metrics in harmonic gauge (even 2.5PM in [10] and 2.5PN metrics in [8]). Our method differs from those since it is in continuity of the IAU2000. Hence, we use the "exponential parametrization" as in DSX [7] or in the IAU2000 recommendations [6]. Moreover, we give the general formal solution including all c^{-4} terms and satisfying the "strong spatial isotropy condition" $g_{00}g_{ij} = -\delta_{ij} + O(c^{-4})$ [7], before giving it in the harmonic gauge (but our solution, including c^{-4} terms in g_{ij}, is no longer isotropic).

Other works have been done to get the GR metric with all c^{-4} terms [11,12]. But in those works, the general solution of the corresponding equations of GR has not been obtained. Moreover, the authors did not use the full harmonic gauge condition $\partial_\alpha(\sqrt{-g}g^{\alpha\beta}) = 0$ (written at the required order) since they consider only the time component of this condition, i.e. $\partial_\alpha(\sqrt{-g}g^{\alpha 0}) = 0$. One shows in this paper the full harmonic gauge condition, i.e. including $\partial_\alpha(\sqrt{-g}g^{\alpha i}) = 0$, allows one to get the formal solution without the assumptions on the source term made in [11,12].

The paper is organized as follows. In Sec. II one specifies notations and conventions used in this paper. In Sec. III we link the orders of the required quantities to the order at which the metric is considered. IAU2000 and DSX91 metrics are recalled in Sec. IV, while the necessity of having a metric at the c^{-4} level for coming up space missions is pointed out in Sec. V. In Sec. VI we give the general isotropic solution, with particular attention to the

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harmonic gauge. Section VII is devoted to discussions and possible extensions of this work.

II. CONVENTIONS AND NOTATIONS

As suggested by the IAU2000 recommendations, we use the signature $(-+++)$ for the space-time metric. Space-time indices go from 0 to 3 and are denoted by greek letters, while space indices (from 1 to 3) are denoted by latin letters. The time variable x^0 is ct. We do not make the unit choice $c = 1$, since we are interested in developments in power of c^{-1}. We use the Einstein summation convention for both types of indices, whatever the position of repeated indices. We introduce the following notations:

(i) $\overset{(n)}{Q}$ is the term of order c^{-n} of any quantity Q. In this paper, we consider only cases where n takes integer (positive or negative) values.

(ii) In all the text, $\overset{(\to n)}{Q}$ is a short way for "Q up to terms of order c^{-n}", i.e. including c^{-n} and neglecting c^{-n-1} terms.

(iii) $[a, b]$ represents any quantity whose lowest-order term is of order c^{-a}, and highest-order considered term is of order c^{-b}. Straightforward considerations lead to the following algebra:

$$[a; b] + [a'; b'] = [\min(a, a'); \min(b, b')] \quad (1)$$

(except the very special case where $a = a'$ and the corresponding terms exactly cancel).

$$[a; b] \times [a'; b'] = [a + a'; \min(a' + b, a + b')] \quad (2)$$

$$\frac{1}{c^n}[a; b] = [a + n; b + n] \quad (3)$$

$$\frac{1}{[a; b]} = [-a; b - 2a], \quad (4)$$

where $\min(x, y, \ldots)$ represents the minimum of the numbers $x, y, \ldots$.

(iv) For any quantity A, writing

$$[A] = [x, y]$$

means the lowest-order term of A is of order c^{-x}, and that its development is considered (or obtained) only until the term of order c^{-y}.

In this paper, we write $\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha}$, $\partial^2_{\alpha\beta} \equiv \frac{\partial^2}{\partial x^\alpha \partial x^\beta}$ (instead of $\partial_\alpha \partial_\beta$, which is not ambiguous since we do not use notations like ∂^α for $\eta^{\alpha\beta}\partial_\beta$, for instance) and $\partial_t \equiv \frac{\partial}{\partial t} = c\partial_0$. We use the "$w - \gamma$ exponential parametrization" such as in DSX [7]. Hence, our metric is presented in the form

$$g_{00} = -e^{-2w/c^2} \qquad g_{0i} = -\frac{4}{c^3} w_i \qquad g_{ij} = \gamma_{ij} e^{2w/c^2}, \quad (5)$$

where w, w_i, and γ_{ij} are the ten functions representing the metric. This is not restrictive since we consider only cases where $g_{00} < 0$ (i.e. that x^0 is a "true" time variable) over all the space-time regions considered in this paper (which are weak gravitational field regions).

