Attractor universe in the scalar-tensor theory of gravitation

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In the scalar-tensor theory of gravitation it seems nontrivial to establish if solutions of the cosmological equations in the presence of a cosmological constant (or a vacuum energy) behave as attractors independently of the initial values. We develop a general formulation in terms of two-dimensional phase space, mainly according to the Brans-Dicke model requiring the scalar field decoupled from the matter Lagrangian in the Jordan frame. We show that there are two kinds of fixed points, one of which is an attractor depending on the coupling constant and equation of state. We find that the static universe in the Jordan frame is an attractor in the presence of a cosmological constant for some range of the coupling constant. We extend our analysis to a power-law potential, finding a new type of power-law inflation caused by the coupling to the matter fluid, also as an attractor.

I. INTRODUCTION

Einstein's general relativity has proved to be the simplest theory for the successful understanding of a number of experiments and observations. Still, on the other hand, there seem to be growing indications that a yet-to-be-discovered scalar field might play fundamental roles in cosmology. A list of the possible sources include the one expected from higher-dimensional theories of gravitation such as superstring/M theory and a scalar field (a volume modulus) which couples to 4-dimensional gravity after compactification, also those in the context of the brane world scenario. We may reasonably expect that realistic consequences of these hypotheses can be implemented in terms of the scalar-tensor theory proposed first by Jordan [1], developed later by Brans and Dicke [2,3].

One of the recent focuses of the scalar field is aimed particularly at the origin of the dark energy which appears to be required from the observed acceleration of the universe, with the renewed interest of today's version of the cosmological constant problem, culminating to the twin questions; the fine-tuning problem and coincidence problem. We argued [4–7] that the scalar-tensor theory is precisely what causes the behavior, $\Lambda \sim t^{-2}$ realized numerically by $10^{-120} \sim (10^{60})^{-2}$ [8], where 10^{60} is today's age of the universe in units of the Planck time $\sim 10^{-43}$ s, hence preparing another simple implementation of the scenario of the decaying cosmological constant as discussed in [9,10].

Before reaching conclusions to be compared with observation, however, we must go through certain complications including details on the choice of the conformal frames among other things. Also many consequences derive from leaving the Brans-Dicke model [2,3], as required ultimately by a single technical aspect on the attractor nature of the cosmological solutions. This motivated us to develop a general framework of studying dynamics of the system including the scalar field taking the unique roles of the conformal transformation properly into account.

After a brief introduction of the action of the scalar-tensor theory in both conformal frames, the Jordan and Einstein frames in Sec. II, we enter Sec. III to develop a formulation in the Einstein frame in which we may trace how the cosmological solutions evolve in two-dimensional phase space. We assume the presence of the exponential potential of the scalar field corresponding to the simple cosmological constant in the Jordan frame. It is crucially important to make a right choice of the new extended time-coordinate other than the conventional cosmic time. Also of central importance is to deal with self-autonomous systems.

In Subsection III A we discuss the fixed points in phase space. We show that there are two different sets of fixed points: one (FP1) is for the well-known universe of scalar-field dominance, and the other (FP2) represents a new type of the universe in which the matter fluid energy is scaled to the potential of the scalar field because of the coupling with the matter fluid. We find that the universe of the latter type always expands in the same way as the radiation-dominant universe in the Einstein frame. In Subsection III B, we present the stability analysis and the behavior of the attractor solutions. Flows of trajectories in phase space are illustrated for the examples taken from the radiation-dominant universe. Basically the same analysis will be repeated in Sec. IV now in the Jordan frame.

In Sec. V we generalize the argument to the power-law potential with a monomial of the scalar field multiplied by the cosmological constant chosen in the preceding section in the Jordan frame. We find a new type of inflation at the

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fixed point FP2 even for the potential which is too steep to cause inflation without the coupling with the matter fluid. Section VI is devoted to the concluding remarks.

In Appendix A, we discuss the effect of the curvature term not included in the preceding sections. In the subsequent three appendices, we add related discussions on the accelerating universe. Appendix B reveals the presence of a complication in the numerical analysis in the Jordan frame, while in Appendix C we discuss the attractor nature of the scale-invariant model as an alternative to the Brans-Dicke model. The final Appendix D will be devoted to offering another simplified approach to the power-law potential.

II. SCALAR-TENSOR THEORY: JORDAN FRAME VS EINSTEIN FRAME

We discuss cosmology in the scalar-tensor theory of gravitation. We assume the presence of a potential $V(\phi)$ in the Jordan frame. The action is

$$S = S_g + S_m, \tag{2.1}$$

where

$$S_g = \int d^4x \sqrt{-g}\left[\frac{\xi}{2}\phi^2 R - \frac{\epsilon}{2}(\nabla\phi)^2 - V(\phi)\right], \tag{2.2}$$

$$S_m = \int d^4x \sqrt{-g}L_m(g,\phi), \tag{2.3}$$

with $\epsilon = \pm 1$ [11]. Note that the matter action S_m is assumed to have no scalar field ϕ according to the Brans-Dicke model [2,3], in which the weak equivalence principle (WEP) is intended to be respected. The scalar-tensor theory defined in this way is equivalent to the traditional Jordan-Brans-Dicke theory with an added potential $U(\varphi) = V(\phi)$, often expressed as

$$S_g = \int d^4x \sqrt{-g}\left[\varphi R - \frac{\omega}{\varphi}(\nabla\varphi)^2 - U(\varphi)\right], \tag{2.4}$$

with the Brans-Dicke constant $\omega = \epsilon/(4\xi)$ and $\varphi = (\xi/2)\phi^2$.

Although we may further extend this type of the scalar-tensor theory with an arbitrary function of the scalar field ϕ multiplied with R, we confine ourselves to the original simple φ, because it features global scale invariance except generally for the $V(\phi)$ term.

The choice $\epsilon = -1$ in (2.2) is closely related to string theory. The D-dimensional action for the zero modes in the closed string sector is given [12] by

$$S = \frac{1}{2}\int d^Dx \sqrt{-g}e^{-2\Phi}[R(g) + 4(\nabla\Phi)^2], \tag{2.5}$$

which, reexpressed according to our own sign convention [13] as in (1.30) of [4], corresponds to the first two terms in (2.2) with $\epsilon = -1$ and $\xi = 1/4$, hence $\omega = -1$, by introducing $\phi = 2e^{-\Phi}$.

We can always move to the Einstein frame by a conformal transformation [1–3,14–17]

$$g_{\mu\nu} \rightarrow g_{*\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{2.6}$$

where

$$\Omega^2 = \xi\phi^2 = \exp(2\zeta\sigma), \tag{2.7}$$

with

$$\zeta^2 \equiv (6 + \epsilon\xi^{-1})^{-1} = (6 + 4\omega)^{-1}, \tag{2.8}$$

which defines a canonical scalar field σ in the Einstein frame;

$$S = \int d^4x \sqrt{-g_*}\left[\frac{1}{2}R_* - \frac{1}{2}(\nabla_*\sigma)^2 - V_*(\sigma)\right]$$
$$+ \int d^4x \sqrt{-g_*}L_m(\psi_*, g_*, \sigma), \tag{2.9}$$

where

$$V_*(\sigma) = \exp(-4\zeta\sigma)V(\phi). \tag{2.10}$$

We mark the quantities in the Einstein frame with the subscript $*$, while those in the Jordan frame are left unmarked, unless otherwise indicated. This is in accordance with the notation used in [4].

We point out that $\epsilon = -1$, apparently indicating a ghost nature of the nondiagonalized field ϕ, is a real difficulty only if ζ^2 turns out to be negative implying a negative energy for the diagonalized field σ. Because of the nonminimal coupling term, which mixes ϕ with the spinless component of the rank-2 tensor field $g_{\mu\nu}$, ϕ is a nondiagonalized field as was argued in Chapter 2.6 of [4]. See also [18]. The conformal transformation to the Einstein frame can be viewed to remove this mixing by means of a diagonalized field σ, as was also discussed near the beginning of Chapter 1.3.2 of [4]. We always assume the condition

$$\zeta^2 > 0. \tag{2.11}$$

This can be obeyed even if $\epsilon = -1$ if $\xi > 1/6$. Imposing $\xi > 0$, which we assume throughout this paper, due to the required positivity of the energy of tensor gravity, we find that $\epsilon = 1$ allows any ξ but with $\zeta^2 < 1/6$, while $\epsilon = -1$ constrains $\xi > 1/6$ and $\zeta^2 > 1/6$, as displayed graphically in Fig. 1 of [6].

The parameters in (2.5) give $\zeta^2 = (D - 2)/4$ which is $1/2$ for $D = 4$. If $\epsilon = -1$ and $\xi = 1/6$, we find $\zeta^2 \rightarrow \infty$, implying no kinetic term in the Einstein frame, hence no degree of freedom. We do not consider this choice any further.

Because we assume that no ϕ field enters L_m in the Jordan frame, we find that the energy-momentum of the matter fluid is conserved in the Jordan frame;

$$\nabla^\nu T^\mu{}_\nu = 0, \tag{2.12}$$

for which WEP is respected. The energy-momentum tensor in the Einstein frame is obtained by

$$T_*{}^\mu{}_\nu = \exp(-4\zeta\sigma)T^\mu{}_\nu, \tag{2.13}$$

which is no longer conserved;

$$\nabla_*^\nu T_*{}^\mu{}_\nu = -\zeta T_* \nabla_*^\mu \sigma, \tag{2.14}$$

where $T_* = T_*{}^\rho{}_\rho$. Note that the universal free-fall (UFF) is still maintained, as an expression of WEP.

III. COSMOLOGY WITH A COSMOLOGICAL CONSTANT: ANALYSIS IN THE EINSTEIN FRAME

We discuss cosmology in the scalar-tensor theory with a cosmological constant $V = V_0$. In this section, we discuss it in the Einstein frame, though the analysis in the Jordan frame is given in the next section.

A. The basic equations and the fixed points

The metric of the isotropic and homogeneous universe is given by the Friedmann-Robertson-Walker (FRW) form:

$$ds_*^2 = -dt_*^2 + a_*^2 ds_3^2, \tag{3.1}$$

where ds_3^2 is the metric of maximally symmetric three-dimensional space with the curvature constant $k = 0$ or ± 1.

