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Quantization of the Jackiw-Teitelboim model

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We study the phase space structure of the Jackiw-Teitelboim model in its connection variables formulation where the gauge group of the field theory is given by local $SL(2,\mathbb{R})$ [or SU(2) for the Euclidean model], i.e. the de Sitter group in two dimensions. In order to make the connection with two-dimensional gravity explicit, a partial gauge fixing of the de Sitter symmetry can be introduced that reduces it to space-time diffeomorphisms. This can be done in different ways. Having no local physical degrees of freedom, the reduced phase space of the model is finite dimensional. The simplicity of this gauge field theory allows for studying different avenues for quantization, which may use various (partial) gauge fixings. We show that reduction and quantization are noncommuting operations: the representation of basic variables as operators in a Hilbert space depends on the order chosen for the latter. Moreover, a representation that is natural in one case may not even be available in the other leading to inequivalent quantum theories.

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I. INTRODUCTION

The Jackiw-Teitelboim (JT) model [1,2] is one of the simplest but nontrivial formulations of general relativity (GR) in two-dimensional space-time with cosmological constant k. Its action is given by

$$S_{\rm JT} = \frac{1}{2}\int d^2x\sqrt{-g}\,\psi(R-2k). \tag{1.1}$$

It is invariant under space-time diffeomorphisms and leads to the Liouville equation

$$R - 2k = 0. \tag{1.2}$$

It contains only a finite number of degrees of freedom, namely one (here we assume the space-time topology $M = S^1\times\mathbb{R}$). The model may be quantized in the original variables of Jackiw and Teitelboim in a canonical framework including the two first class constraints corresponding to space-time diffeomorphism invariance [1,3].

On the other hand, one may take profit of its equivalence with a BF theory [4–6], which has a structure similar to the first order formulation of four-dimensional GR in Ashtekar's variables. Here, instead of being the four-dimensional Lorentz group, the gauge group is that of two-dimensional de Sitter or anti-de Sitter symmetry SO (1, 2), or SO(3) in the Euclidean de Sitter case—or better their covering groups, $SL(2,\mathbb{R}) \approx SU(1,1)$ or SU(2), respectively. The fields are a gauge connection 1-form ω and a scalar ϕ in the adjoint representation. A quantization in the Euclidean case was presented in [7], using spin network and spinfoam techniques [8].

The canonical formulation of the BF theory gives rise to three first class constraints whose Poisson bracket algebra reproduces the three-dimensional Lie algebra of the gauge group. The quantization may follow various roads, using some complete or partial gauge fixing [5,6,9–12], or no gauge fixing at all [7,13]. In [11], a time gauge has been used, which consists in the vanishing of the connection component $\omega^0{}_x$—which is interpreted as the space component of the *zweibein* (2-bein) form e^0—with the purpose of simulating the time gauge fixing of four-dimensional gravity leading to the Ashtekar variables formulation [8]. This partial gauge fixing leads to a reduction of the number of first class constraints to two, corresponding to the space-time diffeomorphism invariance—namely, one constraint generating the space diffeomorphisms and the other one playing the role of the Hamiltonian constraint.

Despite the extensive literature studying the JT model, there is, to our knowledge, no complete treatment of the quantization of the Lorentzian sector in its first order formulation using loop variables (for reviews on other methods see [14,15]). The main difficulty is technical: the fact that the gauge group in that case is noncompact precludes the possibility of using the standard quantization techniques that are applicable in the Riemannian case. The purpose of the present paper is twofold.

On the one hand, we study in detail the quantization of the model in the Lorentzian sector. This is achieved through a minimalistic application of the techniques developed in [16] which can also be introduced from the point of view of [17]. It is by now known [18] that the general case of a gauge theory with noncompact internal

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gauge symmetries presents important difficulties that are not completely resolved by the methods proposed in [16]. Interestingly, those difficulties vanish in the two-dimensional case and the quantization presented here is well defined.

On the other hand, we propose a new partial gauge fixing defined by the vanishing of one component of the scalar field ϕ, to compare the corresponding quantum theory with the theories already constructed [7] or under construction [12]. The question makes sense since it is well known [19,20] that, even in theories with a finite number of degrees of freedom, it may exist inequivalent quantization of the same classical theory if some assumption of the von Neumann theorem is invalid, such as for instance the existence of pairs of self-adjoint operators "p, q" obeying Heisenberg commutation relations. In a gauge theory, inequivalent quantizations can also arise from the possible inequivalence of the two customary procedures of quantization consisting in either reducing the unconstrained phase space and then quantizing or quantizing first and then imposing the constraints at the quantum level (Dirac procedure). This is particularly important in our case where, even though the number of physical degrees of freedom is finite, the unreduced phase space of the system is infinite dimensional. Consequently, in the second quantization procedure—the Dirac procedure—the von Newman theorem has no bearing and infinitely many inequivalent quantizations exist in principle.

II. THE JACKIW-TEITELBOIM MODEL IN THE *BF* FORMULATION

It is possible to extend the JT model in order to include degenerate geometry configurations by formulating it in a first order formulation taking the form of a *BF* theory [13] in two dimensions. The equivalence between the original JT model and its *BF* extension is a subtle issue of similar nature as the equivalence between metric formulations of 2 + 1 gravity and first order formulations such the *BF* and the Chern-Simons formulations (see [21] for a discussion in three dimensions; the problem is analyzed in the context of two-dimensional dilaton gravity in [22]). Similar problems arise in 3 + 1 gravity. However, in most cases such extensions of a theory lead to important simplifications in the quantum theory. In four and three dimensions the latter extensions are naturally suitable for the implementation of the loop quantum gravity techniques for quantization. As we would like to explore these techniques in two dimensions, we take the *BF* extension of the JT model as the definition of our model in this paper.

The gauge group G is de Sitter or anti-de Sitter in Riemannian or Lorentzian space-time. The infinitesimal generators are

$$J_0 := P_0, \qquad J_1 := P_1, \qquad J_2 := \Lambda,$$

with commutation relations

$$[J_0, J_1] = kJ_2, \qquad [J_0, J_2] = -J_2, \qquad [J_1, J_2] = \sigma J_1,$$

where k is the cosmological constant and σ is the metric signature, equal to 1 in the Riemannian case and to -1 in the Lorentzian case. A redefinition of the Lie algebra basis allows one to reduce its commutation rules to

$$[J_i, J_j] = f_{ij}{}^k J_k, \quad \text{with } f_{01}{}^2 = 1, \qquad f_{12}{}^0 = \sigma, \qquad f_{20}{}^1 = 1, \qquad (i, j, \ldots = 0, 1, 2), \tag{2.1}$$

which, for $\sigma = -1$ or 1, is the Lie algebra of SO(3) or SO(1, 2). We shall consider in the following their covering groups SU(2) or SL(2, $\mathbb{R}$). The invariant Killing form η_{ij} has the form of a Euclidian or Minkowskian three-dimensional metric:

$$\eta_{ij} := -\frac{\sigma}{2} f_{ik}{}^l f_{jl}{}^k = \mathrm{diag}(\sigma, 1, 1). \tag{2.2}$$

The fields are an SU(2) or SL(2, $\mathbb{R}$) connection 1-form ω^i and a scalar field ϕ^i in the adjoint representation of the gauge group. From this point on, we shall denote the internal gauge group with G, and we will use $\mathfrak{g}$ to designate its Lie algebra. We will use the explicit SL(2, $\mathbb{R}$) and SU(2) when we specialize to the Lorentzian or Riemannian models, respectively. The components ω^0 and ω^1 are interpreted as the *zweibein* components and ω^2 as the rotation, respectively, Lorentz connection.

