PHYSICAL REVIEW D **79,** 084003 (2009)

Covariant Galileon

C. Deffayet*

AstroParticule & Cosmologie, UMR 7164-CNRS, Université Denis Diderot-Paris 7, CEA, Observatoire de Paris, 10 rue Alice Domon et Léonie Duquet, F-75205 Paris Cedex 13, France

G. Esposito-Farèse†

GReCO, Institut d'Astrophysique de Paris, UMR 7095-CNRS, Université Pierre et Marie Curie-Paris 6, 98bis boulevard Arago, F-75014 Paris, France

A. Vikman‡

CCPP, New York University, Meyer Hall of Physics, 4 Washington Place, New York, New York 10003, USA

(Received 14 January 2009; published 3 April 2009)

We consider the recently introduced ''Galileon'' field in a dynamical spacetime. When the Galileon is assumed to be minimally coupled to the metric, we underline that both field equations of the Galileon and the metric involve up to third-order derivatives. We show that a unique nonminimal coupling of the Galileon to curvature eliminates all higher derivatives in all field equations, hence yielding second-order equations, without any extra propagating degree of freedom. The resulting theory breaks the generalized ''Galilean'' invariance of the original model.

DOI: 10.1103/PhysRevD.79.084003 PACS numbers: 04.50.−h, 11.10.−z, 98.80.−k

I. INTRODUCTION

An interesting scalar-field theory, named *Galileon*, was recently introduced in Ref. [1] inspired by the Dvali-Gabadadze-Porrati (DGP) model [2] and its ability to produce an accelerated expansion of the Universe without introducing any dark energy or cosmological constant [3,4]. More specifically, taking the so-called decoupling limit of the DGP model, one can extract an effective theory for a scalar field π argued to describe the scalar sector of the original model [5,6]. In particular, this scalar-field theory still exhibits interesting properties of the original model, such as the existence of the self-accelerating branch of DGP cosmology. Similarly to the decoupling limit of massive gravity [7], the action of this effective theory contains second-order derivatives acting on the scalar field. But in contrast to massive gravity, in which the equations of motion contain fourth-order derivatives, signaling the presence of ghostlike modes or at least extra degrees of freedom [8,9], the field equation for π is in fact of *second* order. Indeed, it is well known that, according to Ostrogradski's theorem [10], higher-derivative theories contain extra degrees of freedom, and are usually plagued by negative energies and related instabilities (see e.g. [11]). In the decoupling limit of the DGP model,[1] it is thus striking that such additional degrees of freedom do not appear. Moreover, in addition to the usual constant-shift symmetry $\pi \to \pi + c$ in field space, due to the absence of undifferentiated π's in the action, this theory also possesses a symmetry under constant shifts of the gradient $\partial_\mu \pi \to \partial_\mu \pi + b_\mu$. This symmetry implies that the π field equation contains *only* second derivatives (but no first derivative nor any undifferentiated field).

The recent Ref. [1] classified all possible four-dimensional actions for a scalar field, π, which have the same properties as the DGP effective theory discussed above: having Lorentz invariant equations of motion which contain only second derivatives of π on a flat (Minkowski) background. Besides known Lagrangians which are (i) linear in π, $\mathcal{L}_1 = \pi$, (ii) the standard quadratic kinetic Lagrangian for π in the form $\mathcal{L}_2 = \partial_\mu \pi \partial^\mu \pi$, and (iii) a cubic Lagrangian $\mathcal{L}_3 = \Box\pi \partial_\mu \pi \partial^\mu \pi$ which is the one obtained in the decoupling limit of DGP model, Ref. [1] argues there only exist two possible other Lagrangians which share the same property in four spacetime dimensions. The first one, named $\mathcal{L}_4$ is made of a linear combination of four terms each made of a product of four π and a total of six derivatives acting on π. The second one, named $\mathcal{L}_5$, is made of a linear combination of seven terms each made of a product of five π and a total of eight derivatives acting on π. The linear combinations in $\mathcal{L}_4$ and $\mathcal{L}_5$ are uniquely chosen (up to total derivatives) so that the equations of motions derived from the corresponding action only contain second-order derivatives acting on π. The analysis carried in [1] was made in flat spacetime and one can legitimately expect things to change radically when one considers the same theory on a curved dynamical spacetime.

Indeed, first, when varied with respect to π, the Lagrangians $\mathcal{L}_4$ and $\mathcal{L}_5$ are expected to generate third-

*deffayet@iap.fr
†gef@iap.fr
‡alexander.vikman@nyu.edu

[1]Note that this decoupling limit not only exists in massive gravity and in the DGP model, but also in the generalized framework of Ref. [12].

order derivatives acting on the metric. These derivatives, in the form of gradients of the Riemann tensor, appear via commutations of fourth-order covariant derivatives acting on π, which eventually disappear (together with third-order derivatives). Second, when varied with respect to the metric, the same Lagrangians are expected to generate third-order derivatives acting on π, from the covariant derivatives present in the π action. Should such higher derivatives be generated, instabilities are expected to be present, or at least extra degrees of freedom to propagate, as can simply be understood from the counting of initial conditions necessary to have a well posed Cauchy problem. In fact, it is argued in [1] that the Galileon model cannot be covariantized while keeping all its desired properties, in particular, its symmetries and the absence of ghosts.

It is our purpose here to examine this question in some detail. We will first show that the naive expectation summarized above turns out to be true: namely, a naive covariantization of the π action leads to third-order derivatives both in the π and the metric field equations. However, we will show that there is a possible (and unique[2]) nonminimal coupling to curvature that has the property of removing at the same time higher derivatives in both equations of motions (those of π and in the π energy-momentum tensor). The full set of equations concerning the simpler Lagrangian $\mathcal{L}_4$ will be discussed in the bulk of the paper, while the heaviest ones corresponding to the more complex $\mathcal{L}_5$ will be given in the Appendix.