III. ORDERS IN THE METRIC AND CORRESPONDING ORDERS OF THE RICCI TENSOR AND OTHER USEFUL QUANTITIES

The goal of this section is to determine until which order one has to write the quantities required in both the Einstein and gauge equations to get the metric at a given order. Let us consider the components $g_{\alpha\beta}$ have to be determined in such a way that

$$[g_{00}] = [0; x]; \qquad [g_{0i}] = [3; y]; \qquad [g_{ij}] = [0; z]. \quad (6)$$

The inversion relation $g_{\alpha\beta} g^{\beta\gamma} = \delta^\gamma_\alpha$ writes

$$g_{00} g^{00} + g_{0k} g^{0k} = 1 \quad (7)$$

$$g_{00} g^{0i} + g_{0k} g^{ik} = 0 \quad (8)$$

$$g_{0i} g^{00} + g_{ik} g^{0k} = 0 \quad (9)$$

$$g_{0i} g^{0j} + g_{ik} g^{jk} = \delta^j_i. \quad (10)$$

From these relations, it turns easily that g^{00}, g^{0i}, and g^{ij} are zeroth, third and zeroth order quantities, respectively, as are the covariant components. It also turns out that

$$(g_{00} g_{ik} - g_{0i} g_{0k}) g^{0k} + g_{0i} = 0 \quad (11)$$

$$(g_{00} g_{ik} - g_{0i} g_{0k}) g^{jk} = g_{00} \delta^j_i. \quad (12)$$

From (11) and (12), we are able to get

$$[g^{0i}] = [3; y']; \qquad [g^{ij}] = [0; z']. \quad (13)$$

Inserting in (7)

$$[g^{00}] = [0; x']. \quad (14)$$

In these expressions, one has set

$$x' = \min(x, y + 3, z + 6) \quad (15)$$

$$y' = \min(x + 3, y, z + 3) \quad (16)$$

$$z' = \min(x, y + 3, z). \quad (17)$$

Note that inserting (13) in (9) does not allow one to determine g^{00} with the same precision as inserting in (7).

From these results, it is possible to derive the orders at which the Christoffel connection's and the Ricci tensor's components can be obtained. Indeed, from (5)

$$[\partial_k g_{00}] = [2, x]$$

and, from the PN assumptions

$$[\partial_0 g_{00}] = [3, x+1].$$

The same considerations lead to

$$[\partial_0 g_{0i}] = [4, y+1], \qquad [\partial_j g_{0i}] = [3, y].$$

As we shall see, we consider in this paper only cases where g_{ij} in (5) has no terms in c^{-n} with $n < 2$ unless constant terms. Hence, let us consider only cases where

$$[\partial_0 g_{ij}] = [3, z+1], \qquad [\partial_k g_{ij}] = [2, z].$$

It turns out that the Christoffel connection's components can be determined so that

$$[\Gamma^0_{00}] = [3; \min(x+1, y+2, z+5)]$$
$$[\Gamma^0_{0i}] = [2; \min(x, y+3, z+4)]$$
$$[\Gamma^0_{ij}] = [3; \min(x+3, y, z+1)]$$
$$[\Gamma^k_{00}] = [2; \min(x, y+1, z+2)]$$
$$[\Gamma^k_{0i}] = [3; \min(x+3, y, z+1)]$$
$$[\Gamma^k_{ij}] = [2; \min(x+2, y+3, z)].$$

From this, and since $R^{\alpha\beta} = g^{\alpha\sigma}g^{\beta\rho}R_{\sigma\rho}$, the Ricci tensor components can be derived so that

$$[R_{00}] = [R^{00}] = [2; \min(x, y+1, z+2)]$$
$$[R_{0i}] = [R^{0i}] = [3; \min(x+3, y, z+1)]$$
$$[R_{ij}] = [R^{ij}] = [2; \min(x, y+1, z)].$$

If one is interested in the slow motion case, as in satellite navigation for instance, one has to consider

$$x = y + 1 = z + 2$$

and it turns out that the Ricci tensor components have to be developed in such a way that

$$[R^{00}] = [2; z+2]; \qquad [R^{0i}] = [3; z+1]; \qquad [R^{ij}] = [2; z]. \tag{18}$$

On the other hand, if one is interested in the fast motion case, as in light propagation for instance (the case in the current paper), one has instead

$$x = y = z$$

and it turns out that the Ricci tensor components have to be developed in such a way that

$$[R^{00}] = [2; z]; \qquad [R^{0i}] = [3; z]; \qquad [R^{ij}] = [2; z]. \tag{19}$$

Note that in both cases, (15)–(17) show that one has $(x', y', z') = (x, y, z)$.