The theory is the 2D version of *BF* theory, and can be seen as the $g \mapsto 0$ limit (g being the coupling constant) of 2D Yang Mills theory. Its action takes the form

$$S = \int_{\mathcal{M}} \phi^i F^j(\omega) \eta_{ij}, \tag{2.3}$$

and the field equations are

$$d_\omega \phi^i := d\phi^i + f_{jk}{}^i \omega^j \phi^k = 0, \qquad F^i := d\omega^i + \tfrac{1}{2} f_{jk}{}^i \omega^j \omega^k = 0, \tag{2.4}$$

where d_ω is the ω-covariant exterior differential and F^i the curvature 2-form of the connection. At first sight the action is invariant under two kinds of gauge transformations: the conventional Yang-Mills-like local G transformations generated by a Lie algebra valued scalar field λ

$$\delta_\lambda \omega = d_\omega \lambda, \qquad \delta_\lambda \phi = [\phi, \lambda], \tag{2.5}$$

whose exponentiated version gives

$$\omega' = a\omega a^{-1} + a d a^{-1}, \qquad \phi' = a\phi a^{-1}, \tag{2.6}$$

for $a = \exp(\lambda) \in G$. The active diffeomorphisms are generated by a vector field v

$$\delta_v \omega = \mathcal{L}_v \omega, \qquad \delta_v \phi = \mathcal{L}_v \phi, \tag{2.7}$$

$\mathcal{L}_v$ being the Lie derivative. However, on shell a diffeomorphism generated by v is the same transformation as a local G transformation generated by the field

$$\lambda^i = v^a \omega_a^i, \tag{2.8}$$

as can be easily checked by writing these equations in components and using the equations of motion. Therefore, in this theory the diffeomorphisms (acting on the space of solutions) can be considered as a subgroup of the local G gauge transformations.

Solutions are given by flat connections $\omega = gdg^{-1}$ for $g(x) \in G$, and (covariantly) constant ϕ fields. Locally, one can choose a gauge so that the connection $\omega = 0$. In this gauge the equation $d_\omega \phi = 0$ implies that $\phi = \text{constant}$. This is particularly important for the Lorentzian case since it implies that the causal type of ϕ (thought as a vector in three-dimensional Minkowski internal geometry) cannot change in the classical solutions. This conclusion is a global one since no gauge transformation $\phi \to g\phi g^{-1}$ can send a timelike ϕ into a spacelike one or vice versa. In fact, this property is taken over to the quantum theory where we will show that superselection sectors associated to ϕ being spacelike or timelike appear.

III. THE HAMILTONIAN FORMULATION

When $\mathcal{M} = S^1 \times R$, one can quantize the theory in the canonical framework. General topologies can in principle be considered in the path integral approach. The Hamiltonian formulation is obtained through the standard 1 + 1 space-time decomposition. More precisely, one introduces an arbitrary foliation of $\mathcal{M}$ by choosing a time function. In terms of this foliation the action becomes

$$S = \int dt \int_{S^1} dx(\phi_i \dot{\omega}^i + \omega_t^i D\phi_i). \tag{3.1}$$

We use the notations $D\phi^i := \partial\phi^i + f_{jk}{}^i \omega^j \phi^k$, $\omega^i := \omega_x^i$, $\partial := \partial_x$, x being the space coordinate. The Poisson bracket among the phase space variables is

$$\{\omega^j(x), \phi_i(y)\} = \delta_i^j \delta(x-y) \quad \text{or}$$
$$\{\omega^j(x), \phi^i(y)\} = \eta^{ij}\delta(x-y).$$

We have three first class constraints [23–25] corresponding to the three components of the Gauss law $g_i := D\phi^i = 0$. Explicitly these components are

$$\begin{aligned} g_0 &= \partial\phi^0 + \sigma(\omega^1\phi^2 - \omega^2\phi^1) \approx 0, \\ g_1 &= \partial\phi^1 + \omega^2\phi^0 - \omega^0\phi^2 \approx 0, \\ g_2 &= \partial\phi^2 + \omega^0\phi^1 - \omega^1\phi^0 \approx 0. \end{aligned} \tag{3.2}$$

The smeared Gauss constraints $g(\alpha) \equiv \int_{S^1} \mathrm{Tr}[\alpha D\phi]$ for $\alpha \in \mathfrak{g}$ are first class—they satisfy the Poisson bracket identity $\{g(\alpha), g(\beta)\} = g([\alpha, \beta])$—and generate infinitesimal G-gauge transformations:

$$\begin{aligned} \{g(\alpha), \omega^i\} &= \partial\alpha^i + f_{jk}{}^i \omega^j \alpha^k, \\ \{g(\alpha), \phi^i\} &= f_{jk}{}^i \phi^j \alpha^k. \end{aligned} \tag{3.3}$$

There are therefore three local first class constraints for the three configuration variables ω_a^i. Thus, the naive counting of degrees of freedom gives zero physical degrees of freedom. However, the naive counting is only sensitive to local excitations. The theory has indeed global degrees of freedom. In particular, if $M = S^1 \times \mathbb{R}$, an algebraic basis for the gauge invariant (Dirac) observables is given by

$$O_1 = \phi_i \phi^i, \qquad O_2 = \mathrm{Tr}\Big[P\exp\Big(-\int_{S^1}\omega\Big)\Big]. \tag{3.4}$$

The physical phase space being therefore two-dimensional, the theory has a single (global) degree of freedom. We note for further use that the quantity

$$Q := \frac{\omega^i \phi_i}{\phi^i \phi_i} \tag{3.5}$$

transforms as an Abelian connection under the special gauge transformations which leave ϕ invariant:

$$\{g(\alpha_{\text{Abel}}), Q\} \approx \partial a, \quad \text{with } \alpha_{\text{Abel}}^i = a\phi^i. \tag{3.6}$$

This holds up to the constraint $\partial(\phi_i\phi^i) \approx 0$ which follows from (3.2).

A. Partial gauge fixings

Space-time diffeomorphisms are hidden inside the larger gauge group of BF theory that in two dimensions corresponds to the local G transformations given in (2.5). Notice that the former are generated by the two components of a vector field in $\mathcal{M}$ while the latter are generated by the three components of $\lambda \in \mathfrak{g}$. In this section we partially gauge fix the symmetries of BF theory in order to establish a more direct relationship with diffeomorphisms, and hence emphasize the relationship of the model with two-dimensional gravity.

We will partially gauge fix the G gauge symmetry by requiring the fourth constraint,

$$g_3 = \phi_i n^i \approx 0, \tag{3.7}$$

where n^i is a fixed normalized vector in the internal space. Before going into the technical details of the constraint algebra, let us discuss the geometric interpretation of the partial gauge fixing introduced by the above equation. Because of the fact that $\phi \cdot \phi$ is a Dirac observable (a constant of motion), one can separate the analysis into three distinct (dynamically independent) cases in the Lorentzian case: $\phi \cdot \phi > 0$, $\phi \cdot \phi = 0$, and $\phi \cdot \phi < 0$ (in the Riemannian case there is only the first sector). The following discussion is restricted to the Lorentzian sector.

1. The "spacelike" sector: $\phi \cdot \phi > 0$

In that case the good choice of gauge fixing corresponds to $n^i = $ timelike. The condition (3.7) is expected to reduce the group $\mathrm{SL}(2, \mathbb{R})$ to a two-dimensional subgroup. A moment of reflection shows that this is given by the Cartesian product of the $U(1) \subset SL(2, \mathbb{R})$ that leaves invariant n^i, and the little group (the boosts) leaving invariant the vector ϕ^i. All this will become transparent in the following.

The choice $n^i =$ spacelike leads to a degenerate situation for phase space points where $\phi^i \propto n^i$, as on these points the little groups associated to n^i and ϕ^i coincide. This leads to complications that we will not analyze in this work.