The basic equations in the Einstein frame are

$$H_*^2 + \frac{k}{a_*^2} = \frac{1}{3}\left(\frac{1}{2}\dot{\sigma}^2 + V_* + \rho_*\right), \tag{3.2}$$

$$\ddot{\sigma} + 3H_*\dot{\sigma} + \frac{\partial V_*}{\partial \sigma} = \zeta(\rho_* - 3P_*), \tag{3.3}$$

where $H_* = \dot{a}_*/a$, P_*, and ρ_* are the Hubble expansion parameter, the pressure, and the energy density in the Einstein frame, respectively. The dot implies d/dt_* throughout in the Einstein frame. Equation (2.14) in the Einstein frame is then reexpressed as

$$\dot{\rho}_* + 3H_*(P_* + \rho_*) = -\zeta\dot{\sigma}(\rho_* - 3P_*). \tag{3.4}$$

Assuming the equation of state $P_* = (\gamma - 1)\rho_*$, we further reexpress Eqs. (3.2), (3.3), and (3.4) into

$$H_*^2 + \frac{k}{a_*^2} = \frac{1}{3}\left(\frac{1}{2}\dot{\sigma}^2 + V_* + \rho_*\right), \tag{3.5}$$

$$\ddot{\sigma} + 3H_*\dot{\sigma} - 4\zeta V_* = \zeta(4 - 3\gamma)\rho_*, \tag{3.6}$$

$$\dot{\rho}_* + 3\gamma H_*\rho_* = -\zeta(4 - 3\gamma)\dot{\sigma}\rho_*. \tag{3.7}$$

We now introduce a new dimension-free time coordinate τ_* by

$$d\tau_* = 2\sqrt{V_*}dt_*. \tag{3.8}$$

We further introduce $\mathcal{H}_* = a_*'/a_*$, where the prime is for

a differentiation with respect to τ_*. We then put Eqs. (3.5), (3.6), and (3.7) into the new form

$$\mathcal{H}_*^2 + \frac{k}{4V_*a_*^2} = \frac{1}{6}\left[\sigma'^2 + \frac{1}{2}\left(1 + \frac{\rho_*}{V_*}\right)\right], \tag{3.9}$$

$$\sigma'' + 3\mathcal{H}_*\sigma' - \zeta\left[2\sigma'^2 + 1 + \frac{(4 - 3\gamma)\rho_*}{4V_*}\right] = 0, \tag{3.10}$$

$$\rho_*' + 3\gamma\mathcal{H}_*\rho_* = -\zeta(4 - 3\gamma)\sigma'\rho_*. \tag{3.11}$$

Focusing on $k = 0$, we differentiate Eq. (3.9) with respect to τ_* to obtain

$$\mathcal{H}_*' = -\frac{2 - \gamma}{4}\sigma'^2 + \frac{\gamma}{8} + 2\zeta\sigma'\mathcal{H}_* - \frac{3\gamma}{2}\mathcal{H}_*^2. \tag{3.12}$$

Here we have used Eqs. (3.9) and (3.10) as well as the equation

$$(\rho_*/V_*)' = -3\gamma(\mathcal{H}_* - \zeta\sigma')(\rho_*/V_*), \tag{3.13}$$

which is obtained from Eq. (3.11) and the definition (2.10) of V_*.

In the same way we put Eq. (3.10) into

$$\sigma'' = -3\mathcal{H}_*\sigma' + \frac{3\gamma\zeta}{4}(2\sigma'^2 + 1) + 3\zeta(4 - 3\gamma)\mathcal{H}_*^2, \tag{3.14}$$

where Eq. (3.9) has been used to obtain the last term on the right-hand side. A set of Eqs. (3.12) and (3.14) gives a self-autonomous system. In fact, by introducing the variables x and y defined by $x = \sigma'$ and $y = \zeta^{-1}\mathcal{H}_*$, we derive

$$x' = \frac{3\zeta}{4}[2\gamma x^2 - 4xy + 4\zeta^2(4 - 3\gamma)y^2 + \gamma], \tag{3.15}$$

$$y' = \frac{1}{8\zeta}[-2(2 - \gamma)x^2 + 16\zeta^2 xy - 12\zeta^2\gamma y^2 + \gamma]. \tag{3.16}$$

By choosing $x' = y' = 0$, we find four fixed points in this system;

FP $1_\pm$: $(x_F, y_F) = (x_1^{(\pm)}, y_1^{(\pm)})$

$$\equiv \pm\left(\frac{2\zeta}{\sqrt{3 - 8\zeta^2}}, \frac{1}{2\zeta\sqrt{3 - 8\zeta^2}}\right), \tag{3.17}$$

FP$2_\pm$: $(x_F, y_F) = (x_2^{(\pm)}, y_2^{(\pm)})$

$$\equiv \pm\left(\frac{\sqrt{\gamma}}{\sqrt{2(2 - \gamma - 2(4 - 3\gamma)\zeta^2)}}, \frac{\sqrt{\gamma}}{\sqrt{2(2 - \gamma - 2(4 - 3\gamma)\zeta^2)}}\right). \tag{3.18}$$

The fixed points FP1$_\pm$ exist if $\zeta^2 < 3/8$, while the fixed points FP2$_\pm$ exist if $\gamma \geq 4/3$ or if $\gamma < 4/3$ with $\zeta^2 < (2 - \gamma)/[2(4 - 3\gamma)]$ (equivalently, $\gamma > 2(4\zeta^2 - 1)/(6\zeta^2 - 1)$). For $\zeta = 1/2$, two types of fixed points coincide with each other. In Fig. 1, we show in which portion of the ζ^2-γ plane we find the fixed points.

From (3.9) with $k = 0$, we obtain

$$\frac{\rho_*}{V_*} = 2(6\zeta^2 y^2 - x^2) - 1, \qquad (3.19)$$

which is constant at the fixed points.

At the fixed point FP1$_\pm$, we find immediately

$$(\rho_*/V_*)_{\text{FP1}} = 0, \qquad (3.20)$$

while at the fixed point FP2$_+$, we obtain

$$\left(\frac{\rho_*}{V_*}\right)_{\text{FP2}} = \frac{2(4\zeta^2 - 1)}{2 - \gamma - 2(4 - 3\gamma)\zeta^2}. \qquad (3.21)$$

This result is consistent with Eq. (3.13), i.e.

$$\left(\frac{\rho_*}{V_*}\right)' = -3\gamma\zeta(y - x)\left(\frac{\rho_*}{V_*}\right), \qquad (3.22)$$

the right-hand side of which vanishes at the fixed point FP2$_+$, showing that the ratio of the energy density to the potential is constant.

Next we discuss the scale factor and scalar field at the fixed points. We have

$$\sigma' = x_F, \qquad (\ln a_*)' = \zeta y_F, \qquad (3.23)$$

for fixed points.

For FP1$_+$, we have $(x_F, y_F) = (x_1^{(+)}, y_1^{(+)})$, giving

$$\sigma = \frac{2\zeta}{\sqrt{3 - 8\zeta^2}} \tau_* + \sigma_0, \qquad (3.24)$$

$$a_* = a_{*0} \exp\left[\frac{\tau_*}{2\sqrt{3 - 8\zeta^2}}\right]. \qquad (3.25)$$

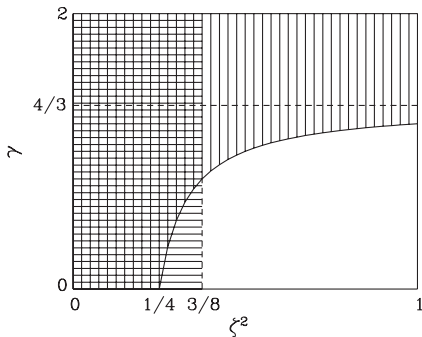By inverting (3.8) and substituting from (3.23) we obtain

FIG. 1. The fixed points FP1$_\pm$ and FP2$_\pm$ exist in the horizontally and vertically shaded regions, respectively.

$$t_* = t_{*0} \exp\left[\frac{4\zeta^2}{\sqrt{3 - 8\zeta^2}} \tau_*\right]. \qquad (3.26)$$

Substituting this back into (3.25) and (3.26) then yields

$$\sigma = \frac{1}{2\zeta} \ln\left(\frac{t_*}{t_{*0}}\right) + \sigma_0, \qquad (3.27)$$

$$a_* = a_0\left(\frac{t_*}{t_{*0}}\right)^{1/(8\zeta^2)}. \qquad (3.28)$$

In order to fix σ_0, we go back to the original equations of motion, i.e., setting $\rho_* = 0$ finding

$$H_*^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\sigma}^2 + V_*\right), \qquad (3.29)$$

which gives the value of the scalar field σ_0 at $t_* = t_{*0}$ as

$$\exp(-4\zeta\sigma_0) = \frac{3 - 8\zeta^2}{64\zeta^4 t_0^2 V_0}. \qquad (3.30)$$

Equation (3.28) shows that the solution with $\zeta < 1/(2\sqrt{2})$ gives a power-law inflation.

For the fixed point FP2$_+$, we have $(x_F, y_F) = (x_2^{(+)}, y_2^{(+)})$, replacing (3.27) and (3.28) and by

$$\sigma = \frac{\sqrt{\gamma}}{\sqrt{2(2 - \gamma - 2(4 - 3\gamma)\zeta^2)}} \tau_* + \sigma_0, \qquad (3.31)$$

$$a_* = a_{*0} \exp\left[\frac{\zeta\sqrt{\gamma}}{\sqrt{2(2 - \gamma - 2(4 - 3\gamma)\zeta^2)}} \tau_*\right], \qquad (3.32)$$

respectively. The cosmic time in the Einstein frame is

$$t_* = t_{*0} \exp\left[\frac{2\zeta\sqrt{\gamma}}{\sqrt{2(2 - \gamma - 2(4 - 3\gamma)\zeta^2)}} \tau_*\right]. \qquad (3.33)$$

Hence we find

$$\sigma = \frac{1}{2\zeta} \ln\left(\frac{t_*}{t_{*0}}\right) + \sigma_0, \qquad (3.34)$$

$$a_* = a_{*0}\left(\frac{t_*}{t_{*0}}\right)^{1/2}. \qquad (3.35)$$

It is important to notice that $a_* \sim t_*^{1/2}$ follows also for dust dominance. In fact this behavior is true for any equation of state. For the remedy of this unfavorable result, the reader is advised to see Sec. 4.4.3 of [4] or Sec. 3.4 of [6].

