Constraints on some $f(R)$ gravity models in the Palatini formalism from a time-varying gravitational constant

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In this work, we studied how the relative variation of the gravitational constant $|\dot{\bar{G}}/\bar{G}|_0$ at the redshift $z = 0$ restricts model parameters in Palatini $f(R)$ gravity. According to results from the observation of big bang nucleosynthesis and the anisotropies in the cosmic microwave background radiation, the ratio that gives the general constraint condition on the $f(R)$ model is taken to be $|\dot{\bar{G}}/\bar{G}|_0 < 10^{-13}\ \mathrm{yr}^{-1}$. Associating with the scalar curvature at present, this constraint condition yields concrete relations among model parameters in the generalized ΛCDM model, the logarithmic Lagrangian, and the exponential gravity model. These constraint relations which can be reconcilable with the value $|\dot{\bar{G}}/\bar{G}|_0 < 10^{-13}\ \mathrm{yr}^{-1}$ are plotted.

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I. INTRODUCTION

Among the multitude of efforts devoted to explaining and modeling the current acceleration of the cosmic expansion discovered with type Ia supernovae [1], so-called $f(R)$ gravity has received much attention. Disposing of the concept of dark energy, the $f(R)$ theory claims that the deviation from Einstein's general relativity on a cosmic scale results in the accelerated expansion of the Universe [2,3]. One form of the $f(R)$ theory is the Palatini formulation in which the metric and the connection are assumed to be independent to each other. This formulation results in a second order field equation which directly reduces to the standard general relativity with a cosmological constant in vacuum [4]. Moreover, the cosmological evolution of radiation, matter, and accelerated epochs appeared successfully in it [5].

However, there are a few free parameters in the $f(R)$ gravity which should be restricted to fit with astronomical data more accurately. Recently, the restrictions on parameters of some simple $f(R)$ models in Palatini formulation, such as $f = R^p + \Lambda$, $f = R + \mu/R^m$, $f = R + \alpha R^2 + \beta/R$, and $f = R + \alpha \ln R + \beta$, are given by the data from cosmological large scale phenomena [6–8]. But, for other complex models with more parameters, one will meet tremendous difficulties in using these data to give constraints on the model parameters. A notable fact is that the $f(R)$ gravity contains a time-varying gravitational constant [9]. So, we can utilize the relative variation of the gravitational constant $|\dot{\bar{G}}/\bar{G}|$ to find the limitation on the model parameters.

Several facts enlighten us to estimate the magnitude of the ratio $|\dot{\bar{G}}/\bar{G}|$ at the redshift $z = 0$. On the cosmological large scale and over 10^{10} yr from the early Universe to the present time, big bang nucleosynthesis and anisotropies in the cosmic microwave background radiation yield upper limits on the long-term averaged variation such that $|\dot{\bar{G}}/\bar{G}| < 10^{-13}\ \mathrm{yr}^{-1}$ [10]. At the same time, the value of the ratio $|\dot{\bar{G}}/\bar{G}|$ is calculated in the $f(R)$ models with the restricted parameters mentioned above [6–8] to be $|\dot{\bar{G}}/\bar{G}| < 10^{-13}\ \mathrm{yr}^{-1}$ at the redshift $z = 0$. Moreover, the motion of the planets in the solar system has provided constraints with $|\dot{\bar{G}}/\bar{G}| < 10^{-14}\ \mathrm{yr}^{-1}$ at 1σ confidence at present [11]. According to these results, we can reliably use the ratio $|\dot{\bar{G}}/\bar{G}| < 10^{-13}\ \mathrm{yr}^{-1}$ at the redshift $z = 0$ to give the limitation on model parameters of the Palatini $f(R)$ models. In Sec. II, the $f(R)$ model in Palatini formulation and the corresponding field equation are reviewed. The general form of the constraint condition is also derived. In Sec. III, we analyze $f(R)$ gravity models, such as the generalization of ΛCDM, logarithmic Lagrangian, and the exponential gravity model, and the constraint relations among the model parameters are given. The conclusion and the discussion are given in Sec. IV.

II. GENERAL CONSTRAINT CONDITIONS

The generalized action of the $f(R)$ gravity is

$$I = \frac{1}{2\kappa} \int f(R)\sqrt{-g}\, d^4x + I_m, \tag{1}$$

where $\kappa = 8\pi G$ is the coupling constant, $f(R)$ is a function of the scalar curvature $R = g^{\mu\nu}R_{\mu\nu}$, and I_m represents the matter action which is supposed to be independent of the affine connection $\Gamma^\alpha_{\mu\nu}$. The curvature tensor $R_{\mu\nu}$ is given by

$$R_{\mu\nu} = -\Gamma^\lambda_{\mu\lambda,\nu} + \Gamma^\lambda_{\mu\nu,\lambda} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho} + \Gamma^\rho_{\mu\nu}\Gamma^\lambda_{\lambda\rho}. \tag{2}$$