To simplify notation we can take $n^i = (1, 0, 0)$ which simply turns (3.7) into simply $g_3 = \phi^0 \approx 0$. With this choice, the matrix $G_{\alpha\beta} = \{g_\alpha, g_\beta\}$ becomes

$$G(x, y) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sigma\phi^2\delta(x-y) \\ 0 & 0 & 0 & \sigma\phi^1\delta(x-y) \\ 0 & \sigma\phi^2\delta(x-y) & -\sigma\phi^1\delta(x-y) & 0 \end{bmatrix}. \tag{3.8}$$

In order to isolate the second class part of the previous constraints we make the following redefinition:

$$\begin{aligned} C_0 &= g_0 = \partial\phi^0 + \sigma(\omega^1\phi^2 - \omega^2\phi^1) \approx 0, \\ C_1 &= \phi^1 g_1 + \phi^2 g_2 = \phi^1(\partial\phi^1 + \omega^2\phi^0 - \omega^0\phi^2) + \phi^2(\partial\phi^2 + \omega^0\phi^1 - \omega^1\phi^0) \approx 0, \\ C_2 &= \phi^1 g_1 - \phi^2 g_2 = \phi^1(\partial\phi^1 + \omega^2\phi^0 - \omega^0\phi^2) - \phi^2(\partial\phi^2 + \omega^0\phi^1 - \omega^1\phi^0) \approx 0, \\ C_3 &= g_3 = \phi^0 \approx 0. \end{aligned} \tag{3.9}$$

With this definition the matrix $C = \{C_\alpha, C_\beta\}$ is block diagonal, namely, up to terms involving constraints:

$$C \approx \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\sigma\phi^2\phi^1\delta(x-y) \\ 0 & 0 & 2\phi^2\sigma\phi^1\delta(x-y) & 0 \end{bmatrix}. \tag{3.10}$$

This implies that the pair (C_2, C_3) is second class and can therefore be explicitly solved. For instance, they can be used to solve for ω^0. Namely,

$$\phi^0 = 0, \qquad \omega^0 = \frac{1}{2}\left(\frac{\partial\phi^1}{\phi^2} - \frac{\partial\phi^2}{\phi^1}\right). \tag{3.11}$$

The first class ones become

$$\begin{aligned} C_0 &= \omega^1\phi^2 - \omega^2\phi^1 = \epsilon_{AB}\omega^A\phi^B \approx 0, \\ C_1 &= \tfrac{1}{2}\partial(\phi^A\phi^A) \approx 0, \end{aligned} \tag{3.12}$$

where $A, B = 1, 2$. It is easy to see that the Dirac bracket among the remaining variables is

$$\{\omega^B(x), \phi^A(x)\}_D = \delta^{BA}\delta(x-y). \tag{3.13}$$

Direct computation shows that the algebra of the first class constraints is Abelian. More precisely, if we define

$$C_0(a) = \int_{S^1} dx a(x) C_0(x), \quad \text{and} \quad C_1(b) = \int_{S^1} dx b(x) C_1(x), \tag{3.14}$$

we have the following Dirac bracket algebra:

$$\{C_0(a), C_0(b)\}_D = 0 = \{C_1(a), C_1(b)\}_D \tag{3.15}$$

and

$$\{C_0(a), C_1(b)\}_D = -\frac{1}{2}\int_{S^1} dx \int_{S^1} dy a(x)\partial b(y)\epsilon_{AB}\phi^A(x) \times \{\omega^B(x), \phi^C\phi^C(y)\}_D = 0. \tag{3.16}$$

The gauge transformations generated by the first class constraints are

$$\begin{aligned} \delta_{(0)}\omega^A &= \{\omega^A, C_0(a)\}_D = -\sigma a\epsilon^{BA}\omega^B, \\ \delta_{(0)}\phi^A &= \{\phi^A, C_0(a)\}_D = -\sigma a\epsilon^{BA}\phi^B, \end{aligned} \tag{3.17}$$

which correspond to local U(1) internal rotations, and

$$\begin{aligned} \delta_{(1)}\omega^A &= \{\omega^A, C_1(a)\}_D = -\phi^A\partial a, \\ \delta_{(1)}\phi^A &= \{\phi^A, C_1(a)\}_D = 0. \end{aligned} \tag{3.18}$$

What is the geometric meaning of the transformation generated by C_1? Recall that $C_1 \equiv \phi^1 g_1 + \phi^2 g_2$ which is nothing other than the expression in the gauge (3.7) of $\phi^i g_i$. This is precisely the generator of the internal "boosts" leaving the "spacelike" field ϕ^i invariant, as anticipated above.

The Riemannian theory is fully described by the equations of this section. The only change is that both C_0 and C_1 generate U(1) transformations in that case.

2. *The "timelike" sector:* $\phi \cdot \phi < 0$

The gauge fixing analogous to the previous case would now be defined with n^i spacelike and we would expect condition (3.7) to reduce the group $SL(2, \mathbb{R})$ to the Cartesian product of the boosts leaving invariant n^i, and of the little group [the U(1) rotations] leaving invariant the vector ϕ^i. It seems however difficult to control the positivity of $\phi \cdot \phi$, which is no more automatic. A more appropriate choice is to take $\phi^i(x) = \phi(x)u^i$, where u is some fixed timelike vector. With the choice $u = (1, 0, 0)$, this amounts to adding to the constraints (3.2)—taken with $\sigma = -1$—the new constraints:

$$g_3 = \phi^1 \approx 0, \qquad g_4 = \phi^2 \approx 0.$$

The 5×5 Poisson bracket matrix $G_{\alpha\beta} = \{g_\alpha, g_\beta\}$, $\alpha, \beta = 0, \cdots, 4$ reads (up to constraints)

$$G(x, y) \approx \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix} \phi(x)\delta(x-y). \quad (3.19)$$

Only g_0 is first class, the remainder four constraints being of second class. Eliminating them through the Dirac procedure, we are left with the strong conditions,

$$\phi^A = 0, \qquad \omega^A = 0, \qquad A = 1, 2,$$

and the Dirac bracket algebra (with $\omega := \omega^0$)

$$\{\omega(x), \phi(y)\}_D = -\delta(x - y). \quad (3.20)$$

The first class constraint reads

$$g_0(x) = \partial\phi \text{ or, in integral form: } g_0(a) = -\int_{S^1} dx \partial a \phi. \quad (3.21)$$

and generates the U(1) gauge rotations which leaves ϕ^i invariant:

$$\{g_0(a), \phi(y)\}_D = 0, \qquad \{g_0(a), \omega(x)\}_D = \partial a.$$

ω plays the role of the U(1) connection. We note that the connection Q (3.5) is equivalent to ω, since $Q = \omega/\phi$ and ϕ is constrained to be a constant.

3. The "null" sector: $\phi \cdot \phi = 0$

We add to the constraints (3.2)—taken with $\sigma = -1$—the partial gauge-fixing constraint

$$g_3 = \phi^0 - 1 \approx 0,$$

plus an additional constraint $g_4 = \phi \cdot \phi \approx 0$ which imposes the null condition. Notice that g_4 commutes with the Gauss constraints and with g_3 so it is automatically first class. The constraint algebra is the same as in the spacelike sector. The first class constraints are again $C_0 = \epsilon_{AB}\phi^A\omega^B$ and $C_1 = \partial(\phi^A\phi_A)/2 - \epsilon_{AB}\phi^A\omega^B$ and $g_4 = \phi^A\phi_A - 1$. The constraint system is clearly reducible as C_1 is a combination of the other. Following the standard procedure we drop the constraint C_1. Classical solutions are maps from S^1 to S^1, due to the action of C_0 only the homotopy class of these maps is physically meaningful. The quantization of the null sector is outside the scope of this paper.[1]

[1]For a study of the topological properties of the classical solutions of a more general type of models of which this case is a particular one, see [15]

4. Diffeomorphisms, Virasoro and Abelian generators

For simplicity the following analysis is performed in the $\phi \cdot \phi > 0$ sector. Let us define

$$\Gamma = \frac{\epsilon_{AB}\phi^B\partial\phi^A}{\phi^C\phi^C}, \quad (3.22)$$

solution of the equation

$$\partial\phi^A - \epsilon^{AB}\Gamma\phi^B = 0, \quad (3.23)$$

an analog to the torsion-free connection of general relativity in the first order formulation. We may check that $\Gamma \approx \omega_0$ and that (3.23), with Γ replaced by ω_0, is a constraint, a combination of the first class constraint C_1 and of the second class one C_2.

One can introduce variables invariant under the gauge group generated by C_0 as follows:

$$\Pi \equiv \frac{1}{2}\phi^A\phi^A \quad \text{and} \quad Q \equiv \frac{\phi^A\omega^A}{\phi^C\phi^C}, \quad (3.24)$$

obeying the following equation:

$$\int_{S^1} dx a(x)\{\Gamma(x), Q(y)\}_D = 0, \quad (3.25)$$

for arbitrary $a(x) \in C^1(S^1)$. Π corresponds, in the $\phi^0 = 0$ gauge, to the invariant O_1 defined in (3.4), whereas Q is the Abelian connection (3.5). The meaning of these quantities will become clearer in the next section.