To fix σ_0, we use the original Friedmann equation

$$H_*^2 = \frac{1}{6}\dot{\sigma}^2 + \frac{V_*}{3}\left(1 + \frac{\rho_*}{V_*}\right), \qquad (3.36)$$

where ρ_*/V_* is a constant given by Eq. (3.21). We then find

$$\exp(-4\zeta\sigma_0) = \frac{(4\zeta^2 - 1)[2 - \gamma - 2(4 - 3\gamma)\zeta^2]}{16\gamma\zeta^4 t_{*0}^2 V_0}.$$

(3.37)

Note that spacetime in this range is static in the Jordan frame, as will be shown in (4.22).

B. Stability analysis and the attractors

Next we analyze stability of the fixed points FP1$_\pm$ and FP2$_\pm$ in the self-autonomous system (3.15) and (3.16).

1. Perturbation analysis

The simplest way is to apply a perturbation analysis. We perturb the variable (x, y) around the fixed point (x_F, y_F) as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_F + \delta x \\ y_F + \delta y \end{pmatrix}.$$

(3.38)

Inserting Eq. (3.38) into the basic equations (3.15) and (3.16), we find a set of the linear differential equations:

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix}' = \begin{pmatrix} A_{xx} & A_{xy} \\ A_{yx} & A_{yy} \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix},$$

(3.39)

where the components of the matrix A is given by

$$\begin{aligned} A_{xx} &= 3\zeta(\gamma x_F - y_F), \\ A_{xy} &= 3\zeta(-x_F + 2\zeta^2(4 - 3\gamma)y_F), \\ A_{yx} &= -\frac{2 - \gamma}{2\zeta}x_F + 2\zeta y_F, \\ A_{yy} &= \zeta(2x_F - 3\gamma y_F). \end{aligned}$$

(3.40)

Assuming $\delta x, \delta y \propto e^{\omega\tau_*}$, we find the equation for the eigenvalue ω as

$$\omega^2 - \mathrm{Tr}A\omega + \det A = 0,$$

(3.41)

where

$$\begin{aligned} \mathrm{Tr}\,A &= \frac{1}{4\zeta^2 - 1}[(x_F - 4\zeta^2 y_F) + \{3\gamma(4\zeta^2 - 1) \\ &\quad - (3 - 8\zeta^2)\}(x_F - y_F)], \end{aligned}$$

(3.42)

$$\begin{aligned} \det A &= \frac{3}{2(4\zeta^2 - 1)}[(2 - \gamma - 2(4 - 3\gamma)\zeta^2)(x_F - 4\zeta^2 y_F)^2 \\ &\quad - 2\gamma\zeta^2(3 - 8\zeta^2)(x_F - y_F)^2]. \end{aligned}$$

(3.43)

In this expression, the first terms in (3.42) and (3.43) vanish for FP1$_\pm$, while the second terms disappear for FP2$_\pm$. The fixed point is stable in the following two cases:
  (i) Equation (3.41) has two negative real roots.
  (ii) Equation (3.41) has two complex conjugate roots with a negative real part.
The condition is

$$\mathrm{Tr}\,A < 0 \quad \text{and} \quad \det A > 0.$$

(3.44)

For FP1$_\pm$, we find

$$(\mathrm{Tr}A)_{\mathrm{FP1}_\pm} = [4(3\gamma + 2)\zeta^2 - 3(\gamma + 1)]y_1^{(\pm)},$$

(3.45)

$$(\det A)_{\mathrm{FP1}_\pm} = 3\gamma\zeta^2(3 - 8\zeta^2)(4\zeta^2 - 1)(y_1^{(\pm)})^2.$$

(3.46)

For the expanding universe (FP1$_+$), which we are interested in, the above condition (3.44) gives

$$\zeta^2 < \tfrac{1}{4}.$$

(3.47)

For FP2$_\pm$, we have

$$(\mathrm{Tr}\,A)_{\mathrm{FP2}_\pm} = -y_2^{(\pm)},$$

(3.48)

$$(\det A)_{\mathrm{FP2}_\pm} = -3\gamma\zeta^2(3 - 8\zeta^2)(4\zeta^2 - 1)(y_2^{(\pm)})^2.$$

(3.49)

The stability condition for the expanding universe (the fixed point FP2$_+$) gives

$$\zeta^2 > \tfrac{1}{4}.$$

(3.50)

In this way we find that FP1$_+$ and FP2$_+$ are the attractors for $\zeta^2 < 1/4$ and for $\zeta^2 > 1/4$ (also with $\zeta^2 < (2 - \gamma)/[2(4 - 3\gamma)]$ for the existence of FP2$_+$), respectively. For $\zeta^2 = 1/4$, two types of fixed points merge with each other, sharing the same behaviors, but like a saddle point rather than an attractor. Figure 2 shows in which portion of the ζ^2-γ plane we have the attractor fixed points.

2. Phase-space analysis

We also study stability by use of a phase-space analysis of the dynamical system with (3.15) and (3.16) [19–22]. We may discuss global stability rather than a local one in the perturbative approach, as we will show shortly. There is a limitation, however, because dependence on the values of γ, ζ^2 is not as simple as shown in (3.45) and in the subsequent equations. We must develop the computation for each of these parameters separately, though without any difficulty in principle. For this reason, we show the following examples of radiation dominance ($\gamma = 4/3$) illustrat-
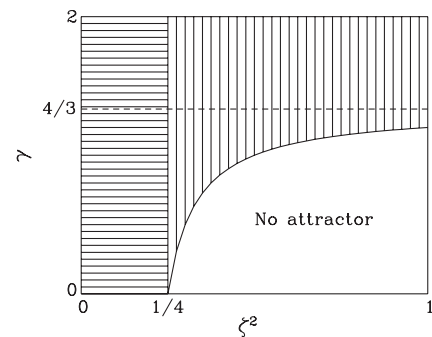

FIG. 2. The attractor fixed points (FP1$_+$ and FP2$_+$) in the same parameter space of ζ^2 and γ, also the same shading pattern as in Fig. 1.

ing generic features shared commonly by this type of analyses with any values of the parameters.

For the sake of convenience we start with reproducing (3.15) and (3.16) for $\gamma = 4/3$;

$$x' = \zeta(2x^2 - 3xy + 1), \qquad (3.51)$$

$$y' = \frac{1}{2\zeta}\left(-\frac{1}{3}x^2 + 4\zeta^2 xy - 4\zeta^2 y^2 + \frac{1}{3}\right), \qquad (3.52)$$

which determine how the point (x, y) representing the solution moves with time. The fixed points are (3.17) for FP1$_\pm$, while $(x_2^{(\pm)}, y_2^{(\pm)}) = \pm(1, 1)$ from (3.18) for FP2$_\pm$. We also focus upon FP2$_\pm$.

A basis for the required analysis is prepared first by drawing the "null curves" for $x' = 0$ and $y' = 0$. By choosing the vanishing left-hand sides of (3.51) and (3.52), we find that the former curves, solid (blue), are in fact hyperboloids, whereas the latter ones, dashed (red) curves are either hyperbolic or elliptic depending on $\zeta^2 > 1/3$ or $\zeta^2 < 1/3$, respectively, as shown in Figs. 3 and 4, used separately for the analyses of the two choices for ζ^2. The crossings represent fixed points. Note that there are four fixed points (FP1$_\pm$, FP2$_\pm$) for $\zeta^2 < 1/3$, while two fixed points (FP2$_\pm$) for $\zeta^2 > 1/3$.

In Fig. 3(a), we draw two sets of hyperboloids for $\zeta^2 > 1/3$. It is rather easy to determine which side of each curve implies positive or negative x' and y', as shown by the symbols like $+_x$ and $+_y$, for $x' > 0$ and $y' > 0$, respectively. We then determine in what direction a point, or better called a trajectory, "flows" with time in a given

region in x-y space bounded by the null curves, as illustrated symbolically by arrows (green). In Fig. 3(b), we show a typical trajectory represented by a dotted curve which enters the diagram first near the lower-left corner, making a big loop outside the frame of the diagram, then reentering again, and finally spirals into the crossing at FP2$_+$ ($x = y = 1$), which corresponds to (3.34) and (3.35). This is the way we now establish our previous numerical results obtained on the basis of the heuristic approach [4,6] to be an authentic attractor in a strict sense.

We notice, on the other hand, that a trajectory, or the solution, may not always converge to a fixed point, straying instead toward infinity, as will be illustrated in Appendix B. Obviously, however, not reaching the point of $x' = y' = 0$ implies the destination not corresponding to the steady and lasting solution, as given by (3.34) and (3.35), for example. In other words, any solution that survives a long time must come from the attractor.

There is another fixed point FP2$_-$ ($x = y = -1$) to which no trajectory flows into as long as we start with a positive ρ, the same sign as V_0. Note that this fixed point corresponds to the contracting universe. The flows shown in Fig. 3(c) indicate that this is a repeller to be interpreted as the time-reversed point against the attractor at FP2$_+$.