In the Palatini formulation of the $f(R)$ model, the metric $g_{\mu\nu}$ and the affine connection $\Gamma^\alpha_{\mu\nu}$ are independent variables. Varying Eq. (1) with respect to the metric $g_{\mu\nu}$ and the affine connection $\Gamma^\alpha_{\mu\nu}$, respectively, we get

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$$R_{\mu\nu}F - \frac{1}{2}g_{\mu\nu}f = \kappa T_{\mu\nu}, \tag{3}$$

and

$$\nabla_{(\Gamma)\lambda}(F\sqrt{-g}g^{\mu\nu}) = 0. \tag{4}$$

Where $F = \partial f/\partial R$, $T_{\mu\nu} = -2\delta I_m/(\sqrt{-g}\delta g^{\mu\nu})$ is the energy-momentum tensor of matter, and $\nabla_{(\Gamma)\lambda}$ represents the covariant derivative with respect to the connection $\Gamma^{\lambda}_{\mu\nu}$. Contracting Eq. (3), we get

$$RF - 2f = \kappa T. \tag{5}$$

Expressing the generalized Ricci tensor $R_{\mu\nu}$ in terms of the Ricci tensor $R_{(g)\mu\nu}$ associated with the metric $g_{\mu\nu}$ and expressing covariant derivatives $\nabla_{(\Gamma)\lambda}$ in terms of the Levi-Civita connection, we obtain the generalized field equation

$$G_{\mu\nu} = \frac{\kappa}{F}(T_{\mu\nu} + \tau_{\mu\nu}), \tag{6}$$

where $G_{\mu\nu} = R_{(g)\mu\nu} - g_{\mu\nu}R_{(g)}/2$ is the Einstein tensor and

$$\tau_{\mu\nu} = \left(\frac{3\nabla_\sigma F\nabla^\sigma F}{4\kappa F} + \frac{f - FR - 2\nabla_\sigma\nabla^\sigma F}{2\kappa}\right)g_{\mu\nu} + \frac{\nabla_\mu\nabla_\nu F}{\kappa} - \frac{3\nabla_\mu F\nabla_\nu F}{2\kappa F}. \tag{7}$$

The right-hand side of Eq. (6) indicates that the effective gravitational constant is modified by the factor $1/F$ and a new source $\tau_{\mu\nu}$ appeared which can be regarded as the dark energy. Specifically, in the case of the Hilbert action with $f = R$, Eq. (6) reduces to the usual Einstein-Hilbert gravity $G_{\mu\nu} = \kappa T_{\mu\nu}$. Motivated by recent observations, we adopt the metric in the spatially flat Friedman-Robertson-Walker (FRW) form $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$ and the perfect-fluid energy-momentum tensor $T^\nu_\mu = \mathrm{diag}(-\rho, p, p, p)$. The state parameter in the equation of state $p = \omega\rho$ has the value $\omega = 0$ for matter and $\omega = 1/3$ for radiation. Therefore the generalized cosmic evolution equations are given by

$$3H^2 = 8\pi\bar{G}\rho_m + \frac{RF - f}{2F} + 3H\frac{\dot{\bar{G}}}{\bar{G}} - \frac{3}{4}\left(\frac{\dot{\bar{G}}}{\bar{G}}\right)^2, \tag{8}$$

and

$$3H^2 + 2\dot{H} = -8\pi\bar{G}p_m + \frac{3}{4}\left(\frac{\dot{\bar{G}}}{\bar{G}}\right)^2 + \frac{RF - f}{2F} + \frac{\ddot{\bar{G}}}{\bar{G}} + 2H\frac{\dot{\bar{G}}}{\bar{G}}, \tag{9}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter, and the effective gravitational constant is defined as [12]

$$\bar{G} \equiv \frac{G}{F}. \tag{10}$$

We can infer from Eqs. (8) and (9) that the time variation of the effective gravitational constant $\bar{G}$ leads to the accelerated expansion of the Universe at present time, and the dark energy can be regard as the time-varying gravitational constant in part [13].

Generally, a variable gravitational constant can induce many observable effects of astronomy. Inversely, data derived from these effects determine the value range of the variable gravitational constant and therefore restrict the value range of the denominator F. Since the conservation of the energy and momentum, the term ρa^3 is a constant for matter. Using Eq. (5) and the constant ρa^3, the first order time derivative of the scalar curvature is given by

$$\dot{R} = \frac{3H(2f - RF)}{RF' - F}, \tag{11}$$

where $F' = \partial F/\partial R$. According to the definition Eq. (10), the relative value of the time-varying effective gravitational constant $\dot{\bar{G}}/\bar{G}$ is given by

$$\frac{\dot{\bar{G}}}{\bar{G}} = -\frac{F'}{F}\dot{R}. \tag{12}$$

The density parameter of matter is defined as [14]

$$\Omega_m \equiv \frac{\kappa\rho_m}{3FH^2}. \tag{13}$$

Using Eqs. (5) and (13), Eq. (8) can be rewritten as

$$R = 3\left(2H - \frac{\dot{\bar{G}}}{\bar{G}}\right)^2 - 9H^2\Omega_m, \tag{14}$$