II. MINIMALLY VS NONMINIMALLY COUPLED GALILEON

As shown in [1], any linear combination of $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, defined in the Introduction, $\mathcal{L}_4$ and $\mathcal{L}_5$, defined as follows[3]

$$\mathcal{L}_4 = (\Box\pi)^2(\pi_{;\mu}\pi^{;\mu}) - 2(\Box\pi)(\pi_{;\mu}\pi^{;\mu\nu}\pi_{;\nu}) - (\pi_{;\mu\nu}\pi^{;\mu\nu})(\pi_{;\rho}\pi^{;\rho}) + 2(\pi_{;\mu}\pi^{;\mu\nu}\pi_{;\nu\rho}\pi^{;\rho}), \quad (1)$$

$$\begin{aligned}\mathcal{L}_5 = {} & (\Box\pi)^3(\pi_{;\mu}\pi^{;\mu}) - 3(\Box\pi)^2(\pi_{;\mu}\pi^{;\mu\nu}\pi_{;\nu}) - 3(\Box\pi) \\ & \times (\pi_{;\mu\nu}\pi^{;\mu\nu})(\pi_{;\rho}\pi^{;\rho}) + 6(\Box\pi)(\pi_{;\mu}\pi^{;\mu\nu}\pi_{;\nu\rho}\pi^{;\rho}) \\ & + 2(\pi_{;\mu}{}^{\nu}\pi_{;\nu}{}^{\rho}\pi_{;\rho}{}^{\mu})(\pi_{;\lambda}\pi^{;\lambda}) + 3(\pi_{;\mu\nu}\pi^{;\mu\nu}) \\ & \times (\pi_{;\rho}\pi^{;\rho\lambda}\pi_{;\lambda}) - 6(\pi_{;\mu}\pi^{;\mu\nu}\pi_{;\nu\rho}\pi^{;\rho\lambda}\pi_{;\lambda}), \end{aligned} \quad (2)$$

(where a semicolon denotes the covariant derivative ∇_μ associated with the metric $g_{\mu\nu}$ and π is a scalar field, the Galileon) has the property that, considered on flat spacetime where $\nabla_\mu = \partial_\mu$, the derived equations of motion for π only contain second-order derivatives of π. Actually, the various terms written in Eqs. (1) and (2) are not independent in flat spacetime. For instance, the combination $(\Box\pi)^2(\pi_{;\mu}\pi^{;\mu}) + 2(\Box\pi)(\pi_{;\mu}\pi^{;\mu\nu}\pi_{;\nu}) - (\pi_{;\mu\nu}\pi^{;\mu\nu}) \times (\pi_{;\rho}\pi^{;\rho}) - 2(\pi_{;\mu}\pi^{;\mu\nu}\pi_{;\nu\rho}\pi^{;\rho})$ is a total derivative in Minkowski spacetime, so that $\mathcal{L}_4$ may be rewritten more simply as $\mathcal{L}_4 = 2(\pi_{;\rho}\pi^{;\rho})[(\Box\pi)^2 - (\pi_{;\mu\nu}\pi^{;\mu\nu})] +$ tot. div. However, this expression does differ from Eq. (1) in curved background, and we will come back to this below. Similar rewritings also exist for Eq. (2) in Minkowski spacetime, while giving different expressions in curved background.

Following Ref. [1] a minimal coupling to the metric and matter of the Galileon could be defined by the action

$$S = \int d^4x\sqrt{-g}(R + \mathcal{L}_\pi + \mathcal{L}_{\text{matter}}), \quad (3)$$

where $\mathcal{L}_\pi$ is a linear combination (with arbitrary constant coefficients c_i) of $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$, and $\mathcal{L}_5$,

$$\mathcal{L}_\pi = \sum_{i=1}^{i=5} c_i \mathcal{L}_i, \quad (4)$$

R is the Ricci scalar for the metric $g_{\mu\nu}$, and $\mathcal{L}_{\text{matter}}$ is the Lagrangian of matter fields minimally coupled to a metric $\tilde{g}_{\mu\nu}$ made of the Einstein frame metric $g_{\mu\nu}$ and of the Galileon π, e.g. in a conformal way like in $\tilde{g}_{\mu\nu} = e^{2\pi}g_{\mu\nu}$. It is clear that neither $\mathcal{L}_1$ nor $\mathcal{L}_2$ are able to generate equations of motion containing derivatives of order higher than two when varied with respect to π or $g_{\mu\nu}$. The DGP-motivated Lagrangian $\mathcal{L}_3$ is known to generate a second-order field equation for π, and it cannot yield either higher derivatives when varied with respect to the metric. Indeed, it contains at most first derivatives of $g_{\mu\nu}$, and those are multiplied by first derivatives of π. As we will see now, the situation is strikingly different for $\mathcal{L}_4$ and $\mathcal{L}_5$.

Let us first discuss the case of $\mathcal{L}_4$. When one varies $\mathcal{L}_4$ with respect to π, we obtain the equation of motion $\mathcal{E}_4 = 0$, where[4]

$$\begin{aligned}\mathcal{E}_4 \equiv {} & 2(\pi_{;\mu}\pi^{;\mu})(\pi_{;\nu}{}^{\nu}{}_{\rho}{}^{\rho} - \pi_{;\nu\rho}{}^{\nu\rho}) + 2\pi^{;\mu}\pi^{;\nu}(2\pi_{;\mu\rho\nu}{}^{\rho} \\ & - \pi_{;\mu\nu\rho}{}^{\rho} - \pi_{;\rho}{}^{\rho}{}_{\mu\nu}) + 10(\Box\pi)\pi^{;\mu}(\pi_{;\mu\nu}{}^{\nu} - \pi_{;\nu}{}^{\nu}{}_{\mu}) \\ & + 12\pi_{;\mu}\pi^{;\mu\nu}(\pi_{;\rho}{}^{\rho}{}_{\nu} - \pi_{;\nu\rho}{}^{\rho}) + 8\pi^{;\mu}\pi^{;\nu\rho}(\pi_{;\nu\rho\mu} \\ & - \pi_{;\mu\nu\rho}) - 4(\Box\pi)^3 - 8(\pi_{;\mu}{}^{\nu}\pi_{;\nu}{}^{\rho}\pi_{;\rho}{}^{\mu}) \\ & + 12(\Box\pi)(\pi_{;\mu\nu}\pi^{;\mu\nu}), \end{aligned} \quad (5)$$

where the first two terms contain fourth-order derivatives, the following three terms contain third-order derivatives, and the last three terms contain second-order derivatives. One notices in fact that the fourth- and third-order deriva-

[2]This nonminimal coupling is unique up to total derivatives, and provided all π's are differentiated.

