Next-to-leading term of the renormalized stress-energy tensor of the quantized massive scalar field in Schwarzschild spacetime. The back reaction

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The next-to-leading term of the renormalized stress-energy tensor of the quantized massive field with an arbitrary curvature coupling in the spacetime of the Schwarzschild black hole is constructed. It is achieved by functional differentiation of the DeWitt-Schwinger effective action involving coincidence limit of the Hadamard-Minakshisundaram-DeWitt-Seely coefficients a_3 and a_4. It is shown, by comparison with the existing numerical results, that inclusion of the second-order term leads to substantial improvement of the approximation of the exact stress-energy tensor even in the closest vicinity of the event horizon. The back reaction of the quantized field upon the Schwarzschild black hole is briefly discussed.

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I. INTRODUCTION

If the Compton length, $\lambda_c = \hbar/mc$, associated with a quantized massive field is much smaller than a characteristic radius of curvature, L, (where the latter means, as usual, any length scale of the background geometry) then the nonlocal contribution to the renormalized effective action, W_R, can be neglected and its series expansion in m^{-2} can be constructed using the DeWitt-Schwinger method. Since in the renormalization prescription one has to absorb the first three terms of the expansion into the classical action of the quadratic gravity with the cosmological term, the lowest nonvanishing term of the W_R is to be constructed from the (integrated) coincidence limit of the fourth Hadamard-Minakshisundaram-DeWitt-Seely coefficient, $[a_3]$, whereas the next-to-leading term is constructed form $[a_4]$. Generally one has

$$W_R = \frac{1}{32\pi^2} \sum_{n=3}^{\infty} \frac{(n-3)!}{(m^2)^{n-2}} \int d^4x \sqrt{g} [a_n]. \qquad (1)$$

For the technical details of this approach the reader is referred, for example, to Refs. [1,2] and the references cited therein.

It is a well known fact that the complexity of $[a_n]$ increases rapidly with n making calculations of the coefficients for $n > 2$ a highly nontrivial task. It is expected therefore that the applicability of the series (1), truncated at some definite n, will be limited to the simplest geometries with symmetries. On the other hand, however, as the coefficients depend on the background geometry, and, possibly, on a ''potential'' term, they can be used to construct the renormalized stress-energy tensor, T_a^b, by functional differentiation of W_R with respect to the metric. Such a tensor can be defined in a wide class of geometries, and, by construction, it gives a unique opportunity to study the back reaction on the metric in a self-consistent way. Of course, the results of such calculations should be interpreted with care as the particle creation, which is a nonlocal process, is ignored.

The coefficient $[a_2]$ has been calculated by DeWitt [3] whereas $[a_3]$ has been obtained by Sakai and Gilkey [4,5]; the fifth coefficient $[a_4]$ has been calculated in Refs. [6–8]. The results for $[a_4]$ are rather hard to compare as there are various simplification strategies that can be employed, and, unfortunately, some of the results contain not only typographical errors. Moreover, a compact or even tricky notation is of little help in situations when the main task is to calculate the stress-energy tensor in a specific spacetime. Therefore, in order to construct the approximation to the renormalized stress-energy tensor we have independently calculated $[a_4]$ for a massive scalar field with an arbitrary curvature coupling satisfying the equation

$$(-\Box + \xi R + m^2)\phi = 0, \qquad (2)$$

where ξ is the parameter describing the curvature coupling and R is the curvature scalar, using the fully covariant method of DeWitt [3] and checked the calculations constructing $[a_4]$ in the Riemann normal coordinates [9]. The thus calculated coefficients have been compared among themselves and with their known values in concrete geometries. For example, when specialized to $n = 4$ the coefficient $[a_4]$ precisely reproduces the coefficient obtained from Dowker's general formula for $[a_n]$ in the de Sitter (dS) spacetime [10]

$$[a_4]^{\rm dS} = -\frac{6}{(4a^2)^4} \sum_{k=0}^{4} \frac{|(2^{2k-1}-1)B_{2k}|}{k!(4-k)!} = -\frac{1}{105a^8}, \qquad (3)$$

where B_{2k} are Bernoulli numbers and a is the radius of the curvature. It is zero in the optical version of the Nariai metric, as expected. Moreover, as an additional partial check, we have also calculated the basic ingredient of the DeWitt method $[\Box^5\sigma]$ in two different ways, where the biscalar $\sigma(x, x')$ is half the square of the geodetic distance between points x and x'. Subsequently, making use of the standard formula

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$$T^{ab} = \frac{2}{\sqrt{g}} \frac{\delta W_R}{\delta g_{ab}}, \tag{4}$$

we have constructed the next-to-leading (i.e., m^{-4}) term of the renormalized stress-energy tensor in a general spacetime. To the best of our knowledge it is the first attempt to go beyond the first order (i.e., m^{-2}) in the calculations of this type.

There are several reasons of carrying out the second-order calculations (besides natural curiosity as this is a practically unexplored region of the quantum field theory in curved background). First, it should be emphasized that although the m^{-2} expansion is used, it does not mean that the second-order term is negligible. Indeed, it may comprise a considerable fraction of the first-order term, leading, as we shall show in this paper, to improvement of the approximation. Further, the higher order terms may dramatically change the type of the solutions of the semiclassical Einstein field equations. An interesting example in this regard is given by the Bertotti-Robinson geometry. It can be shown that although the Bertotti-Robinson geometry is a self-consistent solution of the semiclassical Einstein field equations with the source term given solely by the leading term of the renormalized stress-energy tensor [11] it does not remain so when the next-to-leading term is taken into account. Finally, let us observe that the coincidence limits of $a_n(x, x')$ appear naturally in the formulas for the field fluctuation in both the massive and massless case.

The DeWitt method is easily programmable, and the number of terms that appear at intermediate stages of calculations can be reduced significantly by a carefully chosen simplification strategy. On the other hand, the calculations carried out in the Riemann normal coordinates are extremely fast [12]. The calculations of the coefficient $[a_4]$ and its functional derivatives with respect to the metric tensor have been carried out with the aid of FORM [13] and its multithread version TFORM [14].

The thus obtained approximate stress-energy tensor can be applied in any spacetime provided the temporal changes of the geometry are small and $\lambda_c/L \ll 1$. The effective action approach that we employ in this paper requires the metric to be positively defined. Consequently, the stress-energy tensor can be obtained by analytic continuation of its Euclidean counterpart at the final stage of calculations.

The first-order (i.e., m^{-2}) approximation to the renormalized stress-energy tensor of the massive scalar, spinor, and vector field in the general spacetime has been constructed in Refs. [15,16]. These results generalize the analogous results obtained earlier by Frolov and Zel'nikov [2,17,18] for the vacuum type-D metrics as well as the analytic approximation obtained by Anderson, Hiscock, and Samuel for the massive scalar field in a general static and spherically-symmetric geometries [19]; see also Popov's paper [20]. The Anderson, Hiscock, and Samuel approximation is equivalent to the Schwinger-DeWitt expansion; to obtain the lowest (i.e., m^{-2}) terms, one has to use the sixth-order WKB expansion of the mode functions.

The range of applicability of such a stress-energy tensor is dictated by the limitations of the validity of the renormalized effective action. Numerical calculations reported in Refs. [19,21] confirm that the Schwinger-DeWitt method provides a good approximation of the renormalized stress-energy tensor of the massive scalar field with an arbitrary curvature coupling as long as the mass of the field remains sufficiently large.

The stress-energy tensors constructed in Refs. [15,16,19] have been applied in a number of physically interesting cases, such as various black holes [15,16,21–24], their interiors [25], and wormholes [26]. In this paper, we shall calculate the renormalized stress-energy tensor of the massive scalar field (in a large mass limit) with an arbitrary curvature coupling in the geometry of the Schwarzschild black hole up to m^{-4} terms and explicitly demonstrate that inclusion of the next-to-leading term leads to substantial improvement of the approximation. That means that the second-order term is not negligible and should be included in any serious calculations. We shall also analyze the back reaction problem and briefly study the quantum-corrected Schwarzschild black hole. Throughout the paper a natural system of units is adopted, although in some formulas the constants $\hbar$, c, and G have been, for clarity, restored.

II. THE STRESS-ENERGY TENSOR

Now let us return to Eq. (1) and retain only the first two terms. The approximate stress-energy tensor constructed from the coefficients $[a_3]$ and $[a_4]$ is, therefore, given by

$$T^{ab} = \frac{1}{32\pi^2 m^2} \frac{2}{\sqrt{g}} \frac{\delta}{\delta g_{ab}} \int d^4x \sqrt{g}[a_3] + \frac{1}{32\pi^2 m^4} \frac{2}{\sqrt{g}} \times \frac{\delta}{\delta g_{ab}} \int d^4x \sqrt{g}[a_4] \equiv T^{(1)}_{ab} + T^{(2)}_{ab}. \tag{5}$$

Since the coefficients $[a_3]$ and $[a_4]$ are, respectively, the operators of dimension six and eight constructed from the Riemann tensor, its covariant derivatives up to some prescribed order and contractions, the result of the functional differentiation of the effective action with respect to the metric tensor is rather complicated. Moreover, one expects that any attempt to employ the thus obtained results for a concrete line element would be, computationally, a real challenge [27]. For example, for a general static and spherically-symmetric geometry described by a line element of the form

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{6}$$

the expression describing the next-to-leading term of the stress-energy tensor, when fully expanded, consists of 2582 primitive terms for $T_t^{(2t)}$, 2026 for $T_r^{(2)r}$, and 2066 for $T_\theta^{(2)\theta}$. This can be contrasted with the number of the primitive terms in the tensor $T_a^{(1)b}$:615 for $T_t^{(1)t}$, 463 for $T_r^{(1)r}$, and 634 $T_\theta^{(1)\theta}$. Fortunately, the final result for a simple metric is, as we shall see, quite simple. Here we shall confine ourselves to the simplified case $h(r) = 1/f(r)$, which, nevertheless, covers numerous situations of physical interest.

