Analogs of the double-Reissner-Nordström solution in magnetostatics and dilaton gravity: Mathematical description and basic physical properties

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In this paper we consider a magnetic analog of the double-Reissner-Nordström solution and construct the corresponding magnetic potential A_φ in the explicit form. The behavior of the resulting solution under the Harrison transformation then naturally singles out *the asymmetric black diholes*—configurations composed of two nonextreme black holes possessing unequal masses, and charges equal in magnitude but opposite in sign—as its most general subclass for which equilibrium of the black-hole constituents can be achieved with the aid of the external magnetic (or electric) field. We also generalize the double-Reissner-Nordström solution to dilaton gravity with arbitrary dilaton coupling, yielding the four-dimensional *double-Gibbons-Maeda* spacetime. The study of some physical properties of the solutions obtained leads, in particular, to very simple formulas for the areas of the horizons and surface gravities.

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I. INTRODUCTION

As was shown in [1–5], static axisymmetric solutions of the Einstein-Maxwell-dilaton theory can be generated from the known stationary solutions of the Einstein equations or directly from the static solutions of Einstein-Maxwell theory. The solution-generating procedure permitting one to do this was described in detail in [4,5], and an interesting outcome of its use are various dihole spacetimes describing a dilatonic pair of black holes endowed with electric or magnetic dipole moments (the black-hole constituents carry charges which are equal in magnitude but opposite in sign[1]). Thus, the Davidson-Gedalin solution [2] representing a dilatonic generalization of Bonnor's magnetic dipole solution [6] can be interpreted, according to a thorough analysis carried out by Emparan [3], as a pair of magnetically charged extreme Reissner-Nordström black holes (a dilatonic *extreme* dihole). Moreover, Emparan demonstrated that the extremal constituents in Bonnor's and Davidson-Gedalin diholes can be balanced by placing the diholes into the external magnetic field. The nonextremal electric diholes were constructed and studied by Emparan and Teo [5]. It has been shown recently [7] that the Emparan-Teo solutions can be rewritten in a parametrization involving Komar masses and charges of the constituents as arbitrary parameters; the paper [7] also clarifies the construction of magnetic analogs of those solutions.

All the extreme and nonextreme diholes considered in [2,3,5], in addition to being static and axisymmetric, have one more symmetry—they are equatorially symmetric; i.e., the corresponding metric functions are invariant under the transformation $z \to -z$ [8], and this is a reflection of the fact that these solutions are composed of identical (up to the sign of the charge) constituents. It would therefore be of interest to go beyond the known families of *symmetric* dihole solutions and consider a more general case of *asymmetric* black diholes—the ones consisting of a pair of black-hole constituents with unequal masses but equal and opposite charges. In the present paper we will demonstrate a remarkable general property of such solutions possessing zero total magnetic/electric charge to be able to describe equilibrium configurations of two nonextreme Reissner-Nordström black holes [9,10] by placing these into the external magnetic/electric field. We will accomplish this task within the framework of the magnetostatic analog of the double-Reissner-Nordström solution [11,12] by constructing the corresponding magnetic potential A_φ whose behavior on the symmetry axis provides one with the restrictions on the charges of black holes defining equilibrium configurations in the presence of the external field. We mention that, in the case of extremal black holes with nonzero total charge, the inability of the additional Bonnor-Melvin-type magnetic field [13,14] to regularize the symmetry axis was proved by Liang and Teo [4], and later on [5], Emparan and Teo conjectured that the same must be true for the nonextreme black holes with unequal charges as well. In our paper the validity of the latter conjecture will find its justification thanks to the use of the explicit formula for the magnetic potential A_φ obtained. Moreover, we shall derive the general conditions needed for the constituents of asymmetric black diholes to be in equilibrium in the external magnetic field.

The second major objective pursued by the present paper is the construction of the four-dimensional double-Gibbons-Maeda solution for two arbitrary static dilatonic black holes endowed with electric or magnetic charges. It is very advantageous to have such a solution on hand to be able to study analytically the general properties of inter-

[1]In this sense the solution obtained by Liang and Teo [4] is not, strictly speaking, of the dihole type since, in general, the charges of the constituents are not equal in absolute values.

acting black holes in the presence of a dilaton field, and we will generate the desired solution with the aid of a procedure described by Emparan and Teo in [5]. We remark that recently several formal and hardly manageable mathematical relations for some physical properties of the interacting Reissner-Nordström black holes have been presented in [15], so in our paper we shall pay special attention to elaborating the thermodynamic characteristics of both the double-Reissner-Nordström and the double-Gibbons-Maeda solutions in a concise, explicit form.

The paper is organized as follows. In Sec. II we review the double-Reissner-Nordström solution and construct its magnetic analog. The main original results of this section are the remarkably simple formulas for the horizon areas of interacting Reissner-Nordström black holes and the analytical expression of the magnetic potential A_φ for two magnetically charged static black holes. In Sec. III we consider a family of asymmetric diholes arising as a special subclass of the double-Reissner-Nordström solution for which the equilibrium of black-hole constituents can be achieved by means of the Harrison transformation [16]. Here we find a simple formula for the magnetic parameter of the exterior field which determines the equilibrium of two static black holes endowed with unequal masses and equal but opposite charges. Section IV is devoted to the derivation of the dilatonic generalization of the double-Reissner-Nordström solution which we interpret as a nonlinear superposition of two four-dimensional Gibbons-Maeda solutions [1]; in this section we also briefly comment on the physical properties of the double-Gibbons-Maeda spacetime. Concluding remarks are given in Sec. V.