[3]We use the sign conventions of Ref. [13], notably the mostly plus signature.

[4]Equation (5) may also be written in a slightly different form by using the identity $\pi^{;\mu}\pi^{;\nu}(\pi_{;\mu\rho\nu}{}^{\rho} - \pi_{;\mu\rho}{}^{\rho}{}_{\nu}) + \pi_{;\mu}\pi^{;\mu\nu}(\pi_{;\rho}{}^{\rho}{}_{\nu} - \pi_{;\nu\rho}{}^{\rho}) - \pi^{;\mu}\pi^{;\nu\rho}(\pi_{;\nu\rho\mu} - \pi_{;\mu\nu\rho}) = 0$.

tives disappear on a flat spacetime (as they should according to Ref. [1]). Indeed, commuting the derivatives, we find that one can rewrite $\mathcal{E}_4$ as

$$\begin{aligned}\mathcal{E}_4 = &-4(\Box\pi)^3 - 8(\pi_{;\mu}{}^{\nu}\pi_{;\nu}{}^{\rho}\pi_{;\rho}{}^{\mu}) + 12(\Box\pi)\\ &\times(\pi_{;\mu\nu}\pi^{;\mu\nu}) - (\pi_{;\mu}\pi^{;\mu})(\pi_{;\nu}R^{;\nu})\\ &+ 2(\pi_{;\mu}\pi_{;\nu}\pi_{;\rho}R^{\mu\nu;\rho}) + 10(\Box\pi)(\pi_{;\mu}R^{\mu\nu}\pi_{;\nu})\\ &- 8(\pi_{;\mu}\pi^{;\mu\nu}R_{\nu\rho}\pi^{;\rho}) - 2(\pi_{;\mu}\pi^{;\mu})(\pi_{;\nu\rho}R^{\nu\rho})\\ &- 8(\pi_{;\mu}\pi_{;\nu}\pi_{;\rho\sigma}R^{\mu\rho\nu\sigma}). \end{aligned} \tag{6}$$

We are left over with derivatives of the Ricci tensor and scalar and hence with third-order derivatives of the metric. One can think of a nonminimal coupling to the metric which would get rid of those terms, in a form of a linear combination of the two terms $\mathcal{L}_{4,1}$ and $\mathcal{L}_{4,2}$ defined as[5]

$$\mathcal{L}_{4,1} = (\pi_{;\mu}\pi^{;\mu})(\pi_{;\nu}\pi^{;\nu})R, \tag{7a}$$

$$\mathcal{L}_{4,2} = (\pi_{;\lambda}\pi^{;\lambda})(\pi_{;\mu}R^{\mu\nu}\pi_{;\nu}). \tag{7b}$$

In fact there is a unique combination of those two terms, namely $\mathcal{L}_{4,2} - \frac{1}{2}\mathcal{L}_{4,1}$ which added to $\mathcal{L}_4$ eliminates all the third derivatives in the π equations of motion. Specifically, if we add to action

$$S_4 = \int d^4x\sqrt{-g}\,\mathcal{L}_4 \tag{8}$$

the action

$$\begin{aligned} S_4^{\text{nonmin}} &\equiv \int d^4x\sqrt{-g}(\pi_{;\lambda}\pi^{;\lambda})\pi_{;\mu}\Big[R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\Big]\pi_{;\nu}\\ &= \int d^4x\sqrt{-g}(\pi_{;\lambda}\pi^{;\lambda})(\pi_{;\mu}G^{\mu\nu}\pi_{;\nu}), \end{aligned} \tag{9}$$

$G^{\mu\nu}$ denoting the Einstein tensor, we obtain the equations of motion for π in the form $\mathcal{E}'_4 = 0$, where $\mathcal{E}'_4$ is given by

$$\begin{aligned}\mathcal{E}'_4 = &-4(\Box\pi)^3 - 8(\pi_{;\mu}{}^{\nu}\pi_{;\nu}{}^{\rho}\pi_{;\rho}{}^{\mu})\\ &+ 12(\Box\pi)(\pi_{;\mu\nu}\pi^{;\mu\nu}) + 2(\Box\pi)(\pi_{;\mu}\pi^{;\mu})R\\ &+ 4(\pi_{;\mu}\pi^{;\mu\nu}\pi_{;\nu})R + 8(\Box\pi)(\pi_{;\mu}R^{\mu\nu}\pi_{;\nu})\\ &- 4(\pi_{;\lambda}\pi^{;\lambda})(\pi_{;\mu\nu}R^{\mu\nu}) - 16(\pi_{;\mu}\pi^{;\mu\nu}R_{\nu\rho}\pi^{;\rho})\\ &- 8(\pi_{;\mu}\pi_{;\nu}\pi_{;\rho\sigma}R^{\mu\rho\nu\sigma}). \end{aligned} \tag{10}$$

We see that this equation does not contain derivatives of order higher than 2, and that it obviously reduces to the original form of Ref. [1] in flat spacetime. On the other hand, notice that it involves first-order derivatives of π in curved spacetime. This breaks the "Galilean" symmetry under $\partial_\mu\pi \to \partial_\mu\pi + b_\mu$, $\pi \to \pi + c$ which is a covariant generalization of the transformation $\pi \to \pi + b_\mu x^\mu + c$ (where b_μ and c are constants) defined in Ref. [1] in Minkowski spacetime. Note also the complex mixing of the field degrees of freedom implied by the presence of second derivatives of both π and $g_{\mu\nu}$ in this equation.