The constraints C_0 and C_1 are scalar densities of weight one. This is why they are naturally smeared with scalar functions a and b in order to produce coordinate independent quantities $C_0(a)$ and $C_1(b)$, respectively. We would like now to define an equivalent set of constraints that would be suitably smeared with vector fields of S^1. In order to do this, one needs first to define vector density constraints which can be achieved by multiplying the original ones by density one phase space functions. Without further motivation we redefine the constraints as

$$\begin{aligned} V_1 &= -\Gamma C_0 = -\frac{\epsilon_{AB}\phi^B\partial\phi^A}{\phi^C\phi^C}\epsilon_{DE}\omega^D\phi^E \approx 0, \\ V_2 &= -QC_1 = -\frac{1}{2}\frac{\phi^A\omega^A}{\phi^C\phi^C}\partial(\phi^B\phi^B) \approx 0. \end{aligned} \quad (3.26)$$

These are vector densities of weight one (or scalar densities of weight two). As long as we are away from configurations for which $\Gamma = 0$ or $Q = 0$, the previous constraints define the same constraint surface. Assuming we have two vector fields α and β, we define the smeared versions of the previous constraints in the obvious manner. Then one has that

$$\{V_1(\alpha), V_2(\beta)\}_D = 0 \quad (3.27)$$

and

$$\{V_1(\alpha), V_1(\beta)\}_D = V_1([\alpha, \beta]),$$
$$\{V_2(\alpha), V_2(\beta)\}_D = V_2([\alpha, \beta]), \tag{3.28}$$

where $[\alpha, \beta]$ is the vector field commutator. Therefore V_1 and V_2 commute with respect to each other, each of them satisfying a classical Virasoro algebra. They look like diffeomorphism generators, however none of the two generates diffeomorphisms of S^1. The combination that does this is

$$D = V_1 + V_2. \tag{3.29}$$

The analog of the Hamiltonian constraint of gravity then is

$$H = V_1 - V_2. \tag{3.30}$$

These satisfy the "gravity" algebra

$$\{D(\alpha), D(\beta)\}_D = D([\alpha, \beta]),$$
$$\{H(\alpha), H(\beta)\}_D = D([\alpha, \beta]), \tag{3.31}$$
$$\{H(\alpha), D(\beta)\}_D = H([\alpha, \beta]).$$

The constraint $D(\alpha)$ generate standard diffeomorphisms as can be checked by a direct calculation, namely,

$$\{D(\alpha), \phi^A\}_D = \alpha\Gamma\epsilon^{AB}\phi^B \approx -\alpha\partial\phi^A = \mathcal{L}_\alpha\phi^A, \tag{3.32}$$

$$\{D(\alpha), \omega^A\}_D = \alpha\Gamma\epsilon^{AB}\omega_B - \partial(\alpha Q)\phi^A$$
$$\approx -(\alpha\partial\omega^A + \partial\alpha\omega^A) = -\mathcal{L}_\alpha\omega^A, \tag{3.33}$$

where the weak equalities in (3.32) and (3.33) mean the insertion of the constraint equations. Similarly, H generates time evolution up to space diffeomorphisms and field equations.

B. Full reduction

Notice, from (3.24), that $(Q(x), \Pi)$ are conjugate U(1)-invariant fields, i.e., they strongly commute with C_0), and

$$\{Q(x), \Pi(y)\}_D = \delta(x - y). \tag{3.34}$$

Since Π also commutes with C_1, it represents a strong Dirac observable of the model. So we introduce

$$\mathcal{P} \equiv \int_{S^1} \Pi. \tag{3.35}$$

On the other hand, Q does not commute with C_1 but it transforms as an Abelian connection—cf. (3.6):

$$\delta_{(1)}Q = \{Q, C_1(a)\}_D = -\partial a. \tag{3.36}$$

Therefore, it is quite easy to define a strong Dirac observable

$$\mathcal{X} \equiv \int_{S^1} Q \tag{3.37}$$

so that

$$\{\mathcal{X}, \mathcal{P}\}_D = 1. \tag{3.38}$$

The connection Q is an $\mathbb{R}$-connection (associated to the boost structure group) in the $\phi \cdot \phi > 0$ sector, while it becomes a U(1) connection in the $\phi \cdot \phi < 0$ sector.

C. Time-gauge reduction

In this section we describe the results obtained in [11] where the "temporal gauge" was considered. This gauge fixing is based in the one taken in four dimensions, through which we can obtain a compact gauge group as the residual symmetry, as described in [8]. It consists in making the *zweibein* component $\chi := \omega_x^0$ vanish and it is implemented as an extra constraint $\chi \approx 0$. The action then reads

$$S = \int dt \int_{S^1} dx(\phi_i\dot{\omega}^i + \omega_t^i D\phi_i + B\chi), \tag{3.39}$$

and again the Dirac method is used in order to eliminate the second class constraints. The remaining constraints, namely $\mathcal{G}_0'$ and $\mathcal{G}_1'$, are

$$\mathcal{G}'_0(x) = (\omega_x^1)\mathcal{G}_0(x)$$
$$= \sigma\omega_x^1\partial_x\left(\frac{\partial_x\phi^2}{\omega_x^1}\right) + k(\omega_x^1)^2\phi^2 - \omega_x^1\omega_x^2\phi^1, \tag{3.40}$$

$$\mathcal{G}'_1(x) = \omega_x^1\mathcal{G}_1(x) = \omega_x^1\partial_x\phi^1 + \omega_x^2\partial_x\phi^2. \tag{3.41}$$

The Dirac bracket algebra of these constraints is closed:

$$\{\mathcal{G}_0'(\epsilon), \mathcal{G}_0'(\eta)\}_D = \sigma\mathcal{G}_1'([\epsilon, \eta]),$$
$$\{\mathcal{G}_0'(\epsilon), \mathcal{G}_1'(\eta)\}_D = -\mathcal{G}_0'([\epsilon, \eta]), \tag{3.42}$$
$$\{\mathcal{G}_1'(\epsilon), \mathcal{G}_1'(\eta)\}_D = -\mathcal{G}_1'([\epsilon, \eta]),$$

where $[\epsilon, \eta] = (\epsilon\partial_x\eta - \eta\partial_x\epsilon)$, which confirms that $\mathcal{G}_0'$ and $\mathcal{G}_1'$ are first class. In fact, time diffeomorphisms are generated by $\mathcal{G}_0'$ up to constraints, up to field equations ("on-shell realization"), and up to a compensating local Lorentz transformation which takes care of the time-gauge condition. The second unbroken invariance is that of space diffeomorphisms, generated by $\mathcal{G}_1'$.

Observe also that $\mathcal{G}_0'$ and $\mathcal{G}_1'$ are scalar densities of weight 1 and this ensures that they form a closed Lie algebra (3.42)—in contrast with gravity in higher dimensions where the algebra closes with field dependent structure "constants" [8,26,27]. Such a feature is characteristic of two-dimensional theories with general covariance, such as the bosonic string in the approach of [28].

A new redefinition

$$\mathcal{C}_+ = \frac{\sqrt{-\sigma}}{2}\mathcal{G}_0' - \frac{1}{2}\mathcal{G}_1', \tag{3.43}$$

$$\mathcal{C}_- = -\frac{\sqrt{-\sigma}}{2}\mathcal{G}_0' - \frac{1}{2}\mathcal{G}_1', \tag{3.44}$$

leads to the algebra

$$\{\mathcal{C}_+(\epsilon), \mathcal{C}_+(\eta)\}_D = \mathcal{C}_+([\epsilon, \eta]),$$
$$\{\mathcal{C}_-(\epsilon), \mathcal{C}_-(\eta)\}_D = \mathcal{C}_-([\epsilon, \eta]), \tag{3.45}$$
$$\{\mathcal{C}_+(\epsilon), \mathcal{C}_-(\eta)\}_D = 0,$$

which shows a factorization in two classical Virasoro algebras, as in the $\phi^0 = 0$ gauge of Sec. III A.