We encounter another complication, on the other hand, if $\zeta^2 < 1/3$, for which we have four crossings as illustrated in Fig. 4(a). Focusing upon the behaviors in the upper-right quadrant, we have magnified views at each of the two, one in (b) around $x = y = 1$, corresponding to the fixed-point solution of FP2 given by (3.34) and (3.35) above, and
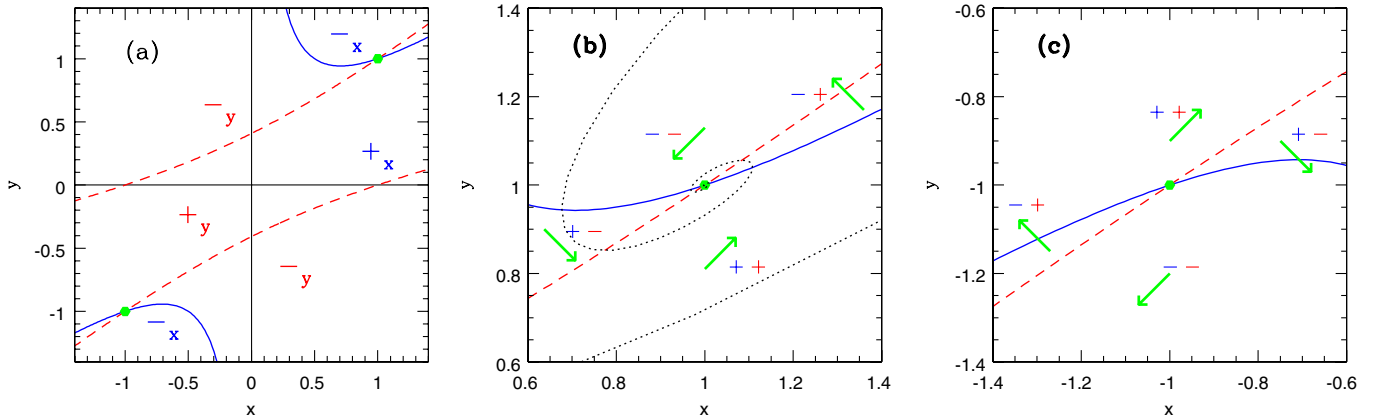

FIG. 3 (color online). An example of $\zeta^2 > 1/3$ in the phase diagram of $x = \sigma'$ and $y = \zeta^{-1}\mathcal{H}_*$. In the overall diagram (a), we chose an example of $\zeta^2 = 1/2$ to draw solid (blue) and dashed (red) curves for the null curves for $x' = 0$ and $y' = 0$, respectively. They separate regions of different signs of them, denoted by $+_x$ and $+_y$, for $x' > 0$ and $y' > 0$, respectively, for example. The crossings, or fixed points, occur between two hyperboloids, marked with a blob at FP2$_+$ ($x = y = 1$) as an attractor, and the one at FP2$_-$ ($x = y = -1$) as a repeller (no fixed points FP1$_\pm$ for this parameter). In each of the smaller regions bounded by the null curves, as shown in (b) and (c), close-up views near the attractor and the repeller, respectively, we have directions of flows of solutions, or trajectories, shown symbolically by the arrows together with a nearby pair of signs for the increment (+) or decrement (−) of the x and y components denoted in this order. A dotted curve in (b) is a trajectory starting from the initial values $\ln\rho = 0.1$, $\varphi = 0.1$, $\dot\varphi = 0$ at the initial Jordan frame "time" $\ln t = 1$, where $\varphi = (\xi/2)\phi^2$ with $\xi = 1/4$. The trajectory flows basically in accordance with the arrows, entering the frame of diagram near the lower-left corner, traversing to go outside beyond the right edge, reentering at the top, spiraling finally into the attractor.
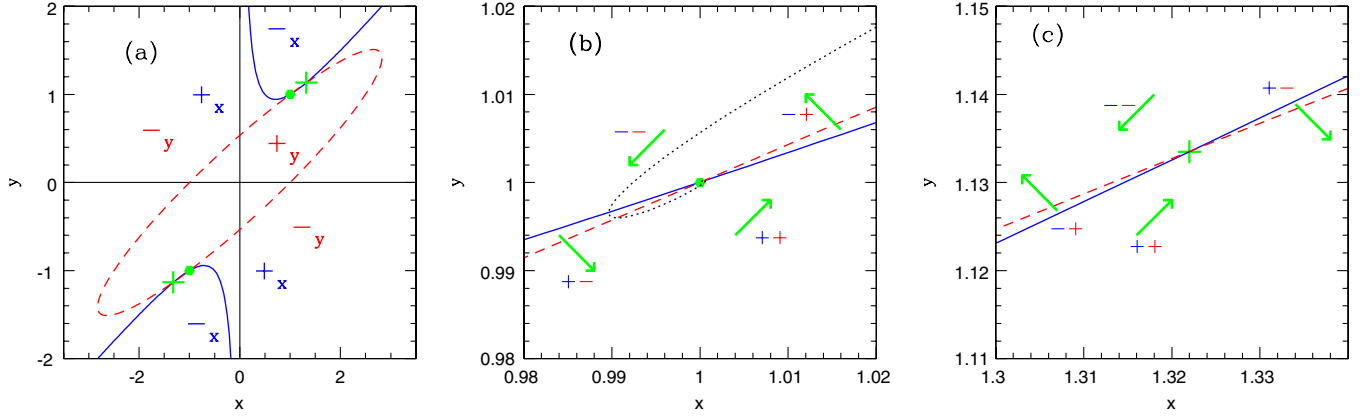
FIG. 4 (color online).   An example of $1/4 < \zeta^2 < 1/3$. In the overall diagram (a), we chose an example of $\zeta^2 = 0.2916$, $\xi = 0.3890$. Unlike in Fig. 3, the dashed (red) null curve is an ellipsoid, thus producing four crossings. In addition to the attractor at FP2$_+$ ($x = y = 1$), denoted by a blob (green), we have another at FP1$_+$ ($x = 1.322$, $y = 1.134$) marked by a cross (green), both accompanied with the time-reversal counterparts in the left-lower quadrant (FP1$_-$ and FP2$_-$), which we ignore for brevity. In a close-up view (b) around FP2$_+$, we show an example of a trajectory with the same initial values as considered in Fig. 3(b), certainly spiraling into the fixed point in accordance with the arrows of flows, while we find no trajectories in (c) around another crossing of the type FP1$_+$. The behaviors of the arrows in the right-upper and left-lower strips remind us of a saddle-point potential resulting in unstable motion.

another shown in (c) now categorized into the type of FP1. The flows in (b) do indicate that the solution in all directions tends finally to the crossing, a real attractor, while according to those illustrated in (c) there are some narrow strips sandwiched between two null curves in which the trajectory tends away from the crossing. This reminds us of a saddle-point potential rather than the purely attractive or repulsive potential in mechanics, hence implying an unstable solution which may not survive a long time, leading ultimately to the same drift toward infinity, as was remarked toward the end of the discussion of the preceding example in Fig. 3. For $1/4 < \zeta^2 < 1/3$, therefore, we may fail to reach the attractor solution with some probability depending on the initial values or on what portion in the x-y plane we started off. We may constrain ourselves to $\zeta^2 > 1/3$, though another *a posteriori* attitude might be suggested: Given what we are at present, right initial values, or right initial locations in phase space, must have been chosen to reach the attractor solution (3.34) and (3.35).

We apply the same analysis for any values of γ, finding similar results. We summarize the power index of scale factor of the attractor solutions (FP1$_+$ for $\zeta < 1/2$, and FP2$_+$ for $\zeta > 1/2$) in Fig. 5. We find that the power-law inflation appears when $\zeta^2 < 1/8$. The power exponent of the scale factor in the Einstein frame is always $1/2$ for FP2$_+$, which does not depend on the equation of state of the matter fluid, as derived in (3.35).

IV. COSMOLOGY IN THE JORDAN FRAME

We repeat the similar analysis for the same system as in Sec. III now in the Jordan frame.

From (2.1) together with the FRW metric

$$ds^2 = -dt^2 + a^2 ds_3^2, \tag{4.1}$$

we derive

$$3\xi\phi^2\left(H^2 + \frac{k}{a^2}\right) + 6\xi H\phi\dot\phi = \frac{\epsilon}{2}\dot\phi^2 + V_0 + \rho, \tag{4.2}$$

$$\xi(\phi\ddot\phi + \dot\phi^2 + 3H\phi\dot\phi) = \zeta^2(4V_0 + \rho - 3P), \tag{4.3}$$

$$\dot\rho + 3H(P + \rho) = 0, \tag{4.4}$$

where

$$H = \frac{\dot a}{a}, \tag{4.5}$$

while $P = P_* \exp(4\zeta\sigma)$ and $\rho = \rho_* \exp(4\zeta\sigma)$ are the
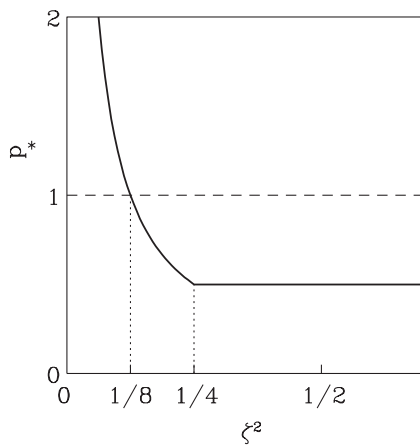

FIG. 5.   The power index p_* of the scale factor in the Einstein frame for the attractor solutions. The power-law inflation appears when $\zeta^2 < 1/8$. The power exponent is always $1/2$ for FP2$_+$, independent of γ.

pressure and the energy density, respectively, in the Jordan frame. Note that the quantities in the Jordan frame are all denoted unmarked in contrast to those marked with the subscript $*$ in the Einstein frame. Accordingly, the dots in (4.2), (4.3), and (4.4) imply differentiation with respect to t instead of t_*.

