PHYSICAL REVIEW D **79,** 084008 (2009)

Quantum gravity at a Lifshitz point

Petr Hořava

Berkeley Center for Theoretical Physics and Department of Physics, University of California, Berkeley, California, 94720-7300, USA, and Physics Division, Lawrence Berkeley National Laboratory, Berkeley, California 94720-8162, USA

(Received 28 January 2009; published 6 April 2009)

We present a candidate quantum field theory of gravity with dynamical critical exponent equal to $z = 3$ in the UV. (As in condensed-matter systems, z measures the degree of anisotropy between space and time.) This theory, which at short distances describes interacting nonrelativistic gravitons, is power-counting renormalizable in $3 + 1$ dimensions. When restricted to satisfy the condition of detailed balance, this theory is intimately related to topologically massive gravity in three dimensions, and the geometry of the Cotton tensor. At long distances, this theory flows naturally to the relativistic value $z = 1$, and could therefore serve as a possible candidate for a UV completion of Einstein's general relativity or an infrared modification thereof. The effective speed of light, the Newton constant and the cosmological constant all emerge from relevant deformations of the deeply nonrelativistic $z = 3$ theory at short distances.

DOI: 10.1103/PhysRevD.79.084008 PACS numbers: 04.60.−m

I. INTRODUCTION

In recent decades, string theory has become the dominant paradigm for addressing questions of quantum gravity. There are many indications suggesting that string theory is sufficiently rich to contain the answers to many puzzles, such as the information paradox or the statistical interpretation of black hole entropy. Yet, string theory is also a rather large theory, possibly with a huge landscape of vacua, each of which leads to a scenario for the history of the universe which may or may not resemble ours. Given this richness of string theory, it might even be logical to adopt the perspective in which string theory is not a candidate for a unique theory of the universe, but represents instead a natural extension and logical completion of quantum field theory. In this picture, string theory would be viewed—just as quantum field theory—as a powerful technological framework, and not as a single theory.

If string theory is such an apparently vast structure, it seems natural to ask whether quantum gravitational phenomena in $3 + 1$ spacetime dimensions can be studied in a self-contained manner in a "smaller" framework. A useful example of such a phenomenon is given by Yang-Mills gauge theories in $3 + 1$ dimensions. While string theory is clearly a powerful technique for studying properties of Yang-Mills theories, their embedding into string theory is not required for their completeness: In $3 + 1$ dimensions, they are UV complete in the framework of quantum field theory.

In analogy with Yang-Mills, we are motivated to look for a "small" theory of quantum gravity in $3 + 1$ dimensions, decoupled from strings. One attempt to construct such a small theory is offered by loop quantum gravity. In this paper, we present a new strategy for addressing this problem. Compared to the previous approaches to quantum gravity, the novelty of our approach is that it takes advantage of theoretical concepts developed in recent decades in condensed-matter physics, in particular in the theory of quantum critical phenomena.

In the context of quantum field theory, the main obstacle against perturbative renormalizability of Einstein's theory of gravity in $3 + 1$ dimensions is well understood (see, e.g., [1] for an excellent introduction). The main problem is that the gravitational coupling constant G_N is dimensionful, with a negative dimension $[G_N] = -2$ in mass units. The Feynman rules also involve the graviton propagator, which scales with the four-momentum $k_\mu \equiv (\omega, \mathbf{k})$ schematically as

$$\frac{1}{k^2}, \tag{1}$$

where $k = \sqrt{\omega^2 - \mathbf{k}^2}$. At increasing loop orders, the Feynman diagrams of this theory require counterterms of ever-increasing degree in curvature. The resulting theory can still be treated as an effective field theory, but it requires a UV completion. Usually, this completion is assumed to take the form of string or M-theory.

An improved UV behavior can be obtained if relativistic higher-derivative corrections are added to the Lagrangian (see [2] for a review of higher-derivative gravity). Terms quadratic in curvature not only yield new interactions (with a dimensionless coupling), they also modify the propagator. Schematically, we get

$$\begin{aligned} &\frac{1}{k^2} + \frac{1}{k^2} G_N k^4 \frac{1}{k^2} + \frac{1}{k^2} G_N k^4 \frac{1}{k^2} G_N k^4 \frac{1}{k^2} + \cdots \\ &\quad = \frac{1}{k^2 - G_N k^4}. \end{aligned} \tag{2}$$

At high energies, the propagator is dominated by the $1/k^4$ term. This cures the problem of UV divergences, and in fact the calculations in Euclidean signature suggest that the theory exhibits asymptotic freedom. However, this cure simultaneously produces a new pathology, which prevents this modified theory from being a solution to the problem of quantum gravity: The resummed propagator (2) exhibits two poles:

$$\frac{1}{k^2 - G_N k^4} = \frac{1}{k^2} - \frac{1}{k^2 - 1/G_N}. \tag{3}$$

One describes candidate massless gravitons, but the other corresponds to ghost excitations and implies violations of unitarity, at least in perturbation theory.

Recently, a new class of gravity models was introduced in [3]. These models exhibit scaling properties which are anisotropic between space and time. Such an anisotropic scaling is common in condensed-matter systems, where the degree of anisotropy between space and time is characterized by the "dynamical critical exponent" z. (Relativistic systems automatically satisfy $z = 1$ as a consequence of Lorentz invariance.) In models of gravity with anisotropic scaling, the problem of renormalizability of gravity is put in a new context. Consider, for example, the case of gravity with $z = 2$ studied in [3]. As a consequence of the nonrelativistic value of z, the dimension of the gravitational coupling constant changes. The critical dimension in which the coupling is dimensionless shifts to $2 + 1$, making the system a suitable candidate for describing the world-volume theory on a bosonic membrane.

The techniques used in the construction of gravity models with anisotropic scaling in [3] closely parallel methods developed in the theory of dynamical critical systems [4,5] and quantum criticality [6]. The prototype of the class of condensed-matter models relevant here is the theory of a Lifshitz scalar in $D + 1$ dimensions [7,8], first proposed as a description of tricritical phenomena involving spatially modulated phases (and reviewed in [3,9], see also [10,11]). The action of the Lifshitz scalar is

$$S = \int dt d^D \mathbf{x} \{(\dot{\Phi})^2 - (\Delta\Phi)^2\}, \tag{4}$$

where the overdot denotes the time derivative, and $\Delta \equiv \partial_i \partial_i$ is the spatial Laplacian. This action describes a free-field fixed point with anisotropic scaling and $z = 2$. At this fixed point, we can add a relevant deformation to the action,

$$- c^2 \int dt d^D \mathbf{x} \partial_i \Phi \partial_i \Phi. \tag{5}$$

Under the influence of this deformation, the theory flows in the infrared to $z = 1$, with Lorentz invariance emerging as an accidental symmetry at long distances. Note that from the short-distance point of view, the emergent long-distance speed of light c originates from the dimensionful coupling constant associated with the relevant deformation (5) of the $z = 2$ fixed point [12].

In our approach to quantum gravity, we consider systems whose scaling at short distances exhibits a strong anisotropy between space and time, with $z > 1$. This will improve the short-distance behavior of the theory. The price we have to pay is that our theory will not exhibit relativistic invariance at short distances. In fact, many developments in string theory suggest that giving up Lorentz invariance as a fundamental symmetry may not be so unreasonable. Indeed, it is difficult to imagine how Lorentz symmetry can survive as a fundamental symmetry in a framework in which the space itself is viewed as an emergent property of the theory. In string theory, quantum mechanics appears to be more fundamental than the symmetries of special or general relativity. As a result, we adopt the perspective that Lorentz symmetry should appear as an emergent symmetry at long distances, but can be fundamentally absent at high energies.

Despite being fundamentally nonrelativistic at short distances, our models of gravity with anisotropic scaling will describe propagating polarizations of the metric. Restoring the explicit factors of the speed of light, the propagator for such gravitons will schematically take the form

$$\frac{1}{\omega^2 - c^2 \mathbf{k}^2 - G(\mathbf{k}^2)^z}, \tag{6}$$

where G is a coupling constant. (Generally, the denominator will also contain other powers of $\mathbf{k}^2$ between 1 and z, which we omit here to keep this introductory discussion simple.)

At high energies, the propagator is dominated by the anisotropic term $1/(\omega^2 - G(\mathbf{k}^2)^z)$. The high-energy behavior of the theory is controlled by a free-field fixed point with anisotropic scaling. For a suitably chosen z, this modification improves the short-distance behavior, shifting the critical dimension at which the theory is power-counting renormalizable. The $c\mathbf{k}^2$ term in (6) becomes important only at lower energies: This term originates from a relevant deformation of the anisotropic UV fixed point, with c a dimensionful coupling. The propagator (6) is reproduced by the resummation of the high-energy propagator in the theory deformed by this relevant operator,

$$\begin{aligned} &\frac{1}{\omega^2 - c^2 \mathbf{k}^2 - G(\mathbf{k}^2)^z} \\ &= \frac{1}{\omega^2 - G(\mathbf{k}^2)^z} + \frac{1}{\omega^2 - G(\mathbf{k}^2)^z} c^2 \mathbf{k}^2 \frac{1}{\omega^2 - G(\mathbf{k}^2)^z} + \cdots. \end{aligned} \tag{7}$$

At low energies, the theory naturally flows to $z = 1$. The relativistic scaling of space and time is "accidentally restored," in the technical sense of renormalization theory. In this low-energy regime, it is natural to adopt the perspective of a theory with relativistic scaling and absorb c into the redefinition of the time coordinate, effectively setting $c = 1$. From the perspective of the $z = 1$ IR fixed point, the higher-curvature terms which dominate the UV fixed point represent small corrections to the $z = 1$ scaling, and the propagator (6) can be interpreted as

$$\frac{1}{\omega^2 - \mathbf{k}^2 - G(\mathbf{k}^2)^z} = \frac{1}{\omega^2 - \mathbf{k}^2} + \frac{1}{\omega^2 - \mathbf{k}^2} G(\mathbf{k}^2)^z \times \frac{1}{\omega - \mathbf{k}^2} + \cdots. \tag{8}$$

Unlike in relativistic higher-derivative theories mentioned above, higher time derivatives are not generated, and the usual problem of higher-derivative gravities with perturbative unitarity is eliminated.