In the following, we will often consider metrics in harmonic gauge, defined by $g^{\alpha\beta}\Gamma^\sigma_{\alpha\beta} = 0$ [or equivalently by $\partial_\alpha(\sqrt{-g}g^{\alpha\sigma}) = 0$, with g the covariant components metric determinant]. One finds

$$[g^{\alpha\beta}\Gamma^0_{\alpha\beta}] = [3; \min(x+1, y, z+1)] \tag{20}$$

$$[g^{\alpha\beta}\Gamma^k_{\alpha\beta}] = [2; \min(x, y+1, z)]. \tag{21}$$

IV. THE 1.5PN METRIC

We recall here results from DSX [7] on the 1PN metric. In fact, the DSX metric is a 1.5PN metric since it is explicitly written at the 1.5PN order (i.e. the paper determines $\overset{(\to 5)}{g}_{00}$, $\overset{(\to 4)}{g}_{0i}$, and $\overset{(\to 3)}{g}_{ij}$, while the 1PN order requires $\overset{(\to 4)}{g}_{00}$, $\overset{(\to 3)}{g}_{0i}$, and $\overset{(\to 2)}{g}_{ij}$ only). But since the 1.5PN metric is the same as the 1PN one (in the sense $\overset{(5)}{g}_{00} = \overset{(4)}{g}_{0i} = \overset{(3)}{g}_{ij} = 0$), several papers are talking about 1PN order while using explicitly 1.5PN metric (which is not logically incorrect, since 1.5PN computations include 1PN ones).

A space-time satisfies the isotropy condition if and only if there is (at least) one coordinate system in which the metric components in (5) satisfy $g_{ij} \propto \delta_{ij}$. It is shown in [7] that, from the Einstein equation, the isotropy condition is satisfied when c^{-4} terms are discarded, and even that there is (at least) one coordinate system which satisfies the strong spatial isotropy condition

$$\gamma_{ij} = \delta_{ij} + O(c^{-4}). \tag{22}$$

Then the 1.5PN metric can always be written, in a well-suited coordinate system,

$$g_{00} = -e^{-2w/c^2} + O(c^{-6}) \tag{23}$$

$$g_{0i} = -\frac{4}{c^3}w_i + O(c^{-5}) \tag{24}$$

$$g_{ij} = \delta_{ij}e^{2w/c^2} + O(c^{-4}). \tag{25}$$

It follows (from Sec. III)

$$g^{00} = -e^{2w/c^2} + O(c^{-6}) \tag{26}$$

$$g^{0i} = -\frac{4}{c^3}w_i + O(c^{-5}) \tag{27}$$

$$g^{ij} = \delta_{ij}e^{-2w/c^2} + O(c^{-4}). \tag{28}$$

Thus, from Sec. III, one has to know $\overset{(\to 5)}{R}{}^{00}$, $\overset{(\to 4)}{R}{}^{0j}$, and $\overset{(\to 3)}{R}{}^{ij}$ to constrain the metric functions at the 1.5PN order. It turns out that

$$R^{00} = -\frac{1}{c^2}\triangle w - \frac{1}{c^4}(3\partial^2_{tt}w + 4\partial^2_{tk}w_k) + O(c^{-6}) \tag{29}$$

$$R^{0i} = -\frac{2}{c^3}(\triangle w_i - \partial^2_{ki}w_k - \partial^2_{ti}w) + O(c^{-5}) \tag{30}$$

$$R^{ij} = -\frac{1}{c^2}\delta_{ij}\triangle w + O(c^{-4}), \tag{31}$$

where $\triangle = \partial^2_{jj}$. The Einstein field equation

$$R^{\alpha\beta} = \frac{8\pi G}{c^4}\left(T^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} g_{\rho\sigma} T^{\rho\sigma}\right) \tag{32}$$

reduces to four equations on w and w_i. Let us recall that T^{00}, T^{0i}, and T^{ij} are, respectively, quantities of orders c^{+2}, c^{+1}, and c^0 [7]. These four equations correspond to the (00) and (0i) equations, namely,

$$\triangle w + \frac{3}{c^2}\partial^2_{tt} w + \frac{4}{c^2}\partial^2_{tj} w_j = -4\pi G\sigma + O(c^{-4}) \tag{33}$$

$$\triangle w_i - \partial^2_{ij} w_j - \partial^2_{ti} w = -4\pi G\sigma^i + O(c^{-2}), \tag{34}$$

where $\sigma = c^{-2}(T^{00} + T^{kk})$ and $\sigma^i = c^{-1} T^{0i}$ (factors c^{-2} and c^{-1} are introduced in order that σ and σ^i are zeroth order quantities). Note that the (ij) equation leads to

$$\triangle w = -4\pi G \frac{T^{00}}{c^2} + O(c^{-2}). \tag{35}$$

This is nothing but a lower order version of Eq. (33). It expresses that the lowest order of w is the Newtonian potential generated by the source.