It is interesting to note that the full action $S_4 + S_4^{\text{nonmin}}$, Eqs. (8) and (9), can be rewritten in a much more compact form thanks to integrations by parts and commutations of derivatives:

$$\begin{aligned} S_4 + S_4^{\text{nonmin}} = \int d^4x\sqrt{-g}(\pi_{;\lambda}\pi^{;\lambda})\Big[&2(\Box\pi)^2\\ &- 2(\pi_{;\mu\nu}\pi^{;\mu\nu}) - \frac{1}{2}(\pi_{;\mu}\pi^{;\mu})R\Big]. \end{aligned} \tag{11}$$

The nonminimal coupling to the Ricci tensor (7b) is indeed automatically taken into account by the rewriting of Eq. (1) discussed below Eq. (2).

Let us now investigate the variation of action S_4, Eq. (8), with respect to the metric.[6] It results in an energy-momentum tensor[7] $\mathcal{T}_4^{\mu\nu} \equiv (-g)^{-1/2}\delta S_4/\delta g_{\mu\nu}$ of the form given by

$$\begin{aligned}\mathcal{T}_4^{\mu\nu} = &(\pi^{;\mu}\pi^{;\nu})\pi^{;\lambda}(2\pi_{;\lambda\rho}{}^{\rho} - \pi^{;\rho}{}_{\rho\lambda}) + (\pi_{;\lambda}\pi^{;\lambda})\pi^{;\mu}(\pi_{;\rho}{}^{\rho\nu} - \pi^{;\nu\rho}{}_{\rho}) + (\pi_{;\lambda}\pi^{;\lambda})\pi^{;\nu}(\pi_{;\rho}{}^{\rho\mu} - \pi^{;\mu\rho}{}_{\rho})\\ &- \pi^{;\lambda}\pi^{;\rho}(\pi^{;\mu}\pi_{;\lambda\rho}{}^{\nu} + \pi^{;\nu}\pi_{;\lambda\rho}{}^{\mu}) + (\pi_{;\lambda}\pi^{;\lambda})(\pi_{;\rho}\pi^{;\mu\nu\rho}) + (\pi_{;\lambda}\pi_{;\rho}\pi_{;\sigma}\pi^{;\lambda\rho\sigma})g^{\mu\nu} - (\pi_{;\lambda}\pi^{;\lambda})(\pi_{;\rho}\pi_{;\sigma}{}^{\sigma\rho})g^{\mu\nu}\\ &+ (\pi^{;\mu}\pi^{;\nu})[3(\pi_{;\lambda\rho}\pi^{;\lambda\rho}) - 2(\Box\pi)^2] + (\pi^{;\mu\nu})\pi_{;\lambda}(2\pi^{;\lambda\rho}\pi_{;\rho} + \pi^{;\lambda}\Box\pi) + 3(\Box\pi)\pi_{;\lambda}(\pi^{;\lambda\mu}\pi^{;\nu} + \pi^{;\lambda\nu}\pi^{;\mu})\\ &- 4\pi_{;\lambda}\pi^{;\lambda\rho}(\pi_{;\rho}{}^{\mu}\pi^{;\nu} + \pi_{;\rho}{}^{\nu}\pi^{;\mu}) - 2(\pi_{;\lambda}\pi^{;\lambda\mu})(\pi_{;\rho}\pi^{;\rho\nu}) - \tfrac{1}{2}(\pi_{;\lambda}\pi^{;\lambda})[(\Box\pi)^2 + (\pi_{;\rho\sigma}\pi^{;\rho\sigma})]g^{\mu\nu}\\ &+ \pi_{;\lambda}\pi_{;\rho}[3\pi^{;\lambda\sigma}\pi_{;\sigma}{}^{\rho} - 2(\Box\pi)\pi^{;\lambda\rho}]g^{\mu\nu}. \end{aligned} \tag{12}$$

Notice that this energy-momentum tensor contains third-order derivatives of π, and even if flat spacetime $g_{\mu\nu} = \eta_{\mu\nu}$ were a solution of Einstein's equations. This shows

[5]These terms are the only possible nonvanishing ones made of contractions of four gradients of π and one curvature tensor, thereby involving a total of 6 derivatives.

[6]Note that this variation was not computed in Ref. [1], because it chose the π energy-momentum tensor to be negligible, as can be justified in an effective theory. Our aim is here to exhibit the higher derivatives it contains and to find a cure, assuming it is a fundamental classical field theory.

[7]This definition of the energy-momentum tensor corresponds to choosing units such that $8\pi G/c^4 = 1$.

that once one lets the metric be dynamical, new degrees of freedom will propagate even on a Minkowski background. One can show that this energy-momentum tensor is conserved on shell. Indeed, one finds (after many commutations of covariant derivatives) that

$$\nabla_\mu \mathcal{T}_4^{\mu\nu} = \tfrac{1}{2}\pi^{;\nu}\mathcal{E}_4. \tag{13}$$

This means, in particular, that the third derivatives present in the expression of $\mathcal{T}_4^{\mu\nu}$ are killed by the application of an extra covariant derivative.

Remarkably, it turns out that the addition to action (8) of the nonminimal coupling (9) is enough to eliminate all third derivatives appearing in the π energy-momentum tensor. Indeed varying the sum of actions (8) and (9), one finds now the energy-momentum tensor