In this paper, we use these ideas to formulate a theory of gravity which would be power-counting renormalizable in $3 + 1$ dimensions. Given the arguments above, this implies that $z = 3$. We develop the theory of gravity at such a "$z = 3$ Lifshitz point" in Sec. II. Under the additional condition of "detailed balance," this theory turns out to be intimately related to topological gravity in three dimensions and the geometry of the Cotton tensor. We discuss various properties of the $z = 3$ UV fixed points, and study the relevant deformations which induce the flow at low energies to $z = 1$, the value of the dynamical exponent in general relativity.

In addition to $z = 3$ gravity in $3 + 1$ dimensions and its infrared flow to $z = 1$, in Sec. III we briefly discuss the case of $z = 4$ in $4 + 1$ and $3 + 1$ dimensions. We also point out that another example of gravity with $z \neq 1$ has already appeared in the literature, under the name of "ultralocal theory" of gravity. Section III D contains a brief discussion of possible applications of anisotropic models of gravity in the context of AdS/CFT correspondence.

II. QUANTUM GRAVITY IN 3 + 1 DIMENSIONS AT A $z = 3$ LIFSHITZ POINT

Our aim is to construct a theory of gravity in $3 + 1$ dimensions with anisotropic scaling using the traditional framework of quantum field theory, i.e., path-integral methods or canonical quantization. Such an anisotropic theory of gravity, characterized by dynamical critical exponent $z = 2$, was introduced in [3]. The main novelty of the present paper is that we are now interested in the case of $z = 3$, which will lead to a power-counting renormalizable theory in $3 + 1$ dimensions. Our construction parallels that of [3], which also contains additional details involving the general class of gravity models with anisotropic scaling.

A. Fields, scalings, and symmetries

The quantum fields of our theory will include the spatial metric field $g_{ij}(\mathbf{x}, t)$, which upon quantization describes propagating, interacting gravitons. In this paper, we will define this theory on a fixed spacetime manifold $\mathcal{M}$, and will not consider the possibility of summing over distinct topologies of spacetime. On $\mathcal{M}$, we will use coordinates

$$(t, \mathbf{x}) \equiv (t, x^i), \qquad i = 1, \ldots D, \tag{9}$$

with D denoting the dimension of space. For most of the paper, we will be interested in the case of $D = 3$, but some of our arguments will be more instructive if we keep D arbitrary. Our notation throughout will be strictly nonrelativistic, unless stated otherwise. For example, the covariant derivative ∇_i is defined with respect to the *spatial* metric g_{ij}, and we use $R^i{}_{jk\ell}$, $R_{ij} \equiv R^k{}_{ikj}$, and $R \equiv R^i_i$ to denote the Riemann tensor, the Ricci tensor, and the Ricci scalar of the spatial metric g_{ij} and its associated connection ∇_i.

1. Anisotropic scaling in gravity

The theory will be constructed so that it is compatible with anisotropic scaling with dynamical critical exponent z,

$$\mathbf{x} \rightarrow b\mathbf{x}, \qquad t \rightarrow b^z t. \tag{10}$$

In order for the theory to be power-counting renormalizable in $3 + 1$ spacetime dimensions, we will choose $z = 3$, but for now we keep z arbitrary.

The scaling in (10) is of course not diffeomorphism invariant (nor is it invariant under the gauge symmetries that we will impose on our system below), and should be interpreted in the following sense: The theory will be designed such that it has a solution which describes an ultraviolet free-field fixed point with scaling properties given by (10). At this fixed point, we will measure canonical dimensions of all objects in the units of spatial momenta. In particular, the spacetime anisotropy is reflected in the dimensions of time and space coordinates,

$$[\mathbf{x}] = -1, \qquad [t] = -z, \tag{11}$$

at this ultraviolet fixed point.

In addition to the spatial metric g_{ij} [of signature $(+ \cdots +)$], the field content of the theory will be given by a spatial vector N_i, and a spatial scalar N. The fields N and N_i are essentially the "lapse" and "shift" variables familiar from general relativity, where they appear in the process of the $3 + 1$ split of the four-dimensional spacetime metric. (The precise way in which these variables are related to the full spacetime metric can be found in [3].) Using such ADM-like variables is particularly natural because of the fundamentally nonrelativistic nature of our system.

In the case of general z, we postulate the classical scaling dimensions of the fields to be

$$[g_{ij}] = 0, \qquad [N_i] = z - 1, \qquad [N] = 0. \tag{12}$$

In the specific case $z = 3$ of interest here, we have $[N_i] = 2$, while N and g_{ij} are dimensionless.

2. Foliation-preserving diffeomorphisms

In the anisotropic scaling (10), the time dimension plays a privileged role. We will encode this special role of time in the theory by assuming that in addition to being a differentiable manifold, our spacetime $\mathcal{M}$ carries an extra structure—that of a codimension-one foliation [13]. This foliation structure $\mathcal{F}$ is to be viewed as a part of the topological structure of $\mathcal{M}$, before any notion of a Riemannian metric is introduced. The leaves of this foliation are the hypersurfaces of constant time. Coordinate transformations adapted to the foliation are of the form

$$\tilde{x}^i = \tilde{x}^i(x^j, t), \qquad \tilde{t} = \tilde{t}(t). \tag{13}$$

Thus, the transition functions are foliation-preserving diffeomorphisms. We will denote the group of foliation-preserving diffeomorphisms of $\mathcal{M}$ by $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$. In the local adapted coordinate system, the infinitesimal generators of $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ are given by

$$\delta x^i = \zeta^i(t, \mathbf{x}), \qquad \delta t = f(t). \tag{14}$$

We will simplify our presentation by further assuming that the spacetime foliation is topologically given by

$$\mathcal{M} = \mathbf{R} \times \Sigma, \tag{15}$$

with all leaves of the foliation topologically equivalent to a fixed D-dimensional manifold Σ.

Differential geometry of foliations is a well-developed branch of mathematics, and represents the proper mathematical setting for the class of gravity theories studied here. We will not review the geometric theory of foliations in any detail here, instead referring the reader to [14–16]. For example, there are two natural classes of functions that can be defined on a foliation: In addition to functions that are allowed to depend on all coordinates, there is a special class of functions which take constant values on each leaf of the foliation. We will call such functions "projectable."

Foliations can be equipped with a Riemannian structure. A Riemannian structure compatible with our codimension-one foliation of $\mathcal{M}$ consists of three objects: g_{ij}, N_i, and N, with N a projectable function; both N and N_i transform as vectors under the reparametrizations of time. As pointed out above, these fields can be viewed as a decomposition of a Riemannian metric on $\mathcal{M}$ into the metric g_{ij} induced along the leaves, the shift variable N_i, and the lapse field N. The generators of $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ act on the fields via

$$\begin{aligned}
\delta g_{ij} &= \partial_i \zeta^k g_{jk} + \partial_j \zeta^k g_{ik} + \zeta^k \partial_k g_{ij} + f \dot{g}_{ij},\\
\delta N_i &= \partial_i \zeta^j N_j + \zeta^j \partial_j N_i + \dot{\zeta}^j g_{ij} + \dot{f} N_i + f \dot{N}_i,\\
\delta N &= \zeta^j \partial_j N + \dot{f} N + f \dot{N}.
\end{aligned} \tag{16}$$

In [3], these transformation rules were derived by starting with the action of spacetime diffeomorphisms on the relativistic metric in the ADM decomposition, and taking the $c \to \infty$ limit. We also saw in [3] that N_i and N can be naturally interpreted as gauge fields associated with the time-dependent spatial diffeomophisms and the time reparametrizations, respectively. In particular, since N is the gauge field associated with the time reparametrization $f(t)$, it appears natural to restrict it to be a projectable function on the spacetime foliation $\mathcal{F}$.

If we wish instead to treat N as an arbitrary function of spacetime, we have essentially two options. First, we can allow an arbitrary spacetime-dependent N as a background field, but integrate only over space-independent fluctuations of N in the path integral. As the second option, we will encounter situations in which N *must* be allowed to be a general function of spacetime, because it participates in an additional gauge symmetry. When that happens, we will integrate over the fluctuations of N in the path integral. An example of such an extra symmetry is the invariance under anisotropic Weyl transformations discussed in Sec. II C 3 below, and in Sec. 5.2 of [3].

B. Lagrangians

We formally define our quantum field theory of gravity by a path integral,

$$\int \mathcal{D}g_{ij} \mathcal{D}N_i \mathcal{D}N \exp\{iS\}. \tag{17}$$

Here $\mathcal{D}g_{ij}\mathcal{D}N_i\mathcal{D}N$ denotes the path-integral measure whose proper treatment involves the Faddeev-Popov gauge fixing of the gauge symmetry $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$, and S is the most general action compatible with the requirements of gauge symmetry (and further restricted by unitarity). As is often the case, this path integral is interpreted as the analytic continuation of the theory which has been Wick rotated to imaginary time $\tau = it$.

Our next step is to construct the action S compatible with our symmetry requirements. For simplicity, we will assume that all global topological effects can be ignored, freely dropping all total derivative terms and not discussing possible boundary terms in the action. This is equivalent to assuming that our space Σ is compact and its tangent bundle topologically trivial. The refinement of our construction which takes into account global topology and boundary terms is outside of the scope of the present work.

