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Hawking radiation via gravitational anomalies in nonspherical topologies

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We study the method of calculating Hawking radiation via gravitational anomalies in gravitational backgrounds of constant negative curvature. We apply the method to topological black holes and also to topological black holes conformally coupled to a scalar field.

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I. INTRODUCTION

Hawking radiation is an important quantum effect in black hole physics. It arises for quantum fields in a background spacetime with an event horizon. Apart from Hawking's original derivation [1,2], which calculates the Bogoliubov coefficients between in- and out-states for a body collapsing to form a black hole, there are also other approaches [3,4]. One of the most interesting proposals was put forward many years ago by Christensen and Fulling [5], who showed that Hawking radiation can be derived from the trace anomaly in the energy-momentum tensor of quantum fields in a Schwarzschild black hole background.

The idea of Christensen and Fulling [5] was to relate an anomaly in conformal symmetry with the energy-momentum tensors of quantum fields in a black hole background. This relation manifests itself as a contribution of the anomaly to the trace T^α_α of the energy-momentum tensor in a theory where it vanishes classically. Requiring finiteness of the energy-momentum tensor of massless fields as seen by a freely falling observer at the horizon in a $(1 + 1)$-dimensional Schwarzschild background metric and using the anomalous trace equation everywhere, one finds an outgoing flux which is in quantitative agreement with Hawking's result.

The validity of this result is subjected to some limitations. The method has been applied to conformal field theories in $(1 + 1)$ dimensions. Also, the assumption in [5] of massless scalar fields was essential to relate fluxes at the horizon to Hawking radiation. The requirement of massless scalar fields was addressed later in [6]. It was considered a massive tachyon field in the background of a dilatonic $(1 + 1)$-dimensional black hole [7]. It was found that the contribution of the tachyon field to Hawking radiation is due to its coupling to the dilaton field, and the Hawking rate due to the tachyon field is enhanced comparable to conformal matter.

Quite recently, Robinson and Wilczek [8] followed by Iso, Umetsu, and Wilczek [9] proposed a new method to calculate Hawking radiation. Their basic idea is to identify outgoing modes of some matter distribution near the horizon as right-moving modes and ingoing modes as left-moving modes, in the Unruh vacuum [10]. Then, because all the ingoing modes cannot classically affect physics outside the horizon, integrating the other modes they obtain an effective chiral action in the exterior region which is anomalous under gauge and general coordinate transformations. However, the underlying theory is invariant under these symmetries, and these anomalies must be canceled by quantum effects of the classically irrelevant ingoing modes. They have proved that the condition for anomaly cancellation at the horizon determines the Hawking flux of the charge and energy-momentum. The flux is universally determined only by the value of anomalies at the horizon.

The crucial point in the Robinson-Wilczek method, and its generalization to include charge, is to reduce an initially high-dimensional theory to two dimensions, in the vicinity of the horizon, which is a necessary step in order to be able to identify the chiral modes. This is achieved by considering a matter source, just outside the horizon of a static spherically symmetric black hole, parametrized by a scalar field minimally coupled to this background. Performing a partial wave decomposition of the scalar field in terms of the wave functions of the classical wave equation, they find that the effective radial potentials for partial wave modes of the scalar field vanish exponentially fast near the horizon. Thus, physics near the horizon can be described using an infinite collection of massless $(1 + 1)$-dimensional scalar fields, each propagating in a $(1 + 1)$-dimensional spacetime with a metric given by the (t, r) section of the original high-dimensional metric, where t and r are the time and radial coordinates, respectively.

The method was further extended to include rotations [11,12]. It was shown that the reduction procedure goes through, and observing that an angular isometry generates an effective $U(1)$ gauge field in the $(1 + 1)$-dimensional theory, with the azimuthal quantum number m serving as the charge of each partial wave, the known results were obtained with angular momentum acting like a chemical potential for the effective charge.

In this work we will study the Hawking effect via the gravitational anomalies method in a gravitational background of constant negative curvature. At first, we will carry out the dimensional reduction procedure of an action

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given by a scalar field minimally coupled to gravity in the background of a $(3+1)$-dimensional topological black hole (TBH), in order to show that near the horizon the theory is reduced to an effective theory of an infinite collection of $(1+1)$-dimensional scalar fields in a $(1+1)$-dimensional background. Identifying the chiral modes, we will finally show that the flux necessary to cancel the gravitational anomalies is identified with the Hawking flux.

We will also apply the method to a topological black hole coupled to a scalar field. Providing that asymptotically the space is anti-de Sitter (AdS), these black hole solutions are stable, they satisfy the Breitenlohner-Freedman bound [13], and the scalar field is regular at the horizon [14]. In this context, we will discuss the applicability of the method in the case of a scalar field backreacting on the gravitational background. As a first step we will consider the case where the scalar field, which generates the Hawking flux, is nonminimally coupled to gravity.

The paper is organized as follows. In Sec. II we review the basic properties of TBHs, and we perform a mode analysis of a scalar field in the background of a TBH. In Sec. III we describe the reduction procedure to two dimensions for a TBH of genus $\tilde{g}=2$. In Sec. IV we derive the Hawking radiation of a TBH of genus $\tilde{g}=2$, and in Sec. V we carry out the same calculation for a TBH coupled to a scalar field. In Sec. VI we investigate whether a scalar field nonminimally coupled to a black hole background has any effect on the Robinson-Wilczek method. Finally, we summarize in the last section.

II. MODE ANALYSIS OF THE WAVE EQUATION OF A SCALAR FIELD IN THE BACKGROUND OF A TBH

We consider the bulk action

$$I=\frac{1}{16\pi G}\int d^dx\sqrt{-g}\left[R+\frac{(d-1)(d-2)}{l^2}\right] \tag{2.1}$$

in asymptotically AdS_d, where G is Newton's constant, R is the Ricci scalar, and l is the AdS radius. The presence of a negative cosmological constant ($\Lambda=-\frac{(d-1)(d-2)}{2l^2}$) allows for the existence of black holes with topology $\mathbb{R}^2\times\Sigma$, where Σ is a $(d-2)$-dimensional manifold of constant negative curvature. These black holes are known as topological black holes [15,16]. The simplest solution of this kind in four dimensions reads

$$\begin{aligned} ds^2&=-f(r)dt^2+\frac{1}{f(r)}dr^2+r^2d\sigma^2,\\ f(r)&=r^2-1-\frac{2\mu}{r}, \end{aligned} \tag{2.2}$$

where we employed units in which the AdS radius is $l=1$, μ is a constant, which is proportional to the mass and is bounded from below $\mu\geq-\frac{1}{3\sqrt{3}}$, and $d\sigma^2$ is the line element of the two-dimensional manifold Σ, which is locally isomorphic to the hyperbolic manifold H^2 and of the form

$$\Sigma=H^2/\Gamma,\qquad \Gamma\subset O(2,1), \tag{2.3}$$

where Γ is a freely acting discrete subgroup (i.e. without fixed points) of isometries. The line element $d\sigma^2$ of Σ is

$$d\sigma^2=d\theta^2+\sinh^2\theta d\varphi^2, \tag{2.4}$$

with $\theta\geq0$ and $0\leq\varphi<2\pi$ being the coordinates of the hyperbolic space H^2 or pseudosphere, which is a noncompact two-dimensional space of constant negative curvature. This space becomes a compact space of constant negative curvature with genus $\tilde{g}\geq2$ by identifying, according to the connection rules of the discrete subgroup Γ, the opposite edges of a $4\tilde{g}$-sided polygon whose sides are geodesics and which is centered at the origin $\theta=\varphi=0$ of the pseudosphere [15–17]. An octagon is the simplest such polygon, yielding a compact surface of genus $\tilde{g}=2$ under these identifications. Thus, the two-dimensional manifold Σ is a compact Riemann two-surface of genus $\tilde{g}\geq2$. Further details on this kind of compactification scheme can be found in [17,18]. The configuration (2.2) is an asymptotically locally AdS spacetime. The horizon structure of (2.2) is determined by the roots of the metric function $f(r)$,

$$f(r)=r^2-1-\frac{2\mu}{r}=0. \tag{2.5}$$

For $-\frac{1}{3\sqrt{3}}<\mu<0$, this equation has two distinct nondegenerate solutions, corresponding to an inner and an outer horizon, r_- and r_+, respectively. For $\mu\geq0$, $f(r)$ has just one nondegenerate root, and so the black hole (2.2) has one horizon, r_h. The horizons for both cases of μ have the nontrivial topology of the manifold Σ. We note that for $\mu=-\frac{1}{3\sqrt{3}}$, $f(r)$ has a degenerate root, but this horizon does not have an interpretation as a black hole horizon [15].