The stress-energy tensor can be written in the form

$$T_a^{(2)b} = \frac{1}{32\pi^2 m^4}\sum_{i=0}^{4}\mathcal{T}_a^{(i)b}\xi^i \tag{7}$$

where

$$\begin{aligned}
\mathcal{T}_t^{(0)t} ={}& \frac{964f^4}{945r^8} + \frac{f^{(8)}f^3}{630} + \frac{17f^{(7)}f^3}{1080r} - \frac{13f^{(6)}f^3}{7560r^2} - \frac{37f^{(5)}f^3}{1260r^3} + \frac{1387f^{(4)}f^3}{6300r^4} - \frac{2683f^{(3)}f^3}{3150r^5} + \frac{1}{315r^8} + \frac{7321f''f^3}{3150r^6} \\
& - \frac{20\,449f'f^3}{4725r^7} - \frac{f^{(4)^2}f^2}{504} + \frac{1}{252}f^{(3)}f^{(5)}f^2 + \frac{19f''f^{(6)}f^2}{1260} + \frac{17f'f^{(7)}f^2}{1260} + \frac{f^{(3)}f^{(4)}f^2}{70r} + \frac{37f''f^{(5)}f^2}{420r} + \frac{19f'f^{(6)}f^2}{180r} \\
& - \frac{103f^{(3)^2}f^2}{840r^2} - \frac{253f''f^{(4)}f^2}{1260r^2} - \frac{11f'f^{(5)}f^2}{126r^2} + \frac{f^{(6)}f^2}{210r^2} + \frac{929f''f^{(3)}f^2}{6300r^3} - \frac{127f'f^{(4)}f^2}{6300r^3} + \frac{f^{(5)}f^2}{90r^3} + \frac{176f''^2f^2}{315r^4} \\
& + \frac{6473f'f^{(3)}f^2}{6300r^4} - \frac{2f^{(4)}f^2}{15r^4} - \frac{47f'f''f^2}{10r^5} + \frac{22f^{(3)}f^2}{45r^5} + \frac{5159f'^2f^2}{900r^6} - \frac{44f''f^2}{45r^6} + \frac{44f'f^2}{45r^7} - \frac{13f^2}{15r^8} + \frac{1}{210}f''^2f^{(4)}f \\
& - \frac{1}{105}f'f^{(3)}f^{(4)}f + \frac{1}{35}f'f''f^{(5)}f + \frac{1}{42}f'^2f^{(6)}f - \frac{f'^3f^{(3)}}{18r^2} - \frac{13f'f^{(3)^2}f}{840r} + \frac{f''^2f^{(3)}f}{210r} + \frac{127f'f''f^{(4)}f}{1260r} + \frac{19f'^2f^{(5)}f}{140r} \\
& - \frac{11f''^3f}{140r^2} - \frac{f^{(3)^2}f}{360r^2} - \frac{229f'f''f^{(3)}f}{420r^2} - \frac{149f'^2f^{(4)}f}{630r^2} + \frac{f''f^{(4)}f}{180r^2} + \frac{f'f^{(5)}f}{60r^2} + \frac{2573f'f''^2f}{6300r^3} + \frac{577f'^2f^{(3)}f}{2100r^3} + \frac{f'f^{(4)}f}{45r^3} \\
& - \frac{19f''^2f}{90r^4} + \frac{5993f'^2f''f}{6300r^4} - \frac{37f'f^{(3)}f}{90r^4} + \frac{f^{(4)}f}{225r^4} - \frac{1361f'^3f}{700r^5} + \frac{17f'f''f}{10r^5} - \frac{f^{(3)}f}{450r^5} - \frac{103f'^2f}{60r^6} - \frac{113f''f}{450r^6} + \frac{259f'f}{225r^7} \\
& - \frac{148f}{945r^8} - \frac{f''^4}{1680} - \frac{f'^2f^{(3)^2}}{420} + \frac{1}{420}f'f''^2f^{(3)} + \frac{f'^3f^{(5)}}{210} + \frac{f'f''^3}{630r} - \frac{f'^2f''f^{(3)}}{315r} + \frac{f'^3f^{(4)}}{63r} + \frac{f''^3}{945r^2} - \frac{37f'^2f''^2}{1260r^2} \\
& - \frac{f'f''f^{(3)}}{315r^2} + \frac{f'^2f^{(4)}}{210r^2} + \frac{23f'^3f''}{225r^3} + \frac{3f'^4}{175r^4} - \frac{f''^2}{900r^4} - \frac{4f'^2f''}{45r^4} + \frac{f'f^{(3)}}{450r^4} + \frac{4f'^3}{15r^5} - \frac{f'f''}{225r^5} - \frac{26f'^2}{225r^6} + \frac{8f'}{945r^7}
\end{aligned} \tag{8}$$

$$\begin{aligned}
\mathcal{T}_t^{(1)t} ={}& -\frac{100f^4}{21r^8} - \frac{f^{(8)}f^3}{70} - \frac{f^{(7)}f^3}{7r} + \frac{2f^{(6)}f^3}{45r^2} + \frac{71f^{(5)}f^3}{210r^3} - \frac{1709f^{(4)}f^3}{630r^4} + \frac{351f^{(3)}f^3}{35r^5} - \frac{7517f''f^3}{315r^6} + \frac{1586f'f^3}{45r^7} - \frac{26f^3}{3r^8} \\
& + \frac{23f^{(4)^2}f^2}{840} - \frac{3}{140}f^{(3)}f^{(5)}f^2 - \frac{11}{84}f''f^{(6)}f^2 - \frac{17}{140}f'f^{(7)}f^2 + \frac{127f^{(3)}f^{(4)}f^2}{420r} - \frac{697f''f^{(5)}f^2}{1260r} - \frac{563f'f^{(6)}f^2}{630r} \\
& + \frac{77f^{(3)^2}f^2}{45r^2} + \frac{3583f''f^{(4)}f^2}{1260r^2} + \frac{823f'f^{(5)}f^2}{630r^2} - \frac{23f^{(6)}f^2}{315r^2} - \frac{25f''f^{(3)}f^2}{9r^3} - \frac{79f'f^{(4)}f^2}{315r^3} - \frac{f^{(5)}f^2}{6r^3} - \frac{4183f''^2f^2}{630r^4} \\
& - \frac{778f'f^{(3)}f^2}{63r^4} + \frac{35f^{(4)}f^2}{18r^4} + \frac{2029f'f''f^2}{35r^5} - \frac{20f^{(3)}f^2}{3r^5} - \frac{4527f'^2f^2}{70r^6} + \frac{28f''f^2}{3r^6} + \frac{4f'f^2}{r^7} + \frac{496f^2}{45r^8} + \frac{1}{42}f''f^{(3)^2}f \\
& - \frac{11}{420}f''^2f^{(4)}f - \frac{101}{420}f'f''f^{(5)}f - \frac{3}{14}f'^2f^{(6)}f + \frac{137f'f^{(3)^2}f}{252r} + \frac{13f''^2f^{(3)}f}{90r} - \frac{41f'f''f^{(4)}f}{210r} - \frac{317f'^2f^{(5)}f}{315r} \\
& + \frac{979f''^3f}{1260r^2} + \frac{f^{(3)^2}f}{20r^2} + \frac{919f'f''f^{(3)}f}{140r^2} + \frac{283f'^2f^{(4)}f}{90r^2} - \frac{7f''f^{(4)}f}{90r^2} - \frac{23f'f^{(5)}f}{90r^2} - \frac{443f'f''^2f}{63r^3} - \frac{221f'^2f^{(3)}f}{45r^3} \\
& + \frac{5f''f^{(3)}f}{18r^3} - \frac{2f'f^{(4)}f}{9r^3} + \frac{139f''^2f}{45r^4} - \frac{604f'^2f''f}{63r^4} + \frac{539f'f^{(3)}f}{90r^4} - \frac{4f^{(4)}f}{45r^4} + \frac{243f'^3f}{10r^5} - \frac{1133f'f''f}{45r^5} + \frac{f^{(3)}f}{15r^5} \\
& + \frac{1889f'^2f}{90r^6} + \frac{191f''f}{45r^6} - \frac{278f'f}{15r^7} + \frac{766f}{315r^8} + \frac{f''^4}{210} + \frac{f'^2f^{(3)^2}}{40} - \frac{2}{105}f'f''^2f^{(3)} + \frac{1}{210}f'^2f''f^{(4)} - \frac{3f'^3f^{(5)}}{70} - \frac{4f'f''^3}{315r} \\
& + \frac{31f'^2f''f^{(3)}}{315r} - \frac{5f'^3f^{(4)}}{63r} - \frac{11f''^3}{630r^2} + \frac{347f'^2f''^2}{1260r^2} + \frac{787f'^3f^{(3)}}{1260r^2} + \frac{11f'f''f^{(3)}}{210r^2} - \frac{23f'^2f^{(4)}}{315r^2} - \frac{53f'^3f''}{35r^3} + \frac{f'^2f^{(3)}}{18r^3} \\
& + \frac{137f'^4}{1260r^4} + \frac{f''^2}{45r^4} + \frac{6f'^2f''}{5r^4} - \frac{2f'f^{(3)}}{45r^4} - \frac{187f'^3}{45r^5} + \frac{4f'f''}{45r^5} + \frac{88f'^2}{45r^6} - \frac{44f'}{315r^7} - \frac{8}{315r^8} + \frac{13}{105}f'f^{(3)}f^{(4)}f
\end{aligned} \tag{9}$$