Since $ds^2 = \Omega^{-2} ds_*^2$ with $\Omega^2 = \xi \phi^2 = \exp(2\zeta \sigma)$, we have the relations between the variables in Jordan frame and those in the Einstein frame as

$$dt = \Omega^{-1} dt_*, \qquad (4.6)$$

$$a = \Omega^{-1} a_*. \qquad (4.7)$$

Note that we do not change the coordinate system when we perform a conformal transformation. However, since we use a cosmic time in each frame, we have to change the time coordinate between t_* and t according to (4.6). The Hubble expansion parameters H and H_* are defined by each cosmic time as

$$H = \frac{1}{a} \frac{da}{dt}, \qquad H_* = \frac{1}{a_*} \frac{da_*}{dt_*}. \qquad (4.8)$$

Hence we have the relation

$$H_* = \Omega^{-1} \left(H + \frac{d \ln \Omega}{dt} \right). \qquad (4.9)$$

Introducing the new time coordinate τ and new scalar field Φ, which are defined by

$$d\tau = 2\xi^{-1/2} V_0^{1/2} \phi^{-1} dt, \qquad (4.10)$$

$$\Phi = \ln \phi, \qquad (4.11)$$

respectively, we rewrite Eqs. (4.2), (4.3), and (4.4) as

$$6\mathcal{H}^2 + \frac{3k\xi e^{2\Phi}}{2V_0 a^2} = \frac{1}{\zeta^2} (\Phi')^2 + \frac{1}{2} \left(1 + \frac{\rho}{V_0} \right), \qquad (4.12)$$

$$\Phi'' - 2(\Phi')^2 + 3\mathcal{H}\Phi' = \zeta^2 \left[1 + \frac{(4 - 3\gamma)\rho}{4V_0} \right], \qquad (4.13)$$

$$\rho' + 3\gamma (\mathcal{H} - \Phi')\rho = 0, \qquad (4.14)$$

where

$$\mathcal{H} \equiv \frac{a'}{a} + \Phi', \qquad (4.15)$$

and the prime is the derivative with respect to τ. We also assume the equation of state, $P = (\gamma - 1)\rho$.

Now, we consider only the spatially flat universe, i.e., $k = 0$ (See Appendix A for $k \neq 0$). Taking the derivative of Eq. (4.12) and using Eqs. (4.12), (4.13), and (4.14), we find

$$\mathcal{H}' = \frac{\gamma - 2}{4\zeta^2} (\Phi')^2 + 2\mathcal{H}\Phi' - \frac{3\gamma}{2} \mathcal{H}^2 + \frac{\gamma}{8}, \qquad (4.16)$$

$$\Phi'' = \frac{3\gamma}{2} (\Phi')^2 - 3\mathcal{H}\Phi' + 3(4 - 3\gamma)\zeta^2 \mathcal{H}^2 + \frac{3\gamma}{4} \zeta^2. \qquad (4.17)$$

This is again a self-autonomous system with two variables x and y:

$$x' = \frac{3\zeta}{4} [2\gamma x^2 - 4xy + 4\zeta^2 (4 - 3\gamma)y^2 + \gamma], \qquad (4.18)$$

$$y' = \frac{1}{8\zeta} [-2(2 - \gamma)x^2 + 16\zeta^2 xy - 12\gamma \zeta^2 y^2 + \gamma], \qquad (4.19)$$

where $x = \zeta^{-1} \Phi'$ and $y = \zeta^{-1} \mathcal{H}$. These equations turn out to be precisely the same as Eqs. (3.15) and (3.16), respectively, implying the same dynamical system, sharing the same fixed points: FP1$_\pm$ (3.17) and FP2$_\pm$ (3.18).

The energy density is given by

$$\frac{\rho}{V_0} = 12\mathcal{H}^2 - \frac{2}{\zeta^2} (\Phi')^2 - 1 = 12\zeta^2 y^2 - 2x^2 - 1, \qquad (4.20)$$

precisely the same as (3.19) for the Einstein frame.

We only show the explicit solutions of the fixed points, because the dynamical properties such as an attractor is the same as that in the Einstein frame. For FP1$_\pm$, we have

$$a = a_0 (\tau - \tau_0)^{(1/4\zeta^2) - 1}, \qquad \phi = \pm \frac{2\zeta}{\sqrt{3 - 8\zeta^2}} (\tau - \tau_0),$$

$$\rho = 0. \qquad (4.21)$$

This gives a power-law inflation for $\zeta^2 < 1/8$.

For FP2$_\pm$, we have

$$a = a_0, \qquad (4.22)$$

$$\phi = \pm \frac{\sqrt{\gamma}}{\sqrt{2(2 - \gamma - 2(4 - 3\gamma)\zeta^2)}} (\tau - \tau_0), \qquad (4.23)$$

$$\rho = \frac{2(4\zeta^2 - 1)}{2 - \gamma - 2(4 - 3\gamma)\zeta^2} V_0, \qquad (4.24)$$

where a_0 and τ_0 are integration constants. The spacetime is a static Minkowski space, in contrast with the expansion in the Einstein frame as shown by (3.35).

The vacuum solution in radiation dominance in [9] may be interpreted as the limit $\zeta^2 \to 1/4$ in (4.22), (4.23), and (4.24). The constant a was also derived in [23].

V. COSMOLOGY WITH A POWER-LAW POTENTIAL

The analysis in the preceding section can be readily extended to include the power-law potential

$$V(\phi) = \lambda_\alpha \phi^\alpha \qquad (5.1)$$

in the Jordan frame. An example is provided by extending the action (2.5) in string theory to

$$S = \frac{1}{2} \int d^D x \sqrt{-g} e^{-2\Phi} [R(g) - 2\Lambda + 4(\nabla \Phi)^2], \quad (5.2)$$

in which we have added the term of Λ, to be corresponded to (5.1) by choosing $\alpha = 2$ and $\lambda_2 = \Lambda/4$.

The action equivalent to (5.1) but expressed in the Einstein frame is given by

$$S = \int d^4 x \sqrt{-g_*} \left[\frac{1}{2} R_*(g_*) - \frac{1}{2} (\nabla_* \sigma)^2 - V_*(\sigma) \right]$$
$$+ \int d^4 x \sqrt{-g_*} L(\sigma, \psi_*, g_*), \quad (5.3)$$

where

$$V_* = \exp[-\zeta(4 - \alpha)\sigma] V_0, \quad (5.4)$$

with $V_0 = \lambda_\alpha / |\xi|^{\alpha/2}$. Since this is of the same form as Eq. (2.9), we repeat the same analysis as before. One of the points to be kept in mind is that the conformal transformation is always the same as (2.6) with (2.7). Hence we have the same energy-momentum conservation law (3.4), as well. Notice also $V_* \to$ const hence an ordinary type of inflation in the Einstein frame, as $\alpha \to 4$.

The basic equations for cosmology are then

$$H_* + \frac{k}{a_*^2} = \frac{1}{3}(\dot{\sigma}^2 + V_* + \rho_*), \quad (5.5)$$

$$\ddot{\sigma} + 3H_*\dot{\sigma} + \frac{\partial V_*}{\partial \sigma} = \zeta(\rho_* - 3P_*), \quad (5.6)$$

$$\dot{\rho}_* + 3H_*(P_* + \rho_*) = -\zeta\dot{\sigma}(\rho_* - 3P_*). \quad (5.7)$$

Introducing a new dimension-free time coordinate as

$$d\tau_* = 2\sqrt{V_*} dt_*, \quad (5.8)$$

also assuming the equation of state and focusing on $k = 0$ as before, we obtain formally the same equations as (3.9), (3.10), and (3.11), but replace (3.13) by

$$(\rho_*/V_*)' = -[3\gamma \mathcal{H}_* - \zeta(3\gamma - \alpha)\sigma'](\rho_*/V_*). \quad (5.9)$$

Computing in the same way as before, we come to replacing (3.15) and (3.16) by

$$x' = \frac{\zeta}{4}[2(3\gamma - \alpha)x^2 + 12\zeta^2(4 - 3\gamma)y^2 - 12xy + 3\gamma - \alpha], \quad (5.10)$$

$$y' = \frac{1}{8\zeta}[-2(2 - \gamma)x^2 - 12\gamma\zeta^2 y^2 + 4\zeta^2(4 - \alpha)xy + \gamma], \quad (5.11)$$

respectively, where $x = \sigma'$ and $y = \zeta^{-1}\mathcal{H}_*$ as before.

We find four fixed points as the previous case:

$$\text{FP 1}_\pm: (x_F, y_F) = (x_1^{(\pm)}, y_1^{(\pm)})$$
$$\equiv \pm \frac{1}{\sqrt{2(6 - \zeta^2(4 - \alpha)^2)}} \left(\zeta(4 - \alpha), \frac{1}{\zeta} \right), \quad (5.12)$$

$$\text{FP2}_\pm: (x_F, y_F) = (x_2^{(\pm)}, y_2^{(\pm)})$$
$$\equiv \pm \frac{1}{\sqrt{6[3\gamma(2 - \gamma) - 2(4 - 3\gamma)(3\gamma - \alpha)\zeta^2]}}$$
$$\times (3\gamma, 3\gamma - \alpha). \quad (5.13)$$

The fixed points $\text{FP1}_\pm$ exist if

$$4\left(1 - \frac{\sqrt{6}}{4\zeta} \right) < \alpha < 4\left(1 + \frac{\sqrt{6}}{4\zeta} \right). \quad (5.14)$$

For fixed points $\text{FP2}_\pm$, the constraint for existence is a little more complicated, as we find

$$\alpha > 3\gamma \left[1 - \frac{2 - \gamma}{2(4 - 3\gamma)} \frac{1}{\zeta^2} \right] \quad \text{for } \gamma < \frac{4}{3}, \quad (5.15)$$

$$\text{any values of } \zeta \text{ and } \alpha \quad \text{for } \gamma = \frac{4}{3}, \quad (5.16)$$

$$\alpha < 3\gamma \left[1 + \frac{2 - \gamma}{2(3\gamma - 4)} \frac{1}{\zeta^2} \right] \quad \text{for } \gamma > \frac{4}{3}. \quad (5.17)$$

Next we analyze the solutions of these fixed points. In what follows, we consider only the expanding universe ($y_F > 0$), i.e. FP1_+, while FP2_+ for $\alpha < 3\gamma$ and FP2_- for $\alpha > 3\gamma$.