We will examine the eigenmodes of the classical wave equation of a scalar field Φ of mass m_Φ, in the background of the topological black hole (2.2), and perform a partial wave decomposition of the wave functions. The classical wave equation in the background of (2.2), without any identifications of the pseudosphere (i.e. H^2), is

$$\nabla^2\Phi=m_\Phi^2\Phi, \tag{2.6}$$

where ∇^2 is the Laplace-Beltrami operator defined by

$$\nabla^2\equiv\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu), \tag{2.7}$$

and hence the wave equation is

$$\left[-\frac{1}{f}\partial_t^2+\frac{1}{r^2}\partial_r(r^2f\partial_r)+\frac{1}{r^2\sinh\theta}\partial_\theta(\sinh\theta\partial_\theta)+\frac{1}{r^2\sinh^2\theta}\partial_\varphi^2\right]\Phi=m_\Phi^2\Phi. \tag{2.8}$$

We factorize out the angular and radial dependence of the field as

$$\Phi(t,r,\theta,\varphi) = \frac{R(t,r)}{r} Y(\theta,\varphi). \tag{2.9}$$

With this factorization and using separation of variables, we get two differential equations. The angular wave equation is

$$-\left[\frac{1}{\sinh\theta}\partial_\theta(\sinh\theta\partial_\theta) + \frac{1}{\sinh^2\theta}\partial_\varphi^2\right]Y(\theta,\varphi) = \lambda Y(\theta,\varphi), \tag{2.10}$$

while the radial wave equation is

$$\partial_t^2 R(t,r) - f\left[(\partial_r f)\partial_r - \frac{\partial_r f}{r} + f\partial_r^2 - m_\Phi^2 - \frac{\lambda}{r^2}\right]R(t,r) = 0, \tag{2.11}$$

where λ is a separation constant. The angular wave equation has the solution [17]

$$Y_l^m(\theta,\varphi) = P_l^m(\cosh\theta)e^{im\varphi} = P^m_{-(1/2)\pm i\xi}(\cosh\theta)e^{im\varphi}, \tag{2.12}$$

where P_l^m are the associated Legendre functions and

$$\begin{aligned} m &= 0, \pm1, \pm2, \pm3, \ldots, \qquad \lambda = -l(l+1), \\ l &= -\tfrac{1}{2} \pm i\xi, \qquad \lambda = \xi^2 + \tfrac{1}{4}. \end{aligned} \tag{2.13}$$

The radial wave equation, after separating the time dependence by writing $R(t,r) = R(r)e^{i\omega t}$, becomes

$$\omega^2 R(r) + f\left[(\partial_r f)\partial_r - \frac{\partial_r f}{r} + f\partial_r^2 - m_\Phi^2 - \frac{\xi^2 + \frac{1}{4}}{r^2}\right]R(r) = 0. \tag{2.14}$$

There is no general solution to this equation, but we can write it in a very simple form using the tortoise coordinate r_* defined by

$$\frac{\partial r_*}{\partial r} = \frac{1}{f(r)}. \tag{2.15}$$

The radial wave equation in terms of the tortoise coordinate r_* becomes

$$[\partial_{r_*}^2 + \omega^2 - f(r(r_*))V(r(r_*))]R(r(r_*)) = 0, \tag{2.16}$$

with

$$V(r) = \frac{1}{r^2}\left[r\frac{df(r)}{dr} + \xi^2 + \frac{1}{4}\right] + m_\Phi^2, \tag{2.17}$$

which for the $f(r)$ of the background (2.2) is

$$V(r) = 2 + \frac{\xi^2 + \frac{1}{4}}{r^2} + \frac{2\mu}{r^3} + m_\Phi^2. \tag{2.18}$$

In conclusion, the eigenmodes of the classical wave equation in the background of (2.2) are

$$\Phi(t,r,\theta,\varphi) = \frac{R_\xi(t,r)}{r} P^m_{-(1/2)\pm i\xi}(\cosh\theta)e^{im\varphi}. \tag{2.19}$$

Without any identifications of the pseudosphere, the spectrum of the angular wave equation is continuous; thus ξ takes any real value $\xi \geq 0$. Since the two-dimensional manifold Σ is a quotient space of the form H^2/Γ and is a compact space of constant negative curvature, the spectrum of the angular wave equation is discretized and thus ξ takes discrete real values, $\xi \geq 0$. On the simplest such manifold of constant negative curvature, which is a compact surface of genus $\tilde{g} = 2$, the angular wave functions (2.12) must satisfy four periodicity conditions, and the compatibility of these four periodicity conditions is what generates the discrete spectrum [17]. In general, there are no explicit analytical results in the literature for the angular eigenvalues $\lambda(l)$ and for the angular eigenfunctions, although some numerical results exist [17]. Therefore, in the next sections we will elaborate only on the case of Σ being a compact two-dimensional manifold of genus $\tilde{g} = 2$ with constant negative curvature.

III. DIMENSIONAL REDUCTION FOR A TOPOLOGICAL BLACK HOLE

We consider matter in the background of the topological black hole (2.2) of genus $\tilde{g} = 2$ given by a complex scalar field $\phi(x)$ with an action of the form

$$S = S_{\text{free}} + S_{\text{int}}, \tag{3.1}$$

where S_{free} is the free part of the action,

$$S_{\text{free}} = -\frac{1}{2}\int d^4x\sqrt{-g}\,\phi^*\nabla^2\phi, \tag{3.2}$$

and S_{int} is the part of the action which includes a mass term, potential terms, and interaction terms. We perform a partial wave decomposition of ϕ in terms of the eigenmodes (2.19),

$$\phi(t,r,\theta,\varphi) = \sum_{m=-\infty}^{+\infty} \frac{R_{\xi m}(t,r)}{r}\mathcal{Y}_\xi^m(\theta,\varphi), \tag{3.3}$$

where, for convenience, we chose a different normalization of the eigenmodes by defining the functions $\mathcal{Y}_\xi^m$ [19] as

$$\begin{aligned} \mathcal{Y}_\xi^m(\theta,\varphi) &\equiv \left(\frac{2\pi}{\xi\tanh(\pi\xi)}\right)^{1/2}\mathcal{P}^{m0}_{-(1/2)+i\xi}(\cosh\theta)e^{im\varphi} \\ &= \left(\frac{2\pi}{\xi\tanh(\pi\xi)}\right)^{1/2} \\ &\quad\times \frac{\Gamma(i\xi+\frac{1}{2})}{\Gamma(i\xi+m+\frac{1}{2})}P^m_{-(1/2)+i\xi}(\cosh\theta)e^{im\varphi}, \end{aligned} \tag{3.4}$$

with $m = 0, \pm1, \pm2, \pm3, \ldots$ and ξ taking discrete real values, $\xi \geq 0$. In this definition we have used the functions $\mathcal{P}_l^{mn}$, which form the canonical basis for the irreducible representations of the group SL(2, C) and can be viewed as playing the same role for the group SU(1,1) (see Appendix A). These functions are related to the associated Legendre functions through

$$\mathcal{P}_l^{m0}(\cosh\theta) = \frac{\Gamma(l+1)}{\Gamma(l+m+1)} P_l^m(\cosh\theta). \tag{3.5}$$

The functions $\mathcal{Y}_\xi^m$ form a complete set of functions on the manifold Σ, they satisfy four periodicity conditions [17], and their orthogonality condition is Eq. (A15) of Appendix A, that is,

$$\int_0^\infty d\theta \int_0^{2\pi} d\varphi \sinh\theta \mathcal{Y}_\xi^m(\theta,\varphi)(\mathcal{Y}_{\xi'}^{m'}(\theta,\varphi))^* = \delta_{\xi\xi'}\delta_{mm'}. \tag{3.6}$$

Furthermore, it is proved in Appendix A that they satisfy the equation

$$\Delta_\Omega \mathcal{Y}_\xi^m(\theta,\varphi) = -(\xi^2 + \tfrac{1}{4})\mathcal{Y}_\xi^m(\theta,\varphi), \tag{3.7}$$

where Δ_Ω is the differential operator

$$\Delta_\Omega = \frac{1}{\sinh\theta}\partial_\theta(\sinh\theta\partial_\theta) + \frac{1}{\sinh^2\theta}\partial_\varphi^2. \tag{3.8}$$