$$
\begin{aligned}
\mathcal{T}_t^{(2)t} ={}& -\frac{26f^4}{r^8}+\frac{f^{(8)}f^3}{30}+\frac{f^{(7)}f^3}{3r}-\frac{14f^{(6)}f^3}{45r^2}-\frac{11f^{(5)}f^3}{9r^3}+\frac{487f^{(4)}f^3}{45r^4}-\frac{551f^{(3)}f^3}{15r^5}+\frac{307f''f^3}{5r^6}+\frac{754f^3}{9r^8}-\frac{29f^{(4)^2}f^2}{120}\\
&-\frac{13}{60}f^{(3)}f^{(5)}f^2+\frac{13}{60}f''f^{(6)}f^2+\frac{17}{60}f'f^{(7)}f^2-\frac{202f^{(3)}f^{(4)}f^2}{45r}-\frac{11f''f^{(5)}f^2}{15r}+\frac{149f'f^{(6)}f^2}{90r}-\frac{1447f^{(3)^2}f^2}{180r^2}\\
&-\frac{1223f''f^{(4)}f^2}{90r^2}-\frac{589f'f^{(5)}f^2}{90r^2}+\frac{17f^{(6)}f^2}{45r^2}+\frac{173f''f^{(3)}f^2}{9r^3}+\frac{44f'f^{(4)}f^2}{9r^3}+\frac{37f^{(5)}f^2}{45r^3}+\frac{362f''^2f^2}{15r^4}+\frac{2081f'f^{(3)}f^2}{45r^4}\\
&-\frac{143f^{(4)}f^2}{15r^4}-\frac{706f'f''f^2}{3r^5}+\frac{1328f^{(3)}f^2}{45r^5}+\frac{3301f'^2f^2}{15r^6}-\frac{152f''f^2}{15r^6}-\frac{716f'f^2}{5r^7}-\frac{1864f^2}{45r^8}-\frac{1}{10}f''f^{(3)^2}f\\
&-\frac{1}{20}f''^2f^{(4)}f-f'f^{(3)}f^{(4)}f+\frac{1}{2}f'^2f^{(6)}f-\frac{91f'f^{(3)^2}f}{20r}-\frac{179f''^2f^{(3)}f}{180r}-\frac{229f'f''f^{(4)}f}{45r}+\frac{38f'^2f^{(5)}f}{45r}-\frac{3f''^3f}{4r^2}\\
&-\frac{9f^{(3)^2}f}{20r^2}-\frac{649f'f''f^{(3)}f}{30r^2}-\frac{596f'^2f^{(4)}f}{45r^2}+\frac{19f''f^{(4)}f}{90r^2}+\frac{119f'f^{(5)}f}{90r^2}+\frac{1367f'f''^2f}{30r^3}+\frac{1463f'^2f^{(3)}f}{45r^3}-\frac{40f''f^{(3)}f}{9r^3}\\
&-\frac{2f'f^{(4)}f}{15r^3}-\frac{151f''^2f}{10r^4}+\frac{17f'^2f''f}{r^4}-\frac{437f'f^{(3)}f}{15r^4}+\frac{32f^{(4)}f}{45r^4}-\frac{1493f'^3f}{15r^5}+\frac{656f'f''f}{5r^5}-\frac{1187f'^2f}{15r^6}-\frac{409f''f}{15r^6}\\
&+\frac{1010f'f}{9r^7}-\frac{742f}{45r^8}-\frac{f''^4}{40}-\frac{f'^2f^{(3)^2}}{8}+\frac{1}{10}f'f''^2f^{(3)}-\frac{1}{10}f'^2f''f^{(4)}+\frac{f'^3f^{(5)}}{10}+\frac{f'f''^3}{15r}-\frac{67f'^2f''f^{(3)}}{90r}-\frac{11f'^3f^{(4)}}{45r}\\
&+\frac{11f''^3}{90r^2}-\frac{2f'^2f''^2}{5r^2}-\frac{167f'^3f^{(3)}}{90r^2}-\frac{11f'f''f^{(3)}}{30r^2}+\frac{17f'^2f^{(4)}}{45r^2}+\frac{41f'^3f''}{5r^3}-\frac{8f'^2f^{(3)}}{9r^3}-\frac{37f'^4}{10r^4}-\frac{8f''^2}{45r^4}-\frac{5f'^2f''}{r^4}\\
&+\frac{16f'f^{(3)}}{45r^4}+\frac{212f'^3}{9r^5}+\frac{1}{4}f'f''f^{(5)}f-\frac{32f'f''}{45r^5}-\frac{38f'^2}{3r^6}+\frac{44f'}{45r^7}+\frac{2}{15r^8}-\frac{18f'f^3}{r^7}-\frac{7f^{(3)}f}{9r^5}
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
\mathcal{T}_t^{(3)t} ={}& \frac{932f^4}{9r^8}+\frac{2f^{(6)}f^3}{3r^2}+\frac{4f^{(5)}f^3}{3r^3}-\frac{44f^{(4)}f^3}{3r^4}+\frac{124f^{(3)}f^3}{3r^5}+\frac{12f''f^3}{r^6}-\frac{2632f'f^3}{9r^7}-\frac{1352f^3}{9r^8}+\frac{2f^{(4)^2}f^2}{3}+\frac{1}{3}f''f^{(6)}f^2\\
&+\frac{38f^{(3)}f^{(4)}f^2}{3r}+\frac{20f''f^{(5)}f^2}{3r}+\frac{4f'f^{(6)}f^2}{3r}+\frac{71f^{(3)^2}f^2}{6r^2}+\frac{21f''f^{(4)}f^2}{r^2}+\frac{11f'f^{(5)}f^2}{r^2}-\frac{2f^{(6)}f^2}{3r^2}-\frac{158f''f^{(3)}f^2}{3r^3}\\
&-\frac{56f'f^{(4)}f^2}{3r^3}-\frac{4f^{(5)}f^2}{3r^3}-\frac{82f''^2f^2}{3r^4}-\frac{166f'f^{(3)}f^2}{3r^4}+\frac{52f^{(4)}f^2}{3r^4}+\frac{1100f'f''f^2}{3r^5}-\frac{136f^{(3)}f^2}{3r^5}-\frac{226f'^2f^2}{r^6}-\frac{88f''f^2}{r^6}\\
&+\frac{1744f'f^2}{3r^7}-\frac{32f^2}{3r^8}-\frac{1}{3}f''f^{(3)^2}f+\frac{1}{6}f''^2f^{(4)}f+\frac{8}{3}f'f^{(3)}f^{(4)}f+\frac{7}{6}f'f''f^{(5)}f-\frac{11f''^2f^{(3)}f}{3r}+\frac{16f'f''f^{(4)}f}{r}\\
&+\frac{14f'^2f^{(5)}f}{3r}-\frac{113f''^3f}{9r^2}+\frac{13f^{(3)^2}f}{6r^2}-\frac{28f'f''f^{(3)}f}{3r^2}+\frac{14f'^2f^{(4)}f}{r^2}+\frac{f''f^{(4)}f}{r^2}-\frac{7f'f^{(5)}f}{3r^2}-\frac{124f'f''^2f}{r^3}-\frac{88f'^2f^{(3)}f}{r^3}\\
&+\frac{70f''f^{(3)}f}{3r^3}+\frac{20f'f^{(4)}f}{3r^3}+\frac{80f''^2f}{3r^4}+\frac{176f'^2f''f}{3r^4}+\frac{154f'f^{(3)}f}{3r^4}-\frac{8f^{(4)}f}{3r^4}+\frac{1378f'^3f}{9r^5}-\frac{868f'f''f}{3r^5}+\frac{4f^{(3)}f}{r^5}\\
&+\frac{102f'^2f}{r^6}+\frac{76f''f}{r^6}-\frac{856f'f}{3r^7}+\frac{520f}{9r^8}+\frac{f''^4}{12}+\frac{f'^2f^{(3)^2}}{4}-\frac{1}{3}f'f''^2f^{(3)}+\frac{1}{3}f'^2f''f^{(4)}-\frac{2f'f''^3}{9r}+\frac{2f'^2f''f^{(3)}}{3r}\\
&+\frac{4f'^3f^{(4)}}{3r}-\frac{4f''^3}{9r^2}-\frac{8f'^2f''^2}{3r^2}-\frac{f'^3f^{(3)}}{r^2}+\frac{4f'f''f^{(3)}}{3r^2}-\frac{2f'^2f^{(4)}}{3r^2}-\frac{52f'^3f''}{3r^3}+\frac{14f'^2f^{(3)}}{3r^3}+\frac{169f'^4}{9r^4}+\frac{2f''^2}{3r^4}\\
&+\frac{16f'^2f''}{3r^4}-\frac{4f'f^{(3)}}{3r^4}-\frac{532f'^3}{9r^5}+\frac{8f'f''}{3r^5}+\frac{36f'^2}{r^6}-\frac{32f'}{9r^7}-\frac{4}{9r^8}+f^{(3)}f^{(5)}f^2+\frac{53f'f^{(3)^2}f}{6r}
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
\mathcal{T}_t^{(4)t} = {}& \frac{242f^4}{3r^8} + \frac{704f^3f'}{3r^7} - \frac{72f^3f''}{r^6} - \frac{8f^3f^{(3)}}{r^5} + \frac{4f^3f^{(4)}}{r^4} - \frac{728f^3}{3r^8} + \frac{108f^2f'^2}{r^6} - \frac{244f^2f'f''}{r^5} + \frac{18f^2f'f^{(3)}}{r^4} - \frac{464f^2f'}{r^7} \\
& + \frac{11f^2f''^2}{r^4} + \frac{40f^2f''f^{(3)}}{r^3} + \frac{4f^2f''f^{(4)}}{r^2} + \frac{144f^2f''}{r^6} + \frac{4f^2f^{(3)^2}}{r^2} + \frac{16f^2f^{(3)}}{r^5} - \frac{8f^2f^{(4)}}{r^4} + \frac{244f^2}{r^8} - \frac{272ff'^3}{3r^5} \\
& - \frac{124ff'^2f''}{r^4} + \frac{72ff'^2f^{(3)}}{r^3} + \frac{16ff'^2f^{(4)}}{r^2} - \frac{72ff'^2}{r^6} + \frac{108ff'f''^2}{r^3} + \frac{106ff'f''f^{(3)}}{r^2} + \frac{8ff'f''f^{(4)}}{r} + \frac{248ff'f''}{r^5} \\
& + \frac{8ff'f^{(3)^2}}{r} - \frac{20ff'f^{(3)}}{r^4} - \frac{16ff'f^{(4)}}{r^3} + \frac{224ff'}{r^7} + \frac{100ff''^3}{3r^2} + \frac{22ff''^2f^{(3)}}{r} + ff''^2f^{(4)} + 2ff''f^{(3)^2} - \frac{40ff''f^{(3)}}{r^3} \\
& - \frac{4ff''f^{(4)}}{r^2} - \frac{72ff''}{r^6} - \frac{4ff^{(3)^2}}{r^2} - \frac{8ff^{(3)}}{r^5} + \frac{4ff^{(4)}}{r^4} - \frac{248f}{3r^8} - \frac{80f'^4}{3r^4} + \frac{8f'^3f''}{r^3} + \frac{176f'^3}{3r^5} + \frac{7f'^2f''^2}{r^2} + \frac{4f'^2f''f^{(3)}}{r} \\
& + \frac{4f'^2f''}{r^4} - \frac{8f'^2f^{(3)}}{r^3} - \frac{36f'^2}{r^6} + \frac{f'f''^3}{3r} + \frac{1}{2}f'f''^2f^{(3)} + \frac{16f^2f'f^{(4)}}{r^3} - \frac{2f'f''f^{(3)}}{r^2} - \frac{4f'f''}{r^5} + \frac{2f'f^{(3)}}{r^4} + \frac{16f'}{3r^7} - \frac{f''^4}{8} \\
& + \frac{2f''^3}{3r^2} - \frac{f''^2}{r^4} + \frac{2}{3r^8} + \frac{8f'^3f^{(3)}}{r^2} - \frac{10ff''^2}{r^4}
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
\mathcal{T}_r^{(0)r} = {}& -\frac{386f^4}{945r^8} + \frac{23f^{(7)}f^3}{7560r} + \frac{37f^{(6)}f^3}{1512r^2} - \frac{37f^{(5)}f^3}{1260r^3} - \frac{103f^{(4)}f^3}{1260r^4} + \frac{1121f^{(3)}f^3}{3150r^5} - \frac{3097f''f^3}{3150r^6} + \frac{7823f'f^3}{4725r^7} - \frac{f^{(4)^2}f^2}{2520} \\
& + \frac{f^{(3)}f^{(5)}f^2}{1260} - \frac{f''f^{(6)}f^2}{1260} + \frac{f'f^{(7)}f^2}{1260} + \frac{f^{(3)}f^{(4)}f^2}{630r} + \frac{f''f^{(5)}f^2}{84r} + \frac{29f'f^{(6)}f^2}{1260r} - \frac{f^{(3)^2}f^2}{168r^2} + \frac{23f''f^{(4)}f^2}{252r^2} + \frac{4f'f^{(5)}f^2}{45r^2} \\
& - \frac{183f''f^{(3)}f^2}{700r^3} - \frac{283f'f^{(4)}f^2}{1260r^3} + \frac{f^{(5)}f^2}{90r^3} - \frac{127f''^2f^2}{3150r^4} + \frac{1441f'f^{(3)}f^2}{6300r^4} + \frac{2f^{(4)}f^2}{45r^4} + \frac{3937f'f''f^2}{3150r^5} - \frac{2f^{(3)}f^2}{9r^5} \\
& - \frac{1511f'^2f^2}{900r^6} + \frac{4f''f^2}{9r^6} - \frac{4f'f^2}{9r^7} + \frac{f^2}{3r^8} + \frac{1}{210}f'^2f^{(6)}f - \frac{f'f^{(3)^2}f}{360r} - \frac{f''^2f^{(3)}f}{630r} + \frac{f'f''f^{(4)}f}{84r} + \frac{17f'^2f^{(5)}f}{420r} + \frac{11f''^3f}{1260r^2} \\
& + \frac{f^{(3)^2}f}{840r^2} - \frac{13f'f''f^{(3)}f}{1260r^2} - \frac{f'^2f^{(4)}f}{126r^2} - \frac{f''f^{(4)}f}{420r^2} + \frac{f'f^{(5)}f}{420r^2} - \frac{1291f'f''^2f}{6300r^3} - \frac{169f'^2f^{(3)}f}{1260r^3} + \frac{f'f^{(4)}f}{45r^3} + \frac{f''^2f}{18r^4} \\
& + \frac{1249f'^2f''f}{2100r^4} - \frac{f'f^{(3)}f}{18r^4} - \frac{3289f'^3f}{6300r^5} - \frac{41f'f''f}{90r^5} + \frac{7f^{(3)}f}{450r^5} + \frac{113f'^2f}{180r^6} + \frac{41f''f}{450r^6} - \frac{73f'f}{225r^7} + \frac{68f}{945r^8} - \frac{f''^4}{1680} \\
& - \frac{f'^2f^{(3)^2}}{420} + \frac{1}{420}f'f''^2f^{(3)} + \frac{f'^3f^{(5)}}{210} + \frac{f'f''^3}{630r} - \frac{f'^2f''f^{(3)}}{315r} + \frac{f'^3f^{(4)}}{63r} + \frac{f''^3}{945r^2} - \frac{37f'^2f''^2}{1260r^2} - \frac{f'^3f^{(3)}}{18r^2} - \frac{f'f''f^{(3)}}{315r^2} \\
& + \frac{f'^2f^{(4)}}{210r^2} + \frac{23f'^3f''}{225r^3} + \frac{3f'^4}{175r^4} - \frac{f''^2}{900r^4} - \frac{4f'^2f''}{45r^4} + \frac{f'f^{(3)}}{450r^4} + \frac{4f'^3}{15r^5} - \frac{f'f''}{225r^5} - \frac{26f'^2}{225r^6} + \frac{8f'}{945r^7} + \frac{1}{315r^8}
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
\mathcal{T}_r^{(1)r} = {} & \frac{40f^4}{21r^8} - \frac{f^{(7)}f^3}{35r} - \frac{8f^{(6)}f^3}{35r^2} + \frac{71f^{(5)}f^3}{210r^3} + \frac{191f^{(4)}f^3}{210r^4} - \frac{281f^{(3)}f^3}{63r^5} + \frac{3277f''f^3}{315r^6} - \frac{134f'f^3}{9r^7} + \frac{10f^3}{3r^8} + \frac{f^{(4)^2}f^2}{280} \\
& - \frac{1}{140}f^{(3)}f^{(5)}f^2 + \frac{1}{140}f''f^{(6)}f^2 - \frac{1}{140}f'f^{(7)}f^2 + \frac{41f^{(3)}f^{(4)}f^2}{1260r} - \frac{109f''f^{(5)}f^2}{1260r} - \frac{3f'f^{(6)}f^2}{14r} + \frac{118f^{(3)^2}f^2}{315r^2} \\
& - \frac{59f''f^{(4)}f^2}{140r^2} - \frac{437f'f^{(5)}f^2}{630r^2} + \frac{1033f''f^{(3)}f^2}{315r^3} + \frac{25f'f^{(4)}f^2}{9r^3} - \frac{f^{(5)}f^2}{6r^3} - \frac{199f''^2f^2}{630r^4} - \frac{1234f'f^{(3)}f^2}{315r^4} - \frac{17f^{(4)}f^2}{30r^4} \\
& - \frac{1583f'f''f^2}{105r^5} + \frac{152f^{(3)}f^2}{45r^5} + \frac{4841f'^2f^2}{210r^6} - \frac{224f''f^2}{45r^6} + \frac{44f'f^2}{45r^7} - \frac{188f^2}{45r^8} - \frac{1}{420}f''^2f^{(4)}f + \frac{1}{210}f'f^{(3)}f^{(4)}f \\
& + \frac{1}{420}f'f''f^{(5)}f - \frac{3}{70}f'^2f^{(6)}f + \frac{169f'f^{(3)^2}f}{1260r} - \frac{f''^2f^{(3)}f}{42r} - \frac{11f'f''f^{(4)}f}{630r} - \frac{107f'^2f^{(5)}f}{315r} - \frac{71f''^3f}{1260r^2} - \frac{23f^{(3)^2}f}{1260r^2} \\
& + \frac{187f'f''f^{(3)}f}{252r^2} + \frac{137f'^2f^{(4)}f}{315r^2} + \frac{23f''f^{(4)}f}{630r^2} - \frac{23f'f^{(5)}f}{630r^2} + \frac{647f'f''^2f}{315r^3} + \frac{361f'^2f^{(3)}f}{315r^3} + \frac{f''f^{(3)}f}{18r^3} - \frac{f'f^{(4)}f}{3r^3} \\
& - \frac{23f''^2f}{45r^4} - \frac{2738f'^2f''f}{315r^4} + \frac{107f'f^{(3)}f}{90r^4} + \frac{109f'^3f}{14r^5} + \frac{61f'f''f}{9r^5} - \frac{13f^{(3)}f}{45r^5} - \frac{71f'^2f}{6r^6} - \frac{59f''f}{45r^6} + \frac{262f'f}{45r^7} - \frac{326f}{315r^8} \\
& + \frac{f''^4}{210} + \frac{f'^2f^{(3)^2}}{40} - \frac{2}{105}f'f''^2f^{(3)} + \frac{1}{210}f'^2f''f^{(4)} - \frac{3f'^3f^{(5)}}{70} - \frac{4f'f''^3}{315r} + \frac{31f'^2f''f^{(3)}}{315r} - \frac{5f'^3f^{(4)}}{63r} - \frac{11f''^3}{630r^2} \\
& + \frac{347f'^2f''^2}{1260r^2} + \frac{787f'^3f^{(3)}}{1260r^2} + \frac{11f'f''f^{(3)}}{210r^2} - \frac{23f'^2f^{(4)}}{315r^2} - \frac{53f'^3f''}{35r^3} + \frac{f'^2f^{(3)}}{18r^3} + \frac{137f'^4}{1260r^4} + \frac{f''^2}{45r^4} + \frac{6f'^2f''}{5r^4} - \frac{2f'f^{(3)}}{45r^4} \\
& - \frac{187f'^3}{45r^5} + \frac{4f'f''}{45r^5} + \frac{88f'^2}{45r^6} - \frac{44f'}{315r^7} - \frac{8}{315r^8}
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
\mathcal{T}_r^{(2)r} = {} & \frac{10f^4}{r^8} + \frac{f^{(7)}f^3}{15r} + \frac{8f^{(6)}f^3}{15r^2} - \frac{11f^{(5)}f^3}{9r^3} - \frac{137f^{(4)}f^3}{45r^4} + \frac{281f^{(3)}f^3}{15r^5} - \frac{473f''f^3}{15r^6} + \frac{314f'f^3}{15r^7} - \frac{290f^3}{9r^8} - \frac{f^{(4)2}f^2}{120} \\
& + \frac{1}{60}f^{(3)}f^{(5)}f^2 - \frac{1}{60}f''f^{(6)}f^2 + \frac{1}{60}f'f^{(7)}f^2 - \frac{7f^{(3)}f^{(4)}f^2}{15r} + \frac{f'f^{(6)}f^2}{2r} - \frac{553f^{(3)2}f^2}{180r^2} - \frac{173f''f^{(4)}f^2}{90r^2} + \frac{59f'f^{(5)}f^2}{90r^2} \\
& - \frac{611f''f^{(3)}f^2}{45r^3} - \frac{518f'f^{(4)}f^2}{45r^3} + \frac{37f^{(5)}f^2}{45r^3} + \frac{107f''^2f^2}{15r^4} + \frac{1061f'f^{(3)}f^2}{45r^4} + \frac{101f^{(4)}f^2}{45r^4} + \frac{302f'f''f^2}{5r^5} - \frac{88f^{(3)}f^2}{5r^5} \\
& - \frac{1741f'^2f^2}{15r^6} + \frac{76f''f^2}{5r^6} + \frac{572f'f^2}{15r^7} + \frac{692f^2}{45r^8} + \frac{1}{20}f''^2f^{(4)}f - \frac{1}{10}f'f^{(3)}f^{(4)}f - \frac{1}{20}f'f''f^{(5)}f + \frac{1}{10}f'^2f^{(6)}f \\
& - \frac{211f'f^{(3)2}f}{180r} + \frac{73f''^2f^{(3)}f}{180r} - \frac{34f'f''f^{(4)}f}{45r} + \frac{26f'^2f^{(5)}f}{45r} + \frac{17f''^3f}{60r^2} + \frac{17f^{(3)2}f}{180r^2} - \frac{53f'f''f^{(3)}f}{10r^2} - \frac{161f'^2f^{(4)}f}{45r^2} \\
& - \frac{17f''f^{(4)}f}{90r^2} + \frac{17f'f^{(5)}f}{90r^2} - \frac{109f'f''^2f}{30r^3} - \frac{13f'^2f^{(3)}f}{45r^3} - \frac{8f''f^{(3)}f}{9r^3} + \frac{74f'f^{(4)}f}{45r^3} - \frac{f''^2f}{10r^4} + \frac{703f'^2f''f}{15r^4} - \frac{137f'f^{(3)}f}{15r^4} \\
& - \frac{653f'^3f}{15r^5} - \frac{524f'f''f}{15r^5} + \frac{31f^{(3)}f}{15r^5} + \frac{1231f'^2f}{15r^6} + \frac{101f''f}{15r^6} - \frac{1838f'f}{45r^7} + \frac{302f}{45r^8} - \frac{f''^4}{40} - \frac{f'^2f^{(3)2}}{8} + \frac{1}{10}f'f''^2f^{(3)} \\
& - \frac{1}{10}f'^2f''f^{(4)} + \frac{f'^3f^{(5)}}{10} + \frac{f'f''^3}{15r} - \frac{67f'^2f''f^{(3)}}{90r} - \frac{11f'^3f^{(4)}}{45r} + \frac{11f''^3}{90r^2} - \frac{2f'^2f''^2}{5r^2} - \frac{167f'^3f^{(3)}}{90r^2} - \frac{11f'f''f^{(3)}}{30r^2} \\
& + \frac{17f'^2f^{(4)}}{45r^2} + \frac{41f'^3f''}{5r^3} - \frac{8f'^2f^{(3)}}{9r^3} - \frac{37f'^4}{10r^4} - \frac{8f''^2}{45r^4} - \frac{5f'^2f''}{r^4} + \frac{16f'f^{(3)}}{45r^4} + \frac{212f'^3}{9r^5} - \frac{32f'f''}{45r^5} - \frac{38f'^2}{3r^6} + \frac{44f'}{45r^7} \\
& + \frac{2}{15r^8}
\end{aligned} \tag{15}
$$