1. The kinetic term

The kinetic term in the action will be given by the most general expression which is (i) quadratic in first time derivatives $\dot{g}_{ij}$ of the spatial metric, and (ii) invariant under the gauge symmetries of foliation-preserving diffeomorphisms $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$. The object that transforms covariantly under $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ is not $\dot{g}_{ij}$, but instead the second fundamental form

$$K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i). \tag{18}$$

This tensor measures the extrinsic curvature of the leaves of constant time in the spacetime foliation $\mathcal{F}$. In terms of K_{ij} and its trace $K \equiv g^{ij} K_{ij}$, the kinetic term is given by

$$S_K = \frac{2}{\kappa^2}\int dt d^D\mathbf{x}\sqrt{g}N(K_{ij}K^{ij} - \lambda K^2). \tag{19}$$

This kinetic term contains two coupling constants: κ and λ. The dimension of κ depends on the spatial dimension D: Since the dimension of the volume element is

$$[dt d^D\mathbf{x}] = -D - z, \tag{20}$$

and each time derivative contributes $[\partial_t] = z$, the scaling dimension of κ is

$$[\kappa] = \frac{z - D}{2}. \tag{21}$$

As intended, this coupling will be dimensionless in $3 + 1$ spacetime dimensions if $z = 3$.

The presence of an additional, dimensionless coupling λ reflects the fact that each of the two terms in (19) is separately invariant under $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$. In other words, the requirement of $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ symmetry allows the generalized De Witt "metric on the space of metrics"

$$G^{ijk\ell} = \tfrac{1}{2}(g^{ik}g^{j\ell} + g^{i\ell}g^{jk}) - \lambda g^{ij}g^{k\ell} \tag{22}$$

to contain a free parameter λ. It is this generalized De Witt metric that defines the form quadratic in K_{ij} which appears in the kinetic term (see [3]).

In general relativity, the requirement of invariance under all spacetime diffeomophisms forces $\lambda = 1$. In our theory with $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ gauge invariance, λ represents a dynamical coupling constant, susceptible to quantum corrections.

It is interesting to note that the kinetic term S_K is universal, and independent of both the desired value of z and the dimension of spacetime. The only place where the value of z shows up in S_K is in the scaling dimension of the integration measure (20), which in turn determines the dimension (21) of κ. The main difference between theories with different z will be in the pieces of the action which are independent of time derivatives.

2. The potential

The logic of effective field theory suggests that the complete action should contain all terms compatible with the imposed symmetries, which are of dimension equal to or less than the dimension of the kinetic term, $[K_{ij}K^{ij}] = 2z$. In addition to S_K, which contains the two independent terms of second order in the time derivatives of the metric, the general action will also contain terms that are independent of time derivatives. Since our framework is fundamentally nonrelativistic, we will refer to all terms in the action which are independent of the time derivatives (but do depend on spatial derivatives) simply as the "potential."

There is a simple way to construct potential terms invariant under our gauge symmetry $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$: Starting with any scalar function $V[g_{ij}]$ which depends only on the metric and its spatial derivatives, the following potential term,

$$S_V = \int dt d^D\mathbf{x}\sqrt{g}NV[g_{ij}], \tag{23}$$

will be invariant under $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$.

Throughout this paper, our strategy is to focus first on the potential terms of the same dimension as $[K_{ij}K^{ij}]$, at first ignoring all possible relevant terms of lower dimensions in V. This is equivalent to focusing first on the high-energy limit, where such highest-dimension terms dominate. Once the high-energy behavior of the theory is understood, one can restore the relevant terms, and study the flows of the theory away from the UV fixed point that such relevant operators induce in the infrared.

With our choice of $D = 3$ and $z = 3$, there are many examples of terms in V of the same dimension as the kinetic term in (19). Some such terms are quadratic in curvature,

$$\nabla_k R_{ij}\nabla^k R^{ij}, \qquad \nabla_k R_{ij}\nabla^i R^{jk}, \qquad R\Delta R, \qquad R^{ij}\Delta R_{ij}; \tag{24}$$

they will not only add interactions but also modify the propagator. Other terms, such as

$$R^3, \qquad R^i_j R^j_k R^k_i, \qquad RR_{ij}R^{ij}, \tag{25}$$

are cubic in curvature, and therefore represent pure interacting terms. Some of the terms of the correct dimension are related by the Bianchi identity and other symmetries of the Riemann tensor, or differ only up to a total derivative. Additional constraints on the possible values of the couplings will likely follow from the requirements of stability and unitarity of the quantum theory. However, the list of independent operators appears to be prohibitively large, implying a proliferation of couplings which makes explicit calculations rather impractical.

C. UV theory with detailed balance

In order to reduce the number of independent coupling constants, we will impose an additional symmetry on the theory. The reason for this restriction is purely pragmatic, to limit the proliferation of independent couplings mentioned in the previous paragraph. The way in which this restriction will be implemented, however, is very reminiscent of methods used in nonequilibrium critical phenomena and quantum critical systems. As a result, it is natural to suspect that there might also be conceptual reasons behind restricting the general class of classical theories to conform to this framework in systems with gravity as well.

We will require the potential term to be of a special form,

$$S_V = \frac{\kappa^2}{8}\int dt d^D\mathbf{x}\sqrt{g}NE^{ij}\mathcal{G}_{ijk\ell}E^{k\ell}, \tag{26}$$

and will further demand that E^{ij} itself follow from a variational principle,

$$\sqrt{g}E^{ij} = \frac{\delta W[g_{k\ell}]}{\delta g_{ij}} \tag{27}$$

for some action W. The two copies of E^{ij} in (26) are contracted by $\mathcal{G}_{ijk\ell}$, the inverse of the De Witt metric (22). Loosely borrowing terminology from nonequilibrium dynamics, we will say that theories whose potential is of the form (26) with (27) for some W satisfy the "detailed balance condition."

In the context of condensed matter, the virtue of the detailed balance condition is in the simplification of the renormalization properties. Systems which satisfy the detailed balance condition with some D-dimensional action W typically exhibit a simpler quantum behavior than a generic theory in $D+1$ dimensions. Their renormalization can be reduced to the simpler renormalization of the associated theory described by W, followed by one additional step—the renormalization of the relative couplings between the kinetic and potential terms in S. Examples of this phenomenon include scalar fields [17] or Yang-Mills gauge theories [9,18]. Investigating the precise circumstances under which this "quantum inheritance principle" holds for gravity systems will be important for understanding the quantum properties of gravity models with nonrelativistic values of z.

Since we are primarily interested in theories which are spatially isotropic, W must be the action of a relativistic theory in Euclidean signature. (Obvious generalizations to theories with additional spatial anisotropies are clearly possible, but will not be pursued in this paper.) In [3], a theory of gravity in $D+1$ dimensions satisfying the detailed balance condition was constructed, with W the Einstein-Hilbert action

$$W = \frac{1}{\kappa_W^2}\int d^D\mathbf{x}\sqrt{g}(R - 2\Lambda_W). \tag{28}$$

The potential S_V of this theory takes the form

$$S_V = \frac{\kappa^2}{8\kappa_W^4}\int dt d^D\mathbf{x}\sqrt{g}N\left(R^{ij} - \frac{1}{2}Rg^{ij} + \Lambda_W g^{ij}\right) \times \mathcal{G}_{ijk\ell}\left(R^{k\ell} - \frac{1}{2}Rg^{k\ell} + \Lambda_W g^{k\ell}\right). \tag{29}$$

At short distances, the curvature term in W dominates over Λ_W, and the resulting potential S_V is quadratic in the curvature tensor: The theory exhibits anisotropic scaling with $z=2$ in the UV. Turning on Λ_W in W leads to lower-dimension terms in S_V which dominate at long distances, and the theory undergoes a classical flow to $z=1$ in the IR. The anisotropic scaling in the UV shifts the critical dimension of this theory, which is now renormalizable by power counting in $2+1$ dimensions. In dimensions higher than $2+1$, the theory with potential (29) is merely a low-energy effective field theory, and can be expected to break down at the scale set by the dimensionful coupling κ_W.

Here we are interested in constructing a theory which satisfies detailed balance, and exhibits the short-distance scaling with $z=3$ leading to power-counting renormalizability in $3+1$ dimensions. Therefore, E^{ij} must be of third order in spatial derivatives. As it turns out, there is a unique candidate for such an object: the Cotton tensor

$$C^{ij} = \varepsilon^{ik\ell}\nabla_k\left(R^j_\ell - \tfrac{1}{4}R\delta^j_\ell\right). \tag{30}$$

This tensor not only exhibits all the required symmetries, it also follows from a variational principle.

1. Properties of the Cotton tensor

The Cotton tensor enjoys several symmetry properties which may not be immediately obvious from its definition in (30):

(i) It is symmetric and traceless,

$$C^{ij} = C^{ji}, \qquad g_{ij}C^{ij} = 0. \tag{31}$$

(ii) It is transverse (or covariantly conserved),

$$\nabla_i C^{ij} = 0. \tag{32}$$

(iii) It is conformal, with conformal weight $-5/2$. More precisely, under local spatial Weyl transformations

$$g_{ij} \to \exp\{2\Omega(\mathbf{x})\}g_{ij}, \tag{33}$$

it transforms as

$$C^{ij} \to \exp\{-5\Omega(\mathbf{x})\}C^{ij}, \tag{34}$$

with no terms containing derivatives of $\Omega(\mathbf{x})$.

The Cotton tensor plays an important role in geometry. Recall that in dimensions $D>3$, the property of conformal flatness of a Riemannian metric is equivalent to the vanishing of the Weyl tensor $C_{ijk\ell}$, defined as the completely traceless part of the Riemann tensor:

$$C_{ijk\ell} = R_{ijk\ell} - \frac{1}{D-2}(g_{ik}R_{j\ell} - g_{i\ell}R_{jk} - g_{jk}R_{i\ell} + g_{j\ell}R_{ik}) + \frac{1}{(D-1)(D-2)}(g_{ik}g_{j\ell} - g_{i\ell}g_{jk})R. \tag{35}$$

In $D=3$, however, the Weyl tensor vanishes identically, and another object has to take over the role in the criterion of conformal flatness of 3-manifolds. This object is the Cotton tensor, of third order in spatial derivatives.