Substituting the partial wave decomposition of ϕ in the free part of the action, we get

$$S_{\text{free}} = -\frac{1}{2}\int dtdrd\theta d\varphi r^2 \sinh\theta\bigg\{\bigg(\sum_{m'=-\infty}^{+\infty}\frac{R_{\xi'm'}}{r}\mathcal{Y}_{\xi'}^{m'}\bigg)^* \times\bigg[-\frac{1}{f}\partial_t^2 + \frac{1}{r^2}\partial_r(r^2 f\partial_r) + \frac{1}{r^2}\Delta_\Omega\bigg] \times\bigg(\sum_{m=-\infty}^{+\infty}\frac{R_{\xi m}}{r}\mathcal{Y}_\xi^m\bigg)\bigg\}, \tag{3.9}$$

and with the help of the property (3.7),

$$S_{\text{free}} = -\frac{1}{2}\sum_{m,m'}\int dtdrd\theta d\varphi \sinh\theta\bigg[R_{\xi'm'}^*\bigg(-\frac{1}{f}\bigg)\partial_t^2 R_{\xi m} + \frac{R_{\xi'm'}^*}{r}\partial_r\bigg(r^2 f\partial_r\bigg(\frac{R_{\xi m}}{r}\bigg)\bigg) - \frac{R_{\xi'm'}^*}{r}\frac{R_{\xi m}}{r}\bigg(\xi^2 + \frac{1}{4}\bigg)\bigg] \times \mathcal{Y}_\xi^m(\theta,\varphi)(\mathcal{Y}_{\xi'}^{m'}(\theta,\varphi))^*. \tag{3.10}$$

Performing the integrations on θ and φ, using the normalization condition (3.6), we have

$$S_{\text{free}} = -\frac{1}{2}\sum_{m=-\infty}^{\infty}\int dtdr\bigg[R_{\xi m}^*\bigg(-\frac{1}{f}\bigg)\partial_t^2 R_{\xi m} + \frac{R_{\xi m}^*}{r}\partial_r\bigg(r^2 f\partial_r\bigg(\frac{R_{\xi m}}{r}\bigg)\bigg) - \frac{R_{\xi m}^*}{r}\frac{R_{\xi m}}{r}\bigg(\xi^2+\frac{1}{4}\bigg)\bigg]. \tag{3.11}$$

Next, we will make a transformation to the tortoise coordinates (t, r_*), defined by (2.15), and consider only the region near the event horizon. In the case of $-\frac{1}{3\sqrt{3}} < \mu < 0$, this is the outer horizon r_+, and in the case of $\mu \geq 0$, this is the horizon r_h. In order to include in our analysis both of these cases, we denote both r_+ and r_h as r_H.

But first, we will determine the behavior of the radial coordinate r and of the metric function $f(r)$ in tortoise coordinates in the region near the horizon r_H. The Taylor expansion of the metric function $f(r)$ around the event horizon is

$$f(r) = 2\kappa(r - r_H) + \sum_{n=2}^{\infty}\frac{f^{(n)}(r_H)}{n!}(r - r_H)^n, \tag{3.12}$$

where $\kappa \equiv \frac{1}{2}(\partial_r f)|_{r_H}$ is the surface gravity. In the region near the event horizon, we can keep only the first two terms of the Taylor expansion,

$$f(r) \approx 2\kappa(r - r_H). \tag{3.13}$$

Transforming to the tortoise coordinates (t, r_*) and integrating both sides of (2.15) using the approximation (3.13), we get

$$r_* \approx \int\frac{1}{2\kappa(r - r_H)}dr + C, \tag{3.14}$$

and so

$$r(r_*) \approx Ae^{2\kappa r_*} + r_H, \tag{3.15}$$

where C is an arbitrary integration constant and $A \equiv e^{-2\kappa C}$. Finally, Eqs. (3.13) and (3.15) give

$$f(r(r_*)) \approx 2\kappa A e^{2\kappa r_*}. \tag{3.16}$$

The last two equations describe the behavior of r and $f(r)$ in tortoise coordinates in the region near the event horizon. Note that the limit $r \to r_H$ is equivalent to the limit $r_* \to -\infty$ in tortoise coordinates, which means that the event horizon in tortoise coordinates is located at $(-\infty)$. In addition, note that near the event horizon $f(r(r_*))$ vanishes exponentially fast, hence $f(r(r_*))$ is a suppression factor near the event horizon.

Now, we can return to Eq. (3.11), transform to tortoise coordinates, and consider only the region near the horizon. After using the fact that $f(r(r_*))$ is a suppression factor near the horizon and keeping only dominant terms, the free part of the action becomes, in tortoise coordinates and in the region near the horizon,

$$(S_{\text{free}})_* = \sum_{m=-\infty}^{\infty} -\frac{1}{2}\int dtdr_* R_{\xi m}^*[-\partial_t^2 + f\partial_r(f\partial_r)]R_{\xi m}, \tag{3.17}$$

where the upper star denotes complex conjugation, the lower star denotes the tortoise coordinates, and f, $R_{\xi m}$, $R_{\xi m}^*$ are implicit functions of r_*. Transforming back to the original coordinates we find

$$S_{\rm free} = \sum_{m=-\infty}^{\infty} -\frac{1}{2}\int dt dr R^{*}_{\xi m}\left[-\frac{1}{f}\partial_t^2 + \partial_r(f\partial_r)\right]R_{\xi m}. \tag{3.18}$$

Concerning the part $S_{\rm int}$ of the action, which includes a mass term, potential terms, and interaction terms, after performing a partial wave decomposition in terms of the functions $\mathcal{Y}^m_\xi$ and upon transforming to the tortoise coordinates, one finds that all of its terms contain the factor $f(r(r_*))$ and vanish exponentially fast near the horizon. Thus, the total action S is obtained,

$$S = \sum_{m=-\infty}^{\infty} -\frac{1}{2}\int dt dr R^{*}_{\xi m}\left[-\frac{1}{f}\partial_t^2 + \partial_r(f\partial_r)\right]R_{\xi m}. \tag{3.19}$$

According to this action, physics in the region near the horizon can be effectively described by an infinite collection of (1 + 1)-dimensional free massless complex scalar fields, each propagating in a (1 + 1)-dimensional spacetime, which is given by the (t, r) part of the (3 + 1)-dimensional metric of the topological black hole of genus $\tilde{g} = 2$, that is,

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2. \tag{3.20}$$

IV. HAWKING RADIATION FROM TOPOLOGICAL BLACK HOLES

In the reduced (1 + 1)-dimensional background (3.20), outgoing modes of the (1 + 1)-dimensional fields near the horizon behave as right-moving modes and ingoing modes behave as left-moving modes. If we neglect the ingoing modes in the region near the horizon, because they cannot classically affect physics outside the horizon, then the effective two-dimensional theory becomes chiral. As it is known [20–24] a two-dimensional chiral theory exhibits a gravitational anomaly. The consistent gravitational anomaly for right-handed fields reads [20,22]

$$\nabla_\mu T^\mu_\nu = \frac{1}{96\pi\sqrt{-g_{(2)}}}\epsilon^{\beta\delta}\partial_\delta\partial_\alpha\Gamma^\alpha_{\nu\beta}, \tag{4.1}$$

and the covariant gravitational anomaly takes the form

$$\nabla_\mu \tilde{T}^{\mu\nu} = \frac{\epsilon^{\nu\mu}}{96\pi\sqrt{-g_{(2)}}}\partial_\mu R, \tag{4.2}$$

where T^μ_ν and $\tilde{T}^\mu_\nu$ are the consistent and covariant energy-momentum tensors, respectively, $\epsilon^{01} = -\epsilon^{10} = 1$, and R and $g_{(2)}$ are the Ricci scalar and the metric determinant of the reduced metric (3.20), respectively. The consistent gravitational anomaly satisfies the Wess-Zumino consistency condition, but the consistent energy-momentum tensor T^μ_ν does not transform covariantly under general coordinate transformations. The covariant energy-momentum tensor $\tilde{T}^\mu_\nu$, on the contrary, transforms covariantly under general coordinate transformations, but the covariant gravitational anomaly does not satisfy the Wess-Zumino consistency condition. Consistent and covariant expressions are related by local counterterms [21,23,24]. In [9,11] the consistent expressions for the anomalies were taken, whereas the imposed boundary conditions involved the covariant form. A reformulation of this approach was given in [25], where only covariant expressions were used, rectifying this conceptual issue. Furthermore, a more technically simplified way to obtain the Hawking flux was suggested in [26–28], where the calculation involved only the expressions for the anomalous covariant Ward identities and the covariant boundary conditions. We will follow this approach to derive the Hawking flux.