Hence, to the 1.5PN approximation correspond four GR equations, and the gauge choice (22) keeps 1 degree of freedom. This gauge invariance is characterized by an arbitrary differentiable function λ:

$$w' = w - \frac{1}{c^2}\partial_t \lambda \tag{36}$$

$$w'_i = w_i + \tfrac{1}{4}\partial_i \lambda. \tag{37}$$

IAU2000 metric

The IAU2000 resolution corresponds to a particular gauge choice (the harmonic gauge) in order to fix the last degree of freedom. The harmonic gauge writes, in both the 1.5PN DSX metric [$(x, y, z) = (5, 4, 3)$ in (6)] and the IAU2000 metric [$(x, y, z) = (4, 4, 3)$ in (6)]

$$g^{\alpha\beta}\Gamma^0_{\alpha\beta} = O(c^{-5}) \tag{38}$$

$$g^{\alpha\beta}\Gamma^i_{\alpha\beta} = O(c^{-4}). \tag{39}$$

Equation (38) writes

$$\partial_t w + \partial_j w_j = O(c^{-2}). \tag{40}$$

Since, in some sense, the spatial coordinates are ''fixed'' by the strong isotropy condition, Eq. (40) can be understood as fixing the time coordinate. Equation (39) reduces to a triviality since the strong isotropy condition makes spatial coordinates always harmonic modulo $O(c^{-4})$. Hence, in harmonic gauge, the four metric field equations reduce to

$$\Box_m w = -4\pi G\sigma + O(c^{-4}) \tag{41}$$

$$\triangle w_i = -4\pi G\sigma^i + O(c^{-2}), \tag{42}$$

where $\Box_m = \triangle - c^{-2}\partial^2_{tt}$.

V. WHY SHOULD ONE CONSIDER ALL THE c^{-4} METRIC TERMS?

TIPO, for instance, projects to measure interplanetary distances using onboard clocks and a one-way laser link. The clock's accuracy is expected to be of order 10^{-11} s (corresponding to the millimeter level in term of distances). It is then necessary to use a metric allowing to calculate the time transfer with such accuracy or even a bit more (for safety, in order to take all generic effects into account and also that some numerical coefficients may be sensitively larger than unity).

If space-time would be Minkowskian, the time transfer would be $t_{\rm Mink} = c^{-1}\int dl$, where dl is the infinitesimal spatial length on the photon's trajectory. When space-time is developed in PN expansion, to each new order n corresponds a correction on the time transfer. For $n = 3/2$, the correction comes essentially from the g_{0i} terms in the metric, i.e. is related to the orbital and rotational velocities of the sources (Sun and planets). For $n = 2$, the correction comes essentially from the g_{00} and the g_{ij} terms in the metric, i.e. is related to second order Schwarzschild-like terms and (among other things) nonlinear effects involving two bodies [6].

Let us first consider the $n = 2$ terms. The order of magnitude of the solar Schwarzschild correction corresponding to this order is given by

$$\delta t_2 \sim \int \frac{dl}{c}\left(\frac{r_s}{r}\right)^2 = \frac{r_s}{c}\int \frac{dl}{r}\frac{r_s}{r} \tag{43}$$

with r_s the solar Schwarzschild radius, of about 3 km. Let us point out that it is important to evaluate the integral of $(r_s/r)^2$ over all the time transfer, and not only its maximal value, i.e. its value at impact parameter b (minimal distance with respect to the body, here the Sun), as it is often made. Indeed, the maximum value acts only on a part of the time transfer, and the smaller the impact parameter, the smaller the part of the time transfer on which the maximum value is acting. Let r_p be a distance of the order of the astronomical unit. We get

$$\delta t_2 \sim (10^{-5}\ {\rm s})\frac{r_s}{b}\int_0^{x_p} \frac{dx}{\cosh x}, \tag{44}$$

where $\cosh x_p = r_p/b$. Evaluating this expression with $b \sim 4R_s \ll r_p$, R_s being the Sun's radius, we get $\delta t_2 \sim 10^{-11}$ s. This shows that the second order terms are required to reach the level of the expected data precision. There are other terms at this level, for instance coupling terms involving products like $M_S M_{pl}$, M_S, and M_{pl} being the masses of the Sun and of a planet, which will be numerically significantly smaller than the previously estimated contribution, the planetary masses being considerably smaller than the Sun's one.

Let us now estimate the $n = 3/2$ terms. Since these terms come from the g_{0i} components of the metric, their relative contributions to the time transfer on any portion of the trajectory are expected to be of the order

$$\epsilon \sim |2g_{0i}/g_{00}| \sim 8\frac{GMv}{rc^3}$$

from the expression of g_{0i} [6]. M is the mass of the considered body, v its external (orbital) or internal (rotation) velocity, and r its distance from the light ray. The order of magnitude of the cumulated effect on the flying time is then expected to be given by

$$\delta t_{3/2} \sim \int_{r=b}^{r=r_p} 4\frac{r_s}{r}\frac{v}{c}\frac{dl}{c}\left(= 4\frac{r_s}{c}\frac{v}{c}\ln\left[\sqrt{\left(\frac{r_p}{b}\right)^2 - 1} + \frac{r_p}{b}\right]\right),$$