For FP1_+, we find

$$\sigma = \frac{\zeta(4 - \alpha)}{\sqrt{2(6 - \zeta^2(4 - \alpha)^2)}} \tau_* + \sigma_0, \quad (5.18)$$

$$a_* = a_{*0} \exp \left[\frac{\tau_*}{\sqrt{2(6 - \zeta^2(4 - \alpha)^2)}} \right]. \quad (5.19)$$

The cosmic time t_* is given by

$$t_* = t_{*0} \exp \left[\frac{\zeta^2(4 - \alpha)^2 \tau_*}{\sqrt{2(6 - \zeta^2(4 - \alpha)^2)}} \right]. \quad (5.20)$$

Hence the solution of FP1_+ is described by

$$\sigma = \frac{2}{\zeta(4 - \alpha)} \ln \left(\frac{t_*}{t_{*0}} \right) + \sigma_0, \quad (5.21)$$

$$a_* = a_{*0} \left(\frac{t_*}{t_{*0}} \right)^{p_*}, \quad (5.22)$$

where

$$p_* = \frac{2}{\zeta^2(4-\alpha)^2}. \tag{5.23}$$

In this case, matter does not contribute to the expansion of the universe, i.e. at the fixed point, $\rho_* = 0$. The cosmic expansion becomes inflationary if $p_* > 1$, i.e.,

$$4\left(1 - \frac{\sqrt{2}}{4\zeta}\right) < \alpha < 4\left(1 + \frac{\sqrt{2}}{4\zeta}\right). \tag{5.24}$$

The singularity of (5.23) at $\alpha \to 4$ is obviously related to the limit $V_* \to$ const as noticed in (5.4).

For FP2$_\pm$, we have

$$\sigma = \pm \frac{3\gamma\tau_*}{\sqrt{6[3\gamma(2-\gamma) - 2(4-3\gamma)(3\gamma-\alpha)\zeta^2]}} + \sigma_0, \tag{5.25}$$

$$a_* = a_{*0} \exp\left[\frac{\pm(3\gamma-\alpha)\zeta\tau_*}{\sqrt{6[3\gamma(2-\gamma) - 2(4-3\gamma)(3\gamma-\alpha)\zeta^2]}}\right]. \tag{5.26}$$

The cosmic time t_* is given by

$$t_* = t_{*0} \exp\left[\frac{\pm 3\gamma(4\zeta-\alpha)\tau_*}{2\sqrt{6[3\gamma(2-\gamma) - 2(4-3\gamma)(3\gamma-\alpha)\zeta^2]}}\right]. \tag{5.27}$$

Hence the solution of FP2$_+$ is described by

$$\sigma = \frac{2}{\zeta(4-\alpha)} \ln\left(\frac{t_*}{t_{*0}}\right) + \sigma_0, \tag{5.28}$$

$$a_* = a_{*0}\left(\frac{t_*}{t_{*0}}\right)^{p_*}, \tag{5.29}$$

where

$$p_* = \frac{2(3\gamma-\alpha)}{3\gamma(4-\alpha)} = \frac{2}{3\gamma} + \Delta p_*, \tag{5.30}$$

where

$$\Delta p_* = \frac{2(3\gamma-4)}{3\gamma(4-\alpha)}, \tag{5.31}$$

which describes the deviation from the conventional power exponent with the adiabatic index γ. It is due to the interaction between the matter fluid and the scalar field ϕ. It is precisely this interaction that keeps the ratio ρ_*/V_* constant at FP2$_\pm$. The value is obtained from Eq. (3.19);

$$\frac{1}{6}\left(\frac{\rho_*}{V_*}\right)_{\text{FP2}} = 2\zeta^2(y_2^{(\pm)})^2 - \frac{(x_2^{(\pm)})^2}{3} - \frac{1}{6}$$
$$= \frac{2[(4-\alpha)^2 + (3\gamma-4)(4-\alpha) - 3\gamma/\zeta^2]}{2(3\gamma-4)^2 + 2(3\gamma-4)(4-\alpha) + 3\gamma(2-\gamma)/\zeta^2}, \tag{5.32}$$

which is consistent with (5.9) for the energy density.

In order for this energy density to be positive, we have to impose the following condition:

$$\alpha < \tfrac{1}{2}[(3\gamma+4) - \sqrt{(3\gamma-4)^2 + 12\gamma/\zeta^2}] \quad \text{for FP2}_+,$$

$$\alpha > \tfrac{1}{2}[(3\gamma+4) + \sqrt{(3\gamma-4)^2 + 12\gamma/\zeta^2}] \quad \text{for FP2}_-.$$

We also find a simple result $p_* = 1/2$ follows either by $\alpha = 0$ for any γ or by $\gamma = 4/3$ for any α. This result may be described by

$$p_* = \tfrac{1}{2} + \Delta p_*', \tag{5.33}$$

where

$$\Delta p_*' = \frac{(3\gamma-4)\alpha}{6\gamma(4-\alpha)} = \frac{\alpha}{4}\Delta p_*. \tag{5.34}$$

We are now looking into more details of the power-law inflation. We find a power-law inflation by σ if the power exponent of the potential V_* is sufficiently small, i.e. $|\zeta(4-\alpha)| < \sqrt{2}$, and if there is no coupling between the matter fluid and the scalar field σ in the Einstein frame. This type of inflation is realized in the fixed point FP1$_+$ even when we include the coupling with the matter fluid, which has nothing to do with inflation at FP1$_+$. However, the power-law inflation may occur also for FP2$_\pm$, in which the coupling to the matter fluid is non-negligible. From the condition $p_* > 1$ for the power exponent of the scale factor in the Einstein frame, we find the following conditions:

$$\alpha < \frac{6\gamma}{3\gamma-2} \quad \text{or} \quad \alpha/2 > 2 \quad \text{for } \gamma < \frac{2}{3}, \tag{5.35}$$

$$\alpha > 4 \quad \text{for } \gamma = \frac{2}{3}, \tag{5.36}$$

$$4 < \alpha < \frac{6\gamma}{3\gamma-2} \quad \text{for } \frac{2}{3} < \gamma < \frac{4}{3}, \tag{5.37}$$

$$\text{no case} \quad \text{for } \gamma = \frac{4}{3}, \tag{5.38}$$

$$\frac{6\gamma}{3\gamma-2} < \alpha < 4 \quad \text{for } \gamma > \frac{4}{3}. \tag{5.39}$$

This is a new type of inflation. The potential itself is too steep to cause inflation, but the matter fluid assists to cause a faster expansion because of its coupling to the scalar field. Note that including the matter fluid with $\gamma < 2/3$ means that we can have inflation just by this fluid. But the coupling with the scalar field does not assist such a type of inflation. Rather it will restrict the possibility of inflationary expansion.

Next we have to analyze the stability of the fixed points. We can analyze it by perturbations or by use of the phase diagram, as shown in the previous section. Here we give the perturbation analysis. Inserting the perturbations

around the fixed points $x = x_F + \delta x$, $y = y_F + \delta y$ in the basic equations (5.10) and (5.11), we find a set of the linear perturbation equations (3.39) with the components of the matrix being

$$A_{xx} = (3\gamma - \alpha)x_F - 3y_F,$$

$$A_{xy} = -3x_F + 6\zeta^2(4 - 3\gamma)y_F,$$

$$A_{yx} = -\frac{2 - \gamma}{2\zeta^2}x_F + \frac{(4 - \alpha)}{2}y_F, \qquad (5.40)$$

$$A_{yy} = \frac{(4 - \alpha)}{2}x_F - 3\gamma y_F.$$

Setting δx, $\delta y \propto e^{\omega\tau_*}$, we find the eigenequation for ω as (3.41) with

$$\text{Tr } A = (3\gamma + 2 - 3\alpha/2)x_F - 3(\gamma + 1)y_F, \qquad (5.41)$$

$$\det A = \frac{1}{2\zeta^2}[(3\gamma - \alpha)(4 - \alpha)\zeta^2 - 3(2 - \alpha)]x_F^2$$
$$+ 3(\alpha\gamma - 10\gamma + 8)x_Fy_F$$
$$+ 3[3\gamma - \zeta^2(4 - 3\gamma)(4 - \alpha)]y_F^2. \qquad (5.42)$$

Using these equations, we analyze the stability of the fixed points as before. The result is shown in Fig. 6. The regions denoted by FP1 and FP1-I give the attractor solution of FP1$_+$, while those by FP2 and FP2-I correspond to the attractors of FP2$_+$. For FP1$_+$, the inflationary solutions exist in the case of a flat potential ($\zeta|4 - \alpha| < 1/2$), but the new type of inflationary solution also appears for FP2$_+$ if $\gamma \neq 4/3$. Each region for inflation is also shown by FP1-I or FP2-I. This new type of inflation is assisted by a coupling between the scalar field σ and the matter fluid.

In the Jordan frame, we show only the behavior of the fixed points FP1$_\pm$ and FP2$_\pm$. After a conformal transformation, we find them

$$a \propto t^p, \qquad (5.43)$$

$$\phi \propto t^q, \qquad (5.44)$$

where

$$(p, q) = \begin{cases} \left(\frac{2(1-(4-\alpha)\zeta^2)}{(2-\alpha)(4-\alpha)\zeta^2}, \frac{2}{2-\alpha}\right) & \text{for FP1}_\pm \\ \left(-\frac{2\alpha}{2-\alpha}, \frac{2}{2-\alpha}\right) & \text{for FP2}_\pm. \end{cases} \qquad (5.45)$$

The scale factor in the Jordan frame for FP2$_\pm$ is not constant except for $\alpha = 0$. It is expanding for $\alpha < 0$ or $\alpha > 2$, while it is contracting for $0 < \alpha < 2$. For $\alpha = 2$, we find $a \propto \exp[-\frac{2}{3\gamma}t]$.

VI. CONCLUDING REMARKS

We have presented a formulation in which we trace the temporal development of cosmological solutions of the scalar-tensor theory in two-dimensional phase space. Thanks to assuming a simple equation of state, in Secs. III, IV, and V, we have obtained two different sets of fixed points, FP1$_\pm$ and FP2$_\pm$. Conditions of obtaining attractors are studied in detail.

We have established the attractor nature of the fixed points. At FP2$_\pm$, the scale factor behaves like a constant and $t_*^{1/2}$ in the Jordan and the Einstein frames, respectively, for $\zeta^2 > 1/4$ or $\xi < 1/2$ ($\omega < -1/2$) with $\epsilon = -1$, when we have a simple cosmological constant in the Jordan frame. This solution is also accompanied with the proportionality between ρ_* and V_*, called a scaling behavior, which is going to be replaced by the "interlacing" behavior, as exemplified in Fig. 5.8. of [4], by further extending the model.