We consider the expression for the two-dimensional covariant gravitational Ward identity, that is, the covariant anomaly (4.2), and taking its $\nu = t$ component, we get

$$\partial_r \tilde{T}^r_t = \frac{1}{96\pi} f\partial_r f'', \tag{4.3}$$

where we have used the facts that the background is static and that the Ricci scalar is $R = -f''(r)$, while a prime denotes differentiation with respect to r. Equation (4.3) can be written as

$$\partial_r \tilde{T}^r_t = \partial_r \tilde{N}^r_t \tag{4.4}$$

or

$$\partial_r(\tilde{T}^r_t - \tilde{N}^r_t) = 0, \tag{4.5}$$

where

$$\tilde{N}^{\,r}_{\,t} = \frac{1}{96\pi}\left(ff'' - \frac{f'^2}{2}\right). \tag{4.6}$$

Solving Eq. (4.4) we find

$$\tilde{T}^{\,r}_{\,t}(r) = a_H + \tilde{N}^r_t(r) - \tilde{N}^r_t(r_H). \tag{4.7}$$

Here a_H is an integration constant. Imposing the covariant boundary condition [11,28]

$$\tilde{T}^{\,r}_{\,t}(r_H) = 0, \tag{4.8}$$

namely, the vanishing of the covariant energy-momentum tensor at the event horizon, yields $a_H = 0$. Hence, the anomalous covariant energy-momentum tensor (4.7) is

$$\tilde{T}^{\,r}_{\,t}(r) = \tilde{N}^r_t(r) - \tilde{N}^r_t(r_H). \tag{4.9}$$

In what follows we restore in our formulas the value of the AdS radius l, which had been set to $l = 1$. So, the metric function is

$$f(r) = \frac{r^2}{l^2} - 1 - \frac{2\mu}{r}. \tag{4.10}$$

We remind the reader that the Hawking flux is measured at infinity, where there is no gravitational anomaly, and in

[8,9,11] it was given by the anomaly-free (or conserved) energy-momentum tensor. In [8,9,11] this required splitting the space into two distinct regions, one near the horizon and the other away from it, and using both the anomalous Ward identity in the vicinity of the horizon and the normal Ward identity in the exterior region. This is redundant if one observes that, for the metric (3.20) and the specific metric function $f(r)$ of the $(3+1)$-dimensional topological black hole of genus $\tilde{g}=2$, the gravitational anomaly vanishes at asymptotic infinity, $r\to\infty$. Indeed, we see that in this limit

$$\frac{\epsilon^{\nu\mu}}{96\pi\sqrt{-g_{(2)}}}\partial_\mu R=-\frac{\epsilon^{\nu 1}}{96\pi}f'''=-\frac{\epsilon^{\nu 1}}{96\pi}\frac{12\mu}{r^4}\to 0 \tag{4.11}$$

and

$$\partial_r\tilde{N}^r_t=\frac{1}{96\pi}\partial_r\left(ff''-\frac{f'^2}{2}\right)=\frac{1}{96\pi}f\partial_r f'' \\ =\frac{1}{96\pi}\left(\frac{12\mu}{l^2r^2}-\frac{12\mu}{r^4}-\frac{24\mu^2}{r^5}\right)\to 0. \tag{4.12}$$

It is also important to notice that although the gravitational anomaly vanishes at infinity, the $\tilde{N}^r_t$ does not, since

$$\tilde{N}^r_t(r\to\infty)=-\frac{l^{-2}}{48\pi}, \tag{4.13}$$

because the spacetime asymptotically is not flat but AdS. The last three equations, and observation of Eq. (4.5), imply that the anomaly-free (or conserved) energy-momentum tensor, which is the energy flux Φ measured at infinity, is given by

$$\Phi=\tilde{T}^r_t(r\to\infty)-\tilde{N}^r_t(r\to\infty) \\ =-\frac{l^{-2}}{48\pi}-\tilde{N}^r_t(r_H)-\left(-\frac{l^{-2}}{48\pi}\right)=-\tilde{N}^r_t(r_H). \tag{4.14}$$

Thus,[1] the energy flux measured at infinity is

$$\Phi=-\tilde{N}^r_t(r_H)=\frac{1}{192\pi}f'^2(r_H), \tag{4.15}$$

or in a different form,

$$\Phi=\frac{\pi}{12}\left(\frac{f'(r_H)}{4\pi}\right)^2. \tag{4.16}$$

A beam of massless blackbody radiation moving in the positive r direction at a temperature T has a flux of the form $\Phi=\frac{\pi}{12}T^2$. Therefore, we see that the flux (4.16) has a form equivalent to blackbody radiation with a temperature

$$T_H=\frac{f'(r_H)}{4\pi}=\frac{\kappa}{2\pi}. \tag{4.17}$$

This temperature is exactly the Hawking temperature of a $(3+1)$-dimensional topological black hole of genus $\tilde{g}=2$ as determined in [16]. Hence, Φ is the Hawking flux.

V. HAWKING RADIATION FROM A TBH CONFORMALLY COUPLED TO A SCALAR FIELD

Another interesting nonspherical background is a TBH conformally coupled to a scalar field. Consider four-dimensional gravity with a negative cosmological constant $(\Lambda=-3l^{-2})$ and a scalar field $\phi(x)$ described by the action

$$I[g_{\mu\nu},\phi]=\int d^4x\sqrt{-g}\left[\frac{R_E+6l^{-2}}{16\pi G}-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-V(\phi)\right], \tag{5.1}$$

where R_E is the Ricci scalar in the Einstein frame, l is the AdS radius, and G is Newton's constant. We take the following self-interaction potential:

$$V(\phi)=-\frac{3}{4\pi G l^2}\sinh^2\sqrt{\frac{4\pi G}{3}}\phi. \tag{5.2}$$

It was proved in [14] that there is a static black hole solution (MTZ black hole) with topology $\mathbb{R}^2\times\Sigma$, where Σ is a two-dimensional manifold of constant negative curvature, which is locally isomorphic to the hyperbolic manifold H^2 and of the form

$$\Sigma=H^2/\Gamma,\qquad \Gamma\subset O(2,1), \tag{5.3}$$

where Γ is a freely acting discrete subgroup (i.e. without fixed points) of isometries. This black hole solution is given by

$$ds^2=\frac{r(r+2G\mu)}{(r+G\mu)^2}\left[-\left(\frac{r^2}{l^2}-\left(1+\frac{G\mu}{r}\right)^2\right)dt^2+\left(\frac{r^2}{l^2}-\left(1+\frac{G\mu}{r}\right)^2\right)^{-1}dr^2+r^2d\sigma^2\right], \tag{5.4}$$

and the scalar field is

$$\phi(r)=\sqrt{\frac{3}{4\pi G}}\operatorname{arctanh}\frac{G\mu}{r+G\mu}. \tag{5.5}$$

Here $d\sigma^2$ is the line element of the two-dimensional manifold Σ,

[1] In the case of an asymptotically flat spacetime treated in [26–28], it was $\tilde{N}^r_t(r\to\infty)=0$ and the gravitational anomaly vanished at infinity, so that the energy flux was calculated as $\Phi=\tilde{T}^r_t(r\to\infty)=-\tilde{N}^r_t(r_H)$. The difference in the case of the $(3+1)$-dimensional TBH of genus $\tilde{g}=2$ is that, although the gravitational anomaly vanishes at infinity, the $\tilde{N}^r_t(r\to\infty)$ is not zero, due to the fact that the spacetime is asymptotically AdS, so one must take it into consideration according to Eq. (4.5), in order to find the correct conserved energy-momentum tensor at infinity. Of course, if we put $\tilde{N}^r_t(r\to\infty)=0$ in Eqs. (4.4), (4.5), (4.9), and (4.14), we retrieve the result for the asymptotically flat case. Note that, finally, for both the asymptotically AdS spacetime and the asymptotically flat spacetime, the energy flux is given by the equation $\Phi=-\tilde{N}^r_t(r_H)$.