where r_s is the Schwarzschild radius of the considered body. In the case of a giant planet (Jupiter), r (under the $\int$) is always of the order of some astronomical units in the considered applications. One finds (orbital effect) $\delta t_{3/2,\mathrm{Jup}} \sim 10^{-12}$ s. If the considered body is an inner planet (Earth, or Venus), r can reach values of the order of 10^4 km. In this case, $\delta t_{3/2,\mathrm{inner}} \sim 10^{-13}$ s (orbital effect). If the considered body is the Sun, its orbital velocity is of the order of $(M_{\mathrm{Jup}}/M_S)v_{\mathrm{Jup}}$, where M_{Jup} and v_{Jup} are the Jovian mass and orbital velocity (the effects of the other planets on the Sun's orbital motion are at best of the same order). Taking $r \sim R_S$, one finds (orbital effect) $\delta t_{3/2,\mathrm{Sun,orbital}} \sim 10^{-11}$ s. The rotational effect of the Sun can be roughly estimated putting $v \sim \omega_S R_S$, ω_S being the Sun's angular rotation velocity. One finds $\delta t_{3/2,\mathrm{Sun,rotation}} \sim 10^{-9}$ s.

These estimations show that both c^{-3} and c^{-4} metric terms have to be taken into account to reach the precision of TIPO-like experiments (this is clear for the Sun, but also for the giant planetary c^{-3} terms, since numerical coefficients larger than unity can make the cumulated effect reaches 10^{-11} s).

It could be of interest to note that the effect of c^{-3} terms is expected to be numerically only 2 orders of magnitude larger than the c^{-4} terms.

VI. THE METRIC INCLUDING ALL THE c^{-4} TERMS

In GR, gravity is defined by a symmetric tensor field $g_{\alpha\beta}$ with ten components. But the Einstein equation actually fixes six components, keeping 4 degrees of freedom due to the invariance of the theory through any change of coordinates (called gauge invariance, or gauge invariant field, of the theory). Because of the strong spatial isotropy condition, 1.5PN GR reduces to four Eqs. (33) and (34), with a gauge invariant field characterized by an arbitrary differentiable function λ (36) and (37). (One could note that 1.5PN GR has the same formal structure as the electromagnetism theory [7,13]).

At the c^{-4} level, g_{ij} is no more isotropic and we get six new equations on g_{ij}, with 3 degrees of freedom corresponding to the gauge choice of spatial coordinates. Hence, we expect six new equations with a new gauge invariant field characterized by an arbitrary 3-vector.

Let us write the covariant components of the metric under the form

$$g_{00} = -e^{-2w/c^2} \tag{45}$$

$$g_{0i} = -\frac{4}{c^3}w_i \tag{46}$$

$$\begin{aligned} g_{ij} &= \delta_{ij}e^{2w/c^2} + \frac{4\tau_{ij}}{c^4} + O(c^{-5}) \\ &= \delta_{ij}\left(1 + \frac{2w}{c^2} + \frac{2w^2}{c^4}\right) + \frac{4\tau_{ij}}{c^4} + O(c^{-5}). \end{aligned} \tag{47}$$

In (45)–(47), x, y, and z defined in (6) are such that $(x, y, z) = (\infty, \infty, 5)$. Hence, from (15)–(17), we can get algebraically $g^{\alpha\beta}$ so that $(x', y', z') = (11, 8, 5)$. However, since w and w_i have to be determined neglecting c^{-3} and c^{-2} terms, respectively, it is sufficient to require $x = y = z = 5$. One has then $x' = y' = z' = 5$ and the contravariant metric writes

$$g^{00} = -e^{2w/c^2} + O(c^{-5}) \tag{48}$$

$$g^{0i} = -\frac{4}{c^3}w_i + O(c^{-5}) \tag{49}$$

$$g^{ij} = \delta_{ij}e^{-2w/c^2} - \frac{4\tau_{ij}}{c^4} + O(c^{-5}). \tag{50}$$

From Sec. III, we must get $\overset{(\to 4)\alpha\beta}{R}$, i.e. $\overset{(\to 4)ij}{R}$ (since $\overset{(\to 4)00}{R}$ and $\overset{(\to 4)0i}{R}$ are already known from 1.5PN formulas), to fix the six new functions τ_{ij}. We get

$$\begin{aligned} R^{ij} = &-\frac{\delta_{ij}}{c^2}\Box_m w + \frac{2}{c^4}(\partial^2_{ti}w_j + \partial^2_{tj}w_i) - \frac{2}{c^4}\partial_i w\partial_j w \\ &+ \frac{4\delta_{ij}}{c^4}w\triangle w + \frac{2}{c^4}(\partial^2_{ki}\tau_{kj} + \partial^2_{kj}\tau_{ik} - \triangle\tau_{ij} \\ &- \partial^2_{ij}\tau_{kk}) + O(c^{-5}). \end{aligned} \tag{51}$$

One gets for the (00) and (0i) equations (32)

$$\triangle w + \frac{3}{c^2}\partial^2_{tt}w + \frac{4}{c^2}\partial^2_{tj}w_j = -4\pi G\sigma + O(c^{-3}) \tag{52}$$

$$\triangle w_i - \partial^2_{ij}w_j - \partial^2_{ti}w = -4\pi G\sigma^i + O(c^{-2}). \tag{53}$$