II. THE DOUBLE-REISSNER-NORDSTRÖM SOLUTION AND ITS ANALOG IN MAGNETOSTATICS

The exact electrostatic five-parameter solution describing the nonlinear superposition of two arbitrary Reissner-Nordström black holes has been worked out in a closed analytical form in Refs. [11,12] using different but mathematically equivalent parametrizations. Since we are interested in the binary black-hole systems, in what follows we shall write out the double-Reissner-Nordström solution in the Varzugin-Chistyakov parametrization [17] employed in [12], which involves as arbitrary parameters the relative distance and the individual Komar masses and charges [18] of the constituents; we shall also slightly change the notations of Ref. [12] to make them more congruent with the recent physical representation of the Emparan-Teo symmetric dihole solution [7]. The Ernst potentials $\mathcal{E}$ and Φ [19] defining the double-Reissner-Nordström solution have the form [12]

$$\mathcal{E}=\frac{\mathcal{A}-\mathcal{B}}{\mathcal{A}+\mathcal{B}},\qquad \Phi=-A_t=\frac{\mathcal{C}}{\mathcal{A}+\mathcal{B}},$$
$$\mathcal{A}=\Sigma\sigma[\nu(R_+ + R_-)(r_+ + r_-) + 4\kappa(R_+R_- + r_+r_-)] - (\mu^2\nu - 2\kappa^2)(R_+ - R_-)(r_+ - r_-),$$
$$\mathcal{B}=2\Sigma\sigma[(\nu m + 2\kappa M)(R_+ + R_-) + (\nu M + 2\kappa m)(r_+ + r_-)] + 2\sigma[\nu\mu(Q-\mu) - 2\kappa(RM - \mu q - \mu^2)](R_+ - R_-)$$
$$+ 2\Sigma[\nu\mu(q+\mu) + 2\kappa(Rm + \mu Q - \mu^2)](r_+ - r_-),$$
$$\mathcal{C}=2\Sigma\sigma\{[\nu(q+\mu) + 2\kappa(Q-\mu)](R_+ + R_-) + [\nu(Q-\mu) + 2\kappa(q+\mu)](r_+ + r_-)\}$$
$$+ 2\sigma[\mu\nu M + 2\kappa(\mu m - RQ + \mu R)](R_+ - R_-) + 2\Sigma[\mu\nu m + 2\kappa(\mu M + Rq + \mu R)](r_+ - r_-), \tag{1}$$

where A_t is the electric component of the electromagnetic four-potential $A_i=(0,0,0,A_t)$, the functions $R_\pm$ and $r_\pm$ are defined as

$$R_\pm=\sqrt{\rho^2+\left(z+\frac{1}{2}R\pm\Sigma\right)^2},$$
$$r_\pm=\sqrt{\rho^2+\left(z-\frac{1}{2}R\pm\sigma\right)^2},$$
$$\Sigma=\sqrt{M^2-Q^2+2\mu Q},\qquad \sigma=\sqrt{m^2-q^2-2\mu q},$$
$$\mu:=\frac{mQ-Mq}{R+M+m}, \tag{2}$$

and R is the distance between the centers of the black holes; the constant objects ν and κ are given by the formulas

$$\nu=R^2-\Sigma^2-\sigma^2+2\mu^2$$
$$\equiv R^2-M^2-m^2+(Q-\mu)^2+(q+\mu)^2, \tag{3}$$
$$\kappa=Mm-(Q-\mu)(q+\mu).$$

Note that the upper Reissner-Nordström constituent is described by the functions $r_\pm$, and it possesses the mass m and charge q (see Fig. 1). The location of the lower constituent is determined by the functions $R_\pm$, its mass and charge being M and Q, respectively. The total mass M_T and total charge Q_T of the system are simple sums of the individual masses and charges, respectively:

$$M_T=M+m,\qquad Q_T=Q+q. \tag{4}$$

The metric functions f and γ of the double-Reissner-Nordström solution, which enter as coefficients into Weyl's

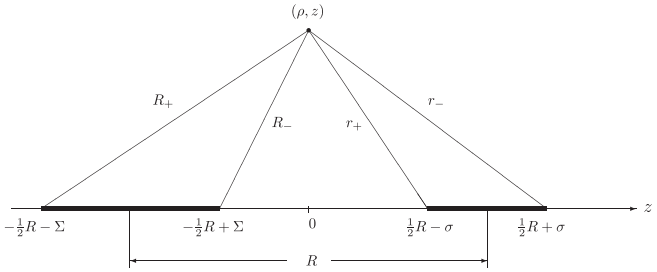

FIG. 1. Location of the black-hole constituents on the symmetry axis.

line element

$$ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f dt^2, \quad (5)$$

are expressible in terms of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ as follows:

$$f = \frac{\mathcal{A}^2 - \mathcal{B}^2 + \mathcal{C}^2}{(\mathcal{A} + \mathcal{B})^2}, \qquad e^{2\gamma} = \frac{\mathcal{A}^2 - \mathcal{B}^2 + \mathcal{C}^2}{K_0^2 R_+ R_- r_+ r_-}, \quad (6)$$
$$K_0 = 4\Sigma\sigma[R^2 - (M-m)^2 + (Q-q-2\mu)^2].$$

Obviously, formulas (1)–(6) describe a pair of interacting nonextreme black-hole constituents when $\Sigma^2 > 0$ and $\sigma^2 > 0$.