$$\begin{aligned}\mathcal{T}_r^{(3)r} = {} & -\frac{364f^4}{9r^8} + \frac{824f^3f'}{9r^7} + \frac{20f^3f''}{r^6} - \frac{28f^3f^{(3)}}{r^5} + \frac{8f^3f^{(4)}}{3r^4} + \frac{4f^3f^{(5)}}{3r^3} + \frac{520f^3}{9r^8} + \frac{258f^2f'^2}{r^6} - \frac{268f^2f'f''}{3r^5} \\ & - \frac{166f^2f'f^{(3)}}{3r^4} + \frac{16f^2f'f^{(4)}}{r^3} + \frac{3f^2f'f^{(5)}}{r^2} - \frac{656f^2f'}{3r^7} - \frac{82f^2f''^2}{3r^4} + \frac{50f^2f''f^{(3)}}{3r^3} + \frac{9f^2f''f^{(4)}}{r^2} + \frac{2f^2f''f^{(5)}}{3r} \\ & - \frac{8f^2f''}{r^6} + \frac{41f^2f^{(3)2}}{6r^2} + \frac{4f^2f^{(3)}f^{(4)}}{3r} + \frac{104f^2f^{(3)}}{3r^5} - \frac{8f^2f^{(4)}}{3r^4} - \frac{4f^2f^{(5)}}{3r^3} + \frac{16f^2}{3r^8} + \frac{1018ff'^3}{9r^5} - \frac{328ff'^2f''}{3r^4} \\ & - \frac{56ff'^2f^{(3)}}{3r^3} + \frac{8ff'^2f^{(4)}}{r^2} + \frac{2ff'^2f^{(5)}}{3r} - \frac{246ff'^2}{r^6} - \frac{20ff'f''^2}{r^3} + \frac{6ff'f''f^{(3)}}{r^2} + \frac{8ff'f''f^{(4)}}{3r} + \frac{1}{6}ff'f''f^{(5)} \\ & + \frac{212ff'f''}{3r^5} + \frac{17ff'f^{(3)2}}{6r} + \frac{1}{3}ff'f^{(3)}f^{(4)} + \frac{30ff'f^{(3)}}{r^4} - \frac{8ff'f^{(4)}}{3r^3} - \frac{ff'f^{(5)}}{3r^2} + \frac{392ff'}{3r^7} - \frac{23ff''^3}{9r^2} - \frac{5ff''^2f^{(3)}}{3r} \\ & - \frac{1}{6}ff''^2f^{(4)} + \frac{32ff''^2}{3r^4} + \frac{14ff''f^{(3)}}{3r^3} + \frac{ff''f^{(4)}}{3r^2} - \frac{12ff''}{r^6} - \frac{ff^{(3)2}}{6r^2} - \frac{20ff^{(3)}}{3r^5} - \frac{200f}{9r^8} + \frac{169f'^4}{9r^4} - \frac{52f'^3f''}{3r^3} \\ & - \frac{f'^3f^{(3)}}{r^2} + \frac{4f'^3f^{(4)}}{3r} - \frac{532f'^3}{9r^5} - \frac{8f'^2f''^2}{3r^2} + \frac{2f'^2f''f^{(3)}}{3r} + \frac{1}{3}f'^2f''f^{(4)} + \frac{16f'^2f''}{3r^4} + \frac{f'^2f^{(3)2}}{4} + \frac{14f'^2f^{(3)}}{3r^3} \\ & - \frac{2f'^2f^{(4)}}{3r^2} + \frac{36f'^2}{r^6} - \frac{2f'f''^3}{9r} - \frac{1}{3}f'f''^2f^{(3)} + \frac{4f'f''f^{(3)}}{3r^2} + \frac{8f'f''}{3r^5} - \frac{4f'f^{(3)}}{3r^4} - \frac{32f'}{9r^7} + \frac{f''^4}{12} - \frac{4f''^3}{9r^2} + \frac{2f''^2}{3r^4} - \frac{4}{9r^8}\end{aligned} \tag{16}$$