The Cotton tensor also plays an important role in physics. In the initial value problem of the Hamiltonian formulation of general relativity, it is natural to ask what set of initial conditions can be freely specified for the metric and its canonical momenta, without violating the constraint part of Einstein's equations. It was shown by York [19–21] that the Cotton tensor plays a central role in answering this question. The correct initial conditions are set by specifying the values of two tensors with the symmetries of the Cotton tensor: One related to the initial value for the conformal structure of the spatial metric, and the other specifying the initial value of the conjugate momenta. For this reason, C^{ij} is often referred to as the "Cotton-York tensor" in the physics literature.

Lastly, the Cotton tensor follows from a variational principle, with action

$$W = \frac{1}{w^2}\int_\Sigma \omega_3(\Gamma). \tag{36}$$

Here w^2 is a dimensionless coupling, and

$$\begin{aligned}\omega_3(\Gamma) &= \mathrm{Tr}(\Gamma\wedge d\Gamma + \tfrac{2}{3}\Gamma\wedge\Gamma\wedge\Gamma)\\ &\equiv \varepsilon^{ijk}(\Gamma^m_{i\ell}\partial_j\Gamma^\ell_{km} + \tfrac{2}{3}\Gamma^n_{i\ell}\Gamma^\ell_{jm}\Gamma^m_{kn})d^3\mathbf{x}\end{aligned} \tag{37}$$

is the gravitational Chern-Simons term, with the Christoffel symbols Γ^i_{jk} treated as known functionals of the metric g_{ij}, and not as independent variables. The variation of (36) with respect to g_{ij} yields the vanishing of the Cotton tensor as the equations of motion.

Without any loss of generality, we will assume that the coupling w^2 is positive; its sign can be changed by flipping the orientation of the 3-manifold Σ. Unlike in Chern-Simons gauge theories with a compact gauge group, the coupling constant of Chern-Simons gravity in $2+1$ dimensions is not quantized, as a result of the absence of large gauge transformations. In our framework, however, we are only interested in the action of a theory in three dimensions in "imaginary time," and require that this Euclidean action be real. This is to be contrasted with the conventional interpretation of the three-dimensional theory, which involves analytic continuation to real time in $2+1$ dimensions, and imposes slightly different reality conditions on the action.

2. $z = 3$ *gravity with detailed balance*

Having reviewed some of the properties of the Cotton tensor, we can now write down the full action of our $z = 3$ gravity theory in $3+1$ dimensions:

$$\begin{aligned}S &= \int dt d^3\mathbf{x}\sqrt{g}N\left\{\frac{2}{\kappa^2}(K_{ij}K^{ij} - \lambda K^2) - \frac{\kappa^2}{2w^4}C_{ij}C^{ij}\right\}\\ &= \int dt d^3\mathbf{x}\sqrt{g}N\left\{\frac{2}{\kappa^2}(K_{ij}K^{ij} - \lambda K^2) - \frac{\kappa^2}{2w^4}\right.\\ &\quad\left.\times\left(\nabla_i R_{jk}\nabla^i R^{jk} - \nabla_i R_{jk}\nabla^j R^{ik} - \frac{1}{8}\nabla_i R\nabla^i R\right)\right\}.\end{aligned} \tag{38}$$

As a result of the uniqueness of the Cotton tensor, the action given in (38) describes the most general $z = 3$ gravity satisfying the detailed balance condition, modulo the possible addition of relevant terms, which will be discussed in Sec. II E.

We can demonstrate that after the Wick rotation to imaginary time, this action can be written—up to a total derivative—as a sum of squares,

$$\begin{aligned}S &= i\int d\tau d^3\mathbf{x}\sqrt{g}N\left\{\frac{2}{\kappa^2}(K_{ij}K^{ij} + \lambda K^2) + \frac{\kappa^2}{2w^4}C_{ij}C^{ij}\right\}\\ &= 2i\int d\tau d^3\mathbf{x}\sqrt{g}N\left(\frac{1}{\kappa}K_{ij} - \frac{\kappa}{2w^2}C_{ij}\right)\\ &\quad\times G^{ijk\ell}\left(\frac{1}{\kappa}K_{k\ell} - \frac{\kappa}{2w^2}C_{k\ell}\right),\end{aligned} \tag{39}$$

First, $C_{ij}G^{ijk\ell}C_{k\ell} = C_{ij}C^{ij}$ because C^{ij} is traceless. As to the cross terms $K_{ij}G^{ijk\ell}C_{k\ell}$, they can be written as a total derivative,

$$\begin{aligned}&\frac{1}{w^2}\int d\tau d^3\mathbf{x}\sqrt{g}NK_{ij}G^{ijk\ell}C_{k\ell}\\ &\quad= \frac{1}{2w^2}\int d\tau d^3\mathbf{x}\sqrt{g}(\dot g_{ij} - \nabla_i N_j - \nabla_j N_i)C_{ij}\\ &\quad= \int d\tau d^3\mathbf{x}\sqrt{g}\left(\dot g_{ij}\frac{\delta W}{\delta g_{ij}} + \frac{1}{w^2}\nabla_i(N_j C^{ij})\right)\\ &\quad= \int d\tau d^3\mathbf{x}\left(\dot{\mathcal{L}} + \frac{1}{w^2}\partial_i(\sqrt{g}N_j C^{ij})\right),\end{aligned}$$

where we used the transverse property (32) of C_{ij}, and $\mathcal{L}$ is the Lagrangian density of the action W in (36).

Introducing an auxiliary field B^{ij}, it is convenient to rewrite the imaginary-time action as

$$\begin{aligned}S = 2i\int d\tau d^3\mathbf{x}\sqrt{g}N\Big\{&B^{ij}\left(\frac{1}{\kappa}K_{ij} - \frac{\kappa}{2w^2}C_{ij}\right)\\ &- B^{ij}\mathcal{G}_{ijk\ell}B^{k\ell}\Big\}.\end{aligned} \tag{40}$$

This form of the action, with all terms at least linear in the auxiliary field B^{ij} and with the linear term proportional to a gradient flow equation, is symptomatic of theories satisfying the detailed balance condition in the context of condensed-matter systems, in particular, in the theory of quantum and dynamical critical phenomena [4,5], stochastic quantization [22,23], and nonequilibrium statistical mechanics [24].

In that condensed-matter context, the property of detailed balance often has one interesting implication. If a quantum critical system in $D+1$ dimensions satisfies detailed balance with some W in D dimensions, the partition function of the theory described by W yields a natural solution of the Schrödinger equation of the theory in $D+1$ dimensions, which plays the role of a candidate ground-state wave function. Similarly, in nonequilibrium statistical mechanics and dynamical critical phenomena, the corresponding statement is essentially the Wick rotation of this correspondence to imaginary time: The partition function of the D-dimensional theory defined by W represents an equilibrium state solution of the dynamical theory with detailed balance in $D+1$ dimensions.

In our case, this correspondence formally suggests that

$$\Psi_0[g_{ij}(\mathbf{x})] = \exp\left\{-\frac{1}{2w^2}\int \mathrm{Tr}\left(\Gamma\wedge d\Gamma + \frac{2}{3}\Gamma\wedge\Gamma\wedge\Gamma\right)\right\} \tag{41}$$

is a solution of the Schrödinger equation of the theory in canonical quantization. One might be tempted to consider (41) a candidate for the ground-state wave function of quantum gravity with $z = 3$. However, it becomes quickly obvious that (41) is an unphysical solution: W is not bounded from below, Ψ_0 is non-normalizable, and any attempts to build a spectrum of excited states above this hypothetical ground state lead inevitably to pathologies.

This is to be compared to relativistic Yang-Mills gauge theory in $3 + 1$ dimensions, which is in fact surprisingly similar to our $z = 3$ theory of gravity, in at least two respects:

(i) it also satisfies detailed balance,

(ii) the corresponding action W in three dimensions is also given by the Chern-Simons action, $\int \omega_3(A)$, with A the Yang-Mills one-form gauge field.

Similarly to (41), the candidate ground-state wave function

$$\Psi_0[A] \sim \exp\left\{-\int \omega_3(A)\right\} \tag{42}$$

is formally a solution of the Schrödinger equation for Yang-Mills theory in $3 + 1$ dimensions, but an equally unphysical one. A very clear and conclusive analysis showing why (42) is unphysical can be found in [25,26]. The fate of the formal solution (41) of the gravity theory is the same as the fate of (42) in Yang-Mills: The failure for (41) to be the true ground-state wave function is not a flaw of the theory, it just means that—just as in $3 + 1$ dimensional Yang-Mills theory—the true ground-state wave function is much harder to find.

In passing, it is amusing to note that essentially the same expression Ψ_0 given in (41) was proposed some time ago as a candidate ground-state wave function of loop quantum gravity, where it is known as the "Kodama wave function." Again, a long list of conclusive reasons why this cannot possibly be the physical wave function of quantum gravity can be found in [25].

3. Anisotropic Weyl invariance at $\lambda = 1/3$

The fact that the Cotton tensor is conformal suggests that, under special circumstances, the classical action of $z = 3$ gravity in $3 + 1$ may be invariant under suitably defined local scale transformations. As we now show, this is indeed the case: With $\lambda = 1/3$, our $z = 3$ theory develops a classical anisotropic Weyl invariance, similar to that observed in [3] in the case of the $z = 2$ theory in $2 + 1$ dimensions with $\lambda = 1/2$.