$$\begin{aligned}\mathcal{T}_4'^{\mu\nu} = {}& 4(\Box\pi)\pi_{;\rho}[\pi^{;\mu}\pi^{;\rho\nu} + \pi^{;\nu}\pi^{;\rho\mu}] - 2(\Box\pi)^2(\pi^{;\mu}\pi^{;\nu}) + 2(\Box\pi)(\pi_{;\lambda}\pi^{;\lambda})(\pi^{;\mu\nu}) + 4(\pi_{;\lambda}\pi^{;\lambda\rho}\pi_{;\rho})(\pi^{;\mu\nu})\\ & - 4(\pi_{;\lambda}\pi^{;\lambda\mu})(\pi_{;\rho}\pi^{;\rho\nu}) + 2(\pi_{;\lambda\rho}\pi^{;\lambda\rho})(\pi^{;\mu}\pi^{;\nu}) - 2(\pi_{;\lambda}\pi^{;\lambda})(\pi^{;\mu}{}_{\rho}\pi^{;\rho\nu}) - 4\pi^{;\lambda}\pi_{;\lambda\rho}[\pi^{;\rho\mu}\pi^{;\nu} + \pi^{;\rho\nu}\pi^{;\mu}]\\ & - (\Box\pi)^2(\pi_{;\lambda}\pi^{;\lambda})g^{\mu\nu} - 4(\Box\pi)(\pi_{;\lambda}\pi^{;\lambda\rho}\pi_{;\rho})g^{\mu\nu} + 4(\pi_{;\lambda}\pi^{;\lambda\rho}\pi_{;\rho\sigma}\pi^{;\sigma})g^{\mu\nu} + (\pi_{;\lambda}\pi^{;\lambda})(\pi_{;\rho\sigma}\pi^{;\rho\sigma})g^{\mu\nu}\\ & + (\pi_{;\lambda}\pi^{;\lambda})(\pi^{;\mu}\pi^{;\nu})R - \tfrac{1}{4}(\pi_{;\lambda}\pi^{;\lambda})(\pi_{;\rho}\pi^{;\rho})g^{\mu\nu}R - 2(\pi_{;\lambda}\pi^{;\lambda})\pi_{;\rho}[R^{\rho\mu}\pi^{;\nu} + R^{\rho\nu}\pi^{;\mu}] + \tfrac{1}{2}(\pi_{;\lambda}\pi^{;\lambda})(\pi_{;\rho}\pi^{;\rho})R^{\mu\nu}\\ & + 2(\pi_{;\lambda}\pi^{;\lambda})(\pi_{;\rho}R^{\rho\sigma}\pi_{;\sigma})g^{\mu\nu} - 2(\pi_{;\lambda}\pi^{;\lambda})(\pi_{;\rho}\pi_{;\sigma}R^{\mu\rho\nu\sigma}),\end{aligned} \tag{14}$$

which now contains at most second derivatives. As expected, identity (13) is also verified by $\mathcal{T}_4'^{\mu\nu}$ and $\mathcal{E}_4'$, given in Eq. (10) above.

As we will see now, things proceed along the same line for the Lagrangian $\mathcal{L}_5$. Indeed, varying the action

$$S_5 = \int d^4x\sqrt{-g}\,\mathcal{L}_5 \tag{15}$$

with respect to π, we find, after commutation of covariant derivatives, that the π equations of motion do not contain derivatives or order higher than 2 acting on π but contain third-order derivatives of the metric in the form of first derivatives of the curvature. Those first derivatives are found to be given by the combination

$$\begin{aligned}& - 3(\Box\pi)(\pi_{;\mu}\pi^{;\mu})(\pi_{;\nu}R^{;\nu}) + 3(\pi_{;\mu}\pi^{;\mu\nu}\pi_{;\nu})(\pi_{;\rho}R^{;\rho})\\ & + 6(\Box\pi)(\pi_{;\mu}\pi_{;\nu}\pi_{;\rho}R^{\mu\nu;\rho}) + 6(\pi_{;\mu}\pi^{;\mu})\\ & \times(\pi_{;\nu}\pi_{;\rho\sigma}R^{\rho\sigma;\nu}) - 12(\pi_{;\mu}\pi_{;\nu}\pi^{;\rho}\pi_{;\rho\sigma}R^{\mu\sigma;\nu})\\ & - 6(\pi_{;\mu}\pi_{;\nu}\pi_{;\rho}\pi_{;\sigma\lambda}R^{\mu\sigma\nu\lambda;\rho}).\end{aligned} \tag{16}$$

One can think of eliminating those terms by adding to the action a linear combination of the following seven nontrivial contractions with the curvature tensors[8]:

$$\mathcal{L}_{5,1} = (\pi_{;\lambda}\pi^{;\lambda})(\pi_{;\mu}\pi_{;\nu}\pi_{;\rho\sigma}R^{\mu\rho\nu\sigma}), \tag{17a}$$
$$\mathcal{L}_{5,2} = (\pi_{;\mu}\pi^{;\mu})(\pi_{;\nu}\pi^{;\nu})(\pi_{;\rho\sigma}R^{\rho\sigma}), \tag{17b}$$
$$\mathcal{L}_{5,3} = (\pi_{;\mu}\pi^{;\mu})(\pi_{;\nu}\pi^{;\nu\rho}R_{\rho\sigma}\pi^{;\sigma}), \tag{17c}$$
$$\mathcal{L}_{5,4} = (\pi_{;\mu}\pi^{;\mu})(\Box\pi)(\pi_{;\nu}R^{\nu\rho}\pi_{;\rho}), \tag{17d}$$
$$\mathcal{L}_{5,5} = (\pi_{;\mu}\pi^{;\mu\nu}\pi_{;\nu})(\pi_{;\rho}R^{\rho\sigma}\pi_{;\sigma}), \tag{17e}$$
$$\mathcal{L}_{5,6} = (\pi_{;\mu}\pi^{;\mu})(\pi_{;\nu}\pi^{;\nu})(\Box\pi)R, \tag{17f}$$
$$\mathcal{L}_{5,7} = (\pi_{;\mu}\pi^{;\mu})(\pi_{;\nu}\pi^{;\nu\rho}\pi_{;\rho})R. \tag{17g}$$

Not all those terms are independent, though. In fact using the contracted Bianchi identity $R^{\mu\nu}{}_{;\nu} = \frac{1}{2}R^{;\mu}$, one can show that the combination $\mathcal{L}_{5,2} + 4\mathcal{L}_{5,3} - \frac{1}{2}\mathcal{L}_{5,6} - 2\mathcal{L}_{5,7}$ is a total derivative, and hence has an invariant action, i.e.,

$$\delta\int d^4x\sqrt{-g}\Big(\mathcal{L}_{5,2} + 4\mathcal{L}_{5,3} - \frac{1}{2}\mathcal{L}_{5,6} - 2\mathcal{L}_{5,7}\Big) = 0. \tag{18}$$

It turns out that there is a unique combination (up to the addition of the above expression), which added to action S_5, Eq. (15), removes all higher derivatives (those of the metric as well as those of π) in the π equations of motion, $\mathcal{E}_5' = 0$. This combination is given by

$$\begin{aligned}S_5^{\rm nonmin} = \int d^4x\sqrt{-g}\Big(&-3\mathcal{L}_{5,1} - 18\mathcal{L}_{5,3} + 3\mathcal{L}_{5,4}\\ & + \frac{15}{2}\mathcal{L}_{5,7}\Big).\end{aligned} \tag{19}$$

The resulting field equation for π is given in Eq. (A1) of the Appendix.