$$d\sigma^2 = d\theta^2 + \sinh^2\theta d\varphi^2, \tag{5.6}$$

where $\theta \geq 0$ and $0 \leq \varphi < 2\pi$ are the coordinates of the hyperbolic space H^2. The mass of this solution is given by

$$M = \frac{\sigma}{4\pi}\mu, \tag{5.7}$$

where σ denotes the area of Σ and $\mu > -l/4G$ is a constant. Performing a conformal transformation with a scalar field redefinition of the form

$$\hat{g}_{\mu\nu} = \left(1 - \frac{4\pi G}{3}\Psi^2\right)^{-1} g_{\mu\nu}, \qquad \Psi = \sqrt{\frac{3}{4\pi G}}\tanh\sqrt{\frac{4\pi G}{3}}\phi, \tag{5.8}$$

the action (5.1) and (5.2) reads

$$I[\hat{g}_{\mu\nu}, \Psi] = \int d^4x\sqrt{-\hat{g}}\left[\frac{\hat{R} + 6l^{-2}}{16\pi G} - \frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi - \frac{1}{12}\hat{R}\Psi^2 - \frac{2\pi G}{3l^2}\Psi^4\right]. \tag{5.9}$$

In this frame the scalar field equation is conformally invariant, since the matter action is invariant under arbitrary local rescalings $\hat{g}_{\mu\nu} \to \lambda^2(x)\hat{g}_{\mu\nu}$ and $\Psi \to \lambda^{-1}\Psi$. The black hole solution (5.4) and (5.5) acquires a simple form once expressed in the conformal frame,

$$d\hat{s}^2 = -\left[\frac{r^2}{l^2} - \left(1 + \frac{G\mu}{r}\right)^2\right]dt^2 + \left[\frac{r^2}{l^2} - \left(1 + \frac{G\mu}{r}\right)^2\right]^{-1}dr^2 + r^2 d\sigma^2, \tag{5.10}$$

with

$$\Psi(r) = \sqrt{\frac{3}{4\pi G}}\frac{G\mu}{r + G\mu}. \tag{5.11}$$

We define

$$f(r) \equiv \frac{r^2}{l^2} - \left(1 + \frac{G\mu}{r}\right)^2, \tag{5.12}$$

and the metric (5.10) is written as

$$d\hat{s}^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\theta^2 + r^2\sinh^2\theta d\varphi^2. \tag{5.13}$$

We consider only the case in which the two-dimensional manifold Σ is a compact two-dimensional manifold of genus $\tilde{g} = 2$, with constant negative curvature, after the identifications that we have mentioned in Sec. II. For non-negative mass $\mu \geq 0$, this solution possesses only one event horizon at

$$r_+ = \frac{l}{2}\left(1 + \sqrt{1 + \frac{4G\mu}{l}}\right), \tag{5.14}$$

and Ψ is regular everywhere. For negative mass $-l/4 < G\mu < 0$, the metric (5.10) develops three horizons, two of which are event horizons located at r_{--} and at r_+,

$$r_{--} = \frac{l}{2}\left(-1 + \sqrt{1 - \frac{4G\mu}{l}}\right), \tag{5.15}$$

$$r_- = \frac{l}{2}\left(1 - \sqrt{1 + \frac{4G\mu}{l}}\right), \tag{5.16}$$

$$r_+ = \frac{l}{2}\left(1 + \sqrt{1 + \frac{4G\mu}{l}}\right), \tag{5.17}$$

which satisfy $0 < r_{--} < -G\mu < r_- < l/2 < r_+$. The scalar field Ψ is singular at $r = -G\mu$. The Ricci scalar of the black hole solution (5.10) in the conformal frame is

$$\hat{R} = -12l^{-2}. \tag{5.18}$$

As before, we consider a complex scalar field $\hat{\phi}(x)$ in the background of the MTZ black hole of genus $\tilde{g} = 2$, with a scalar hair Ψ, in the conformal frame. This field has an action of the form

$$S = S_{\text{free}} + S_{\text{int}}, \tag{5.19}$$

where S_{free} is the free part of the action,

$$S_{\text{free}} = -\frac{1}{2}\int d^4x\sqrt{-\hat{g}}\hat{\phi}^*\left[-\frac{1}{f}dt^2 + \frac{1}{r^2}\partial_r(r^2 f\partial_r) + \frac{1}{r^2}\Delta_\Omega\right]\hat{\phi}, \tag{5.20}$$

and S_{int} is the part of the action which includes a mass term, potential terms, and interaction terms, where we have ignored the interaction of $\hat{\phi}$ with Ψ. We perform a partial wave decomposition of $\hat{\phi}$ in terms of the functions $\mathcal{Y}_\xi^m$,

$$\hat{\phi}(t, r, \theta, \varphi) = \sum_{m=-\infty}^{+\infty}\frac{R_{\xi m}(t, r)}{r}\mathcal{Y}_\xi^m(\theta, \varphi). \tag{5.21}$$

We substitute the partial wave decomposition in the free action and transform to the tortoise coordinates (t, r_*) defined by (2.15). Then, one finds that in the region near the event horizon r_+, which for $-l/4 < G\mu < 0$ is the outer event horizon and for $\mu \geq 0$ is the unique event horizon, the effective radial potentials for partial wave modes of the field contain the suppression factor $f(r(r_*))$ and vanish exponentially fast. The same applies to the mass terms and interaction terms of the part S_{int}. Thus, physics in the region near the horizon can be effectively described by an infinite collection of $(1 + 1)$-dimensional free massless scalar fields, each propagating in a $(1 + 1)$-dimensional spacetime, which is given by the (t, r) part of the $(3 + 1)$-dimensional metric of the MTZ black hole of genus $\tilde{g} = 2$ in the conformal frame, that is,

$$d\hat{s}^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2. \tag{5.22}$$

In this two-dimensional background we identify outgoing modes near the horizon as right-moving modes and ingoing modes as left-moving modes. Neglecting the classically irrelevant ingoing modes in the region near the horizon, the effective two-dimensional theory becomes chiral and a gravitational anomaly appears. The covariant gravitational anomaly for right-handed fields reads

$$\nabla_\mu \tilde{T}^{\mu\nu} = \frac{\epsilon^{\nu\mu}}{96\pi\sqrt{-\hat{g}_{(2)}}}\partial_\mu R, \tag{5.23}$$

where $\tilde{T}^\mu_\nu$ is the covariant energy-momentum tensor, $\epsilon^{01} = -\epsilon^{10} = 1$, and $R = -f''(r)$ and $\hat{g}_{(2)}$ are the Ricci scalar and the metric determinant of the reduced metric (5.22), respectively. Taking the $\nu = t$ component of the two-dimensional covariant anomaly (5.23), we have

$$\partial_r \tilde{T}^r_t = \frac{1}{96\pi} f \partial_r f''. \tag{5.24}$$

This equation is written as

$$\partial_r \tilde{T}^r_t = \partial_r \tilde{N}^r_t \tag{5.25}$$

or

$$\partial_r(\tilde{T}^r_t - \tilde{N}^r_t) = 0, \tag{5.26}$$

where

$$\tilde{N}^{\,r}_{\,t} = \frac{1}{96\pi}\left(ff'' - \frac{f'^2}{2}\right). \tag{5.27}$$

Solving Eq. (5.25) we find

$$\tilde{T}^{\,r}_{\,t}(r) = b_+ + \tilde{N}^r_t(r) - \tilde{N}^r_t(r_+). \tag{5.28}$$

Here b_+ is an integration constant. Implementing the usual covariant boundary condition,

$$\tilde{T}^{\,r}_{\,t}(r_+) = 0, \tag{5.29}$$

yields $b_+ = 0$. Therefore, the anomalous covariant energy-momentum tensor is

$$\tilde{T}^{\,r}_{\,t}(r) = \tilde{N}^r_t(r) - \tilde{N}^r_t(r_+). \tag{5.30}$$

We notice that for the metric (5.22), with a metric function $f(r)$ given by Eq. (5.12), the gravitational anomaly vanishes at asymptotic infinity $r \to \infty$, but the $\tilde{N}^r_t$ does not due to the fact that, asymptotically, the spacetime is AdS. Indeed, in this limit we have

$$\begin{aligned}\frac{\epsilon^{\nu\mu}}{96\pi\sqrt{-\hat{g}_{(2)}}}\partial_\mu R &= -\frac{\epsilon^{\nu 1}}{96\pi} f''' \\ &= -\frac{\epsilon^{\nu 1}}{96\pi}\left(\frac{12G\mu}{r^4} + \frac{24(G\mu)^2}{r^5}\right) \to 0\end{aligned} \tag{5.31}$$

and

$$\begin{aligned}\partial_r \tilde{N}^r_t = \frac{1}{96\pi}\bigg(&\frac{12G\mu}{l^2 r^2} + \frac{24(G\mu)^2}{l^2 r^3} - \frac{12G\mu}{r^4} - \frac{48(G\mu)^2}{r^5} \\ &- \frac{60(G\mu)^3}{r^6} - \frac{24(G\mu)^4}{r^7}\bigg) \to 0,\end{aligned} \tag{5.32}$$