$$\begin{aligned}\mathcal{T}_r^{(4)r} = {} & -\frac{94f^4}{3r^8} - \frac{448f^3f'}{3r^7} + \frac{8f^3f^{(3)}}{r^5} + \frac{280f^3}{3r^8} - \frac{228f^2f'^2}{r^6} + \frac{44f^2f'f''}{r^5} + \frac{34f^2f'f^{(3)}}{r^4} + \frac{304f^2f'}{r^7} + \frac{23f^2f''^2}{r^4} \\ & + \frac{8f^2f''f^{(3)}}{r^3} - \frac{16f^2f^{(3)}}{r^5} - \frac{92f^2}{r^8} - \frac{368ff'^3}{3r^5} + \frac{92ff'^2f''}{r^4} + \frac{40ff'^2f^{(3)}}{r^3} + \frac{264ff'^2}{r^6} + \frac{60ff'f''^2}{r^3} + \frac{18ff'f''f^{(3)}}{r^2} \\ & - \frac{40ff'f''}{r^5} - \frac{36ff'f^{(3)}}{r^4} - \frac{160ff'}{r^7} + \frac{22ff''^3}{3r^2} + \frac{2ff''^2f^{(3)}}{r} - \frac{22ff''^2}{r^4} - \frac{8ff''f^{(3)}}{r^3} + \frac{8ff^{(3)}}{r^5} + \frac{88f}{3r^8} - \frac{80f'^4}{3r^4} \\ & + \frac{8f'^3f''}{r^3} + \frac{8f'^3f^{(3)}}{r^2} + \frac{176f'^3}{3r^5} + \frac{7f'^2f''^2}{r^2} + \frac{4f'^2f''f^{(3)}}{r} + \frac{4f'^2f''}{r^4} - \frac{8f'^2f^{(3)}}{r^3} - \frac{36f'^2}{r^6} + \frac{f'f''^3}{3r} + \frac{1}{2}f'f''^2f^{(3)} \\ & - \frac{2f'f''f^{(3)}}{r^2} - \frac{4f'f''}{r^5} + \frac{2f'f^{(3)}}{r^4} + \frac{16f'}{3r^7} - \frac{f''^4}{8} + \frac{2f''^3}{3r^2} - \frac{f''^2}{r^4} + \frac{2}{3r^8},\end{aligned} \tag{17}$$