A. Some properties of the double-Reissner-Nordström solution

Since in Ref. [12] the thermodynamical characteristics of the double-Reissner-Nordström solution were not considered, below we briefly comment on them within the framework of the well-known Smarr mass formula [20], valid for each black-hole constituent:

$$M = \Sigma + \Phi_\Sigma Q = \frac{1}{4\pi}\kappa_\Sigma A_\Sigma + \Phi_\Sigma Q,$$
$$m = \sigma + \Phi_\sigma q = \frac{1}{4\pi}\kappa_\sigma A_\sigma + \Phi_\sigma q, \quad (7)$$

where the quantities Φ_Σ, A_Σ, and κ_Σ describing the lower black hole are, respectively, the constant value of the potential $A_t = -\Phi$ on the horizon, the area of the horizon, and the surface gravity; the analogous characteristics of the upper black hole are Φ_σ, A_σ, and κ_σ. Using (1), one easily finds that

$$\Phi_\Sigma = \frac{Q - 2\mu}{M + \Sigma}, \qquad \Phi_\sigma = \frac{q + 2\mu}{m + \sigma}. \quad (8)$$

At the same time, the calculation of A_Σ and A_σ, which can be carried out, for instance, with the aid of the formulas [5]

$$A_\Sigma = 4\pi\Sigma(\rho f^{-1} e^\gamma)|_{\rho=0,\,-\Sigma\le z+(1/2)R\le\Sigma},$$
$$A_\sigma = 4\pi\sigma(\rho f^{-1} e^\gamma)|_{\rho=0,\,-\sigma\le z-(1/2)R\le\sigma}, \quad (9)$$

is by far more difficult since the initial expressions resulting from the straightforward calculations are extremely complicated and unwieldy. Fortunately, after a laborious and prolonged work we eventually arrive at the following elegant formulas for A_Σ and A_σ:

$$A_\Sigma = \frac{4\pi[(R+M+m)(M+\Sigma) - Q(Q+q)]^2}{(R+\Sigma)^2 - \sigma^2},$$
$$A_\sigma = \frac{4\pi[(R+M+m)(m+\sigma) - q(Q+q)]^2}{(R+\sigma)^2 - \Sigma^2}, \quad (10)$$

and we remark that in obtaining (10) a concise expression for the horizon area of the Emparan-Teo nonextreme dihole [5] was a guiding line for us. Lastly, the form of κ_Σ and κ_σ can be read off from (7) and (10):

$$\kappa_\Sigma = \frac{\Sigma[(R+\Sigma)^2 - \sigma^2]}{[(R+M+m)(M+\Sigma) - Q(Q+q)]^2},$$
$$\kappa_\sigma = \frac{\sigma[(R+\sigma)^2 - \Sigma^2]}{[(R+M+m)(m+\sigma) - q(Q+q)]^2}. \quad (11)$$

Formulas (10) and (11) clearly illustrate that the affirmation made in a recent paper [15] about the absence of simple expressions for the surface gravities and horizon areas in the general case of the double-Reissner-Nordström solution is not correct.

B. The case of magnetically charged black holes

Because of the one-to-one correspondence existing between the electrostatic and magnetostatic Einstein-Maxwell fields [13], the Ernst potentials of the magnetic analog of the double-Reissner-Nordström solution have the form

$$\mathcal{E} = \frac{\mathcal{A} - \mathcal{B}}{\mathcal{A} + \mathcal{B}}, \qquad \Phi = \frac{i\mathcal{C}}{\mathcal{A} + \mathcal{B}}, \quad (12)$$

where the functions $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are the same as in formulas (1), and the corresponding metric functions f and γ are given by Eqs. (6). At the same time, finding the remaining magnetic potential A_φ [the φ component of the four-potential $A_i = (0, 0, A_\varphi, 0)$] represents a difficult technical problem because, unlike in the electrostatic case, its form cannot be read off directly from the expression of Φ. Formally, A_φ can be obtained by solving the following system of first-order differential equations:

$$\frac{\partial A_\varphi}{\partial \rho} = i\rho f^{-1}\frac{\partial \Phi}{\partial z}, \qquad \frac{\partial A_\varphi}{\partial z} = -i\rho f^{-1}\frac{\partial \Phi}{\partial \rho}. \quad (13)$$

However, in practice, the integration of (13) looks infeasible not only in our five-parameter case, but even in the by far simpler case of identical black holes carrying opposite charges [5]. The way out of this problematic situation consists of circumventing the resolution of the system (13) by finding A_φ as the real part of Kinnersley's potential Φ_2 [21], whose construction within Sibgatullin's integral method [22] is analogous to the construction of the Ernst potentials $\mathcal{E}$ and Φ. In Ref. [23] an explicit expression for Φ_2 was obtained in the general case of the analytically

extended multisoliton solution [see formulas (3.13) and (3.14) of [23]]. Then, setting $N = 2$ in the latter formulas and taking into account that in the magnetostatic case the potential Φ_2 is equal to A_φ exactly, one is able, after expanding the determinants and performing the reparametrization within the lines of Ref. [12], to arrive at the desired expression for the magnetic potential. The calculation of A_φ, combined with the subsequent tedious search for considerable simplifications, has finally led us to the following result:

$$A_\varphi = Q + q + \frac{\mathcal{I} - z\mathcal{C}}{\mathcal{A} + \mathcal{B}},$$

$$\begin{aligned}\mathcal{I} = {} & 4\Sigma\sigma\{[mR(Q-\mu) - \mu(\kappa + m^2) + \mu(Q-\mu)^2]r_+r_- - [MR(q+\mu) + \mu(\kappa + M^2) - \mu(q+\mu)^2]R_+R_-\} \\ & + \mu(\nu - 2\kappa)[(\kappa + \mu^2)(R_+ - R_-)(r_+ - r_-) - \Sigma\sigma(R_+ + R_-)(r_+ + r_-)] - (\nu - 2\Sigma\sigma - 2\mu^2)[M\sigma(Q-\mu) \\ & - m\Sigma(q+\mu)](R_+r_- - R_-r_+) + (\nu + 2\Sigma\sigma - 2\mu^2)[M\sigma(Q-\mu) + m\Sigma(q+\mu)](R_-r_- - R_+r_+) \\ & + \Sigma[m\mu\nu + 2M\mu\kappa + 2R\kappa(q+\mu)][2\sigma(r_+ + r_-) - R(r_+ - r_-)] + \sigma[M\mu\nu + 2m\mu\kappa - 2R\kappa(Q-\mu)] \\ & \times [2\Sigma(R_+ + R_-) + R(R_+ - R_-)] + \Sigma\sigma[\nu(Q-\mu) + 2\kappa(q+\mu)][2\sigma(r_+ - r_-) - R(r_+ + r_-)] \\ & + \Sigma\sigma[\nu(q+\mu) + 2\kappa(Q-\mu)][2\Sigma(R_+ - R_-) + R(R_+ + R_-)] - 2\mu(\nu - 2\kappa)\{\sigma[MR + \mu(Q - q - 2\mu)] \\ & \times (R_+ - R_-) - \Sigma[mR + \mu(Q - q - 2\mu)](r_+ - r_-) + \Sigma\sigma[(M-m)(r_+ + r_- - R_+ - R_-) + 2(\nu + 2\kappa)]\}. \end{aligned} \quad (14)$$

The magnetostatic analog of the double-Reissner-Nordström solution is now completely defined by formulas (1)–(6), (12), and (14), Q and q being the individual magnetic charges of the constituents.

III. ASYMMETRIC BLACK DIHOLES

The extreme and nonextreme black dihole solutions considered in Refs. [3,5] are composed of two identical constituents and, as can be easily shown, they are symmetric with respect to the equatorial ($z = 0$) plane. In this connection it would be of apparent interest to extend Emparan's notion of black dihole spacetimes to the equatorially nonsymmetric case and consider *the asymmetric black diholes*—the systems of two black holes which possess unequal masses and charges equal in magnitude but opposite in sign. Let us substantiate the importance of asymmetric diholes by demonstrating a remarkable general property shared by these configurations—the possibility of balance of their black-hole constituents in the external field introduced via the Harrison transformation [16]. We shall treat this special balance effect within the framework of the magnetostatic case for which the application of the above-mentioned transformation is physically somewhat more transparent than in the case of electrostatics, but of course our conclusions will be equally valid for both cases in view of Bonnor's theorem [13].

The action of the Harrison transformation on the magnetostatic analog of the double-Reissner-Nordström solution leads to the metric functions $\tilde{f}$, $\tilde{\gamma}$ and the magnetic potential $\tilde{A}_\varphi$ of the form

$$\begin{aligned} \tilde{f} &= \lambda^2 f, \qquad e^{2\tilde{\gamma}} = \lambda^4 e^{2\gamma}, \\ \tilde{A}_\varphi &= 2B^{-1}[\lambda^{-1}(1 + \tfrac{1}{2}BA_\varphi) - 1], \\ \lambda &= (1 + \tfrac{1}{2}BA_\varphi)^2 + \tfrac{1}{4}B^2\rho^2 f^{-1}, \end{aligned} \quad (15)$$

where f, γ, and A_φ are given by formulas (6) and (14) of the previous section, and B is a real constant defining the exterior "uniform" Bonnor–Melvin-type magnetic field [13,14]. When $B = 0$, one recovers from (15) the magnetic version of the double-Reissner-Nordström solution; in this (asymptotically flat) case, equilibrium of two nonextreme charged black holes with positive masses does not seem possible [11,24] because of an irremovable strut (conical deficit) on the part of the symmetry axis separating the constituents. The question is whether the strut can be removed for some $B \neq 0$. We remind the reader that the balance of aligned Reissner-Nordström black holes implies vanishing of the metric function γ on all the parts of the symmetry axis outside the horizons; i.e., in our two-body case, the balance equations have the form

$$\tilde{\gamma}^{(\mathrm{I})} = 0, \qquad \tilde{\gamma}^{(\mathrm{II})} = 0, \qquad \tilde{\gamma}^{(\mathrm{III})} = 0, \quad (16)$$

where segments I, II, and III of the symmetry axis are defined as $\rho = 0$, $z > \frac{1}{2}R + \sigma$ (the upper part), $\rho = 0$, $-\frac{1}{2}R + \Sigma < z < \frac{1}{2}R - \sigma$ (the intermediate part), and $\rho = 0$, $z < -\frac{1}{2}R - \Sigma$ (the lower part), respectively. Taking into account (15), another way of writing (16) is