but

$$\tilde{N}^{\,r}_{\,t}(r \to \infty) = -\frac{l^{-2}}{48\pi}. \tag{5.33}$$

Using the same arguments as in Sec. IV, we see that the anomaly-free energy-momentum tensor, and thus the energy flux Φ measured at infinity, is

$$\begin{aligned}\Phi &= \tilde{T}^r_t(r \to \infty) - \tilde{N}^r_t(r \to \infty) \\ &= -\frac{l^{-2}}{48\pi} - \tilde{N}^r_t(r_+) - \left(-\frac{l^{-2}}{48\pi}\right) = -\tilde{N}^r_t(r_+) \\ &= \frac{1}{192\pi} f'^2(r_+),\end{aligned} \tag{5.34}$$

or in a different form,

$$\Phi = \frac{\pi}{12}\left(\frac{f'(r_+)}{4\pi}\right)^2. \tag{5.35}$$

This flux is equivalent to a flux of blackbody radiation with a temperature

$$T_H = \frac{f'(r_+)}{4\pi} = \frac{\kappa}{2\pi}, \tag{5.36}$$

where $\kappa \equiv \frac{1}{2}(\partial_r f)|_{r_+}$ is the surface gravity. The temperature T_H is identical to the Hawking temperature of the MTZ black hole as determined in [29,30]. Hence, Φ is identified with the Hawking flux of the MTZ black hole, which is a $(3+1)$-dimensional topological black hole conformally coupled to a scalar field.

VI. ROBINSON-WILCZEK METHOD WITH A SCALAR FIELD NONMINIMALLY COUPLED TO THE BLACK HOLE BACKGROUND

In the previous section we showed that the scalar field Ψ, which is coupled to the black hole, does not explicitly contribute to the Hawking radiation. The reason is that the scalar field does not introduce any new conserved charge, and its only effect is to alter the form of the background black hole solution to a maximal Reissner-Nordström-AdS black hole. In the Robinson-Wilczek method this is expected since the scalar field is time independent, and therefore it cannot generate a flux. For this reason, if we had tried to perform the usual reduction procedure only with the scalar field Ψ, assuming that it gives Hawking radiation, we would have found that its action in the vicinity of the event horizon vanishes due to the suppression factor $f(r(r_*))$ (see Appendix B). However, it is interesting to investigate what happens if the scalar field, which parametrizes the matter, backreacts on the geometry. In this direction we will discuss the consequences that possibly occur in the standard Robinson-Wilczek method when this scalar field is nonminimally coupled to gravity.

We consider, for simplicity, a static, spherically symmetric, four-dimensional spacetime

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2, \tag{6.1}$$

where $f(r)$ is a function which admits at least one event horizon. The event horizon is located at $r = r_H$ where $f(r_H) = 0$ and the surface gravity is $\kappa \equiv \frac{1}{2}(\partial_r f)|_{r_H}$. We also consider an interacting scalar field $\phi(x)$, which is nonminimally coupled to the black hole background (6.1). The action of this scalar field is

$$S[\phi] = \frac{1}{2}\int d^4x\sqrt{-g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \sum_{n=2}^{\infty}\lambda_n\phi^n - \alpha R\phi^2\right], \tag{6.2}$$

where λ_n are a set of arbitrary coupling constants (for example, $\lambda_2 \equiv m^2$ gives the mass), R is the Ricci scalar, and α is a coupling constant to gravity. In particular,

$$\alpha = \begin{cases} 0 & \text{for minimally coupled } \phi(x) \\ \frac{D-2}{4(D-1)} & \text{for conformally coupled } \phi(x), \end{cases} \tag{6.3}$$

for a D-dimensional spacetime. For the spacetime (6.1) it is $D = 4$ and $\alpha = 1/6$, if $\phi(x)$ is conformally coupled to the black hole background. We can write the action (6.2) as the sum of three different terms, each having a different physical meaning,

$$S = S_{\text{free}} + S_{\text{int}} + S_c, \tag{6.4}$$

where the first term is

$$S_{\text{free}} = \frac{1}{2}\int d^4x\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = -\frac{1}{2}\int d^4x\sqrt{-g}\phi\nabla^2\phi, \tag{6.5}$$

and ∇^2 is the Laplace-Beltrami operator. S_{free} is the free part of the action. The second term is

$$S_{\text{int}} = -\frac{1}{2}\int d^4x\sqrt{-g}\sum_{n=2}^{\infty}\lambda_n\phi^n, \tag{6.6}$$

and describes the interactions of the scalar field. The third term is

$$S_c = -\frac{1}{2}\int d^4x\sqrt{-g}\alpha R\phi^2, \tag{6.7}$$

and it is the part of the action, S, which describes the nonminimal coupling of the scalar field to the black hole background (6.1). We will mainly focus our attention on S_c. The Ricci scalar is

$$R = -f''(r) - \frac{4f'(r)}{r} - \frac{2f(r)}{r^2} + \frac{2}{r^2}, \tag{6.8}$$

where a prime denotes differentiation with respect to r. The partial wave decomposition of the scalar field in terms of the spherical harmonics is

$$\phi(x) = \sum_{l,m}\frac{u_{lm}(t,r)}{r}Y_l^m(\theta,\varphi), \tag{6.9}$$

and substituting to the action (6.7), we find, after performing the integrations on θ, φ with the help of the normalization and orthogonality conditions of the spherical harmonics, that

$$S_c = -\frac{1}{2}\int dtdr\alpha R\sum_{l_1,l_2}\sum_{m_1,m2}u_{l_1m_1}u_{l_2m_2}\delta_{l_1l_2}\delta_{m_1m_2}. \tag{6.10}$$

Using the expression (6.8) for R, we get

$$S_c = \frac{1}{2}\int dtdr\alpha\left[f''(r) + \frac{4f'(r)}{r} + \frac{2f(r)}{r^2} - \frac{2}{r^2}\right] \times\sum_{l_1,l_2}\sum_{m_1,m2}u_{l_1m_1}u_{l_2m_2}{}^{(2)}C^{\{m_1,m2\}}_{\{l_1,l_2\}}, \tag{6.11}$$

where ${}^{(2)}C^{\{m_1,m2\}}_{\{l_1,l_2\}} \equiv \delta_{l_1l_2}\delta_{m_1m_2}$. A transformation to tortoise coordinates (t, r_*), defined by Eq. (2.15), transforms S_c to

$$S_{c*} = \frac{1}{2}\int dtdr_*\left\{f(r(r_*))\alpha\left[f''(r(r_*)) + \frac{4f'(r(r_*))}{r(r_*)} + \frac{2f(r(r_*))}{r^2(r_*)} - \frac{2}{r^2(r_*)}\right] \times\sum_{l_1,l_2}\sum_{m_1,m2}u_{l_1m_1}u_{l_2m_2}{}^{(2)}C^{\{m_1,m2\}}_{\{l_1,l_2\}}\right\}, \tag{6.12}$$

where now r, $f(r)$, $u_{l_1m_1}$, $u_{l_2m_2}$ are thought as implicit functions of r_* and the prime still denotes differentiation with respect to r. In the region near the event horizon we have proved that

$$r(r_*) \approx Ae^{2\kappa r_*} + r_H \tag{6.13}$$

and

$$f(r(r_*)) \approx 2\kappa Ae^{2\kappa r_*}. \tag{6.14}$$

Hence, the limit $r \to r_H$ is equivalent to $r_* \to -\infty$ in tortoise coordinates and $f(r(r_*))$ is a suppression factor near the horizon. Now, we examine how each term of S_{c*} behaves in the vicinity of the horizon, using Eqs. (6.13) and (6.14), in order to find which terms are dominant. We easily see that in this region

$$f(r(r_*))\frac{2f(r(r_*))}{r^2(r_*)} \to 0, \tag{6.15}$$

$$-f(r(r_*))\frac{2}{r^2(r_*)}\to 0. \tag{6.16}$$

The other two terms need special attention, so for the region near the event horizon we write

$$f'(r(r_*)) = \frac{\partial f(r(r_*))}{\partial r} = \frac{\partial f(r(r_*))}{\partial r_*}\frac{\partial r_*}{\partial r} = \frac{\partial f(r(r_*))}{\partial r_*}\frac{1}{f(r(r_*))} = \frac{\partial}{\partial r_*}[\ln f(r(r_*))] \approx 2\kappa. \tag{6.17}$$