To see that, we decompose the metric by pulling out the overall scale factor,

$$g_{ij} = g^{1/3}\tilde{g}_{ij} = e^{\phi}\tilde{g}_{ij}, \qquad N_i = e^{\phi}\tilde{N}_i, \tag{43}$$

where $\det\tilde{g}_{ij} = 1$. With this decomposition, the $z = 3$ action (38) becomes

$$\begin{aligned} S = \frac{1}{2}\int dt d^3\mathbf{x}\bigg\{&\frac{e^{3\phi/2}}{\kappa^2 N}[(\dot{\tilde{g}}_{ij} - \tilde{\nabla}_i\tilde{N}_j - \tilde{\nabla}_j\tilde{N}_i)\\ &\times(\tilde{g}^{ik}\tilde{g}^{j\ell} - \lambda\tilde{g}^{ij}\tilde{g}^{k\ell})(\dot{\tilde{g}}_{k\ell} - \tilde{\nabla}_k\tilde{N}_\ell - \tilde{\nabla}_\ell\tilde{N}_k)\\ &+ 3(1-3\lambda)(\dot{\phi} - \tilde{g}^{ij}\tilde{N}_i\partial_j\phi)^2\\ &- 4(1-3\lambda)\tilde{\nabla}_i\tilde{N}_j\tilde{g}^{ij}(\dot{\phi} - \tilde{g}^{ij}\tilde{N}_i\partial_j\phi)]\\ &- \frac{\kappa^2}{w^4}\frac{N}{e^{3\phi/2}}\tilde{C}^{ij}\tilde{g}_{ik}\tilde{g}_{j\ell}\tilde{C}^{k\ell}\bigg\}, \end{aligned} \tag{44}$$

where $\tilde{\nabla}_i$ is the covariant derivative associated with $\tilde{g}_{ij}$, $\tilde{C}^{ij}$ is the Cotton tensor of $\tilde{g}_{ij}$, and we used the conformal property $\tilde{C}^{ij} = e^{5\phi/2}C^{ij}$ of the Cotton tensor which follows from (34).

By inspection, the action (44) will be invariant under local Weyl transformations of the metric

$$g_{ij} \to \exp\{2\Omega(t,\mathbf{x})\}g_{ij}, \tag{45}$$

if we allow the Weyl rescalings to act on N and N_i by

$$N \to \exp\{3\Omega(t,\mathbf{x})\}N, \qquad N_i \to \exp\{2\Omega(t,\mathbf{x})\}N_i, \tag{46}$$

and if we also set $\lambda = 1/3$. This conformal choice of λ eliminates all terms with derivatives of ϕ in (44). Note that, with the addition of the new gauge symmetry (45) and (46) to $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$, the lapse field N can no longer be a projectable function on the foliation, and must be allowed to depend on x^i as well.

It is reassuring to find that the spacetime-dependent anisotropic Weyl transformations (45) and (46) in fact represent the local version of the rigid anisotropic scaling (10) with dynamical exponent $z = 3$. To see that, recall that N, N_i, and g_{ij} can be reassembled into a spacetime metric in $3 + 1$ dimensions, with $g_{00} \sim -N^2$. The scaling rules (46) that we found by requiring the Weyl invariance of the kinetic term then imply $g_{00} \to \exp\{6\Omega(t,\mathbf{x})\}g_{00}$. In the flat background given by $N = 1$, $N_i = 0$, and g_{ij} the flat Euclidean metric, the Weyl transformations with constant Ω reduce precisely to the anisotropic scaling (10) with the value of the dynamical critical exponent $z = 3$, which was the starting point of our construction of gravity at a $z = 3$ Lifshitz point.

Quantum corrections can be expected to generate violations of local anisotropic Weyl invariance. Lessons from relativistic models suggest that such conformal anomalies vanish in theories with a sufficient degree of supersymmetry, and it should be interesting to investigate the conditions which lead to similar cancellations of conformal anomalies in the nonrelativistic models.

D. At the free-field fixed point

The action (38) of the $z = 3$ theory with detailed balance contains three dimensionless coupling constants: κ, λ, and w. However, only one of them, w, controls the strength of interactions. The noninteracting limit corresponds to sending $w \longrightarrow 0$, while keeping λ and the ratio

$$\gamma = \frac{\kappa}{w} \tag{47}$$

fixed. This limit yields a two-parameter family of free-field fixed points, parametrized by λ and γ.

In preparation for the study of the full interacting theory, it is useful to first investigate the properties of this family of free-field fixed points. The linearization of the $z = 3$ theory is performed in exactly the same way as in [3] for the analogous case of the $z = 2$ gravity and we will therefore be relatively brief, referring the reader to [3] for further details.

We expand the theory in small fluctuations h_{ij}, n, and n_i around the flat background,

$$g_{ij} \approx \delta_{ij} + wh_{ij}, \qquad N \approx 1 + wn, \qquad N_i \approx wn_i. \tag{48}$$

The reference background is a solution of the equations of motion of the $z = 3$ theory (38). Keeping only quadratic terms in the action, n drops out from the theory. A natural gauge choice is

$$n_i = 0. \tag{49}$$

This fixes most of the $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ gauge symmetry, leaving time-independent spatial diffeomorphisms $\mathrm{Diff}(\Sigma)$ unfixed. The residual $\mathrm{Diff}(\Sigma)$ gauge symmetry can be conveniently fixed by setting

$$\partial_i h_{ij} - \lambda \partial_j h = 0, \tag{50}$$

where $h \equiv h_{ii}$. Imposing this condition at some fixed time slice $t = t_0$ effectively fixes the residual $\mathrm{Diff}(\Sigma)$ invariance. The Gauss constraint

$$\partial_i \dot{h}_{ij} - \lambda \partial_j \dot{h} = 0 \tag{51}$$

(which follows from varying the original action with respect to n_i) then ensures that (50) stays valid at all times.

In order to diagonalize the linearized equations of motion and read off the dispersion relation of the propagating modes implied by our gauge choice (49) and (50), it is convenient to first redefine the variables by introducing

$$H_{ij} \equiv h_{ij} - \lambda \delta_{ij} h; \tag{52}$$

the gauge condition (50) implies that H_{ij} is transverse. We then decompose the transverse tensor H_{ij} into its transverse traceless part $\tilde{H}_{ij}$ and its trace H,

$$H_{ij} = \tilde{H}_{ij} + \frac{1}{2}\left(\delta_{ij} - \frac{\partial_i \partial_j}{\partial^2}\right) H. \tag{53}$$

This choice of variables diagonalizes the equations of motion in our gauge. Since the kinetic term is universal, its analysis in the $z = 3$ theory is identical to that presented for $z = 2$ in Sec. 4.5 of [3]. In our gauge and in terms of the new variables, the kinetic term takes the form

$$S_K \approx \frac{1}{2\gamma^2} \int dt d^3\mathbf{x} \left\{ \dot{\tilde{H}}_{ij} \dot{\tilde{H}}_{ij} + \frac{1 - \lambda}{2(1 - 3\lambda)} \dot{H}^2 \right\}. \tag{54}$$

It would appear that the dependence of the kinetic term of H on λ can be absorbed into a rescaling of H, but we choose not to do so, because it would obscure the geometric origin of H in the full nonlinear theory.

On the other hand, the potential term of the $z = 3$ theory reduces to

$$S_V \approx -\frac{\gamma^2}{8} \int dt d^3\mathbf{x} \tilde{H}_{ij} (\partial^2)^3 \tilde{H}_{ij}. \tag{55}$$

Because of the conformal properties of the Cotton tensor, the potential term in the Gaussian approximation depends only on $\tilde{H}_{ij}$ and not on H.

As pointed out in [3], the kinetic term (54) indicates that two values of λ play a special role. At $\lambda = 1/3$, the theory becomes compatible with the local anisotropic Weyl invariance discussed in Sec. II C 3 above. At that value of λ, the scalar mode H is a gauge artifact. The kinetic term for H also appears singular at $\lambda = 1$. As explained in [3], this happens because at this special value of λ, the linearized theory exhibits an extra gauge invariance, which can be used to eliminate physical excitations of H as well.

The transverse traceless tensor $\tilde{H}_{ij}$ contains two propagating physical polarizations. These gravitons satisfy a nonrelativistic gapless dispersion relation,

$$\omega^2 = \frac{\gamma^4}{4} (\mathbf{k}^2)^3. \tag{56}$$

For values of λ outside of the two special values 1 and $1/3$, the scalar mode H will represent a physical degree of freedom, with its linearized equation of motion given simply by $\ddot{H} = 0$. When the theory is deformed by relevant operators, the equation of motion for H will contain terms with spatial derivatives up to fourth order, which is not enough to yield a propagator with good ultraviolet properties. It appears that, in order to make the theory power-counting renormalizable at generic values of λ not equal to 1 or $1/3$, either the scalar mode would have to be eliminated by an extra gauge symmetry, or super-renormalizable terms which give short-distance spatial dynamics to the scalar mode need to be added to the potential. We will briefly return to this point in Sec. III B.

E. Relevant deformations and the infrared flow to $z = 1$

So far we have concentrated on terms of the highest-dimension terms in S. These terms will dominate the short-distance dynamics. At long distances, relevant deformations by operators of lower dimensions will become important, in addition to the renormalization-group (RG) flows of the dimensionless couplings.

One could relax the condition of detailed balance, and simply ask that the action S in $3 + 1$ dimensions be a general combination of all marginal and relevant terms. The action of the theory would then take the form

$$S = \int dt d^3\mathbf{x}\sqrt{g} \sum_{[\mathcal{O}_J]=6} \lambda_J \mathcal{O}_J + \int dt d^3\mathbf{x}\sqrt{g} \sum_{[\mathcal{O}_A]<6} \lambda_A \mathcal{O}_A, \tag{57}$$

where the index J goes over all independent marginal terms compatible with $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$, while A parametrizes all independent relevant operators compatible with this symmetry. λ_J and λ_A are the corresponding coupling constants.

It would be desirable to analyze the quantum properties, in particular, the RG flow patterns, of this general family of models. However, the proliferation of operators with dimensions ≤ 6 makes this analysis difficult, and we will again resort to theories which satisfy the additional property of detailed balance.