$$\lambda^4_{(\mathrm{I,II,III})} e^{2\gamma(\mathrm{I,II,III})} = 1, \quad (17)$$

where $\lambda_{(\mathrm{I,II,III})}$ and $\gamma^{(\mathrm{I,II,III})}$ stand for the values of λ and γ on segments I, II, III of the symmetry axis and refer to the "seed" double-Reissner-Nordström solution. From (1) and (6) it follows that $\gamma^{(\mathrm{I,III})} = 0$ identically, and also that $\rho^2 f^{-1}|_{\rho=0} = 0$; hence, the system of the balance equations finally takes the form

$$A_\varphi^{(\mathrm{I})} = 0, \qquad (1 + \tfrac{1}{2}BA_\varphi^{(\mathrm{II})})^8 e^{2\gamma(\mathrm{II})} = 1, \qquad A_\varphi^{(\mathrm{III})} = 0, \quad (18)$$

$A_\varphi^{(\mathrm{I,II,III})}$ being the values of the magnetic potential (14) on

the respective parts of the z axis. From (14) we find the required axis values of the potential A_φ:

$$A_\varphi^{(\mathrm{I})} \equiv 0, \qquad A_\varphi^{(\mathrm{II})} = 2q, \qquad A_\varphi^{(\mathrm{III})} = 2(Q+q), \tag{19}$$

and it follows from (19) that the first condition in (18) is satisfied automatically, while the fulfilment of the third condition in (18) is secured by the vanishing of the total charge:

$$Q_T \equiv Q + q = 0. \tag{20}$$

Now finding $\gamma^{(\mathrm{II})}$ with the aid of (1) and (6), and solving the second equation in (18) for B, we arrive at the following value of the latter parameter at which two magnetically charged black holes will be balanced by the external magnetic field:

$$B = \frac{1}{q}\left(\pm\sqrt[4]{\frac{\nu+2\kappa}{\nu-2\kappa}} - 1\right), \qquad Q + q = 0. \tag{21}$$

Note that for physical reasons ($B \to 0$ when $R \to \infty$) one has to choose the upper sign in (21). The zero total charge condition naturally singles out *the asymmetric black dihole* configurations as the ones whose black-hole constituents can achieve equilibrium in the external magnetic field, the precise "balance" value of a such field being given by formulas (21).

Once the significance of asymmetric black diholes is established, we will turn to their more detailed description, returning again to the electrostatic picture and restricting our consideration to the asymptotically flat case (no external field). Then, the choice $q = -Q$ in the formulas (1)–(3) and (6) of the previous section permits us to introduce the nondimensional parameter μ in the expressions for Σ and σ:

$$\Sigma = \sqrt{M^2 - Q^2(1-2\mu)}, \qquad \sigma = \sqrt{m^2 - Q^2(1-2\mu)},$$
$$\mu := \frac{M+m}{R+M+m}. \tag{22}$$

The expressions for ν and κ, with $q = -Q$, take the form

$$\nu = R^2 - M^2 - m^2 + 2Q^2(1-\mu)^2,$$
$$\kappa = Mm + Q^2(1-\mu)^2, \tag{23}$$

while for the electric potential A_t and metric functions f and γ, we obtain

$$f = \frac{\mathcal{A}^2 - 4\mathcal{B}^2 + 4Q^2\mathcal{C}^2}{(\mathcal{A}+2\mathcal{B})^2}, \qquad A_t = -\frac{2Q\mathcal{C}}{\mathcal{A}+2\mathcal{B}}, \qquad e^{2\gamma} = \frac{\mathcal{A}^2 - 4\mathcal{B}^2 + 4Q^2\mathcal{C}^2}{K_0^2 R_+ R_- r_+ r_-},$$
$$K_0 = 4\Sigma\sigma[R^2 - (M-m)^2 + 4Q^2(1-\mu)^2],$$
$$\mathcal{A} = \Sigma\sigma[\nu(R_+ + R_-)(r_+ + r_-) + 4\kappa(R_+R_- + r_+r_-)] - (Q^2\mu^2\nu - 2\kappa^2)(R_+ - R_-)(r_+ - r_-),$$
$$\mathcal{B} = \Sigma\sigma[(m\nu + 2M\kappa)(R_+ + R_-) + (M\nu + 2m\kappa)(r_+ + r_-)] + Q^2(\mu - \mu^2)(\nu - 2\kappa)[\sigma(R_+ - R_-) - \Sigma(r_+ - r_-)]$$
$$- 2R\kappa[M\sigma(R_+ - R_-) - m\Sigma(r_+ - r_-)],$$
$$\mathcal{C} = \Sigma\sigma(1-\mu)(\nu - 2\kappa)(r_+ + r_- - R_+ - R_-) + \sigma[M\mu\nu + 2\kappa(m\mu - R + R\mu)](R_+ - R_-)$$
$$+ \Sigma[m\mu\nu + 2\kappa(M\mu - R + R\mu)](r_+ - r_-),$$
$$R_\pm = \sqrt{\rho^2 + \left(z + \frac{1}{2}R \pm \Sigma\right)^2}, \qquad r_\pm = \sqrt{\rho^2 + \left(z - \frac{1}{2}R \pm \sigma\right)^2}. \tag{24}$$

From the above formulas (22) and (23) one immediately draws the following three interesting conclusions about the properties of asymmetric black diholes:

(i) Since Σ and σ cannot take the same values for different positive values of M and m, the asymmetric diholes with two extreme components ($\Sigma = \sigma = 0$) do not exist.