Thus, we find for $r_* \to -\infty$

$$f(r(r_*))\frac{4f'(r(r_*))}{r((r_*))}\to 0. \tag{6.18}$$

Similarly, we write

$$\begin{aligned} f(r(r_*))\frac{\partial^2 f(r(r_*))}{\partial r^2} &= f(r(r_*))\frac{\partial}{\partial r}\left[\frac{\partial}{\partial r_*}\ln f(r(r_*))\right] \\ &= f(r(r_*))\frac{\partial}{\partial r_*}\left[\frac{\partial}{\partial r_*}\ln f(r(r_*))\right]\frac{\partial r_*}{\partial r} \\ &= f(r(r_*))\frac{\partial}{\partial r_*}\left[\frac{\partial}{\partial r_*}\ln f(r(r_*))\right]\frac{1}{f(r(r_*))} \\ &= \frac{\partial}{\partial r_*}\left[\frac{\partial}{\partial r_*}\ln f(r(r_*))\right]. \end{aligned} \tag{6.19}$$

Hence, we get

$$f(r(r_*))f''(r(r_*))\to 0. \tag{6.20}$$

From Eqs. (6.15), (6.16), (6.18), and (6.20) the action (6.12) in the region near the event horizon becomes $S_{c*} = 0$ and therefore

$$S_c = 0. \tag{6.21}$$

Regarding the part $S_{\rm int}$ of the total action, which describes the interactions of the scalar field $\phi(x)$, after performing a partial wave decomposition of $\phi(x)$ in terms of the spherical harmonics and upon transforming to the tortoise coordinates, one finds [8,9,11] that it vanishes exponentially fast near the event horizon

$$S_{\rm int} = 0, \tag{6.22}$$

due to the presence of the suppression factor $f(r(r_*))$. Concerning the free part $S_{\rm free}$ of the total action, after performing a partial wave decomposition of $\phi(x)$ of the form of (6.9), transforming to the tortoise coordinates, and keeping only dominant terms [8,9,11], we find in the region near the event horizon

$$S_{\rm free} = -\frac{1}{2}\sum_{l,m}\int dt dr u_{lm}\left[-\frac{1}{f}\partial_t^2 + \partial_r(f\partial_r)\right]u_{lm}. \tag{6.23}$$

Adding Eqs. (6.21), (6.22), and (6.23) we get the total action for the region near the event horizon,

$$S = \sum_{l,m} -\frac{1}{2}\int dt dr u_{lm}\left[-\frac{1}{f}\partial_t^2 + \partial_r(f\partial_r)\right]u_{lm}. \tag{6.24}$$

Thus, physics near the horizon can be described using an infinite set of (1 + 1)-dimensional massless scalar fields, each propagating in a (1 + 1)-dimensional background with a metric

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2. \tag{6.25}$$

In conclusion, the nonminimal coupling of the scalar field to the gravitational background does not introduce any special modification to the reduction procedure, since the part of the action S_c, which describes this nonminimal coupling, vanishes in the region near the event horizon. Then, the standard Robinson-Wilczek method proceeds in exactly the same way as in the case of a minimally coupled scalar field. Of course, the preceding analysis can be generalized for D-dimensional spacetimes ($D > 4$), which have a metric of the type

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_{D-2}^2, \tag{6.26}$$

with the difference that we must expand the scalar field in terms of the $(D-2)$-dimensional spherical harmonics and integrate over a $(D-2)$-dimensional sphere.

We should clarify one point here. The wave equation of a scalar field minimally coupled or nonminimally coupled to gravity in higher than two dimensions will develop a potential which, away from the horizon, will modify Hawking radiation. Therefore, the actual Hawking radiation observed at infinity is calculated through the greybody factors. However, in the Robinson-Wilczek method the thermal Hawking flux results from the infinite (1 + 1)-dimensional fields which act as the thermal source of this flux.

VII. SUMMARY

We studied the method of calculating Hawking radiation via gravitational anomalies in gravitational backgrounds of constant negative curvature. At first we discussed the mode analysis of the scalar wave equation in the background of a topological black hole. In the case of (3 + 1)-dimensional topological black holes of genus $\tilde{g} = 2$, we performed the dimensional reduction procedure to two dimensions and we showed that near the horizon the matter scalar field is reduced to an infinite collection of (1 + 1)-dimensional free massless scalar fields. To calculate Hawking radiation from the topological black holes of genus $\tilde{g} = 2$, we followed the covariant anomalies approach proposed in [26–28], which we modified in order to include asymptotically nonflat spacetimes, because it is conceptually simpler and technical problems connected with a complicated horizon structure of the topological black holes of genus $\tilde{g} = 2$ can be avoided.

We also applied this method to a (3 + 1)-dimensional topological black hole of genus $\tilde{g} = 2$ conformally coupled to a scalar field, and we retrieved the correct Hawking flux and temperature. These solutions are interesting because they are examples of a scalar field back-reacting on the geometry. Because the scalar field is static it cannot give an extra contribution to the Hawking flux. However, there exist solutions of BTZ-type black holes coupled to time-dependent scalar fields [31]. These solutions are not analytical so it is not clear how the Robinson-Wilczek method can be applied to these backgrounds.

It is interesting to investigate if the Robinson-Wilczek method can be applied to general backgrounds where the scalar field responsible for the Hawking flux backreacts on the geometry. In this direction, we addressed the problem of using, in the method of gravitational anomalies, a scalar field nonminimally coupled to the gravitational background, instead of a minimally coupled scalar field as is customary. We proved explicitly that the nonminimal coupling does not affect the dimensional reduction procedure or the method in general, since the part of the action which describes the nonminimal coupling vanishes in the region near the event horizon. It is also interesting to examine the applicability of the gravitational anomaly method in fully dynamical backgrounds, but one first has to tackle more fundamental problems like how one can apply the technique of dimensional reduction to time-dependent backgrounds and how one can uniquely define the surface gravity for time-dependent horizons (for a recent discussion on dynamical black holes, see [32]).

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APPENDIX A: THE FUNCTIONS $\mathcal{Y}_\xi^m$

The functions $\mathcal{P}_l^{mn}(z)$ [17,19] form the canonical basis for the irreducible representations of the group SL(2, C) and can be viewed as playing the same role for the group SU(1, 1). They are defined in the complex z plane with a cut located on the real axis between -1 and $+1$. A convenient representation, which can serve as a definition for the functions $\mathcal{P}_l^{mn}(z)$, is

$$\mathcal{P}_l^{mn}(\cosh\theta) = \frac{1}{2\pi i}\int_C dz\left(\cosh\frac{\theta}{2} - z\sinh\frac{\theta}{2}\right)^{l+n} \times\left(\sinh\frac{\theta}{2} + z\cosh\frac{\theta}{2}\right)^{l-n} z^{m-l-1}, \tag{A1}$$

where C is the unit circle prescribed positively, m and n are integers, and l can be complex (typically of the form $l = -\frac{1}{2} \pm i\xi$, $\xi > 0$). The generating function of the $\mathcal{P}_l^{mn}$ is

$$\sum_{m=-\infty}^{\infty} \mathcal{P}_l^{mn}(\cosh\theta)e^{-im\varphi} = e^{-in\varphi}\left(\cosh\frac{\theta}{2} + e^{i\varphi}\sinh\frac{\theta}{2}\right)^{l+n} \times\left(\cosh\frac{\theta}{2} + e^{-i\varphi}\sinh\frac{\theta}{2}\right)^{l-n}. \tag{A2}$$

In the case of $n = 0$, we have

$$\sum_{m=-\infty}^{\infty} \mathcal{P}^{m0}_{-(1/2)\pm i\xi}(\cosh\theta)e^{-im\varphi} = (\cosh\theta + \sinh\theta\cos\varphi)^{-(1/2)\pm i\xi}. \tag{A3}$$

The functions $\mathcal{P}_l^{mn}$ have the following properties:

$$\mathcal{P}_l^{mn}(\cosh\theta) = \mathcal{P}_l^{-m,-n}(\cosh\theta), \tag{A4}$$

$$\mathcal{P}_l^{mn}(\cosh\theta) = (-1)^{m-n}\mathcal{P}^{nm}_{-l-1}(\cosh\theta), \tag{A5}$$

$$[\mathcal{P}_l^{mn}(\cosh\theta)]^* = \mathcal{P}_{l^*}^{mn}(\cosh\theta). \tag{A6}$$