IV. QUANTIZATION

Quantization prescriptions take a classical theory as an input and are supposed to give us a quantum theory. As it is well known (and should be expected) this recipe is not complete and leads often to inequivalent quantum theories. Consequently, it is instructive to have explicit examples available to illustrate this point. Here we show that different natural choices lead to inequivalent quantum theories in our ultrasimple gauge field theory.

A. Dirac quantization

The physical Hilbert space is given by gauge invariant square integrable functions of G. They are therefore class functions $f(g) = f(aga^{-1})$ for all g, $a \in G$. There are important differences between the quantization of the Riemannian and Lorentzian cases. Therefore, we shall treat each case separately in this section.

1. The Riemannian case, $G = \mathbf{SU}(2)$

This case is treated in great detail in [7,29]. We briefly review the results here. As in the case of loop quantum gravity (LQG), one shifts emphasis from smooth connections to holonomies

$$g_\gamma[\omega] = P\exp\left(-\int_\gamma \omega\right)$$

along oriented paths $\gamma \subset \Sigma$. In the context of the Dirac quantization program one first introduces an auxiliary Hilbert $\mathcal{H}_{\rm aux}$ space where the holonomy $g_\gamma[\omega]$ and the scalar fields ϕ^i are represented as operators. In the present case, once holonomies and ϕ^i have been chosen as fundamental variables, there is a natural choice of $\mathcal{H}_{\rm aux}$ where diffeomorphisms are unitarily implemented. In dimension higher than two, this representation is unique (up to unitary equivalence); however, for technical reasons the uniqueness theorem [30] does not apply to the two-dimensional case: uniqueness remains an open question.

This Hilbert space is given by the Cauchy completion in an appropriate topology of the algebra of functionals of the connection that depend on the holonomy of ω along paths that are edges of arbitrary graphs in Σ (the details of this construction turn out not to be important for our simple model). In a second step one promotes the Gauss constraints (3.2) to self-adjoint operators satisfying the appropriate quantum constraint algebra, and finally looks for the physical Hilbert space $\mathcal{H}_{\rm phys} \subset \mathcal{H}_{\rm aux}$ defined by the kernel of the quantum constraints.

Because of the simplicity of our model, these steps can be shortcut and we can directly construct the physical Hilbert space $\mathcal{H}_{\rm phys}$ in one stroke. The logic is as follows: The Gauss constraints generate infinitesimal SU(2) gauge transformations. Elements of $\mathcal{H}_{\rm phys}$ are gauge invariant functions of the holonomy. As in LQG this restricts the set of possible graphs on which states are defined to closed ones. In our case there is a unique close graph corresponding to the entire initial value surface Σ. Therefore, the physical Hilbert space is given by functions of the holonomy $g[A] \in \mathrm{SU}(2)$ around $\Sigma = S^1$ which are in the kernel of the Gauss constraints (3.2). Those are square integrable functions which are invariant under $\psi(g) = \psi(aga^{-1})$ for any $a \in \mathrm{SO}(3)$ (this invariance is the residual gauge action on the based point from where the holonomy around the Universe is defined). The latter are the so-called class functions $\psi(g) \in \mathcal{L}^2(\mathrm{SO}(3))/\mathcal{G} \subset \mathcal{L}^2(\mathrm{SO}(3))$. Using the Peter-Weyl theorem they can be written as

$$\Psi(g) = \sum_j (2j+1)\psi_j \,\mathrm{Tr}(D^j[g]), \tag{4.1}$$

where $D^j[g]$ are unitary irreducible representations of SU (2), and $\psi_j = \int dg\, \overline{\mathrm{Tr}(D^j[g])}\Psi[g]$ with dg the Haar measure. Consequently, there is a natural construction of the physical Hilbert space based on the following choice of inner product:

$$\langle \Psi, \Phi\rangle = \int dg \overline{\Psi(g)}\Phi(g).$$

Any gauge invariant function $(\psi(g) = \psi(aga^{-1}) \in \mathcal{H}_{\rm phys})$ can be expanded in terms of the characters $\chi_j(g) = \mathrm{Tr}[D^j(g)]$. This are the strict analog of the so-called spin-network states of LQG. A fact that will become more important in the Lorentzian case is that class functions can be thought of as functions of the (unique up to conjugation) Cartan subgroup $H \subset \mathrm{SO}(3)$. More concretely, any SU(2) rotation can be characterized by a U(1) rotation by an angle $\nu \in [0, \pi]$ around an axis defined by a unit vector $\hat{n} \in R^3$. The latter is entirely defined by a point on the unit 2-sphere labeled by spherical coordinates $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. One can express the SU(2) Haar measure in these coordinates as follows:

$$dg = \frac{1}{2\pi^2}\sin^2(\nu)(\sin(\theta)d\theta d\phi)d\nu.$$

Now any class function depends only on the coordinate ν, namely $\psi(g) = \psi(\nu)$. Therefore, the inner product above takes the form

$$\langle \psi, \phi\rangle = \frac{2}{\pi}\int_0^\pi d\nu \sin^2(\nu)\bar{\psi}(\nu)\phi(\nu), \tag{4.2}$$

where we have explicitly performed the θ and ϕ integration. In the Lorentzian case, the analog of the last step (in a formal manipulation) leads to a trivial divergence due to the noncompactness of the gauge group. It is clear that one can regularize this divergence by simply dropping such integration. We will see this in detail in the following section.

Incidentally, notice that the characters can be expressed in terms of linear combinations of powers of $\chi_{1/2}(g)$ which shows that the most general gauge invariant functional of the generalized connection is the observable (3.4) $O_2 = \mathrm{Tr} P \exp(-\int_{S^1} \omega)$. The Dirac observable O_1 is quantized by the self-adjoint operator $O_1 = -\hbar^2 \Delta$, where Δ denotes the Laplacian on SU(2), the characters $\chi_j(g) = \mathrm{Tr}[D^j(g)]$ are its eigenstates with eigenvalues $\hbar^2 j(j+1)$, namely

$$O_1|j\rangle = \hbar^2 j(j+1)|j\rangle, \tag{4.3}$$

where we have used Dirac bracket notation $\chi_j(g) \rightarrow |j\rangle$. The Dirac observable O_2 acts by multiplication, in the spin-network basis its action is given by

$$O_2|j\rangle = c_j^{(+)}|j + \tfrac{1}{2}\rangle + c_j^{(-)}|j - \tfrac{1}{2}\rangle, \tag{4.4}$$

where $c_j^{(\pm)}$ are the corresponding Clebsh-Gordon coefficients. The gauge invariant combination O_3 of ϕ^i and $g_\Sigma[\omega]$ can be quantized using the commutator of O_1 and O_2.

2. The Lorentzian case, $G = \mathbf{SL}(2, \mathbb{R})$

The Lorentzian case is more involved, and, to our knowledge, has not been described in the literature. The main technical complication is the noncompactness of the gauge group which implies that the standard nonperturbative techniques applicable to standard gauge theories with compact groups need to be revised due to the appearance of divergences in the naive treatment. The main technical complication is due to the fact that, unlike the compact case, class functions $\psi(g) = \psi(aga^{-1})$ are not square integrable functions in the Haar measure of SL(2, $\mathbb{R}$). The main difficulty with which one needs to deal is the definition of the physical inner product for functions of SL(2, $\mathbb{R}$)/$\mathcal{G}$ so that appropriate reality conditions are satisfied by the Dirac observables. On the other hand the structure of the theory is richer.

As we saw in the Riemannian case, physical states are characterized by class functions which in turn can be thought off as functions of elements of the Cartan subgroups. The new feature in the Lorentzian sector is that there are two inequivalent (under conjugation) Cartan subgroups in SL(2, $\mathbb{R}$). On the one hand, one has $H_1 = U(1) \in$ SL(2, $\mathbb{R}$), given by rotations fixing an internal time axis, namely

$$H_1 = \left\{ g_\nu = \begin{bmatrix} \cos(\nu) & \sin(\nu) \\ -\sin(\nu) & \cos(\nu) \end{bmatrix} \right\}, \tag{4.5}$$

and $H_2 \in$ SL(2, $\mathbb{R}$) given by the subgroup of boosts fixing some spacelike internal direction, explicitly

$$H_2 = \left\{ g_\eta = \begin{bmatrix} \exp(\eta) & 0 \\ 0 & \exp(-\eta) \end{bmatrix} \right\}. \tag{4.6}$$

This means that conjugation $g \rightarrow aga^{-1}$ for fixed $g \in$ SL(2, $\mathbb{R}$) and arbitrary $a \in$ SL(2, $\mathbb{R}$) generate orbits (gauge orbits) which are labeled by elements of H_1, H_2, respectively[2].