An extension to the power-law potential in Sec. V has shown that the coupling between the scalar field and the matter provides a new type of inflation with γ off the

(a) $\gamma = 1$ (dust)   (b) $\gamma = 4/3$ (radiation)   (c) $\gamma = 2$ (stiff matter)
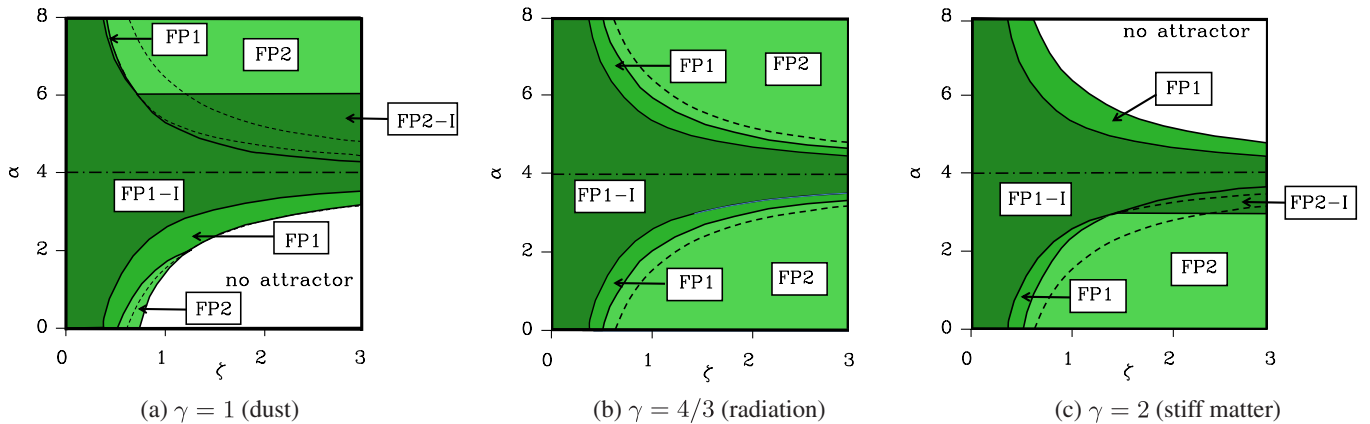
FIG. 6 (color online).   For various matter fluids ((a) dust [$\gamma = 1$], (b) radiation [$\gamma = 4/3$], (c) stiff matter [$\gamma = 2$]), we show the ranges in the ζ-α plane of attractor fixed points (FP1$_+$, FP2$_\pm$). The shaded regions FP1 and FP1-I are those for FP1$_+$, while FP2 and FP2-I are those for FP2$_\pm$, respectively. The dark shaded regions FP1-I and FP2-I are the ranges for power-law inflation. There is no attractor solution in the unshaded region.

conventional choice 4/3, even if the potential of the scalar field is too steep to cause inflationary expansion by itself.

We have also learned that, even confining ourselves to the fixed-point solutions, reaching an attractor can be somewhat complicated if we have another fixed point, as we faced in the example of Fig. 4, though a simple recipe is shown to be applied by adopting either of the two attitudes suggested; limiting the range of ζ^2 or being content with an *a posteriori* consideration that initial values of x and y had happened to be in favor of reaching the attractor. The first appears to be the case in our realistic choice ($\zeta = 1.5823$) to fit the observed accelerating universe shown in Fig. 5.8 of [4], for example. The second of the above suggestions might also apply to drifting to infinities, as will be demonstrated in Appendix B.

If there is a spatial curvature, the result will be changed. However, we can show that the fixed point is still an attractor if the universe expands very fast, i.e., if it is an inflationary universe, as shown in Appendix A.

The most intriguing result in the present scalar-tensor theory is, however, that the *static* universe in the Jordan frame is an attractor, an unavoidable fate in the presence of the cosmological constant. There is no smooth limit as $V_0 \rightarrow 0$. As we reiterate, the Jordan frame features truly constant masses of microscopic fields, according to the Brans-Dicke model, originally intended to qualify this frame to be physical, allowing a nonstatic universe in the absence of the cosmological constant. Its presence alters the entire situation, forcing us to accept $ma =$ const. This entails eventually that the universe in the Einstein frame expands in the *same* rate as the (time-dependent) microscopic length standard. This *crisis* will be evaded only by leaving the Brans-Dicke model, as was elaborated in [4,6] together with the ensuing consequences. The view that this crisis hinges upon the attractor nature of the solutions is now reinforced even more strongly by our study in this article.

We also emphasize that the extension to the power-law potential leaves the above crisis unsolved. The exponent 1/2 in radiation dominance in the Einstein frame is a unique consequence of $\gamma = 4/3$ independent of the way V_0 is modified by the scalar field. The argument on the power-law inflation also remains unaffected, as we point out, by the structure of the mass term.

APPENDIX A: THE EFFECT OF CURVATURE TERM ($k \neq 0$)

In this appendix, we study the curvature effect. It may be convenient to analyze the equations rewritten by new variables because the fixed points are constant. We discuss the cases with a cosmological constant and with a power-law potential separately.

1. The case with a cosmological constant

We show the curvature effects both in the Einstein and in the Jordan frames in this order.

a. Curvature term in the Einstein frame

The curvature term is proportional to $e^{4\zeta\sigma}/a_*^2$. We evaluate the time evolution of this term by

$$\frac{d\ln(e^{4\zeta\sigma}/a_*^2)}{d\tau_*} = 2(2x - y). \tag{A1}$$

We find the behavior near the fixed point (x_F, y_F) as

$$\frac{d\ln(e^{4\zeta\sigma}/a_*^2)}{d\tau_*} = 2(2x_F - y_F)$$

$$= \begin{cases} \frac{8\zeta^2 - 1}{2\zeta\sqrt{3 - 8\zeta^2}} & \text{for FP1}_+ \\ \frac{\sqrt{\gamma}}{\sqrt{2(2 - \gamma - 2(4 - 3\gamma)\zeta^2)}} & \text{for FP2}_+. \end{cases} \tag{A2}$$

Hence we find that the curvature term decreases in time for the inflationary solution at FP1$_+$ ($\zeta^2 < 1/8$). It can be ignored. However, it will grow in time if the universe expands without acceleration. It will become important as the same as the usual case.

b. Curvature term in the Jordan frame

The time evolution of the curvature term in the Jordan frame is given by

$$\frac{d\ln(\phi^2/a^2)}{d\tau} = 2(\Phi' - \phi\mathcal{H}) = 2(2\Phi' - \mathcal{H})$$

$$= 2(2x - y). \tag{A3}$$

We find the behavior near the fixed point (x_F, y_F) as

$$\frac{d\ln(\phi^2/a^2)}{d\tau} = 2(2x_F - y_F), \tag{A4}$$

which is exactly the same as Eq. (A2). Hence the curvature term near the fixed point FP1$_+$ is not important for the inflationary solution ($\zeta^2 < 1/8$).

2. The case with a power-law potential

In this case, we can repeat the same analysis. The curvature term is proportional to $e^{(4-\alpha)\zeta\sigma}/a_*^2$. So the time evolution near the fixed point is given by

$$\frac{d\ln(e^{(4-\alpha)\zeta\sigma}/a_*^2)}{d\tau_*} = \zeta[2(2x_F - y_F) - \alpha x_F]$$

$$= \begin{cases} \dfrac{2\zeta^2(2-\alpha/2)^2 - 1}{\zeta\sqrt{2(6-\zeta^2(\alpha-4\zeta)^2}} & \text{for FP1}_+ \\ \dfrac{\zeta[6\gamma - \alpha(3\gamma-2)]}{\sqrt{6[3\gamma(2-\gamma)-2(4-3\gamma)(3\gamma-\alpha)\zeta^2]}} & \text{for FP2}_+. \end{cases}$$

(A5)

The curvature term is not important for the inflationary solution near the fixed point FP1$_+$ ($\zeta^2(4-\alpha)^2 < 2$) and near the fixed point FP2$_+$ ($\alpha > 6\gamma/(3\gamma-2)$ with $\gamma > 4/3$) or FP2$_-$ ($\alpha < 6\gamma/(3\gamma-2)$ with $2/3 < \gamma < 4/3$).

APPENDIX B: ADDITIONAL SOLUTIONS FOR RADIATION DOMINANCE IN THE JORDAN FRAME

We often relied on the numerical approach to solve the cosmological equations in the Jordan frame because the solution is characterized by the simplest aspect of the static universe. We encounter, however, another complicated aspect to be discussed in what follows.

We started conveniently from

$$6\varphi H^2 = -\tfrac{1}{2}\dot{\phi}^2 + V_0 + \rho - 6H\dot{\varphi}, \qquad (B1)$$

$$\ddot{\varphi} + 3H\dot{\varphi} = 4\zeta^2 V_0, \qquad (B2)$$

$$\dot{\rho} + 4H\rho = 0, \qquad (B3)$$

where $\varphi = (\xi/2)\phi^2$ in terms of which (4.3) has been put into a simplified form in (B2), as in [4]. We also write $V_0 = \Lambda = 1$.

Since H occurs always without derivative, we may eliminate it by using (B3), for example,

$$H = -\frac{1}{4}\frac{\dot{\rho}}{\rho}. \qquad (B4)$$

Equations (B1) and (B2) are then put into

$$3\left(\frac{\dot{\rho}}{\rho}\right)^2 \varphi - 12\frac{\dot{\rho}}{\rho}\dot{\varphi} + 2\xi^{-1}\frac{\dot{\varphi}^2}{\varphi} = 8(V_0 + \rho), \qquad (B5)$$

$$\ddot{\varphi} - \frac{3}{4}\frac{\dot{\rho}}{\rho}\dot{\varphi} = 4\zeta^2 V_0, \qquad (B6)$$

which are to be solved by giving three initial values of ρ, φ, $\dot{\varphi}$.