Let us point out (52) differs from the 1.5PN equation (33) by the order of the neglected term only, while (53) is exactly identical to (34). One gets for the (ij) Einstein equation

$$\frac{2}{c^4}\Theta_{ij}(\tau_{kl}) + \Xi_{ij}(w, w_k) = \frac{8\pi G}{c^4}\Big(\frac{1}{2}\delta_{ij}T^{00} + T^{ij} - \frac{1}{2}\delta_{ij}T^{kk} - \frac{2w}{c^2}\delta_{ij}T^{00}\Big) + O(c^{-5}), \quad (54)$$

where

$$\Theta_{ij}(\tau_{kl}) = \partial^2_{ki}\tau_{kj} + \partial^2_{kj}\tau_{ik} - \triangle\tau_{ij} - \partial^2_{ij}\tau_{kk} \quad (55)$$

$$\Xi_{ij}(w, w_k) = -\frac{\delta_{ij}}{c^2}\Box_m w + \frac{2}{c^4}(\partial^2_{ti}w_j + \partial^2_{tj}w_i) - \frac{2}{c^4}\partial_i w\partial_j w + \frac{4\delta_{ij}}{c^4}w\triangle w. \quad (56)$$

Since w and w_k are required at second and zeroth orders, respectively, in (54), they can be considered as source terms in this equation. We get, using (52) [and since (52) also implies (35)]

$$\Theta_{ij}(\tau_{kl}) = 4\pi G\sigma^{ij} - (\partial^2_{ti}w_j + \partial^2_{tj}w_i) + \partial_i w\partial_j w - 2\delta_{ij}(\partial^2_{tt}w + \partial^2_{tk}w_k) + O(c^{-1}), \quad (57)$$

where $\sigma^{ij} = T^{ij} - \delta_{ij}T^{kk}$. As expected, the operator Θ satisfies an invariant relation, which writes

$$\Theta_{ij}(\tau_{kl} + \partial_k A_l + \partial_l A_k) = \Theta_{ij}(\tau_{kl}),$$

whatever the 3-vector field A_k. Hence, the gauge invariance satisfied by Eqs. (52), (53), and (57), is characterized by

$$w' = w - \frac{1}{c^2}\partial_t\lambda \quad (58)$$

$$w'_i = w_i + \tfrac{1}{4}\partial_i\lambda \quad (59)$$

$$\tau'_{kl} = \tau_{kl} + \partial_k A_l + \partial_l A_k + \tfrac{1}{2}\delta_{kl}\partial_t\lambda, \quad (60)$$

where λ and A_l are arbitrary differentiable fields. The last term in (60) compensates the change of the right-hand side (rhs) of Eq. (57) resulting from (58) and (59).

A. General method to inverse (57)

We can show that there always exists a gauge transformation $\tau'_{kl} = \tau_{kl} + \partial_k A_l + \partial_l A_k$ that reduces the operator Θ to a Laplacian $[\Theta_{ij}(\tau'_{kl}) = -\triangle\tau'_{ij}]$. To be placed on this gauge, A_k must satisfy $\triangle A_j = -\partial_k\tau_{kj} + \frac{1}{2}\partial_j\tau_{kk}$, which is clearly invertible using the Laplacian's Green function. Hence, there always exists a particular gauge where the solution of Eq. (57) is

$$\tau_{ij} = \triangle^{-1}\{-4\pi G\sigma^{ij} + \partial^2_{ti}w_j + \partial^2_{tj}w_i - \partial_i w\partial_j w + 2\delta_{ij}(\partial^2_t w + \partial^2_{tk}w_k)\} + O(c^{-1}), \quad (61)$$

where

$$\triangle^{-1}\{f(t,\vec{x})\} = \int\frac{f(t,\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x'. \quad (62)$$

In other words, we can get the general solution τ'_{kl} of (57) (in any gauge), using first (61), and then considering $\tau'_{kl} = \tau_{kl} + \partial_k V_l + \partial_l V_k$, where V_k is any vector field.

B. The harmonic gauge

From (20) and (21), the harmonic gauge condition writes, at the required order

$$g^{\alpha\beta}\Gamma^{\gamma}_{\alpha\beta} = O(c^{-5}) \quad (63)$$

for both $\gamma = 0$ and $\gamma = i$. One gets

$$\partial_t w + \partial_k w_k = O(c^{-2}) \quad (64)$$

$$\partial_k\tau_{ik} - \tfrac{1}{2}\partial_i\tau_{kk} + \partial_t w_i = O(c^{-1}). \quad (65)$$

Equation (64) simplifies (52) and (53) into

$$\Box_m w = -4\pi G\sigma + O(c^{-3}) \quad (66)$$

$$\triangle w_i = -4\pi G\sigma^i + O(c^{-2}). \quad (67)$$

Using (65), one finds Θ_{ij} (55) writes

$$\Theta_{ij}(\tau_{kl}) = -\triangle\tau_{ij} - \partial_t(\partial_i w_j + \partial_j w_i) + O(c^{-1}). \quad (68)$$

Hence, considering w and w_i as source terms, the solution of the equation (57) in the harmonic gauge is [using (64)]

$$\tau_{ij} = \triangle^{-1}\{-4\pi G\sigma^{ij} - \partial_i w\partial_j w\} + O(c^{-1}). \quad (69)$$

Results on the c^{-4} level metric in harmonic gauge are summarized in the Appendix.