Similarly to Eq. (11), the full action $S_5 + S_5^{\rm nonmin}$ may also be rewritten in various simpler forms thanks to integrations by parts and commutations of derivatives. A particularly elegant one is

$$\begin{aligned}S_5 + S_5^{\rm nonmin} = \frac{5}{2}\int d^4x\sqrt{-g}(\pi_{;\lambda}\pi^{;\lambda})[&(\Box\pi)^3 - 3(\Box\pi)\\ & \times(\pi_{;\mu\nu}\pi^{;\mu\nu}) + 2(\pi_{;\mu}{}^{\nu}\pi_{;\nu}{}^{\rho}\pi_{;\rho}{}^{\mu})\\ & - 6(\pi_{;\mu}\pi^{;\mu\nu}G_{\nu\rho}\pi^{;\rho})].\end{aligned} \tag{20}$$

Varying now action (15) with respect to the metric, we find as previously an energy-momentum tensor $\mathcal{T}_5^{\mu\nu}$ that contains third derivatives of π. However, it turns out that the addition of (19) to (15) eliminates all higher derivatives and generates a new energy-momentum tensor $\mathcal{T}_5'^{\mu\nu}$, whose expression is given in Eq. (A2) of the Appendix, and which contains at most second derivatives (of π and

[8]These terms are the only possible nonvanishing ones made of contractions of five π's (acted on by at least one derivative) and one curvature tensor, involving a total of eight derivatives.

the metric). The analogue of Eq. (13) is consistently satisfied by $\mathcal{T}_5'^{\mu\nu}$ and $\mathcal{E}_5'$.

III. CONCLUSIONS

In this paper, we have shown that all higher-order derivatives appearing in the field equations of the minimally coupled Galileon to a dynamical metric, can be removed by a suitable nonminimal coupling to curvature. This insures that no extra degree of freedom is generated, and thereby defines a class of purely scalar-tensor theories, involving a single scalar degree of freedom, together with the standard graviton and matter fields. However, note that the absence of higher derivatives does not prove the stability of the theory (and conversely, their presence may occur in some specific stable models). This and other issues deserve more investigation. For instance, Ref. [1] considers Lagrangians involving products of more than five π's, which are total derivatives in flat four-dimensional spacetime, and it seems interesting to study their behavior in curved and extradimensional manifolds. Even without assuming the Galileon symmetry of this reference, it is worth studying the general form of scalar-field actions, coupled to gravity, and yielding second-order equations. Sticking with the original motivation of this kind of models, it remains to study their precise phenomenological predictions and their consistency in a cosmological context. We will tackle these questions in a future study.

ACKNOWLEDGMENTS

Our calculations have been cross-checked using several computer programs, notably the XTENSOR package [14] developed by J.-M. Martín-García for MATHEMATICA. It is a pleasure to thank G. Dvali, J. Mourad, O. Pujolas, and I. Sawicki for interesting discussions. The work of A. V. was supported by the James Arthur Fellowship. A. V. would like to thank the AstroParticule & Cosmologie Laboratory for its kind hospitality at the beginning of this project.

APPENDIX: FIELD EQUATIONS DERIVING FROM THE NONMINIMAL EXTENSION OF $\mathcal{L}_5$

We give below the field equation for π, $\mathcal{E}_5' = 0$, deriving from the action $S_5 + S_5^{\text{nonmin}}$, i.e., the sum of Eqs. (15) and (19):

$$\begin{aligned}
\mathcal{E}_5' = {}& -5(\Box\pi)^4 + 30(\Box\pi)^2(\pi_{;\mu\nu}\pi^{;\mu\nu}) + \frac{15}{2}(\Box\pi)^2(\pi_{;\mu}\pi^{;\mu})R + 15(\Box\pi)^2(\pi_{;\mu}R^{\mu\nu}\pi_{;\nu}) - 40(\Box\pi)(\pi_{;\mu}{}^{\nu}\pi_{;\nu}{}^{\rho}\pi_{;\rho}{}^{\mu}) \\
& + 15(\Box\pi)(\pi_{;\mu}\pi^{;\mu\nu}\pi_{;\nu})R - 30(\Box\pi)(\pi_{;\mu}\pi^{;\mu})(\pi_{;\nu\rho}R^{\nu\rho}) - 60(\Box\pi)(\pi_{;\mu}\pi^{;\mu\nu}R_{\nu\rho}\pi^{;\rho}) \\
& - 30(\Box\pi)(\pi_{;\mu}\pi_{;\nu}\pi_{;\rho\sigma}R^{\mu\rho\nu\sigma}) - 15(\pi_{;\mu\nu}\pi^{;\mu\nu})(\pi_{;\rho\sigma}\pi^{;\rho\sigma}) + 30(\pi_{;\mu\nu}\pi^{;\nu\rho}\pi_{;\rho\sigma}\pi^{;\sigma\mu}) - \frac{15}{2}(\pi_{;\mu}\pi^{;\mu})(\pi_{;\nu\rho}\pi^{;\nu\rho})R \\
& - 15(\pi_{;\mu}\pi^{;\mu\nu}\pi_{;\nu\rho}\pi^{;\rho})R - 15(\pi_{;\mu\nu}\pi^{;\mu\nu})(\pi_{;\rho}R^{\rho\sigma}\pi_{;\sigma}) - 30(\pi_{;\mu}\pi^{;\mu\nu}\pi_{;\nu})(\pi_{;\rho\sigma}R^{\rho\sigma}) + 30(\pi_{;\mu}\pi^{;\mu})(\pi_{;\nu}{}^{\rho}R_{\rho}{}^{\sigma}\pi_{;\sigma}{}^{\nu}) \\
& + 60(\pi_{;\mu}\pi^{;\mu\nu}\pi_{;\nu\rho}R^{\rho\sigma}\pi_{;\sigma}) + 30(\pi_{;\mu}\pi^{;\mu\nu}R_{\nu\rho}\pi^{;\rho\sigma}\pi_{;\sigma}) + 15(\pi_{;\mu}\pi^{;\mu})(\pi_{;\nu\rho}\pi_{;\sigma\lambda}R^{\nu\sigma\rho\lambda}) + 30(\pi_{;\mu}\pi_{;\nu}\pi_{;\rho\sigma}\pi^{;\sigma}{}_{\lambda}R^{\mu\rho\nu\lambda}) \\
& - 60(\pi_{;\lambda}\pi^{;\lambda}{}_{\mu}\pi_{;\nu\rho}\pi_{;\sigma}R^{\mu\nu\rho\sigma}) - \frac{15}{2}(\pi_{;\mu}\pi^{;\mu})(\pi_{;\nu}R^{\nu\rho}\pi_{;\rho})R + 15(\pi_{;\mu}\pi^{;\mu})(\pi_{;\nu}R^{\nu\rho}R_{\rho\sigma}\pi^{;\sigma}) \\
& + 15(\pi_{;\mu}\pi^{;\mu})(\pi_{;\nu}\pi_{;\rho}R_{\sigma\lambda}R^{\nu\sigma\rho\lambda}) - \frac{15}{2}(\pi_{;\mu}\pi^{;\mu})(\pi_{;\nu}\pi_{;\rho}R^{\nu}{}_{\sigma\kappa\lambda}R^{\rho\sigma\kappa\lambda}). \qquad \text{(A1)}
\end{aligned}$$