(ii) The formula for the interaction force between the black-hole constituents in an asymmetric dihole is readily obtainable from the general expression derived in [12] for the double-Reissner-Nordström solution, and from (23); it has the form

$$\mathcal{F} = \frac{\kappa}{\nu - 2\kappa} = \frac{Mm + Q^2(1-\mu)^2}{R^2 - (M+m)^2}, \tag{25}$$

which means that no equilibrium states without a supporting strut between the constituents (such states are defined by the equation $\mathcal{F} = 0$) can be achieved, at finite separation, for arbitrary values of Q and any positive values of M and m.

(iii) The absolute value of charge Q in the asymmetric black diholes can considerably exceed the values of M and m due to the presence of the factor $(1-2\mu)$ in the expressions of Σ and σ.

The physical quantities Φ_Σ, Φ_σ, A_Σ, and A_σ characterizing the asymmetric diholes have a somewhat simpler form than in the case of the general double-Reissner-Nordström solution. Thus, for Φ_Σ and Φ_σ we have

$$\Phi_\Sigma = \frac{Q(1-2\mu)}{M+\Sigma}, \qquad \Phi_\sigma = -\frac{Q(1-2\mu)}{m+\sigma}, \tag{26}$$

and for A_Σ and A_σ we get from (10)

$$A_\Sigma = \frac{4\pi(R+M+m)^2(M+\Sigma)^2}{(R+\Sigma)^2-\sigma^2},$$
$$A_\sigma = \frac{4\pi(R+M+m)^2(m+\sigma)^2}{(R+\sigma)^2-\Sigma^2}, \tag{27}$$

so that the charge Q is not explicitly present in (27). We mention that the electric dipole moment of the asymmetric dihole is equal to $-Q(R-M-m)$.

In the limit of equal constituents ($\Sigma=\sigma$, $M=m$), one easily recovers from (26) and (27) the expressions for the electric potential and the horizon area found by Emparan and Teo for a nonextreme symmetric dihole [5].

In conclusion of this section we write out explicitly the expression of the potential A_φ for the magnetic analog of the electrically charged asymmetric black dihole [apparently, the functions $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ of this analog are the same as in formulas (24)]:

$$A_\varphi = \frac{Q(\mathcal{I}-2z\mathcal{C})}{\mathcal{A}+2\mathcal{B}},$$
$$\begin{aligned}\mathcal{I} &= 4\Sigma\sigma[R(1-\mu)-\mu(M+m)](MR_+R_- + mr_+r_-) + \mu(\nu-2\kappa)[(\kappa+\mu^2Q^2)(R_+-R_-)(r_+-r_-)\\
&- \Sigma\sigma(R_++R_-)(r_++r_-)] - (1-\mu)\{[R^2-(\Sigma+\sigma)^2](M\sigma+m\Sigma)(R_+r_- - R_-r_+) + [R^2-(\Sigma-\sigma)^2]\\
&\times(M\sigma-m\Sigma)(R_+r_+ - R_-r_-)\} + \Sigma[m\mu\nu + 2M\mu\kappa - 2R\kappa(1-\mu)][2\sigma(r_++r_-) - R(r_+-r_-)]\\
&+ \sigma[M\mu\nu + 2m\mu\kappa - 2R\kappa(1-\mu)][2\Sigma(R_++R_-) + R(R_+-R_-)] - \Sigma\sigma R(1-\mu)(\nu-2\kappa)\\
&\times(R_++R_-+r_++r_-) - 2(\nu-2\kappa)\{\sigma[MR\mu + (\Sigma^2+2Q^2\mu^2)(1-\mu)](R_+-R_-)\\
&- \Sigma[mR\mu + (\sigma^2+2Q^2\mu^2)(1-\mu)](r_+-r_-) + \mu\Sigma\sigma[(M-m)(r_++r_--R_+-R_-) + 2(\nu+2\kappa)]\},\end{aligned} \tag{28}$$

and we also remark that the explicit form of formula (21) for the parameter B defining the equilibrium of the magnetically charged black-hole constituents in the external magnetic field, taking into account (22) and (23), takes the form

$$B = -\frac{1}{Q}\left(\sqrt[4]{\frac{R^2-(M-m)^2+4Q^2(1-\mu)^2}{R^2-(M+m)^2}} - 1\right), \tag{29}$$

Q being the magnetic charge of the lower constituent.