C. From nonharmonic to harmonic gauge

If (w, w_i, τ_{ij}) is not a harmonic solution, (w', w'_i, τ'_{ij}) defined by (58)–(60) is a harmonic solution if one chooses (λ, A_i) such that

$$\triangle\lambda = -4(\partial_t w + \partial_k w_k) + O(c^{-2})$$

$$\triangle A_i = -\partial_k\tau_{ik} + \frac{1}{2}\partial_i\tau_{kk} - \partial_t w_i + O(c^{-1}).$$

As a corollary, any transformation satisfying $\triangle\lambda = 0$ and $\triangle A_i = 0$ preserves the harmonic gauge condition.

D. The exact Schwarzschild solution

The proposed solution can be checked on the vacuum Schwarzschild solution. This solution has been written in different coordinate systems: Schwarzschild's, isotropic, synchronous (Lemaitre-Rylov's), Eddington-Finkelstein, Kruskal-Szekeres', harmonic, ... Since the harmonic gauge takes a large place in the IAU2000 recommendations (and in the present paper), let us consider the Schwarzschild metric in harmonic coordinates:

$$ds^2 = -\frac{r-m}{r+m}dt^2 + \left[\left(1+\frac{m}{r}\right)^2\delta_{ij} + \frac{r+m}{r-m}\left(\frac{m}{r}\right)^2\frac{x^i x^j}{r^2}\right] \times dx^i dx^j, \tag{70}$$

where one has set $m = GM/c^2$ for compactness. This leads to

$$g_{00} = -1 + 2\frac{m}{r} - 2\frac{m^2}{r^2} + O(c^{-5}) \qquad g_{0i} = 0$$

$$g_{ij} = \left(1 + 2\frac{m}{r} + 2\frac{m^2}{r^2}\right)\delta_{ij} + \frac{m^2}{r^2}\left(\frac{x^i x^j}{r^2} - \delta_{ij}\right) + O(c^{-5})$$

which means w, w_i, and τ_{ij} defined in Eqs. (45)–(47) are given by

$$w = \frac{GM}{r} \qquad w_i = 0 \qquad \tau_{ij} = \frac{1}{4}\frac{G^2M^2}{r^2}\left(\frac{x^i x^j}{r^2} - \delta_{ij}\right).$$

It is easy to check that this solution satisfies both the vacuum version of the field equations (A7)–(A9) and the harmonic gauge conditions (A13) and (A14).

VII. CONCLUSION AND DISCUSSION

In this paper, one has proposed an extension of the IAU2000 metric allowing second order light propagation calculations. In the continuation of the IAU2000 recommendations, the case of specifying the coordinate system by the use of the harmonic gauge condition has been emphasized. It has been argued that these second order terms are required to reach the 10^{-11} s level of precision in Solar System time transfer experiments, a level that should be reached in the foreseeable future. The same kind of arguments suggest that the next (i.e. c^{-5} and c^{-6}) terms should result in negligible contributions. However, on the grounds of some numerical investigations, some authors claim that the c^{-4} terms contribution can be significantly larger than expected by the previous estimates, by about 3 orders of magnitude [14]. One should keep this in mind, and also contemplate that, similarly, the same could be true for the further order terms. Hence, to dispose a safety margin, it could be useful to go beyond the c^{-4} level.

References [11,12] go beyond the IAU2000 metric level, but cannot be seen as an attempt to reach the c^{-5} level, since g_{0i} is limited to $O(c^{-5})$ (i.e. the metric to the c^{-4} level) from the start. Let us consider a solution under the form

$$g_{00} = -e^{-2w/c^2} \tag{71}$$

$$g_{0i} = -\frac{4}{c^3}w_i \tag{72}$$

$$g_{ij} = \delta_{ij}e^{2w/c^2} + \frac{4\tau_{ij}}{c^4} + O(c^{-6}). \tag{73}$$