1. Relevant deformations with detailed balance

In order for the deformed theory to satisfy the detailed balance condition, the relevant deformations themselves must originate from an action principle in D dimensions, subjected to the requirement of diffeomorphism invariance. Adding all possible relevant terms to the Chern-Simons action (36), we get

$$W = \frac{1}{w^2} \int \omega_3(\Gamma) + \mu \int d^3\mathbf{x}\sqrt{g}(R - 2\Lambda_W). \tag{58}$$

This is essentially the action of topologically massive gravity [27,28], a theory which has been argued to be renormalizable [29,30] and possibly finite [31]. The coupling constants μ and Λ_W are of dimension $[\mu] = 1$ and $[\Lambda_W] = 2$.

The relevant operators in the action W of (58) induce relevant terms in the potential term S_V of our $z = 3$ theory. The full action in $3 + 1$ dimensions which satisfies detailed balance with respect to (58) is given by

$$\begin{aligned} S = \int dt d^3\mathbf{x}\sqrt{g}N\bigg\{&\frac{2}{\kappa^2} K_{ij}G^{ijk\ell}K_{k\ell} \\ &- \frac{\kappa^2}{2}\bigg[\frac{1}{w^2}C^{ij} - \frac{\mu}{2}\bigg(R^{ij} - \frac{1}{2}Rg^{ij} + \Lambda_W g^{ij}\bigg)\bigg] \\ &\times \mathcal{G}_{ijk\ell}\bigg[\frac{1}{w^2}C^{k\ell} - \frac{\mu}{2}\bigg(R^{k\ell} - \frac{1}{2}Rg^{k\ell} + \Lambda_W g^{k\ell}\bigg)\bigg]\bigg\}. \end{aligned} \tag{59}$$

It is useful to organize the terms in (59) in the order of their descending dimensions,

$$\begin{aligned} S = \int dt d^3\mathbf{x}\sqrt{g}N\bigg\{&\frac{2}{\kappa^2}(K_{ij}K^{ij} - \lambda K^2) - \frac{\kappa^2}{2w^4}C_{ij}C^{ij} \\ &+ \frac{\kappa^2\mu}{2w^2}\varepsilon^{ijk}R_{i\ell}\nabla_j R^\ell_k - \frac{\kappa^2\mu^2}{8}R_{ij}R^{ij} \\ &+ \frac{\kappa^2\mu^2}{8(1-3\lambda)}\bigg(\frac{1-4\lambda}{4}R^2 + \Lambda_W R - 3\Lambda_W^2\bigg)\bigg\}. \end{aligned} \tag{60}$$

At long distances, the potential is dominated by the last two terms in (60): the spatial curvature scalar and the constant term. These leading terms in the potential combine with the kinetic term, and as a result, the theory flows in the infrared to $z = 1$.

This infrared limit of the deformed theory should be compared to general relativity. As is well known, the Einstein-Hilbert (EH) action in $3 + 1$ dimensions can be rewritten in the ADM formalism (up to a total derivative) as

$$S_{\mathrm{EH}} = \frac{1}{16\pi G_N} \int d^4x\sqrt{g}N\{(K_{ij}K^{ij} - K^2) + R - 2\Lambda\}. \tag{61}$$

In order to compare these two theories, it is natural to express our model in relativistic coordinates by rescaling t,

$$x^0 = ct, \tag{62}$$

with the emergent speed of light given by

$$c = \frac{\kappa^2\mu}{4}\sqrt{\frac{\Lambda_W}{1-3\lambda}}. \tag{63}$$

Here we have assumed that $\Lambda_W/(1 - 3\lambda)$ is positive, which is also required in order for the sign in front of the scalar curvature term in (60) to match general relativity. Note that, from the perspective of the $z = 3$ theory at short distances, the dimension of c is

$$[c] = 2, \tag{64}$$

resulting in $[x^0] = -1$, in accord with the expected relativistic scaling in the infrared.

In the rescaled coordinates (x^0, x^i) suitable at long distances, the infrared limit of (60) then takes the general relativistic form (61), up to higher-derivative corrections which are suppressed at low energies. The effective

Newton constant is given by

$$G_N = \frac{\kappa^2}{32\pi c}, \tag{65}$$

and the effective cosmological constant

$$\Lambda = \tfrac{3}{2}\Lambda_W. \tag{66}$$

It is intriguing that the effective speed of light c, the effective Newton constant G_N, and the effective cosmological constant Λ of the low-energy theory all emerge from the relevant deformations of the deeply nonrelativistic $z = 3$ theory which dominates at short distances.

In theories satisfying the detailed balance condition, the quantum properties of the $D + 1$ dimensional theory are usually closely related to the quantum properties of the associated theory in D dimensions, with action W. It is interesting that in our case of $3 + 1$ dimensional gravity theory with detailed balance, both the Newton constant and the cosmological constant originate from the couplings in the action of topologically massive gravity in three Euclidean dimensions, a theory with excellent ultraviolet properties.

2. Soft violations of the detailed balance condition

There is another possibility that leads to a broader spectrum of relevant deformations of the $z = 3$ theory, without completely abandoning the simplifications implied by the detailed balance condition. Starting with the $z = 3$ theory at short distances, we can add relevant operators directly to the short-distance action S given in (38),

$$S \to S + \int dt d^3\mathbf{x}\sqrt{g}(-M^6 + \mu^4 R + \cdots), \tag{67}$$

with M and μ arbitrary couplings of dimension 1, and "$\cdots$" denote other relevant terms with more than two spatial derivatives of the metric.

This step will break the detailed balance condition, but only softly, by relevant operators of lower dimension than those appearing in the action at short distances as defined in (38). In the UV, the theory still satisfies detailed balance. At long distances, the theory described by (67) again flows to $z = 1$.

III. OTHER DIMENSIONS AND VALUES OF z

Even though the main focus of the present paper is on the theory of gravity with $z = 3$ in $3 + 1$ spacetime dimensions, the ideas are applicable in a broader context. One application of the $z = 2$ gravity in $2 + 1$ dimensions, as a candidate membrane world-volume theory, was discussed in [3]. Here we take at least a brief look at a list of other interesting values of z and spacetime dimensions.

A. Gravity with $z = 4$ in $4 + 1$ dimensions

Power-counting renormalizability in $4 + 1$ dimensions requires $z = 4$. Theories with $z = 4$ satisfying the detailed balance condition in $4 + 1$ dimensions can be constructed from Euclidean gravity actions W quadratic in curvature, familiar from the study of higher-derivative theories in $3 + 1$ dimensions. (See, e.g., [2] for a review of higher-derivative gravity and supergravity.) As in the case of $z = 3$, we begin with first listing all terms of highest order in spatial derivatives, as these are expected to dominate at short distances, near the hypothetical $z = 4$ fixed point that we are attempting to construct. The four-dimensional Euclidean action quadratic in curvature is given by

$$W = \int d^4\mathbf{x}\sqrt{g}(\alpha C_{ijk\ell}C^{ijk\ell} + \beta R^2). \tag{68}$$

This theory has two independent dimensionless couplings α and β. Modulo topological invariants, this is the most general four-derivative action for relativistic gravity in four dimensions. There is no independent $R_{ij}R^{ij}$ term in the action, because

$$\int d^4\mathbf{x}\sqrt{g}(R_{ijk\ell}R^{ijk\ell} - 4R_{ij}R^{ij} + R^2) \tag{69}$$

is a topological invariant (measuring the Euler number of the spatial slices Σ), as a consequence of the Gauss-Bonnet theorem in four dimensions.

We use W to construct the potential S_V of quantum gravity with $z = 4$ in $4 + 1$ dimensions. The high-energy limit of this theory will again be described by

$$\begin{aligned} S &= S_K - S_V \\ &= \frac{1}{2}\int dt d^4\mathbf{x}\sqrt{g}N\bigg\{\frac{4}{\kappa^2}(K_{ij}K^{ij} - \lambda K^2) \\ &\quad - \frac{\kappa^2}{4}\frac{\delta W}{\delta g_{ij}}\mathcal{G}_{ijk\ell}\frac{\delta W}{\delta g_{k\ell}}\bigg\}, \end{aligned} \tag{70}$$

with W now given by (68). κ is dimensionless, as are the two couplings α and β inherited from W. This action can be modified by relevant operators, of dimension <8. If we insist that the deformed theory satisfy detailed balance, such relevant terms in S are generated by adding relevant operators of dimension <4 to W. Either way, the theory in $4 + 1$ dimensions will be dominated at long distances by the lowest-dimension operators in S, which are again given by the scalar curvature R and the cosmological constant term. The theory flows naturally to $z = 1$, with an emergent speed of light, Newton constant, and cosmological constant.

The $z = 4$ theory in $4 + 1$ dimensions is power-counting renormalizable. If the "quantum inheritance principle" holds for the class of models satisfying the detailed balance condition described in (70), the renormalization of α and β would be the same as in the four-dimensional relativistic higher-derivative theory described by (68), which is believed to be asymptotically free [32–34]. As we mentioned in the Introduction, the asymptotic freedom of (68) would seem to make this theory an excellent candidate for solving the problem of quantum gravity in 3 + 1 dimensions, were it not for one persistent flaw: After the Wick rotation to 3 + 1 dimensions, the spectrum of physical states contains ghosts which violate unitarity in perturbation theory.

Our construction of $z = 4$ theory in 4 + 1 dimensions benefits from the asymptotic freedom of the four-dimensional higher-curvature theory (68), but avoids the pitfall of its perturbative nonunitarity. Indeed, we are only interested in the four-dimensional action W in the Euclidean signature, in order to construct the 4 + 1 dimensional action (70).

The only remaining coupling-constant renormalization in the high-energy limit of the theory in 4 + 1 dimensions is the renormalization of κ. However, κ is not an independent coupling associated with interactions; instead, it survives in the noninteracting limit, and parametrizes a family of free-field fixed point as α and β are sent to zero. In this respect, the quantum behavior of this theory would be very similar to the behavior in quantum critical Yang-Mills studied in [9], which inherits asymptotic freedom from relativistic Yang-Mills in four dimensions.