and $f'(r)$, $f''(r)$, $f^{(i)}(r)$ denote the first, second, and ith derivative, respectively. To avoid proliferation of extremely long formulas we display only the time and radial components of the stress-energy tensor as the angular component can be calculated from $\nabla_b T_a^b = 0$, which in the case on hand gives

$$T_\theta^{(2)\theta} = T_\phi^{(2)\phi} = -\frac{1}{4}\frac{f'}{f}(T_t^{(2)t} - T_r^{(2)r}) + \frac{1}{2}T_r^{(2)r\prime}r + T_r^{(2)r}. \tag{18}$$

Making use of the first-order approximation of the stress-energy tensor in the Schwarzschild geometry, one easily obtains [17]

$$T_t^{(1)t} = \frac{M^2}{32\pi^2m^2r^8}\left[\left(16 - \frac{176M}{5r}\right)\eta - \frac{19}{21} + \frac{626M}{315r}\right], \tag{19}$$

$$T_r^{(1)r} = \frac{M^2}{32\pi^2m^2r^8}\left[\left(\frac{48M}{5r} - \frac{32}{5}\right)\eta - \frac{22M}{45r} + \frac{1}{3}\right], \tag{20}$$

and

$$\begin{aligned}T_\theta^{(1)\theta} &= T_\phi^{(1)\phi} \\ &= \frac{M^2}{32\pi^2m^2r^8}\left[\left(-\frac{224M}{5r} + \frac{96}{5}\right)\eta + \frac{734M}{315r} - 1\right],\end{aligned} \tag{21}$$

where $\eta = \xi - 1/6$.

Now, let us consider the second term of the Eq. (5). The second-order calculations are, of course, more involved. Fortunately, there are massive simplifications for the Ricci-flat geometry and the final result in the Schwarzschild geometry is quite simple. Tedious but routine calculations give

$$T_t^{(2)t} = \frac{M^2}{32\pi^2 m^4 r^{10}}\left[\left(144 - \frac{752M}{r} + \frac{6596M^2}{7r^2}\right)\eta - \frac{44}{5} + \frac{22\,664M}{525r} - \frac{27\,166M^2}{525r^2}\right], \tag{22}$$

$$T_r^{(2)r} = \frac{M^2}{32\pi^2 m^4 r^{10}}\left[\left(-\frac{288}{7} - \frac{1164M^2}{7r^2} + \frac{1208M}{7r}\right)\eta + \frac{12}{5} - \frac{776M}{75r} + \frac{5506M^2}{525r^2}\right], \tag{23}$$

and

$$T_\theta^{(2)\theta} = T_\phi^{(2)\phi} = \frac{M^2}{32\pi^2 m^4 r^{10}}\left[\left(\frac{7760M^2}{7r^2} - \frac{6084M}{7r} + \frac{1152}{7}\right)\eta + \frac{1304M}{25r} - \frac{35\,698M^2}{525r^2} - \frac{48}{5}\right]. \tag{24}$$

The constructed tensor is covariantly conserved, regular and it can easily by checked that at the event horizon one has $T_t^{(2)t} = T_r^{(2)r}$. Moreover, it should be noted that although the general expression describing $[a_4]$ involves the terms up to ξ^5, the final result is linear in ξ. Although, generally speaking, there are no limitations placed on the parameter ξ, two of its values are particularly appealing, namely, $\eta = 0$ and $\eta = -1/6$ which lead to the conformal and minimal couplings, respectively.

In Figs. 1–3 the run of the (rescaled) components of the stress-energy tensor $T_b^{(2b)}$ as functions of $z = r/M$ for a few exemplary values of the coupling parameter from the range $0 \le \xi \le 1/6$ is displayed. Although there are strong dependence on ξ, some general features are common for all the curves. Indeed, the $T_t^{(2)t}$ is negative at the event horizon and remains so for $z \lesssim 2.1$ and attains a (positive) maximum. Subsequently it decreases when r approaches a (negative) minimum and falls to zero. Inspection of Fig. 2 shows that $T_r^{(2)r}$ is negative at the event horizon, approaches a (positive) maximum, and fall to zero as $r \to \infty$. Finally, the run of the angular component (Fig. 3) is qualitatively similar to that of $T_t^{(2)t}$. The behavior of the stress-energy for more exotic values of the coupling parameter can easily be inferred from the general formulas (22)–(24). Specifically, at the event horizon one has

$$T_t^{(2)t} = T_r^{(2)r} = \frac{1}{44\,800\pi^2 m^4 (2M)^8}(1250\eta - 53) \tag{25}$$

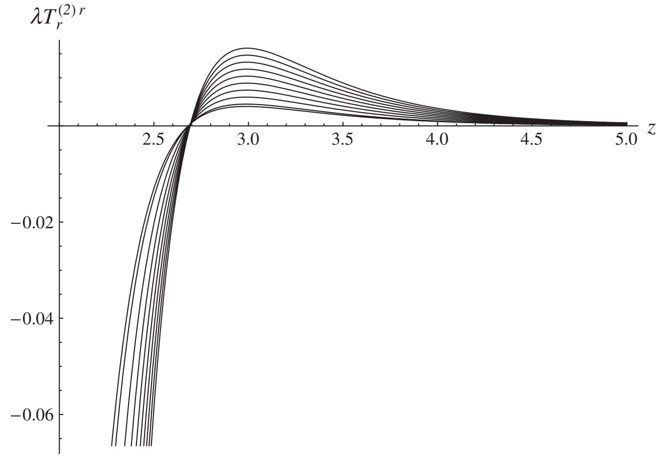

FIG. 2. This graph shows the rescaled $T_r^{(2)r}$ [$\lambda = (8M)^4\pi^2 m^4$] component of the stress-energy tensor of the massive scalar field as a function of $z = r/M$ plotted for a few exemplary values of the coupling parameter ξ. Top to bottom (at the maximum) the curves are plotted for $\xi = 0.2i$ ($i = 0, \ldots, 8$) and for $\xi = 1/6$.

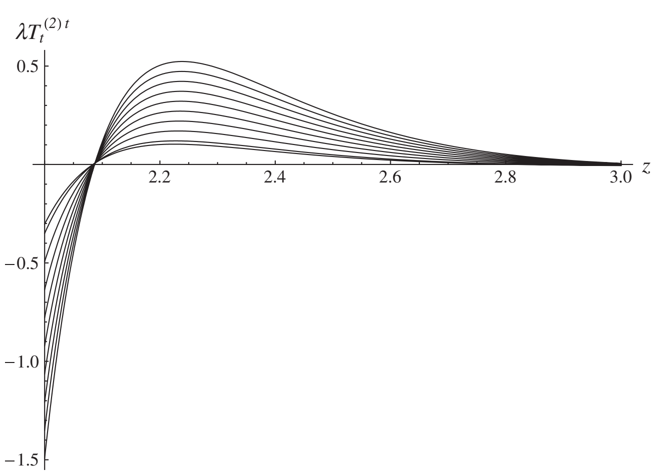

FIG. 1. This graph shows the rescaled $T_t^{(2)t}$ [$\lambda = (8M)^4\pi^2 m^4$] component of the stress-energy tensor of the massive scalar field as a function of $z = r/M$ plotted for a few exemplary values of the coupling parameter ξ. Top to bottom (at the maximum) the curves are plotted for $\xi = 0.2i$ ($i = 0, \ldots, 8$) and for $\xi = 1/6$.