Therefore, gauge invariant states $\Psi(g) = \Psi(aga^{-1})$ can be characterized by two functions

$$\Psi[g] = \begin{cases} \psi_1(\nu) & \text{for } [g] \in H_1 \\ \psi_2(\eta) & \text{for } [g] \in H_2. \end{cases}$$

As mentioned above, the noncompactness of the gauge group implies that gauge invariant states $\Psi(g) = \Psi(aga^{-1})$ are not square integrable with respect to the Haar measure. The reason is that physical states are constant on noncompact adjoint orbits. However, an inner product can be introduced in such a way that these states are normalizable and the appropriate self-adjoint property of observables holds.

This is easily done by mimicking what we did in the previous section when finding an explicit parametrization of the Haar measure in terms of the Cartan subgroup. It is still true that one can write a regular element of SL(2, $\mathbb{R}$) as Abelian "rotations around an axis $\hat{n}$." The main difference is that the unit vector $\hat{n}$ can be either timelike or spacelike. In other words the (rotationally invariant) 2-sphere of directions is now replaced by the [SO(2, 1) invariant] timelike hyperboloid h_0 (given by the points in Minkowski internal space-time $x^i x^j \eta_{ij} = 1$) together with the future and past spacelike hyperboloid $h_\pm$ (given by the points in Minkowski internal space-time[3] $x^i x^j \eta_{ij} = -1$). The second difference is that "rotations around $\hat{n}$" are now standard U(1) rotations (elements of H_1) only when $\hat{n} \in h_\pm$, while they are replaced by boosts (elements of H_2) when $\hat{n} \in h_0$.

Elements of $g \in$ SL(2, $\mathbb{R}$) can be modeled by points on three-dimensional de Sitter space-time (thought of as embedded in a four-dimensional flat space-time of signature $(+ + - -)$) as follows:

$$g = \begin{pmatrix} x^0 + x^3, & x^1 + x^2 \\ x^1 - x^2, & x^0 - x^3 \end{pmatrix} (x^0)^2 - (x^1)^2 + (x^2)^2 - (x^3)^2 = 1.$$

In terms of these coordinates the invariant measure takes the simple form

$$dg = dx^0 dx^1 dx^2 dx^3 \delta((x^0)^2 - (x^1)^2 + (x^2)^2 - (x^3)^2 - 1).$$

Elements $g \in$ SL(2, $\mathbb{R}$) equivalent by conjugation to ele-

[2]More precisely, this is true for the so-called regular elements of SL(2, $\mathbb{R}$) which are an open subset of the full group with complement of measure zero with respect to the Haar measure [31].

[3]The Minkowski metric is $\eta_{ij} = \mathrm{diag}(-1, 1, 1)$.

ments in $H_1 \subset \mathrm{SL}(2,\mathbb{R})$ are characterized by $|\mathrm{Tr}[g]| \le 2$. It is easy to see that these elements can be described in terms of hyperbolic coordinates as

$$x^0 = \cos[\nu] \qquad x^1 = \sin[\nu]\sinh[\rho]\cos[\phi]$$
$$x^2 = \sin[\nu]\cosh[\rho] \qquad x^3 = \sin[\nu]\sinh[\rho]\sin[\phi],$$

where $\rho \in \mathbb{R}^+$ and $\phi \in [0, 2\pi]$ label points on $h_\pm$. For these regular elements the invariant measure becomes

$$dg = \sinh(\rho)\sin^2(\nu)d\nu d\rho d\phi, \qquad \text{for } [g] \in H_1. \quad (4.7)$$

Elements $g \in \mathrm{SL}(2,\mathbb{R})$ equivalent by conjugation to elements in $H_2 \subset \mathrm{SL}(2,\mathbb{R})$ are characterized by $|\mathrm{Tr}[g]| \ge 2$. In terms of hyperbolic coordinates they are characterized by

$$x^0 = \cosh[\eta] \qquad x^1 = \sinh[\eta]\sin[\theta]\cosh[\rho]$$
$$x^2 = \sinh[\eta]\sin[\theta]\sinh[\rho] \qquad x^3 = \sinh[\eta]\cos[\theta],$$

where $\rho, \in \mathbb{R}^+$, and $\theta \in [0, \pi]$ label points on h_0. Explicitly we have $\mathrm{Tr}[g] = 2\cosh[\eta]$. The invariant measure becomes

$$dg = \sin(\theta)\sinh^2(\eta)d\eta d\rho d\phi, \qquad \text{for } [g] \in H_2. \quad (4.8)$$

Indeed the physical Hilbert space $\mathcal{H}_{\mathrm{phys}} = \mathcal{H}_1 \oplus \mathcal{H}_2$ is such that $\Psi(g) = (\psi_1(\nu), \psi_2(\eta)) \in \mathcal{H}_{\mathrm{phys}}$ with $\psi_1(\nu) \in \mathcal{H}_1$ and $\psi_2(\eta) \in \mathcal{H}_2$. The inner product is, respectively,

$$\langle \psi, \phi \rangle_1 = \int_0^{2\pi} d\nu \sin^2(\nu)\bar{\psi}(\nu)\phi(\nu),$$
$$\text{and} \quad \langle \psi, \phi \rangle_2 = \int_0^{\infty} d\eta \sinh^2(\eta)\bar{\psi}(\eta)\phi(\eta),$$

where the integration measure is defined by dropping the redundant integrations from the invariant measures (4.7) and (4.8), respectively. The two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are superselection sectors. Harmonic analysis on $\mathrm{SL}(2,\mathbb{R})$ implies that any function $f(g) \in \mathcal{L}^2(\mathrm{SL}(2,\mathbb{R}))$ can be written as

$$f(g) = \sum_{n\ge1}(2n-1)\mathrm{Tr}(f_n^+ D_n^+[g]) + \sum_{n\ge1}(2n-1) \times \mathrm{Tr}(f_n^- D_n^-[g]) + \int_0^\infty ds\mu(s)\mathrm{Tr}(f_s D_s[g]). \quad (4.9)$$

As in the Euclidean case, physical states can be spanned in terms of characters. From the point of view of the space $\mathcal{L}^2(\mathrm{SL}(2,\mathbb{R}))$ these states are distributional. The restriction of the characters to H_1 and H_2 gives an orthonormal basis of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, of eigenstates of the Dirac observable $O_1 = \phi \cdot \phi = -\hbar^2\Delta$. In $\mathcal{H}_1$ the Laplacian takes the explicit form $\Delta = \sin(\nu)^{-1}\partial_\nu^2\sin(\nu) + 1/4$ and the characters are $\chi_n^\pm(\nu) = \pm\exp(\pm i(n-1)\nu)/(2i\sin(\nu))$ and the eigenvalues of O_1 are given by $\hbar^2(n-\frac{1}{2})(n-\frac{3}{2})$. Therefore, the spin-network states $\chi_n^\pm(\nu) \to |n, \pm\rangle$, which satisfy

$$\langle n, +|m, +\rangle_1 = \langle n, -|m, -\rangle_1 = \delta_{nm} \quad \text{and}$$
$$\langle n, +|m, -\rangle_1 = 0,$$

are a basis for $\mathcal{H}_1$ diagonalizing O_1:

$$O_1|n\rangle = -\hbar^2(n-\tfrac{1}{2})(n-\tfrac{3}{2})|n\rangle. \quad (4.10)$$

The Dirac observable O_2 acts by multiplication by $O_2 = 2\cos(\nu)$ in $\mathcal{H}_1$. In the spin-network basis its action is given by

$$O_2|n, \pm\rangle = |n \pm 1, \pm\rangle + |n \mp 1, \pm\rangle. \quad (4.11)$$

In $\mathcal{H}_2$ the Laplacian takes the explicit form $\Delta = -\sinh(\eta)^{-1}\partial_\eta^2\sinh(\eta) + 1/4$ and the characters are $\chi_s(\eta) = \pm\cos(s\eta)/|\sinh(\eta)|$ and the eigenvalues of O_1 are given by $\hbar^2(s^2 + \frac{1}{4})$. Now spin-network states are labeled by a real parameter $\chi_s(\nu) \to |s\rangle$. We have

$$O_1|s\rangle = \hbar^2(s^2 + \tfrac{1}{4})|s\rangle. \quad (4.12)$$

The Dirac observable $O_2 = \cosh(\eta)$ acts by multiplication. Its action on spin-network states is not a spin-network state.