As mentioned in the bulk of the paper, both the Galileon π and the metric $g_{\mu\nu}$ are differentiated at most twice in this field equation. This is also the case for the variation of the same action $S_5 + S_5^{\text{nonmin}}$ with respect to $g_{\mu\nu}$, i.e., the π energy-momentum tensor, which takes the form

$$\begin{aligned}
\mathcal{T}_5'^{\mu\nu} = {}& -\frac{5}{2}(\Box\pi)^3(\pi^{;\mu}\pi^{;\nu}) - \frac{5}{2}(\Box\pi)^3(\pi_{;\rho}\pi^{;\rho})g^{\mu\nu} + \frac{15}{2}(\Box\pi)^2(\pi_{;\rho}\pi^{;\rho})(\pi^{;\mu\nu}) + \frac{15}{2}(\Box\pi)^2\pi_{;\rho}[\pi^{;\rho\mu}\pi^{;\nu} + \pi^{;\rho\nu}\pi^{;\mu}] \\
& - \frac{15}{2}(\Box\pi)^2(\pi_{;\rho}\pi^{;\rho\sigma}\pi_{;\sigma})g^{\mu\nu} - 15(\Box\pi)(\pi_{;\rho}\pi^{;\rho})(\pi^{;\mu\sigma}\pi_{;\sigma}{}^{\nu}) + 15(\Box\pi)(\pi_{;\rho}\pi^{;\rho\sigma}\pi_{;\sigma})(\pi^{;\mu\nu}) \\
& + \frac{15}{2}(\Box\pi)(\pi_{;\rho\sigma}\pi^{;\rho\sigma})(\pi^{;\mu}\pi^{;\nu}) - 15(\Box\pi)(\pi_{;\rho}\pi^{;\rho\mu})(\pi_{;\sigma}\pi^{;\sigma\nu}) - 15(\Box\pi)\pi^{;\rho}\pi_{;\rho\sigma}[\pi^{;\sigma\mu}\pi^{;\nu} + \pi^{;\sigma\nu}\pi^{;\mu}] \\
& + \frac{15}{2}(\Box\pi)(\pi_{;\rho}\pi^{;\rho})(\pi_{;\sigma\lambda}\pi^{;\sigma\lambda})g^{\mu\nu} + 15(\Box\pi)(\pi_{;\rho}\pi^{;\rho\sigma}\pi_{;\sigma\lambda}\pi^{;\lambda})g^{\mu\nu} + \frac{15}{4}(\Box\pi)(\pi_{;\rho}\pi^{;\rho})(\pi^{;\mu}\pi^{;\nu})R
\end{aligned}$$