We notice, however, that we solve (B5) and (B6) first with respect to $\dot{\rho}/\rho$. This involves solving an *algebraically* quadratic equation for $\dot{\rho}/\rho$, thus producing two *differential* equations, hence resulting in two separate solutions. An example of numerical solutions is shown in Figs. 7(a) and 7(b). In Fig. 7(a), developed basically from Fig. 4.1 of [4], we do find asymptotic behaviors $H \to 0$, $\dot{\phi} \to \sqrt{4V_0/(6\xi - 1)}$, $\rho \to -3V_0(2\xi - 1)/(6\xi - 1)$ corresponding to an attractor solution. Figure 7(b) illustrates, on the other hand, the solution of another equation, but sharing the same initial values of ρ, φ, and $\dot{\varphi}$ as discussed in (a). This one represents, however, a shrinking universe taking place in a short time. This type of the second solution occurs nearly always. It even appears as if we are going to lose an opportunity to reach the fixed-point attractor.

Fortunately, as it turns out, however, the phase-space description is so generous that the resulting two solutions are accommodated as those due to two different initial locations of y, as demonstrated in Fig. 7(c). Again for a typical solution for $\xi = 1/4$, we find two trajectories for the solutions of (B5) and (B6). Both start at $x = 0$ corresponding to our initial value $\varphi' = 0$, but with y separated
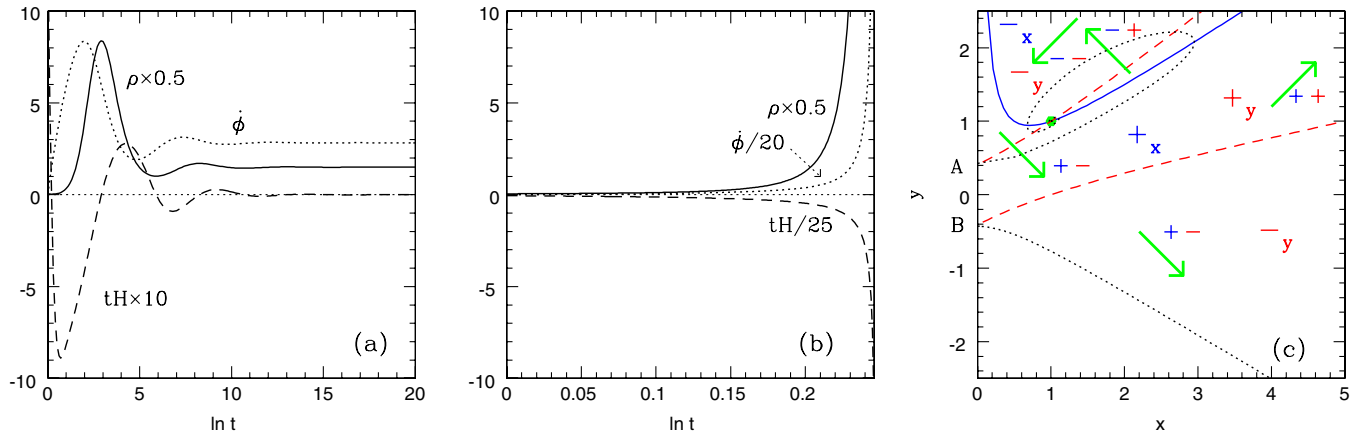

FIG. 7 (color online). Two solutions for the initial values at $\ln t = 0$; $\rho = 0.1$, $\varphi = 0.1$, $\dot{\varphi} = 0$ together with $V_0 = 1$. The asymptotic values mentioned above, hence (3.34) and (3.35) are reached in (a), while these quantities in (b) diverge quickly at $\ln t \to 0.248$, in this example. Behaviors of the solutions shown in (a) and (b) are represented by two different trajectories in phase space of x, y of (c), starting at two different points A and B. The upper curve corresponding to (a) spirals finally down into the attractor denoted by a blob (green) at $x = y = 1$, while the lower curve for (b) runs steadily toward the right-lower infinity, as "guided" by the arrow.

from $y = 0$ nearly the same distance in the opposite directions. The upper curve starts at the point A. After a looplike trip, it finally spirals down into the fixed point at $x = y = 1$, as usual. Another curve starting at B drifts steadily toward $x \to \infty$, $y \to -\infty$. Both trajectories started immediately "outside" the $y' = 0$ boundaries.

As we find from $y = \zeta^{-1}\mathcal{H}_*$ preceding (3.15) and (3.16), the solutions with $y > 0$ and $y < 0$ correspond to the expanding and contracting universes, respectively, in the Einstein frame. From this point of view, what we are in the expanding universe at quite a late epoch must be a consequence of the initial value selected to be $y > 0$, as represented by the upper trajectory in Fig. 7(c). In this sense the solution like Fig. 7(b) is excluded leaving the attractor solution alone.

APPENDIX C: DUST DOMINANCE IN THE SCALE-INVARIANT MODEL

In Secs. III and IV we discussed cosmological solutions mainly in the radiation-dominated universe finding a crisis arising from too much time dependence of particle masses evaded finally by departing from the Brans-Dicke model, even at the risk of WEP violation. The same type of analysis of dust dominance suffers more seriously because it entails $a_* \sim t_*^{1/2}$, as shown by (3.35) even for $\gamma = 1$. As was discussed in [4], the remedy comes simultaneously from the scale-invariant model intended to overcome the crisis for radiation dominance. We sketch below how this model provides attractor solutions also for the dust-dominated universe. See Chapter 4.4.3 of [4] and Section 3.4 of [6] for more details.

The field equations for σ and ρ turn out to be given by (3.6) and (3.7) with the right-hand sides removed to the classical approximation.

As a remarkable difference from (3.35) we find

$$a_* = a_{*0}(t_*/t_{*0})^{2/3}, \tag{C1}$$

in agreement with the conventional law of expansion. Equations (3.51) and (3.52) are replaced by

$$x' = \zeta(2x^2 - 3xy + 1), \tag{C2}$$

$$y' = \frac{1}{2\zeta}\left(-\frac{1}{2}x^2 + \frac{1}{4} + \zeta^2(4x - 3y)y\right), \tag{C3}$$

respectively. The solutions with (3.34) and (C1), and

$$\exp(-4\zeta\sigma_0) = \tfrac{1}{16}\zeta^{-2}, \tag{C4}$$

in place of (3.37) are obtained for $x = 1/\sqrt{2}$, $y = 2\sqrt{2}/3$, corresponding to an attractor yielding $x' = y' = 0$.

A similar distinction between the elliptic and the hyperbolic curves as in radiation dominance occurs also for $\zeta^2 < 3/8$ and $\zeta^2 > 3/8$, respectively. The same recipe should apply as mentioned toward the end of Sec. VI.

We add that the scale invariance coming from the absence of dimensional coupling constants has an advantage that σ serves as a massless Nambu-Goldstone boson which will acquire a small mass after the invariance is finally broken explicitly through loops, as discussed in Sec. 6.3 of [4].

APPENDIX D: ANOTHER APPROACH TO THE POWER-LAW POTENTIAL

It seems also useful to apply (B1)–(B3) to the power-law potential to offer a simplified alternative to derive the same common result $p_* = 1/2$ for radiation dominance as stated immediately after in (5.30).

We multiply V_0 in (B1) and (B3) by ϕ^α. We search for the solution of the type

$$a(t) \sim t^p \tag{D1}$$

and

$$\phi(t) \sim t^\beta. \tag{D2}$$

By substituting them into (B2) modified as above and comparing the exponents of t we obtain

$$\beta = \frac{2}{2 - \alpha}, \tag{D3}$$

implying that $\alpha = 0$ corresponds to $\beta = 1$. In the modified (B1), on the other hand, we find all the terms other than ρ to behave like $t^{2\beta-2}$, while (B3) entails $\rho \sim t^{-4p}$. For a consistent approach we expect $2\beta - 2 = -4p$, or

$$p = \frac{1 - \beta}{2} = -\frac{\alpha}{2(2 - \alpha)}. \tag{D4}$$

This point was not properly recognized when it was erroneously stated in Appendix B of [6] that only $\alpha = 0$ is consistent with the physically acceptable condition $p_* = 1/2$ (the coefficients α and β in Appendix B of [6] are replaced by $\alpha/2$ and $\beta/2$, respectively, according to the present notation).

Now from (4.6) and $\Omega \sim \phi$ combined with (D2), we obtain

$$dt_* = \Omega dt \sim t^\beta dt, \tag{D5}$$

which is integrated to give

$$t_* \sim t^{\beta+1}, \tag{D6}$$

where we have ignored inessential coefficients for simplicity.

In the same context we also use (4.7) to derive

$$a_* = \Omega a \sim t^{\beta+p}. \tag{D7}$$

Combining this with (D4) and (D6) we identify the right-hand side with $t_*^{p_*}$ where

$$p_* = \frac{1}{\beta + 1}(\beta + p) = \frac{1}{2}, \qquad (D8)$$

which turns out to be the same as the result for the purely constant $V_0 = \Lambda$, in agreement with (5.30) with $\alpha = 0$ for any γ. This justifies the present solution to be an attractor.

We also notice that the universe is no longer static in the Jordan frame, as shown in (D4). We may no longer rely on the simplest argument $am = a_* m_* = $ const to leave the BD model. According to Appendix D of [6], particularly its (3.10) and a more general procedure developed there, however, the mass m_* of the matter fields in the Einstein frame is related to m in the Jordan frame as

$$m_* = \Omega^{-1} m \sim t^{-\beta} m. \qquad (D9)$$

From (D3) and (D6) we find

$$t^{-\beta} \sim t_*^{-\beta/(\beta+1)} \sim t_*^{-(1/2)/(1-\alpha/4)}. \qquad (D10)$$

The exponent $-(1/2)/(1 - \alpha/4)$ is not exactly the same as $-1/2$ which would have implied that the universe looks static if measured with respect to the microscopic length standard, but is nevertheless far from zero as expected if the Einstein frame is qualified to be a physical conformal frame for any reasonable choice of α. In this sense the *crisis* for the purely constant V_0 as discussed before is not evaded by multiplying it by the scalar field. Departure from the BD model seems still unavoidable. We also add that the argument for the constant m_* with the assumed scale-invariant model remains unaltered by the multiplied scalar field.

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