The functions $\mathcal{P}_l^{m0}$ are related to the associated Legendre functions P_l^m through

$$\mathcal{P}_l^{m0}(\cosh\theta) = \frac{\Gamma(l+1)}{\Gamma(l+m+1)}P_l^m(\cosh\theta), \tag{A7}$$

and for $l = -\frac{1}{2} + i\xi$, this is

$$\mathcal{P}^{m0}_{-(1/2)+i\xi}(\cosh\theta) = \frac{\Gamma(i\xi + \frac{1}{2})}{\Gamma(i\xi + m + \frac{1}{2})}P^m_{-(1/2)+i\xi}(\cosh\theta). \tag{A8}$$

The functions $\mathcal{P}^{m0}_{-(1/2)+i\xi}$ form a complete set of functions on the pseudosphere and satisfy the orthogonality relation

$$\int_0^\infty d\theta\sinh\theta\,\mathcal{P}^{m0}_{-(1/2)+i\xi}(\cosh\theta)(\mathcal{P}^{m0}_{-(1/2)+i\xi'}(\cosh\theta))^* = \frac{1}{4\pi^2}\xi\tanh(\pi\xi)\delta(\xi - \xi'), \tag{A9}$$

where ξ, $\xi' \geq 0$. When ξ, ξ' take discrete real values, the delta function $\delta(\xi - \xi')$ becomes the Kronecker delta $\delta_{\xi\xi'}$. We also note that the associated Legendre functions satisfy [17,19] the equation

$$\Delta_\Omega[P_l^m(\cosh\theta)e^{im\varphi}] = l(l+1)P_l^m(\cosh\theta)e^{im\varphi}, \tag{A10}$$

or equivalently

$$\Delta_\Omega[P^m_{-(1/2)+i\xi}(\cosh\theta)e^{im\varphi}] = -(\xi^2 + \tfrac{1}{4})P^m_{-(1/2)+i\xi}(\cosh\theta)e^{im\varphi}, \tag{A11}$$

where Δ_Ω is the differential operator

$$\Delta_\Omega = \frac{1}{\sinh\theta}\partial_\theta(\sinh\theta\partial_\theta) + \frac{1}{\sinh^2\theta}\partial^2_\varphi. \tag{A12}$$

We define the functions $\mathcal{Y}_\xi^m$ as

$$\mathcal{Y}_\xi^m(\theta,\varphi) \equiv \left(\frac{2\pi}{\xi\tanh(\pi\xi)}\right)^{1/2}\mathcal{P}^{m0}_{-(1/2)+i\xi}(\cosh\theta)e^{im\varphi} = \left(\frac{2\pi}{\xi\tanh(\pi\xi)}\right)^{1/2} \times \frac{\Gamma(i\xi+\frac{1}{2})}{\Gamma(i\xi+m+\frac{1}{2})}P^m_{-(1/2)+i\xi}(\cosh\theta)e^{im\varphi}. \quad \text{(A13)}$$

From this definition and Eq. (A11) we see that

$$\Delta_\Omega \mathcal{Y}_\xi^m(\theta,\varphi) = -(\xi^2+\tfrac{1}{4})\mathcal{Y}_\xi^m(\theta,\varphi). \quad \text{(A14)}$$

The functions $\mathcal{Y}_\xi^m$ form a complete set of functions on the pseudosphere H^2. For the two-dimensional manifold $\Sigma = H^2/\Gamma$, which is a compact manifold of genus $\tilde{g}=2$, they form a complete set of functions, ξ takes discrete real values and they must satisfy four periodicity conditions, since the functions $\mathcal{P}_l^{m0}(\cosh\theta)$ satisfy four periodicity conditions [17], due to the compactness of Σ. The orthogonality condition of the $\mathcal{Y}_\xi^m$ is found from Eq. (A9) to be

$$\int_0^\infty d\theta\int_0^{2\pi}d\varphi\sinh\theta\mathcal{Y}_\xi^m(\theta,\varphi)(\mathcal{Y}_{\xi'}^{m'}(\theta,\varphi))^* = \delta_{\xi\xi'}\delta_{mm'}. \quad \text{(A15)}$$

APPENDIX B: DIMENSIONAL REDUCTION FOR THE SCALAR HAIR OF THE MTZ BLACK HOLE IN THE CONFORMAL FRAME

We are going to perform the dimensional reduction procedure for the action (5.9). We consider the region near the event horizon r_+, which is the only event horizon for non-negative masses and the outermost event horizon for negative masses. In this region, as we have previously seen, we have

$$r(r_*) \approx Ae^{2\kappa r_*} + r_+ \quad \text{(B1)}$$

and

$$f(r(r_*)) \approx 2\kappa Ae^{2\kappa r_*}. \quad \text{(B2)}$$

Hence, the limit $r\to r_+$ is equivalent to $r_*\to-\infty$ in tortoise coordinates and $f(r(r_*))$ is a suppression factor near the horizon. After transforming to tortoise coordinates, the action (5.9) takes the form

$$I[\hat{g}_{\mu\nu},\Psi]_* = \int dtdr_*d\Omega_\Sigma r^2(r_*)f(r(r_*))\left[\frac{\hat{R}(r(r_*))+6l^{-2}}{16\pi G} - \frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\Psi(r(r_*))\partial_\nu\Psi(r(r_*)) - \frac{1}{12}\hat{R}(r(r_*))\Psi^2(r(r_*)) - \frac{2\pi G}{3l^2}\Psi^4(r(r_*))\right], \quad \text{(B3)}$$

where $d\Omega_\Sigma = \sinh\theta d\theta d\varphi$. Now, we examine the behavior of each term of this action, using Eqs. (B1) and (B2). The scalar field in tortoise coordinates near the event horizon is

$$\Psi(r(r_*)) = \sqrt{\frac{3}{4\pi G}}\frac{G\mu}{r((r_*))+G\mu} \to \sqrt{\frac{3}{4\pi G}}\frac{G\mu}{r_++G\mu}, \quad \text{(B4)}$$

and for the terms of the action (B3) in the region near the event horizon, we get

$$\int dr_*r^2(r_*)f(r(r_*))\left(-\frac{2\pi G}{3l^2}\Psi^4(r(r_*))\right)\to 0, \quad \text{(B5)}$$

$$\int dr_*r^2(r_*)f(r(r_*))\frac{6l^{-2}}{16\pi G}\to 0. \quad \text{(B6)}$$

Substituting $\hat{R}$ from Eq. (5.18) to the third term of the action (B3), we get

$$\int dr_*f(r(r_*))l^{-2}\Psi^2(r(r_*))\to 0. \quad \text{(B7)}$$

Similarly, we find

$$\int dr_*r^2(r_*)f(r(r_*))\frac{\hat{R}(r(r_*))}{16\pi G}\to 0. \quad \text{(B8)}$$

The remaining term of the action to be examined is

$$-\frac{1}{2} \times \int dtdr_*d\Omega_\Sigma r^2(r_*)f(r(r_*))\hat{g}^{\mu\nu}\partial_\mu\Psi(r(r_*))\partial_\nu\Psi(r(r_*)), \quad \text{(B9)}$$

and originates from the part of the action (5.9), which is

$$-\frac{1}{2}\int d^4x\sqrt{-\hat{g}}\hat{g}^{\mu\nu}\partial_\mu\Psi(r)\partial_\nu\Psi(r) = -\frac{1}{2}\int d^4x\sqrt{-\hat{g}}\hat{g}^{rr}\partial_r\Psi(r)\partial_r\Psi(r) = -\frac{1}{2}\int d^4x\sqrt{-\hat{g}}f(r)\frac{3G\mu^2}{4\pi}\frac{1}{(r+G\mu)^4}. \quad \text{(B10)}$$

Therefore, in tortoise coordinates and always near the horizon, from the last equation, we get

$$-\frac{3G\mu^2}{8\pi}\int dtdr_*d\Omega_\Sigma r^2(r_*)f^2(r(r_*))\frac{1}{(r(r_*)+G\mu)^4}\to 0, \quad \text{(B11)}$$

that is,

$$-\frac{1}{2}\int dtdr_*d\Omega_\Sigma r^2(r_*)f(r(r_*))\hat{g}^{\mu\nu}\partial_\mu \times \Psi(r(r_*))\partial_\nu\Psi(r(r_*))\to 0. \quad \text{(B12)}$$

Adding the expressions (B5)–(B8) and (B12), we find that the action (B3) in tortoise coordinates and in the region near the event horizon vanishes. Thus, in the vicinity of the event horizon the action of the conformally coupled scalar field of the MTZ black hole is

$$I[\hat{g}_{\mu\nu}, \Psi] = 0. \tag{B13}$$

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