Upgrading the c^{-4} solution to the c^{-5} level [i.e. determining w and w_i up to $O(c^{-4})$ and $O(c^{-3})$, respectively] requires to compute $\overset{(\to 5)\alpha\beta}{R}$. $\overset{(\to 5)00}{R}$ is known from the 1.5PN case (29). The other components are

$$R^{0i} = \frac{2}{c^3}(-\triangle w_i + \partial^2_{ki}w_k + \partial^2_{ti}w) + \frac{2}{c^5}(-\partial^2_{tk}\tau_{ki} + \partial^2_{ti}\tau_{kk} + \partial_t w\partial_i w) + \frac{4}{c^5}(w_k\partial^2_{ik}w + \partial_k w_k\partial_i w - \partial_k w\partial_i w_k) + \frac{4}{c^5}w(\triangle w_i - \partial^2_{ik}w_k) + O(c^{-6}). \tag{74}$$

$$R^{ij} = [\text{rhs of Eq. (51)}] + O(c^{-6}). \tag{75}$$

This determines the geometrical part of the GR equation at the required level. At this level, the harmonic gauge condition reads $\partial_\alpha(\sqrt{-g}g^{\alpha\beta}) = O(c^{-6})$. This gives

$$\partial_t w + \partial_k w_k + \frac{1}{c^2}\left(\frac{1}{2}\partial_t\tau_{kk} + 2w\partial_t w + 2w_k\partial_k w\right) = O(c^{-3}) \tag{76}$$

$$\partial_k\tau_{ik} - \tfrac{1}{2}\partial_i\tau_{kk} + \partial_t w_i = O(c^{-2}). \tag{77}$$

Inserting (29) and (74)–(77) in Einstein Eq. (32) leads to

$$\Box_m w = -4\pi G\sigma + O(c^{-4}) \tag{78}$$

$$w_i = w_i^{(0)} + \frac{1}{c^2}w_i^{(1)} \tag{79}$$

$$\triangle\tau_{ij} = -4\pi G\sigma^{ij} - \partial_i w\partial_j w + O(c^{-2}), \tag{80}$$

where

$$\triangle w_i^{(0)} = -4\pi G\sigma^i + O(c^{-3}) \tag{81}$$

$$\triangle w_i^{(1)} = \partial^2_{tt}w_i^{(0)} + 2w\triangle w_i^{(0)} - 2w_i^{(0)}\triangle w + 3\partial_i w\partial_k w_k^{(0)} - 4\partial_k w\partial_i w_k^{(0)} + O(c^{-1}). \tag{82}$$

This is the c^{-5} level solution in harmonic gauge. [The harmonic Schwarzschild solution (70) satisfies the vacuum version of Eqs. (76)–(82)].

However, this solution is only formal in the sense the matter tensor also involves the metric. This is not a problem at the c^{-4} level since the matter terms can be developed into multipole expansions. This way, the matter content is described by multipole coefficients, determined from observations in practical applications. But the formalism allowing to expand the matter source terms into multipole expansions is ensured to work until the c^{-4} order only [7,15]. Besides, at the c^{-5} level, PN methods also face problems the way the gravitational field falls off at spatial infinity [16]. This is related to the fact that nonstationary systems generate gravitational radiation.

Further studies would be useful to clarify these issues.

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APPENDIX: c^{-4} METRIC OF GENERAL RELATIVITY IN THE HARMONIC GAUGE

$$g_{00} = -1 + \frac{2w}{c^2} - \frac{2w^2}{c^4} + O(c^{-5}) \tag{A1}$$

$$g_{0i} = -\frac{4}{c^3} w_i + O(c^{-5}) \tag{A2}$$

$$g_{ij} = \delta_{ij}\left(1 + \frac{2w}{c^2} + \frac{2w^2}{c^4}\right) + 4\frac{\tau_{ij}}{c^4} + O(c^{-5}) \tag{A3}$$

$$g^{00} = -1 - \frac{2w}{c^2} - \frac{2w^2}{c^4} + O(c^{-5}) \tag{A4}$$

$$g^{0i} = -\frac{4}{c^3} w_i + O(c^{-5}) \tag{A5}$$

$$g^{ij} = \delta_{ij}\left(1 - \frac{2w}{c^2} + \frac{2w^2}{c^4}\right) - 4\frac{\tau_{ij}}{c^4} + O(c^{-5}) \tag{A6}$$

with (setting $\triangle = \partial^2_{kk}$ and $\Box_m = \triangle - \partial^2_{00} = \triangle - c^{-2}\partial^2_{tt}$)

$$\Box_m w = -4\pi G\sigma + O(c^{-3}) \tag{A7}$$

$$\triangle w_i = -4\pi G\sigma^i + O(c^{-2}) \tag{A8}$$

$$\triangle \tau_{ij} = -4\pi G\sigma^{ij} - \partial_i w \partial_j w + O(c^{-1}) \tag{A9}$$

$$\sigma = c^{-2}(T^{00} + T^{kk}) \tag{A10}$$

$$\sigma^i = c^{-1}T^{0i} \tag{A11}$$

$$\sigma^{ij} = T^{ij} - \delta_{ij}T^{kk}. \tag{A12}$$

The harmonic conditions [used to get (A7)–(A9)] are given by

$$\partial_t w + \partial_k w_k = O(c^{-2}) \tag{A13}$$

$$\partial_k \tau_{ik} - \tfrac{1}{2}\partial_i \tau_{kk} + \partial_t w_i = O(c^{-1}). \tag{A14}$$

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