IV. THE DOUBLE-GIBBONS-MAEDA SOLUTION

To generalize the double-Reissner-Nordström solution to the Einstein-Maxwell-dilaton theory arising from the Lagrangian [1,25]

$$\mathcal{L} = \frac{1}{16\pi}\sqrt{-g}[R - 2(\nabla\phi)^2 - e^{-2\alpha\phi}F^2], \tag{30}$$

where ϕ is the dilaton field and α the arbitrary coupling constant ($\alpha=0$ for the pure Einstein-Maxwell fields, $\alpha=1$ for the low energy effective limit of string theory, and $\alpha=\sqrt{3}$ for the Kaluza-Klein theory), it is advantageous to make use of the solution-generating procedure described in Refs. [3,5]. Recall that, according to this procedure, given a seed electrostatic solution f, γ, and A_t, its dilatonic generalization $\hat{f}$, $\hat{\gamma}$, $\hat{A}_t$, and ϕ is defined by the formulas

$$\hat{f} = f^{1/(1+\alpha^2)}e^{-2\alpha\phi_0}, \qquad \hat{\gamma} = \frac{1}{1+\alpha^2}\gamma + \gamma_0,$$
$$\hat{A}_t = \frac{1}{\sqrt{1+\alpha^2}}A_t, \qquad e^{2\phi} = f^{\alpha/(1+\alpha^2)}e^{2\phi_0}, \tag{31}$$

where $\hat{f}$ and $\hat{\gamma}$ enter the line element

$$ds^2 = \hat{f}^{-1}[e^{2\hat{\gamma}}(d\rho^2+dz^2) + \rho^2d\varphi^2] - \hat{f}dt^2, \tag{32}$$

ϕ_0 is an arbitrary solution of the equation

$$\phi_{0,\rho,\rho} + \rho^{-1}\phi_{0,\rho} + \phi_{0,z,z} = 0, \tag{33}$$

and γ_0 is obtainable from ϕ_0 by solving the system

$$\gamma_{0,\rho} = (1+\alpha^2)\rho(\phi_{0,\rho}^2 - \phi_{0,z}^2),$$
$$\gamma_{0,z} = 2(1+\alpha^2)\rho\phi_{0,\rho}\phi_{0,z}, \tag{34}$$

the integrability condition of which is Eq. (33).

In the case of the double-Reissner-Nordström solution, the functions f and γ are given by formulas (6), while the potential A_t is equal to $-\Phi$ from (1). Therefore, we only need to identify the function ϕ_0 and the corresponding γ_0 to obtain the desired generalization. Since our ϕ_0 has to, in particular, generalize Emparan and Teo's analogous choice for the symmetric black dihole solution [5], we have to take, as ϕ_0, the linear superposition of two arbitrary Schwarzschild solutions with the "masses" Σ and σ, and find the corresponding γ_0 using the results of Ref. [26]. Then, after applying the procedure (31) to the double-Reissner-Nordström solution, we finally obtain

$$\hat{f} = \left[\frac{\mathcal{A}^2 - \mathcal{B}^2 + \mathcal{C}^2}{(\mathcal{A} + \mathcal{B})^2}\right]^{1/(1+\alpha^2)} e^{-2\alpha\phi_0}, \qquad e^{2\hat{\gamma}} = \left[\frac{\mathcal{A}^2 - \mathcal{B}^2 + \mathcal{C}^2}{K_0^2 R_+ R_- r_+ r_-}\right]^{1/(1+\alpha^2)} e^{2\gamma_0},$$
$$\hat{A}_t = -\frac{1}{\sqrt{1+\alpha^2}} \frac{\mathcal{C}}{\mathcal{A} + \mathcal{B}}, \qquad e^{2\phi} = \left[\frac{\mathcal{A}^2 - \mathcal{B}^2 + \mathcal{C}^2}{(\mathcal{A} + \mathcal{B})^2}\right]^{\alpha/(1+\alpha^2)} e^{2\phi_0}, \tag{35}$$

where

$$e^{-2\phi_0} = \left(\frac{R_+ + R_- - 2\Sigma}{R_+ + R_- + 2\Sigma} \cdot \frac{r_+ + r_- - 2\sigma}{r_+ + r_- + 2\sigma}\right)^{\alpha/(1+\alpha^2)},$$
$$e^{2\gamma_0} = \left[\frac{R_+R_- + \rho^2 + (z + \frac{1}{2}R)^2 - \Sigma^2}{2R_+R_-} \cdot \frac{r_+r_- + \rho^2 + (z - \frac{1}{2}R)^2 - \sigma^2}{2r_+r_-} \cdot \frac{R_+r_- + \rho^2 + (z + \frac{1}{2}R + \Sigma)(z - \frac{1}{2}R - \sigma)}{R_-r_- + \rho^2 + (z + \frac{1}{2}R - \Sigma)(z - \frac{1}{2}R - \sigma)}\right.$$
$$\left. \times \frac{R_-r_+ + \rho^2 + (z + \frac{1}{2}R - \Sigma)(z - \frac{1}{2}R + \sigma)}{R_+r_+ + \rho^2 + (z + \frac{1}{2}R + \Sigma)(z - \frac{1}{2}R + \sigma)}\right]^{\alpha^2/(1+\alpha^2)}. \tag{36}$$

This completes the construction of the dilatonic generalization of the electrostatic metric for two charged black holes. In the limit of identical black-hole constituents with opposite electric charges ($\Sigma = \sigma$, $q = -Q$), from the formulas obtained, one easily recovers the Emparan-Teo nonextremal dihole solution [5] in the physical parametrization of Ref. [7]. Moreover, by setting either $m = q = 0$ or $M = Q = 0$ in the above formulas, we arrive at the well-known Gibbons-Maeda solution [1] for a single Reissner-Nordström black hole in the four-dimensional dilaton gravity. So, observing a certain parallelism with the well-known double-Kerr solution [27], it would be most appropriate to call the dilaton two-body configuration considered in this section *the double-Gibbons-Maeda solution*.