$$
\begin{aligned}
&-\frac{15}{2}(\Box\pi)(\pi_{;\rho}\pi^{;\rho})\pi_{;\sigma}[R^{\sigma\mu}\pi^{;\nu}+R^{\sigma\nu}\pi^{;\mu}]+\frac{15}{2}(\Box\pi)(\pi_{;\rho}\pi^{;\rho})(\pi_{;\sigma}R^{\sigma\lambda}\pi_{;\lambda})g^{\mu\nu}-\frac{15}{2}(\Box\pi)(\pi_{;\rho}\pi^{;\rho})(\pi_{;\sigma}\pi_{;\lambda}R^{\mu\sigma\nu\lambda})\\
&-\frac{15}{2}(\pi_{;\rho}\pi^{;\rho})(\pi_{;\sigma\lambda}\pi^{;\sigma\lambda})(\pi^{;\mu\nu})+15(\pi_{;\rho}\pi^{;\rho})(\pi^{;\mu\sigma}\pi_{;\sigma\lambda}\pi^{;\lambda\nu})-15(\pi_{;\rho}\pi^{;\rho\sigma}\pi_{;\sigma})(\pi^{;\mu\lambda}\pi_{;\lambda}{}^{\nu})\\
&-15(\pi_{;\rho}\pi^{;\rho\sigma}\pi_{;\sigma\lambda}\pi^{;\lambda})(\pi^{;\mu\nu})-5(\pi_{;\rho}{}^{\sigma}\pi_{;\sigma}{}^{\lambda}\pi_{;\lambda}{}^{\rho})(\pi^{;\mu}\pi^{;\nu})-\frac{15}{2}(\pi_{;\sigma\lambda}\pi^{;\sigma\lambda})\pi_{;\rho}[\pi^{;\rho\mu}\pi^{;\nu}+\pi^{;\rho\nu}\pi^{;\mu}]\\
&+15\pi_{;\rho}\pi^{;\rho\sigma}\pi_{;\sigma\lambda}[\pi^{;\lambda\mu}\pi^{;\nu}+\pi^{;\lambda\nu}\pi^{;\mu}]+15\pi^{;\rho}\pi_{;\rho\lambda}\pi_{;\sigma}[\pi^{;\lambda\mu}\pi^{;\sigma\nu}+\pi^{;\lambda\nu}\pi^{;\sigma\mu}]-5(\pi_{;\rho}\pi^{;\rho})(\pi_{;\sigma}{}^{\lambda}\pi_{;\lambda}{}^{\kappa}\pi_{;\kappa}{}^{\sigma})g^{\mu\nu}\\
&+\frac{15}{2}(\pi_{;\rho}\pi^{;\rho\sigma}\pi_{;\sigma})(\pi_{;\lambda\kappa}\pi^{;\lambda\kappa})g^{\mu\nu}-15(\pi_{;\rho}\pi^{;\rho\sigma}\pi_{;\sigma\lambda}\pi^{;\lambda\kappa}\pi_{;\kappa})g^{\mu\nu}-\frac{15}{4}(\pi_{;\rho}\pi^{;\rho})\pi_{;\sigma}[\pi^{;\sigma\mu}\pi^{;\nu}+\pi^{;\sigma\nu}\pi^{;\mu}]R\\
&+\frac{15}{4}(\pi_{;\rho}\pi^{;\rho})(\pi_{;\sigma}\pi^{;\sigma\lambda}\pi_{;\lambda})Rg^{\mu\nu}-\frac{15}{2}(\pi_{;\rho}\pi^{;\rho})(\pi_{;\sigma}\pi^{;\sigma\lambda}\pi_{;\lambda})R^{\mu\nu}-\frac{15}{2}(\pi_{;\rho}\pi^{;\rho})(\pi_{;\sigma}R^{\sigma\lambda}\pi_{;\lambda})(\pi^{;\mu\nu})\\
&-\frac{15}{2}(\pi_{;\rho}\pi^{;\rho})(\pi_{;\sigma\lambda}R^{\sigma\lambda})(\pi^{;\mu}\pi^{;\nu})+\frac{15}{2}(\pi_{;\rho}\pi^{;\rho})\pi^{;\sigma}\pi_{;\sigma\lambda}[R^{\lambda\mu}\pi^{;\nu}+R^{\lambda\nu}\pi^{;\mu}]\\
&+\frac{15}{2}(\pi_{;\rho}\pi^{;\rho})\pi_{;\lambda}\pi_{;\sigma}[R^{\lambda\mu}\pi^{;\sigma\nu}+R^{\lambda\nu}\pi^{;\sigma\mu}]+\frac{15}{2}(\pi_{;\rho}\pi^{;\rho})\pi^{;\sigma}R_{\sigma\lambda}[\pi^{;\lambda\mu}\pi^{;\nu}+\pi^{;\lambda\nu}\pi^{;\mu}]\\
&-15(\pi_{;\rho}\pi^{;\rho})(\pi_{;\sigma}\pi^{;\sigma\lambda}R_{\lambda\kappa}\pi^{;\kappa})g^{\mu\nu}+\frac{15}{2}(\pi_{;\rho}\pi^{;\rho})\pi_{;\sigma}\pi_{;\lambda\kappa}[R^{\mu\lambda\sigma\kappa}\pi^{;\nu}+R^{\nu\lambda\sigma\kappa}\pi^{;\mu}]\\
&-\frac{15}{2}(\pi_{;\rho}\pi^{;\rho})\pi_{;\sigma}\pi_{;\lambda}[R^{\mu\sigma\lambda\kappa}\pi_{;\kappa}{}^{\nu}+R^{\nu\sigma\lambda\kappa}\pi_{;\kappa}{}^{\mu}]+\frac{15}{2}(\pi_{;\rho}\pi^{;\rho})\pi^{;\sigma}\pi_{;\sigma\lambda}\pi_{;\kappa}[R^{\mu\lambda\nu\kappa}+R^{\nu\lambda\mu\kappa}]\\
&-\frac{15}{2}(\pi_{;\rho}\pi^{;\rho})(\pi_{;\sigma}\pi_{;\lambda}\pi_{;\kappa\tau}R^{\sigma\kappa\lambda\tau})g^{\mu\nu}.
\end{aligned}
\tag{A2}
$$

[1] A. Nicolis, R. Rattazzi, and E. Trincherini, arXiv:0811.2197.
[2] G. R. Dvali, G. Gabadadze, and M. Porrati, Phys. Lett. B **485**, 208 (2000).
[3] C. Deffayet, Phys. Lett. B **502**, 199 (2001).
[4] C. Deffayet, G. R. Dvali, and G. Gabadadze, Phys. Rev. D **65**, 044023 (2002).
[5] M. A. Luty, M. Porrati, and R. Rattazzi, J. High Energy Phys. 09 (2003) 029.
[6] A. Nicolis and R. Rattazzi, J. High Energy Phys. 06 (2004) 059.
[7] N. Arkani-Hamed, H. Georgi, and M. D. Schwartz, Ann. Phys. (N.Y.) **305**, 96 (2003).
[8] C. Deffayet and J. W. Rombouts, Phys. Rev. D **72**, 044003 (2005).
[9] P. Creminelli, A. Nicolis, M. Papucci, and E. Trincherini, J. High Energy Phys. 09 (2005) 003.
[10] M. Ostrogradski, Mem. Ac. St. Petersbourg VI **4**, 385 (1850).
[11] R. Woodard, Lect. Notes Phys. **720**, 403 (2007).
[12] N. Arkani-Hamed, S. Dimopoulos, G. Dvali, and G. Gabadadze, arXiv:hep-th/0209227; G. Dvali, S. Hofmann, and J. Khoury, Phys. Rev. D **76**, 084006 (2007).
[13] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[14] J.-M. Martín-García, Comput. Phys. Commun. **179**, 597 (2008), the XTENSOR package is freely available at http://metric.iem.csic.es/Martin-Garcia/xAct/.