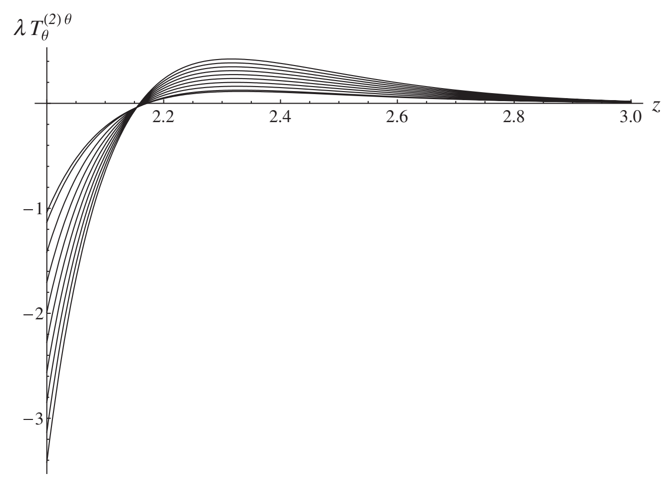

FIG. 3. This graph shows the rescaled $T_\theta^{(2)\theta}$ [$\lambda = (4M)^8\pi^2 m^4$] component of the stress-energy tensor of the massive scalar field as a function of $z = r/M$ plotted for a few exemplary values of the coupling parameter ξ. Top to bottom (at the maximum) the curves are plotted for $\xi = 0.2i$ ($i = 0, \ldots, 8$) and for $\xi = 1/6$.

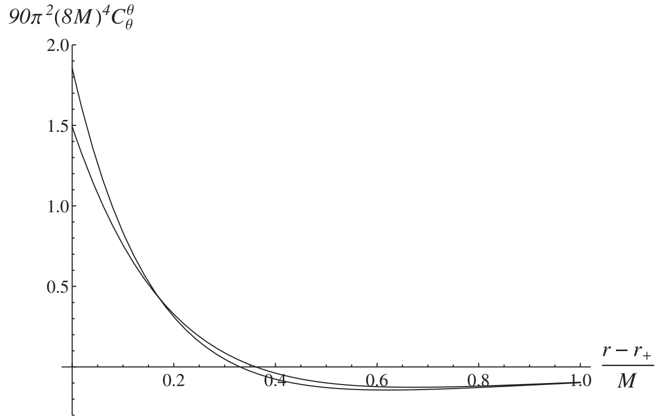

FIG. 4. The curves in this figure display the conformal part of the stress-energy tensor, C_θ^θ, for the massive scalar field with $mM = 2$ in the vicinity of the Schwarzschild black hole. From top to bottom at the event horizon ($r_+ = 2M$) the curves correspond to the first-order approximation and the improved approximation.

and

$$T_\theta^{(2)\theta} = \frac{1}{26\,880\pi^2 m^4 (2M)^8}(1500\eta - 109). \tag{26}$$

It is of interest to compare our approximation with the results of numerical calculations carried out by Anderson, Hiscock, and Samuel and reported in Ref. [19]. They numerically calculated conformal C_θ^θ and nonconformal D_θ^θ contribution to the total stress-energy tensor

$$T_\theta^\theta = C_\theta^\theta + (\xi - \tfrac{1}{6})D_\theta^\theta \tag{27}$$

for $mM = 2$, compared to the thus obtained result with the approximation which is identical with the first-order tensor

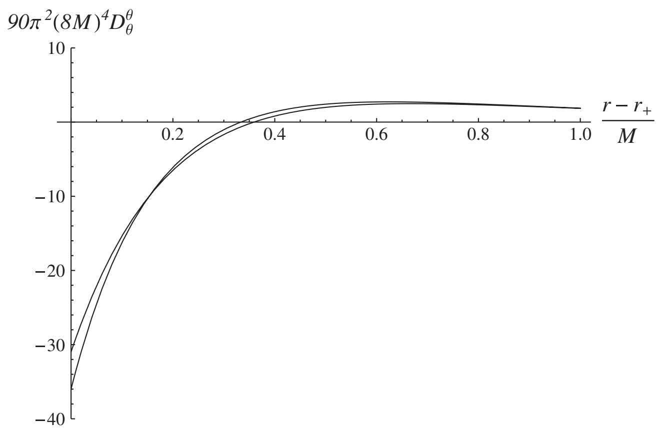

FIG. 5. The curves in this figure display the nonconformal part of the stress-energy tensor, D_θ^θ, for the massive scalar field with $mM = 2$ in the vicinity of the Schwarzschild black hole. From top to bottom at the event horizon ($r_+ = 2M$) the curves correspond to the improved approximation and to the first-order approximation.

(19)–(21) and explicitly demonstrated that the approximation is reasonable. On the other hand, inspection of Figs. 4 and 5 shows that inclusion of the next-to-leading term substantially improves the approximation of the stress-energy tensor even in the closest vicinity of the event horizon. One expects that this approximation is even better for $mM > 2$. A lesson that follows from this demonstration is that the next-to-leading term plays, or at least may play, an important role in the calculations and it can be ignored only after careful examination.

Thus far we have carried out our calculations using the Planck units. It is of some interest to restore the constants c, G, and $\hbar$ in the final expressions describing the renormalized stress-energy tensor. Simple manipulations give

$$T_a^{(i)b} = A_{(i)} \times f_{(i)a}^b(z), \tag{28}$$

where $A_{(1)} = G^2\hbar^3M^2/c^5m^2r^8$, $A_{(2)} = G^2\hbar^5M^2/c^7m^4r^{10}$, and $f_{(i)a}^b(z)$ are dimensionless functions of $z = GM/c^2r$.

Since the Schwinger-DeWitt approximation is local and the geometry at the event horizon is regular, one expects that the stress-energy tensor is also regular there. On the other hand, the stress-energy tensor is regular in the physical sense if it is regular in a coordinate system which is well behaved as $r \to r_+$. For example, the components of the stress-energy tensor T_a^b in a freely falling frame, denoted here as $T_{(0)(0)}$, $T_{(0)(1)}$, and $T_{(1)(1)}$ are

$$T_{(0)(0)} = \frac{\gamma^2(T_r^r - T_t^t)}{f} - T_r^r, \tag{29}$$

$$T_{(1)(1)} = \frac{\gamma^2(T_r^r - T_t^t)}{f} + T_r^r, \tag{30}$$

$$T_{(0)(1)} = -\frac{\gamma\sqrt{\gamma^2 - f}(T_r^r - T_t^t)}{f}, \tag{31}$$

where γ is the energy per unit mass along the geodesic and $f(r) = -g_{tt}(r)$. Inspection of Eqs. (29)–(31) shows that if all components of T_a^b and $(T_r^r - T_t^t)/f$ are finite on the horizon the stress-energy tensor in a freely falling frame is finite as well.

Now, simple calculations show that the difference between radial and time components of the stress-energy factors

$$T_t^{(i)t} - T_r^{(i)r} = \left(1 - \frac{2M}{r}\right)F^{(i)}(r) \tag{32}$$

where

$$F^{(1)}(r) = \frac{M^2}{\pi^2 m^2 r^8}\left(\frac{7}{10}\eta - \frac{13}{336}\right), \tag{33}$$

$$F^{(2)}(r) = \frac{M^2}{\pi^2 m^4 r^{10}}\left[\left(\frac{81}{14} - \frac{485}{28}\frac{M}{r}\right)\eta - \frac{7}{20} + \frac{1021}{1050}\frac{M}{r}\right], \tag{34}$$

$i = 1, 2$, and, consequently, both tensors are regular in a physical sense. Moreover, using our general formula describing the stress-energy tensor it can be shown that it remains so in any static and spherically-symmetric spacetime.

III. THE BACK REACTION PROBLEM

Having constructed the next-to-leading term of the renormalized stress-energy tensor which depends on a general metric one can analyze the back reaction of the quantized field upon the black hole geometry. It should be emphasized once more that accepting the approximation (5) we ignore particle creation which is a nonlocal effect. To simplify our discussion we shall assume that the cosmological constant and the renormalized coupling parameters α and β in the quadratic part of the total action

$$S_q = \int d^4x\sqrt{-g}(\alpha C_{abcd}C^{abcd} + \beta R^2), \tag{35}$$

where C_{abcd} is the Weyl tensor, identically vanish. The semiclassical Einstein field equations have, therefore, a standard form

$$G_a^b[g] = 8\pi\langle T_{ab}[g]\rangle, \tag{36}$$

where $\langle T_{ab}[g]\rangle = \mathcal{O}(\hbar)$ is the renormalized stress-energy tensor.

Since the total stress-energy tensor depends functionally on a wide class of metrics, one can, in principle, construct the self-consistent solution of the system (36). It should be noted however, that since the general stress-energy tensor, $T_a^{(2)b}$, is constructed from $[a_4]$ it contains the terms up to eight derivatives of g_{ab}, and, consequently, there is a real danger that the semiclassical equations may lead to physically unacceptable solutions [28]. Moreover, the tensor $\langle T_a^b\rangle$ is extremely complicated and it is natural that one is forced to refer to some approximations. Here we shall treat the right-hand side of the semiclassical Einstein field equations as perturbation. Restricting to the perturbative solutions of the effective theory may be, therefore, the only one way to obtain the (approximate) physical solutions.

For the quantized massless fields in the Schwarzschild geometry the back reaction program has been initiated by York [29]. Subsequently, it has been applied in numerous papers [30–34], where various aspects of the back reaction of the quantized fields upon the black hole geometry has been studied using the first-order approximation to the stress-energy tensor.

Now, let us introduce the dimensionless parameter ε [35] and make the substitution $\langle T_{ab}[g]\rangle \to \varepsilon\langle T_{ab}[g]\rangle$. Expanding the metric tensor as

$$g_{ab} = g_{ab}^{(0)} + \varepsilon g_{ab}^{(1)} + \mathcal{O}(\varepsilon^2), \tag{37}$$

inserting it into the semiclassical equations (36) and collecting the terms with the like powers of the auxiliary parameter, one obtains

$$G_a^b[g^{(0)}] = 0 \tag{38}$$

and

$$G_a^b[g^{(1)}] = 8\pi(T_a^{(1)b}[g^{(0)}] + T_a^{(2)b}[g^{(0)}]), \tag{39}$$

i.e., the modifications of the geometry caused by the stress-energy tensor calculated in the corrected black hole spacetime, $T_a^{(1)b}[g^{(1)}]$, are ignored as these additional terms would be $\mathcal{O}(\hbar^2)$. In other words we are looking for $\mathcal{O}(\hbar)$ corrections to the classical solution.