Setting $\beta = 0$ in (68) and $\lambda = 1/4$ in (70) would lead to a theory which exhibits an additional gauge invariance, acting on the fields as

$$g_{ij} \to \exp\{2\Omega(t, \mathbf{x})\} g_{ij}, \qquad N \to \exp\{4\Omega(t, \mathbf{x})\} N, \qquad N_i \to \exp\{2\Omega(t, \mathbf{x})\} N_i. \tag{71}$$

These are the local anisotropic Weyl transformations with $z = 4$.

B. $z = 4$ gravity in 3 + 1 dimensions

In three dimensions, the action of Euclidean gravity quadratic in the curvature tensor is

$$W = \frac{1}{M} \int d^3\mathbf{x} \sqrt{g} (R_{ij} R^{ij} + \beta R^2). \tag{72}$$

As in four dimensions, there are again only two independent terms in W, but for a different reason: When $D = 3$, the Riemann tensor is determined in terms of the Ricci tensor, and the Weyl tensor vanishes identically. The two couplings M and M/β are now dimensionful, of dimension 1. In power counting, this makes the theory described by (72) super-renormalizable. When we use W to generate the potential term S_V for $z = 4$ gravity in 3 + 1 dimensions, we consequently end up with a theory whose action again has the form (70), now with W given by (72) and in 3 + 1 dimensions, where it is super-renormalizable by power counting. As in all the previous examples with various values of z, relevant deformations flow the theory to $z = 1$ in the infrared.

Such super-renormalizable terms can also be added to our $z = 3$ theory of gravity described in (38). These terms will give spatial dynamics to the conformal factor of the spatial metric, improving the short-distance properties of the propagator for the scalar mode H of the metric, restoring power-counting renormalizability in the case when H is present as a physical field.

C. The case of $z = 0$: Ultralocal gravity

In the Hamiltonian formulation of general relativity, the Hamiltonian is given by a sum of constraints,

$$H = \int d^D\mathbf{x} (N \mathcal{H}_\perp + N^i \mathcal{H}_i). \tag{73}$$

Notably, the algebra of the Hamiltonian constraints $\mathcal{H}_\perp(\mathbf{x})$ and $\mathcal{H}_i(\mathbf{x})$ in general relativity is not a true Lie algebra—in particular, the constraints do not form the naively expected algebra of spacetime diffeomorphisms. Instead, the structure "constants" of the commutator of $\mathcal{H}_\perp(\mathbf{x})$ with $\mathcal{H}_\perp(\mathbf{y})$ are field dependent, because they contain the components of the spatial metric.

In [35], an alternative theory of gravity was proposed, in which the constraints do form a Lie algebra. In this theory, the commutators of $\mathcal{H}_i$ with themselves and with $\mathcal{H}_\perp$ are the same as in general relativity, but the problematic field-dependent commutator of $\mathcal{H}_\perp(\mathbf{x})$ with $\mathcal{H}_\perp(\mathbf{y})$ is simply replaced by zero. This symmetry can be viewed either as a contraction of the symmetries formed by the Hamiltonian constraints of general relativity, or as a contraction of the algebra of infinitesimal spacetime diffeomorphisms. The contracted symmetry algebra respects a dimension-one foliation of spacetime by a congruence of timelike curves. This congruence can be used to identify the points of space at different times; as a result, the spacetime in this theory of gravity carries a preferred structure of absolute space.

The theory of gravity that realizes this symmetry structure is known as the "ultralocal theory" of gravity. It is interesting to note that ultralocal gravity fits naturally into our framework of gravity models with anisotropic scaling and nontrivial dynamical exponents $z \neq 1$. As shown in [35], the required symmetries force the action of the ultralocal theory to be of the same form $S = S_K - S_V$ as the theories considered here, with the potential term containing only the cosmological constant,

$$S_V = \int dt d^D\mathbf{x} \sqrt{g} \Lambda, \tag{74}$$

and no curvature-dependent terms. There is a clear way to interpret (74) in our framework of gravities with anisotropic scaling: The value of z can be read off as one-half of the number of derivatives appearing in S_V. This is equivalent to declaring (74) to be of the same dimension as the kinetic term S_K. Either way, this approach suggests that the ultralocal theory corresponds formally to the limiting case of $z \to 0$.

Historically, the ultralocal theory of gravity has been studied for at least two additional reasons, besides the context of [35]:

(i) Ultralocal gravity was proposed by Isham in [36], in an attempt to introduce a new formal expansion parameter into general relativity. In [36], the suggested expansion parameter was the coefficient in front of the scalar curvature term in S_V, equal to one in the potential of general relativity and set equal to zero in the ultralocal theory.

(ii) Ultralocal theory is relevant for early universe cosmology in general relativity, because it captures the dynamics of Friedmann-Robertson-Walker solutions in the so-called "velocity dominated" early stages after the big bang, as was first shown by Belinsky, Khalatnikhov, and Lifshitz [37,38].

Unfortunately, the $z \to 0$ limit is rather singular, and the program outlined in (i) was never very successful. As to (ii), the embedding of ultralocal gravity into our framework of gravity with anisotropic scaling raises the possibility of interpreting the cosmological evolution as a flow, from $z \neq 1$ in the early universe to $z = 1$ observed now.

It is remarkable that, even though the action of ultralocal theory is not invariant under all spacetime diffeomorphisms, the theory exhibits "general covariance" [35,39]: In particular, the number of local symmetry generators per spacetime point is $D + 1$, i.e., the same as in general relativity.

D. Bulk-boundary correspondence in gravity at a Lifshitz point

The availability of gravity models with nontrivial values of the dynamical critical exponent z can enhance the spectrum of examples of dualities between gravity in the bulk and field theory on the boundary. This could be particularly relevant for understanding gravity duals of nonrelativistic CFTs.

After the Wick rotation of the $z = 3$ theory in $3 + 1$ dimensions to imaginary time τ, the action of this theory was rewritten in a simple form (40) with the use of an auxiliary field B^{ij}. The same rewriting applies to a much broader class of gravity models which satisfy detailed balance with some D-dimensional action W, such as the $z = 4$ models discussed above. Using this formalism, we can find a large class of classical solutions of such theories, simply by noting that if the following equation holds,

$$\frac{1}{N}(\partial_\tau g_{ij} - \nabla_i N_j - \nabla_j N_i) - \frac{\kappa^2}{2\sqrt{g}} \mathcal{G}_{ijk\ell} \frac{\delta W}{\delta g_{k\ell}} = 0, \quad (75)$$

the full equations of motion are automatically satisfied. While the full equations of motion are of second order in time derivatives and of order $2z$ in spatial derivatives, the simpler equation (75) has its degree reduced by half. (This argument is reminiscent of the BPS condition in supersymmetric theories.) A simple class of solutions to (75) can now be obtained by setting $N = 1$, $N_i = 0$, and taking $g_{ij} = g_{ij}(\mathbf{x})$ to be an arbitrary (τ-independent) solution of the equations of motion,

$$\frac{\delta W}{\delta g_{ij}} = 0, \quad (76)$$

of the D-dimensional theory whose action is W. Clearly, this solution can be trivially continued back to real time, and represents a real static solution of the full theory.

In particular, let us assume that the Euclidean action W is such that it has the Euclidean AdS_D as a solution. This situation is rather generic, and does not pose a very strong restriction on W. With this assumption, the $D + 1$ dimensional theory will have a classical solution which is the direct product of the time dimension and AdS_D,

$$\begin{gathered} N = 1, \qquad N_i = 0, \\ g_{ij} dx^i dx^j = d\rho^2 + \sinh^2\rho \, d\Omega^2_{D-1}. \end{gathered} \quad (77)$$

The boundary of this solution is $S^{D-1} \times \mathbf{R}$. The isometries of the Euclidean AdS_D induce conformal symmetries in the boundary. In addition, there is one more bulk isometry, given by time translations. Thus, the full symmetries are

$$SO(D, 1) \times \mathbf{R}. \quad (78)$$

These symmetries suggest that such a gravity theory in the bulk can serve as a possible holographic dual of dynamical field theories which are already critical in the static limit. Such problems are often encountered in the theory of dynamical critical phenomena. Starting with a universality class of a static critical system in $D - 1$ spatial dimensions, the time-dependent dynamics of the system in D dimensions can also exhibit criticality, with the characteristic property of "critical slowing down" of time-dependent correlation functions. One given static universality class can belong to several different dynamical universality classes. In particular, one universal characteristic of the dynamics is given by the critical exponent z.

If we study such a dynamical critical system on $\mathbf{R}^D$, it will exhibit the anisotropic scaling symmetry given by (10), with $i = 1, \ldots D - 1$. Another possibility is to put this system on $S^{D_1} \times \mathbf{R}$, with the spatial slices of the foliation given by S^{D-1} of a fixed radius. On such a foliation, the scale symmetry (10) is absent, since it would change the radius of the sphere. However, the system still exhibits the symmetries of conformal transformations of S^{D-1} and time translations. Thus, the conformal symmetries left unbroken by the foliation are precisely the bulk isometries (78) of the $\mathrm{AdS}_D \times \mathbf{R}$ solution of gravity theory with anisotropic scaling.

Following [40,41], a nonrelativistic version of the AdS/CFT correspondence has indeed received a lot of attention recently. The focus in this area has been primarily on the CFTs with nontrivial values of z which exhibit conventional relativistic gravity duals. It is natural to broaden this framework, and free the gravity side of the duality of the constraints imposed by relativistic invariance. The gravity models with $z \neq 1$ whose study is initiated in this paper (and in [3]) are potential candidates for describing interesting universality classes on the CFT side, and it would seem unwise to limit the attention to CFTs with relativistic gravity duals.

IV. CONCLUSIONS

In this paper, we presented a class of gravity theories with deeply nonrelativistic scaling at short distances, characterized by dynamical critical exponent z. In particular, we constructed a theory which satisfies the detailed balance condition with $z = 3$ in $3 + 1$ dimensions. This anisotropy between space and time improves the UV behavior of the models, compared to general relativity. Moreover, such theories flow naturally at long distances to an effective theory with relativistic scaling and $z = 1$, and can therefore serve as candidates for a short-distance completion of general relativity or its infrared modifications.