Is there a two-dimensional geometric interpretation of states in the above quantization? The answer to this question is in the affirmative as long as we construct the geometric interpretation in terms of the fundamental variables at hand in the *BF* theory formulation of the JT model. The physical inner product given above Eq. (4.9) can be given a path integral representation [7] which can formally be expressed as

$$\langle \bar{\psi}, \phi \rangle = \int \mathcal{D}\omega\,\delta[F(\omega)]\bar{\psi}(\omega_0)\psi(\omega_1), \quad (4.13)$$

where one integrates over space-time connections ω in a cylinder $M = S^1 \times [0, 1]$, and ω_0 and ω_1 are the pull back of the space-time connection to the corresponding boundaries. The physical inner product between spin-network states can be seen as an evolution of a state of definite $\phi \cdot \phi$ eigenvalue in a 2D manifold where ω is flat.

B. Time-gauge quantization

The purpose of [11,12] is a quantization along the lines of loop quantization in $1 + 3$ dimensions with a time-gauge fixing, in the presumably simpler case of the $(1 + 1)$-dimensional Jackiw-Teitelboim model. However, the procedure still remains somewhat more complicated than in the other cases studied in the present paper. The construction of the kinematical Hilbert space is based on the wave functionals defined on the configuration space spanned by the "holonomies" of the scalar fields ϕ^1 and ϕ^2 defined in Sec. III C, the "polymerlike" scalar product used there leading to nonseparability of the Hilbert space [32]. The conjugate fields ω^1 and ω^2 are represented as functional differential operators, which are diagonal in a spin-network- like orthonormal basis.

The construction of operators representing the classical constraints (3.41) goes through a cell regularization, and the hope is to check the algebra (3.42) at the quantum level, the final task being that of solving the quantum constraints.

C. Quantization in the $\phi \cdot n = 0$ gauge

Recall that the $\phi \cdot n = 0$ gauge in the Lorentzian case, as described classically in Sec. III A, lets us with two first class constraints C_0 and C_1 (3.12) in the $\phi \cdot \phi > 0$ case, C_0 generating U(1) internal rotations in the (ϕ^1, ϕ^2) plane, and C_1 a boost leaving the vector $\phi = (0, \phi^1, \phi^2)$ invariant. In the case $\phi \cdot \phi < 0$, we are left with one constraint g_0 (3.21) generating U(1) internal rotations leaving the vector $\phi = (\phi, 0, 0)$ invariant. In the former case, to the transformation generated by C_1 is associated the Abelian connection Q (3.5). In the latter case, to the U(1) invariance is associated the connection $\omega = \omega^0$. In the spirit of loop quantization, one must thus choose the holonomies H_I of Q or ω along intervals I of S^1 as the configuration space variables (from the connections we go to the generalized connections). These Abelian holonomies are given by

$$H_I = \exp\left(-\int_I dx \mathcal{T}(x)\tau_\star\right), \tag{4.14}$$

with

$$\mathcal{T} = Q, \qquad \tau_\star = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ for the sector } \phi \cdot \phi > 0,$$

$$\mathcal{T} = \omega, \qquad \tau_\star = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ for the sector } \phi \cdot \phi < 0.$$

Notice that these are the generators whose exponentiation leads to group elements of the form (4.6) (a boost) and (4.5) (a rotation), respectively. The holonomy $H = H_{S^1}$ is obviously a Dirac observable, element of the adjoint representation of the group generated by C_1 or g_0.

Let us now discuss the two sectors separately.

1. Sector $\phi \cdot \phi > 0$

It will be convenient to use the invariant (under the action of C_0) variables (3.24), namely

$$\Pi \equiv \frac{1}{2}\phi^A\phi^A \quad \text{and} \quad Q \equiv \frac{\phi^A\omega^A}{\phi^C\phi^C}.$$

The advantage of these variables is that Π is nothing else but O_1 (3.4), one of the Dirac observables, and Q is the Abelian connection discussed above, transforming under the remaining Abelian gauge symmetry generated by the constraint C_1 (boosts) as

$$\{Q, C_1(b)\}_D = -\partial b.$$

[See (3.12) and (3.14)]. A finite transformation of H_I as given by (4.14) reads

$$H_I' = \exp(\tau_\star(b_t - b_s))H_I,$$

where b_s and b_t are the values of b at the ends (source/target) of the interval I. Therefore, the holonomy around the space, $H := H_{S^1}$, being gauge invariant is a Dirac observable. It takes the form

$$H = \exp(-\tau_\star\eta) = \begin{bmatrix} \exp(\eta) & 0 \\ 0 & \exp(-\eta) \end{bmatrix},$$

$$\text{with } \eta = \int_{S^1} dx Q(x) \text{ and } \tau_\star \equiv \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.15}$$

From the classical Dirac bracket algebra (3.13) follows:

$$\{\Pi(x), H\}_D = \tau_\star H, \tag{4.16}$$

and consequently

$$\{\Pi, \eta\}_D = -1. \tag{4.17}$$

This suggests to take Π and η as the classical phase space coordinates, with $\Pi \in \mathbb{R}_+$ (real positive) and $\eta \in \mathbb{R}$. [Owing to the constancy of Π following from the constraint C_1, we can take for Π the value of $\Pi(x)$ at any point $x \in C_1$.]

We can quantize this sector in the conventional Schrödinger scheme. Here, elements of the physical Hilbert space are functions $\Phi(\eta)$—with η defined in (4.15)—belonging to $L_2(\mathbb{R}, d\eta)$, and Π is represented by the operator $\hat{\Pi} = -i\hbar d/d\eta$. Eigenfunctions of $\hat{\Pi}$ are unnormalizable "plane waves" $\exp(i\rho\eta/\hbar)$. Restricting to "wave packets" of positive frequency $\rho > 0$. we recover the set of positive real number as the (now continuous) spectrum of $\hat{\Pi}$. The spectrum of $\Pi = O_1$ is thus constituted by the positive real numbers in agreement with the results of the previous section, but it does not quite coincide with it for small values of s, as there is no discrete gap here [in contrast with (4.12)]. However, there is agreement in the asymptotic regime.

Alternatively, as H is an element of the Abelian boost group, parametrized by the real number η one could explore the quantization based on the so-called polymer representations [19,20] Our task is to construct a physical Hilbert space $\mathcal{H}_{\text{phys}}$ as a representation of the quantum algebra $[\hat{\Pi}, \hat{\eta}] = -i\hbar$ or, better:

$$[\hat{\Pi}, \hat{h}] = \hbar\hat{h}, \qquad \text{where } h = \exp(i\eta), \tag{4.18}$$

corresponding to the classical algebra (4.17). The elements of $\mathcal{H}_{\text{phys}}$ will be taken as functions of the boost group, which we parametrize by real numbers ρ considered as elements of the Bohr compactification [33] $\mathbb{R}_B$ of $\mathbb{R}$. Accordingly, the integration measure for the "almost periodic functions"

$$\Phi_s(\rho) = \exp\left(-\frac{i}{\hbar}s\rho\right), \qquad s \in \mathbb{R},$$

is given by

$$\int_{\mathbb{R}_B} d\mu(\rho)\Phi_s(\rho) = \delta_{s0}. \tag{4.19}$$

The "cylindrical vector space" $\mathcal{H}_{\rm cyl}$ is defined as the set of all finite linear combinations of almost periodic functions, and a Hermitian scalar product is defined with the help of the integration measure (4.19). Hence, in particular,

$$\langle\Phi_s|\Phi_t\rangle = \delta_{st}. \tag{4.20}$$

The action of the operator $\hat{\eta}$ is defined through the action of its almost periodic counterparts $\Phi_t(\rho)$ for any $t \in \mathbb{R}$:

$$\hat{\Phi}_t\Phi_s(\rho) = \Phi_{t+s}(\rho),$$

whereas that of $\hat{\Pi}$ is defined by

$$\hat{\Pi}\Phi_s(\rho) = -i\hbar\frac{d}{d\rho}\Phi(\rho),$$

in agreement with the algebra (4.18).