From Eqs. (19)–(24) one sees that the solution of the back reaction problem reduces to elementary quadratures. First, let us consider the issue of the integration constants which appear in solutions of the differential equations. The zeroth-order equations will yield two integration constants, say, c_1 and c_2, which can be set to -1 and M, respectively. Here M is a "bare" mass of the black hole and c_1 can be determined from the condition $g_{tt}^{(0)}g_{rr}^{(0)} = -1$. On the other hand, the integration constant C_1 appearing in the component $g_{rr}^{(1)}$ of the metric tensor can be absorbed in a process of the finite renormalization of mass. Indeed, it can be demonstrated that with the substitution

$$M = \mathcal{M} - \tfrac{1}{2}\varepsilon C_1 \tag{40}$$

the radial component of the metric tensor can be written as

$$g_{rr}^{-1}(r) = 1 - \frac{2\mathcal{M}}{r} + \frac{\mathcal{M}^2}{5\pi m^2 r^6}\mathcal{P}_r^{(1)}(r) + \frac{\mathcal{M}^2}{\pi m^4 r^8}\mathcal{P}_r^{(2)}(r) + \mathcal{O}(\varepsilon^2), \tag{41}$$

where $\mathcal{P}_r^{(1)}(r)$ and $\mathcal{P}_r^{(2)}(r)$ are given, respectively, by

$$\mathcal{P}_r^{(1)}(r) = \left(\frac{22\mathcal{M}}{3r} - 4\right)\eta - \frac{313\mathcal{M}}{756r} + \frac{19}{84} \tag{42}$$

and

$$\mathcal{P}_r^{(2)}(r) = -\frac{2833\mathcal{M}}{2100r} + \frac{11}{35} + \frac{13\,583\mathcal{M}^2}{9450r^2} - \left(\frac{36}{7} + \frac{1649\mathcal{M}^2}{63r^2} - \frac{47\mathcal{M}}{2r}\right)\eta. \tag{43}$$

Similarly, for g_{tt} to $\mathcal{O}(\hbar)$ one has

$$g_{tt}(r) = -g_{rr}^{-1}(r)\exp(2\varepsilon\psi(r)), \tag{44}$$

where

$$\psi(r) = \frac{\mathcal{M}^2}{\pi m^2 r^6}\left(\frac{7}{15}\eta - \frac{13}{504}\right) + \frac{\mathcal{M}^2}{\pi m^4 r^8}\left[\frac{2042\mathcal{M}}{4725r} - \frac{7}{40} - \left(\frac{485\mathcal{M}}{63r} - \frac{81}{28}\right)\eta\right] + C_2. \tag{45}$$

The integration constant C_2 can be fixed by demanding that the time component of the metric tensor approaches its Minkowskian value as $r \to \infty$, that is equivalent to normalizing the time coordinate at infinity.

As before, it is of some interest to restore the physical constants. Putting $\bar{M} = G\mathcal{M}/c^2$, $l_{Pl} = (\hbar/Gc^3)^{1/2}$, and $\lambda_c = \hbar/mc$ in Eq. (41) the radial component of the metric tensor can schematically be written as

$$g_{rr}^{-1} = 1 - 2z + \varepsilon \frac{\bar{M}^2 \lambda_c^2 l_{Pl}^2}{r^6} W_1(z;\eta) + \varepsilon \frac{\bar{M}^2 \lambda_c^4 l_{Pl}^2}{r^8} W_2(z;\eta), \tag{46}$$

where W_1 and W_2 are simple polynomials depending parametrically on η and their exact form can easily be inferred from Eq. (46). A similar expression can be constructed for g_{tt}, and the result can be schematically written in the form

$$g_{tt} = -1 + 2z + \varepsilon \frac{\bar{M}^2 \lambda_c^2 l_{Pl}^2}{r^6} V_1(z;\eta) + \varepsilon \frac{\bar{M}^2 \lambda_c^4 l_{Pl}^2}{r^8} V_2(z;\eta), \tag{47}$$

where V_i comprise another pair of simple polynomials.

The location of the event horizon of the quantum-corrected Schwarzschild black hole is determined by the equation $g_{tt}(r_+) = 0$. Putting $r_+ = r_+^{(0)} + \varepsilon r_+^{(1)}$ one concludes that r_+ is given by

$$r_+ = 2\bar{M} + \frac{\lambda_c^2 l_{Pl}^2}{480\pi \bar{M}^3}\left(\eta - \frac{29}{504}\right) + \frac{\lambda_c^4 l_{Pl}^2}{2016\pi \bar{M}^5}\left(\frac{17}{1200} - \eta\right). \tag{48}$$

The Euclidean version of the line element (6) obtained with the aid of the Wick rotation has no conical singularity provided the "time" coordinate is periodic with a period β given by

$$\beta = \lim_{r\to r_+} 4\pi (g_{tt} g_{rr})^{1/2} \left(\frac{d}{dr} g_{tt}\right)^{-1}. \tag{49}$$

For the quantum-corrected metric (41)–(45), the period is

$$\beta = 8\pi\mathcal{M} - \frac{\varepsilon}{30\,240\mathcal{M}^3 m^2} - \frac{11\varepsilon}{151\,200\mathcal{M}^5 m^4} \tag{50}$$

to the first order in ε.

The surface gravity, κ, which is proportional to the temperature of the black hole can be calculated (for the Lorentzian metric) from a simple relation

$$\kappa^2 = -\frac{1}{2} k_{a;b} k^{a;b}{}_{|r=r_+} = -\frac{1}{4} (g_{tt} g_{rr})^{-1} \left(\frac{dg_{tt}}{dr}\right)^2_{|r=r_+}, \tag{51}$$

where k^a is a timelike Killing vector. The Hawking temperature, $T_H = \kappa/2\pi$, is, therefore, given by

$$T_H = \frac{1}{4\pi}(-g_{tt} g_{rr})^{-1/2} \left|\frac{dg_{tt}}{dr}\right|_{|r=r_+} = \frac{1}{\beta} = \frac{1}{8\pi\mathcal{M}} + \frac{\varepsilon}{\pi^2 m^2 (4\mathcal{M})^5}\left(\frac{1}{1980} + \frac{11}{9450\mathcal{M}^2 m^2}\right). \tag{52}$$

It should be noted that the temperature, T_H, when expressed in terms of the total mass of the system as seen by a distant observer, is independent of the coupling constant.

On the other hand, one can express the results (41)–(45) in terms of the horizon defined mass, $M_H = r_+/2$ which, of course, differs from the total mass of the system as seen by a distant observer. It can be achieved, for example, by inverting Eq. (48) and the elementary manipulations give

$$\mathcal{M} = M_H - \frac{\varepsilon}{960\pi m^2 M_H^3}\left(\eta - \frac{29}{504}\right) - \frac{\varepsilon}{4032\pi m^4 M_H^5}\left(\frac{17}{1200} - \eta\right). \tag{53}$$

Consequently, one can systematically substitute Eq. (53) in Eqs. (41)–(45), expand and finally linearize the thus obtained results.

Equally well, one can start with a slightly different representation of the line element putting

$$g_{tt}(r) = -\exp(2\psi(r))\left(1 - \frac{2M(r)}{r}\right), \quad g_{rr} = 1 - \frac{2M(r)}{r} \tag{54}$$

expanding the functions $M(r)$ and $\psi(r)$ into the power series

$$M(r) = \sum_{i=0}^{k} M_i(r)\varepsilon^i \tag{55}$$

$$\psi(r) = \sum_{i=1}^{k} \psi_i \varepsilon^i \tag{56}$$

and retaining only the linear terms. Now, accepting the boundary condition $\psi_1(\infty) = 0$, repeating the calculations for the line element (54), with the integration constants appearing in the solution for $M(r)$ determined either from $M_0(\infty) = \mathcal{M}$ and $M_1(\infty) = 0$ or from $M_0(r_+) = r_+/2$ and $M_1(r_+) = 0$, one can easily reconstruct all our previous results.

IV. FINAL REMARKS

In this paper, we report our calculations of the next-to-leading term of the renormalized stress-energy tensor of the quantized massive scalar field in a large mass limit. To achieve this, we have calculated the effective action constructed from the (integrated) coincidence limit of the coefficients $a_3(x, x')$ and $a_4(x, x')$, and, subsequently, we have calculated the approximate stress-energy tensor by functional differentiation of the thus obtained action with respect to the metric tensor. The obtained stress-energy tensor can be employed in any spacetime provided the condition $\lambda_C/L \ll 1$ holds. The general formulas describing the stress-energy tensor are extremely complex, but, when applied to the Schwarzschild geometry, they yield remarkably simple results, which is the main result of this paper. We have compared our improved approximation with the exact stress-energy tensor of the quantized massive scalar field with $mM = 2$ constructed in Ref. [19]. Our results show that inclusion of the next-to-leading term leads to substantial improvement of the approximation. Generally speaking, the contribution of the tensors (8)–(17) is not negligible. Such a contribution can be critical in numerous situations, such as near extreme Reissner-Nordström black holes or the Bertotti-Robinson geometry. The general stress-energy tensor has been used in the analysis of the back reaction of the quantized field upon the geometry of the Schwarzschild black hole.

We indicate a few possible directions of investigations. First, it would be interesting to examine the vacuum polarization effects in more complex backgrounds, as for example, the spacetime of the electrically charged black holes with or without the cosmological constant. Further, the numeric approach to the back reaction would certainly strengthen our understanding of the problem. Finally, it would be interesting to include corrections produced by gravitons. In the Schwarzschild spacetime such corrections decay as r^{-3} [36,37] and are expected to dominate at large distances. This group of problems is actively investigated and the results will be published elsewhere.

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