In this picture, Lorentz invariance is only emergent at long distances, while the fundamental description of the theory is deeply nonrelativistic. At short distances, the spacetime manifold is equipped with an extra structure, of a fixed codimension-one foliation by slices of constant time. This preferred foliation of spacetime defines a global causal structure. The existence of such a preferred causal structure puts some of the fundamental puzzles of general relativity and quantum gravity into a new perspective. In particular, various aspects of the "problem of time" [42,43] traditionally associated with the attempts to quantize general relativity are eliminated: The preferred foliation of spacetime leads to an invariant notion of time, susceptible only to time-dependent reparametrizations.

The existence of the preferred foliation of spacetime also changes the concept of black holes, and consequently the role of the information paradox and the holographic principle. In general relativity, black holes are defined as objects with an event horizon, a notion associated with the full causal structure of the entire spacetime history. Theories of gravity with anisotropic scaling and $z > 1$ at short distances are still expected to have solutions that describe compact objects. Since such theories generically flow at large distances to the relativistic value of $z = 1$, such compact solutions will likely resemble the black hole solutions of general relativity (or its infrared modifications). However, the notion of an event horizon for such solutions is emergent, and holds only approximately in the low-energy regime where the higher-derivative corrections to the equations of motion can be neglected. At short distances, the spacetime is equipped with a preferred time foliation and a causal structure, which precludes the existence of event horizons, at least for foliations without singularities [44].

If the notion of an event horizon is an emergent low-energy concept, the interpretation of the holographic principle also changes. The holographic principle is often interpreted as a stringent bound of the number of degrees of freedom in a given volume of a gravitating system, as implied by the Bekenstein-Hawking entropy carried by black holes of the same size. In a theory which is well approximated by general relativity with $z = 1$ at long distances, but changes its scaling to $z > 1$ with a preferred spacetime foliation at high energies, the notion of a holographic entropy bound applies only to degrees of freedom carrying sufficiently low energies, and should be viewed as an emergent feature of the low-energy dynamics. The high-energy degrees of freedom can evade the bound.

There is an intuitive way to understand the possibility that the holographic bound might be an emergent low-energy bound. Recall that in the Bekenstein-Hawking (BH) entropy formula, the entropy of black holes is given in terms of the area A of the horizon and the fundamental constants c, G_N and $\hbar$ by

$$S_{\mathrm{BH}} = \frac{c^3 A}{4 G_N \hbar}. \tag{79}$$

In particular, the speed of light appears in the numerator. If the speed of light is effectively going to infinity at short distances (which is the behavior found in our anisotropic gravity models), the holographic entropy bound becomes less constraining at higher energies: It only limits the number of possible low-energy degrees of freedom, in the regime where the behavior of the system is approximately relativistic.

This behavior, leading to a radical reduction of the number of degrees of freedom at low energies, is very reminiscent of a similar phenomenon, sometimes referred to as "rigidity," in ordered phases of condensed-matter systems (see for example [45]). Notably, in various examples studied in condensed matter, this rigidity at low energies is often accompanied by an emergent relativistic dispersion relation for the low-energy excitations.

Another aspect of gravity which might be strongly affected by the anisotropic scaling at short distances is cosmology. In the high-energy regime relevant at early times, the effective speed of light in gravity models with anisotropic scaling approaches infinity, and the spacetime manifold exhibits the preferred foliation by constant time slices. This modification of the laws of gravity changes the notion of locality and causality in the early stages of the universe, and can lead to new perspectives on the puzzles usually solved by inflationary scenarios.

ACKNOWLEDGMENTS

Results reported in this paper were presented at the Sowers Workshop on *What is String Theory?* at Virginia Tech (May 2007); at the Banff workshop on *New Dimensions in String Theory* (June 2008); and at the *AdS, Condensed Matter and QCD* workshop at McGill

(October 2008). I wish to thank the organizers for their hospitality, and the participants for stimulating discussions. This work has been supported by NSF Grant No. PHY-0555662, DOE Grant No. DE-AC03-76SF00098, and the Berkeley Center for Theoretical Physics.

---

[1] S. Weinberg, in *General Relativity. An Einstein Centenary Survey*, edited by S.W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1980).

[2] E.S. Fradkin and A.A. Tseytlin, Phys. Rep. **119**, 233 (1985).

[3] P. Hořava, arXiv:0812.4287, [J. High Energy Phys. (to be published)].

[4] P.C. Hohenberg and B.I. Halperin, Rev. Mod. Phys. **49**, 435 (1977).

[5] S.-K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, New York, 1976).

[6] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, England, 1999).

[7] E.M. Lifshitz, Zh. Eksp. Teor. Fiz. **11**, 255 (1941); **11**, 269 (1941).

[8] R.M. Hornreich, M. Luban, and S. Shtrikman, Phys. Rev. Lett. **35**, 1678 (1975).

[9] P. Hořava, arXiv:0811.2217.

[10] E. Ardonne, P. Fendley, and E. Fradkin, Ann. Phys. (N.Y.) **310**, 493 (2004).

[11] P.M. Chaikin and T.C. Lubensky, *Principles of Condensed Matter Physics* (Cambridge University Press, Cambridge, England, 1995).

[12] The idea that Lorentz invariance might be an emergent symmetry has a long history, going back at least to the pioneering paper [46].

[13] In differential geometry, a codimension-q foliation $\mathcal{F}$ on a d-dimensional manifold $\mathcal{M}$ is defined as $\mathcal{M}$ equipped with an atlas of coordinate systems (y^a, x^i) $a = 1, \ldots q$, $i = 1, \ldots d - q$, such that the transition functions take the restricted form $(\tilde{y}^a, \tilde{x}^i) = (\tilde{y}^a(y^b), \tilde{x}^i(y^b, x^j))$. The general theory of foliations is reviewed e.g. in [14–16].

[14] H.B. Lawson, Jr., Bull. Am. Math. Soc. **80**, 369 (1974).

[15] C. Godbillon, *Feuilletages* (Birkhauser, Boston, 1991).

[16] I. Moerdijk and J. Mrčun, *Introduction to Foliations and Lie Groupoids* (Cambridge University Press, Cambridge, England, 2003).

[17] J. Zinn-Justin, Nucl. Phys. **B275**, 135 (1986).

[18] J. Zinn-Justin and D. Zwanziger, Nucl. Phys. **B295**, 297 (1988).

[19] J.W. York, Phys. Rev. Lett. **26**, 1656 (1971).

[20] J.W. York, Phys. Rev. Lett. **28**, 1082 (1972).

[21] J.W. York, J. Math. Phys. (N.Y.) **13**, 125 (1972).

[22] G. Parisi and Y.-S. Wu, Sci. Sin. **24**, 483 (1981).

[23] M. Namiki, *Stochastic Quantization* (Springer, New York, 1992).

[24] M. Le Bellac, F. Mortessagne, and G. Batrouni, *Equilibrium and Non-Equilibrium Statistical Thermodynamics* (Cambridge University Press, Cambridge, England, 2004).

[25] E. Witten, arXiv:gr-qc/0306083.

[26] In contrast, for some theories with detailed balance, $\Psi_0 \sim \exp\{-W/2\}$ does represent a physical normalizable ground-state wave function. Examples include the Lifshitz scalar theory (as discussed, for example, in [10]), and the quantum critical Yang-Mills with $z = 2$ in $4 + 1$ dimensions [9].

[27] S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982).

[28] S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. (N.Y.) **140**, 372 (1982).

[29] S. Deser and Z. Yang, Classical Quantum Gravity **7**, 1603 (1990).

[30] B. Keszthelyi and G. Kleppe, Phys. Lett. B **281**, 33 (1992).

[31] The main difference between (58) and topologically massive gravity stems from the fact that here we are only interested in the Euclidean-signature version of (58), with the real action W. In topologically massive gravity, the Euclidean action W is interpreted as the Wick rotation of the real action from the physical signature $2 + 1$, leading to a slightly different reality condition on W, with w^2 purely imaginary. There has been a recent resurgence of interest in topological massive gravity, initiated by [47]; see also [48].

[32] K.S. Stelle, Phys. Rev. D **16**, 953 (1977).

[33] E.S. Fradkin and A.A. Tseytlin, Nucl. Phys. **B201**, 469 (1982).

[34] E.S. Fradkin and A.A. Tseytlin, Nucl. Phys. **B203**, 157 (1982).

[35] C. Teitelboim, in *General Relativity and Gravitation*, edited by A. Held (Plenum Press, New York, 1980), Vol. 1.

[36] C.J. Isham, Proc. R. Soc. A **351**, 209 (1976).

[37] V.A. Belinsky, I.M. Khalatnikov, and E.M. Lifshitz, Adv. Phys. **19**, 525 (1970).

[38] V.A. Belinsky, I.M. Khalatnikov, and E. Lifshitz, Adv. Phys. **31**, 639 (1982).

[39] M. Henneaux, Bull. Soc. Math. Belg. **31**, 47 (1979).

[40] D.T. Son, Phys. Rev. D **78**, 046003 (2008).

[41] K. Balasubramanian and J. McGreevy, Phys. Rev. Lett. **101**, 061601 (2008).

[42] C.J. Isham, arXiv:gr-qc/9210011.

[43] K.V. Kuchař, in Winnipeg 1991, General Relativity and Relativistic Astrophysics, 1991.

[44] This picture might change if we allow sufficiently singular foliations, for example, if such singularities turn out to be required for a consistent summation over spacetime topologies.

[45] P.W. Anderson, *Basic Notions of Condensed Matter Physics* (Addison-Wesley, Reading, MA, 1984).

[46] S. Chadha and H.B. Nielsen, Nucl. Phys. **B217**, 125 (1983).

[47] W. Li, W. Song, and A. Strominger, J. High Energy Phys. 04 (2008) 082.

[48] E. Witten, arXiv:0706.3359.