We observe that the Φ_s are eigenvectors of $\hat{\Pi}$ with eigenvalue s. Since this operator owes to be positive beyond being self-adjoint, we define $\mathcal{H}_{\rm cyl+}$ as the space generated by the restricted basis $\{\Phi_s; s \geq 0\}$—in analogy with the separation of the positive and negative frequency parts in the relativistic quantum theory of free fields. Finally, the physical Hilbert space $\mathcal{H}_{\rm phys}$ is defined as the Cauchy completion of $\mathcal{H}_{\rm cyl+}$ with respect to the norm induced by the scalar product (4.20). Possessing an uncountable orthonormal basis, $\mathcal{H}_{\rm phys}$ is nonseparable. We note that the spectrum of $\hat{\Pi}$ coincide with the one found in the Schrödinger representation, yet it must be considered as "discrete" as the corresponding eigenvectors have finite norm.

2. *Sector $\phi \cdot \phi < 0$*

In this sector, as discussed in Sec. III A 2, the classical phase space variables are $\omega = \omega^0$ and $\phi = \phi^0$, obeying the canonical Dirac bracket algebra (3.20). The constraint g_0 (3.21) lets ϕ to be a constant, and ω transforms under g_0 as the connection associated to the residual group U(1) of gauge transformations preserving the gauge-fixing conditions $\phi^1 = \phi^2 = 0$. Thus, the classical Dirac observables are ϕ—taken at an arbitrary valor of the space coordinate x—and the holonomy along the space slice:

$$H = \exp(\tau_\star\xi) = \begin{bmatrix} \cos(\xi) & \sin(\xi) \\ -\sin(\xi) & \cos(\xi) \end{bmatrix},$$
$$\xi = \int_{S^1} dx\omega(x), \qquad \tau_\star = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

It is convenient to perform the quantization in terms of the variables ϕ and ξ. From now on we switch emphasis from $H(\xi) \to h(\xi) := \exp(i\xi)$. From the classical Dirac bracket $\{\phi(x), h\}_D = ih$, we define the corresponding quantum commutator as

$$[\hat{\phi}, \hat{h}] = \hbar\hat{h}.$$

With this choice of variables, elements of the physical Hilbert space are continuous functions of U(1), i.e., $\Phi(\theta)$, with $\theta \in [0, 2\pi]$, and $h(\theta)$ acts simply by multiplication. There is a spin-network basis given by the unitary irreducible representations of U(1), explicitly:

$$\Phi_n(\theta) = \frac{1}{\sqrt{2\pi}}\exp(in\theta), \qquad \text{for all } n \in \mathbb{Z}.$$

The spectrum of $\hat{\phi}$ is discrete, given by its eigenvalues $n\hbar$. Thus the spectrum of $O_1 = \hat{\phi}\cdot\hat{\phi}$ is in agreement with the results of the previous section but has eigenvalues that differ from those corresponding to (4.10) for a small value of n. However, the eigenvalues of O_1 approach each other in the large (in Planck units) eigenvalue limit.

D. Quantization after totally reducing

The Dirac bracket algebra given by Eq. (3.38) tells us that the reduced phase space of the Jackiw-Teitelboim model corresponds to that of a system with a single degree of freedom. There are however prequantization conditions that are expected to make the spectra of quantum observables compatible (at least in the large eigenvalue limit) with the results of the previous sections (for details, see [14,15]).

V. CONCLUSIONS

This paper is divided in two parts. In the first part we performed the canonical analysis of the JT model and studied the phase space structure of the theory. We showed that there are dynamically independent sectors corresponding to the cases $\phi \cdot \phi > 0$ (spacelike), $\phi \cdot \phi = 0$ (null), and $\phi \cdot \phi > 0$ (timelike). The system has no local degrees of freedom. However, for $M = S^1 \times \mathbb{R}$ there is one global topological degree of freedom in the timelike and spacelike sectors, respectively. The null sector is special, classical (physically inequivalent) solutions are labeled by a discrete parameter (winding number) [15].

A partial gauge fixing allows, in the spacelike case, to reduce the de Sitter gauge symmetry of the JT model to the two-dimensional diffeomorphism invariance of gravity. We explicitly showed in that case how, after partial gauge fixing, the remaining first class constraints relate to the generators of two Virasoro symmetries, and the familiar diffeomorphism and scalar constraints of gravitational theories.

In the second part we studied the quantization of the JT model using background independent techniques. We first performed the quantization of the model without the introduction of gauge fixing. Even though this was well known in the Riemannian case, the Lorentzian case presented some technical difficulties related to the noncompactness of the gauge group. The difficulties were overcome using group averaging techniques which natu-

TABLE I. Spectrum of the observable O_1 defined in Eq. (3.4), according to various quantization schemes.

	Riemannian Spectrum O_1	Lorentzian Spectrum O_1 in sector I	Spectrum O_1 in sector II
No gauge fixing	$\hbar^2 n(n+1)$ for $n \in \mathbb{Z}/2$	$-\hbar^2(n-\frac{1}{2})(n-\frac{3}{2})$ for $n \in \mathbb{Z}$	$\hbar^2(s^2+\frac{1}{4})$ for $s \in \mathbb{R}^+$
$n \cdot \phi = 0$ gauge	$\hbar^2 n^2$ for $n \in \mathbb{Z}$	$-\hbar^2 n^2$ for $n \in \mathbb{Z}$	$\hbar^2 s^2$ for $s \in \mathbb{R}^+$
Time gauge	?	?	?

rally lead to two possible Hilbert space representations of the fundamental observables. These inequivalent quantum theories have both physical interpretation as they are in one-to-one correspondence with the superselection sectors $\phi \cdot \phi > 0$ (spacelike), and $\phi \cdot \phi > 0$ (timelike).

Alternatively, one can partially (or totally) reduce the gauge freedom of the system at the classical level by introducing gauge conditions, and explore afterwards the quantization of the reduced system. We have explicitly shown that representations of the basic fields obtained following this alternative avenue are not equivalent to the previous ones. The main results of the second part are summarized in Table I, which shows the various inequivalent sectors of the quantum Jackiw-Teitelboim theory. We have called I and II the sectors (phases) of the Lorentzian theory which are described by the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ in the no gauge-fixing part (Sec. IV A), and by the $\phi \cdot \phi < 0$ and $\phi \cdot \phi > 0$ sectors in the $n \cdot \phi = 0$ gauge (Sec. IV C).

We observe that the spectrum of O_1 in the nongauge-fixed and $n \cdot \phi$ gauge coincide in the large eigenvalue limit in both the Riemannian and the Lorentzian phase I. This is compatible with a common semiclassical limit. But, in the Lorentzian phase II the spectrum is that of all positive real numbers, discrete or "continuous" depending on the quantization being "polymeric" or of the "Schrödinger " type. Finally, the spectrum of O_1 is completely changed (continuous) in the fully reduced case, for the Riemannian and for both phases of the Lorentzian theory. The gauge group structure vanishing in this case implies that the kind of representations used in the former cases are not even available. As discussed in Sec. IV B, one could use a polymerlike representation to recover a discrete spectrum but the microscopic Planckian structure is lost in the fully reduced setting.

We have not attempted the quantization of the null sector. This problem seems quite subtle. We notice that the group averaging quantization of Sec. IV A seems to miss that sector. It would be desirable to fully understand the quantum nature of the null sector. At this stage, this is beyond the scope of this paper.

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