The presence of the dilaton field in the double-Gibbons-Maeda solution only slightly changes the form of the corresponding potentials Φ_Σ and Φ_σ in which the additional factor $(1 + \alpha^2)^{-1/2}$ arises, compared to the expressions (8) of the double-Reissner-Nordström solution. On the other hand, the dilaton field has a more substantial effect on the form of the horizon areas, A_Σ and A_σ, due to the presence of the functions ϕ_0 and γ_0 in the expressions for $\hat{f}$ and $\hat{\gamma}$. Nonetheless, for A_Σ and A_σ we have been able to obtain the following compact expressions:

$$A_\Sigma = \frac{4\pi(4\Sigma^2)^{\alpha^2/(1+\alpha^2)}[(R + M + m)(M + \Sigma) - Q(Q + q)]^{2/(1+\alpha^2)}}{(R + \Sigma - \sigma)(R + \Sigma + \sigma)^{(1-\alpha^2)/(1+\alpha^2)}},$$
$$A_\sigma = \frac{4\pi(4\sigma^2)^{\alpha^2/(1+\alpha^2)}[(R + M + m)(m + \sigma) - q(Q + q)]^{2/(1+\alpha^2)}}{(R - \Sigma + \sigma)(R + \Sigma + \sigma)^{(1-\alpha^2)/(1+\alpha^2)}}, \tag{37}$$

which generalize formulas (10) in a nice manner. Notice that, as expected, in the presence of the nonzero dilaton field ($\alpha \neq 0$) the extremal dilatonic holes ($\Sigma = 0$ and/or $\sigma = 0$) develop null singularities.

For the interaction energy in the double-Gibbons-Maeda solution, we find the expression

$$V_{\text{int}} = \frac{(R - M - m)\{(\nu - 2\kappa)^{(1-\alpha^2)/(1+\alpha^2)}[R^2 - (\Sigma + \sigma)^2]^{2\alpha^2/(1+\alpha^2)} - (\nu + 2\kappa)\}}{4(\nu + 2\kappa)}, \tag{38}$$

which follows from the formula [5]

$$V_{\text{int}} = \frac{L}{4}(e^{\hat{\gamma}_0} - 1), \tag{39}$$

L being the coordinate length of the strut, and $\hat{\gamma}_0$ being the value of the metric function $\hat{\gamma}$ on the part of the symmetry axis separating the black-hole constituents.

We end this section by observing that the magnetic analog of the double-Gibbons-Maeda solution is defined by a slight modification of the generating formulas (31), in which one only has to replace the electric potential A_t by the magnetic potential A_φ defined by (14) and set ϕ to $-\phi$; the choice of the functions f, γ, ϕ_0, and γ_0 is the same as in the electrostatic solution.

V. CONCLUSIONS

In the present paper we have considered several applications of the double-Reissner-Nordström solution to the binary systems of charged black holes. We started with a brief review of the electrostatic case, supplementing it with a derivation of remarkably simple formulas for the horizon areas and surface gravities of two interacting Reissner-Nordström black holes. We then obtained a magnetic analog of the double-Reissner-Nordström solution, which re-

quired the construction of the corresponding magnetic potential A_φ with the aid of Sibgatullin's integral method. The analysis of the behavior of magnetically charged black holes in the external magnetic field naturally led us to *asymmetric black diholes* as an important family of binary configurations for which equilibrium of the black-hole constituents is achievable, and we found a precise value of the parameter B of the external field at which the balance of the gravitational and magnetic forces occurs. One would anticipate that the ''balanced'' asymmetric black dihole spacetimes could find interesting applications in the numerical simulations of the nonstationary black-hole configurations as the initial static conditions. The absence of asymmetric dihole configurations with two extreme components is a curious fact established in this paper, which lends additional importance to the Bonnor magnetic dipole metric [6] and to some of its generalizations [3] as the extreme black dihole models. At the same time, it should be mentioned that our paper provides a basis for the analysis of the black dihole configurations involving one extreme and one nonextreme component.

There are various reasons for thinking that the double-Gibbons-Maeda solution, obtained and briefly discussed in Sec. IV, opens new promising opportunities for the study of binary black-hole configurations in the presence of a dilaton field. First of all, this solution describes a pair of *arbitrary* Gibbons-Maeda black holes and, therefore, permits one to study more general binary black-hole systems, with zero or nonzero total charge, than earlier considered in the literature; it is important that the existence of simple formulas (37) for the horizon areas strongly suggests that the overall physical analysis of the double-Gibbons-Maeda spacetime is feasible as well. Furthermore, the availability of the general expression for the magnetic potential A_φ now allows a straightforward application of the dilatonic Harrison transformation [28] to the solution obtained and also makes it possible to search for equilibrium configurations of the constituents in the framework of dilatonic asymmetric black diholes surrounded by the external magnetic field, in analogy with the pure Einstein-Maxwell case. In this respect it is worth mentioning that in Ref. [5] a microscopic description of the entropy of interacting equal black holes carrying opposite charges was given using the ''effective string'' model and the large separation approximation needed for obtaining a weak strut singularity. Although it is clear that a similar analysis is also applicable to our asymmetric black diholes, it seems more advantageous to develop a description which would not be subject to the restriction of large separation; for this purpose, as was already observed in [5], the well-behaved black diholes balanced by the external field are the best solutions to employ. Since the crucial point in the latter solutions is the explicit form of the magnetic potential A_φ which was constructed in the present paper, we expect that the desired improved microscopic description of the entropy will be provided in the future.

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