In this paper, the ratio $|\dot{\bar{G}}/\bar{G}|_0 < 10^{-13}\ \mathrm{yr}^{-1}$ and $H_0 = h \times 10^{-10}\ \mathrm{yr}^{-1}$ with $h \sim 0.7$ are adopted [15]. Because of $|\dot{\bar{G}}/\bar{G}|_0 \ll 2H_0$, the current curvature of the space-time is

$$R_0 = KH_0^2. \tag{15}$$

Where the subscript 0 represents the present value and $K = 12 - 9\Omega_{m0}$. Using Eqs. (5) and (13), one can obtain

$$3H_0^2\Omega_{m0} = \left(\frac{2f - RF}{F}\right)_0. \tag{16}$$

Inserting Eq. (11) in (12) we get

$$\frac{\dot{\bar{G}}}{\bar{G}} = \frac{9H^3\Omega_m F'}{F - RF'}. \tag{17}$$

For convenience, we define a dimensionless number as

$$\varepsilon \equiv H_0^2\left|\frac{F'}{F - RF'}\right|_0, \tag{18}$$

and the magnitude of ε is given by

$$\varepsilon = \frac{1}{9H_0\Omega_{m0}}\left|\frac{\dot{\bar{G}}}{\bar{G}}\right|_0. \tag{19}$$

Requiring $|\dot{\bar{G}}/\bar{G}|_0 < 10^{-13}\ \mathrm{yr}^{-1}$, one can estimate the order of ε as $\varepsilon < 5\times 10^{-4}$. In fact, Eqs. (18) and (19) can only ensure the $f(R)$ gravity models to be consistent with the constraint date [10]. While a cosmologically acceptable model should be satisfied with Eq. (16) in the late time of the evolution of the Universe, this restriction ensures that the Universe is dark-energy-dominated at present time.

III. CONSTRAINTS ON $f(R)$ MODELS

The constraints on the model parameters of the $f(R)$ model in Palatini formalism can be derived with Eqs. (15), (16), (18), and (19) as follows.

A. Theories of type $f = \alpha[R^m + \Lambda(H_0^2)^m]^n$

This model, as the possible generalization of ΛCDM with $n \neq 0$, was discussed in Refs. [16,17], and it can also account for the current acceleration of the Universe. In the case of $m = n = 1$, the standard ΛCDM model is recovered. From Eqs. (16) and (18), we obtain the relation for this model

$$\varepsilon = \frac{1}{K}\left|\frac{2(mn-1)K + (m-1)[(3\Omega_{m0}+K)mn - 2K]}{2(2-mn)K + (2-m)[(3\Omega_{m0}+K)mn - 2K]}\right|, \tag{20}$$

and

$$\Lambda = \frac{1}{2}(3\Omega_{m0}+K)mnK^{m-1} - K^m. \tag{21}$$

Because ε is very small, as an effective approximation, Eq. (20) reduces to

$$2(mn-1)K + (m-1)[(3\Omega_{m0}+K)mn - 2K] = 0. \tag{22}$$

For $\varepsilon = 5\times 10^{-4}$, we have found no significant changes in the results with $\varepsilon = 0$, and parameters approaching infinity are regard as nonphysical ones; we will neglect these values for them in the paper. The following discussions are the same. From Eq. (22), we can obtain the following constraint relation:

$$m = \frac{2K(1-n)}{(3\Omega_{m0}+K)n} + 1. \tag{23}$$

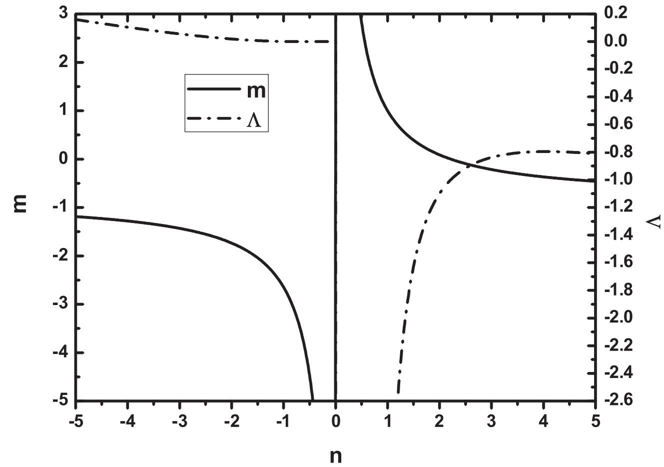

FIG. 1. Constraint relations among the dimensionless parameters of the model $f = \alpha[R^m + \Lambda(H_0^2)^m]^n$ with the density parameter $\Omega_{m0} = 0.3$.

The constraint relations among the model parameters are shown in Fig. 1. For the special case of $mn = 1$, the model becomes $f = \sqrt[m]{R^m + \Lambda(H_0^2)^m}$ [17]. Equations (20) and (21) give $|m-1| = K(K+3\Omega_{m0})\varepsilon/(K-3\Omega_{m0}) < 6\times 10^{-3}$ and $\Lambda = -6(1-\Omega_{m0})$. It means that the parameter m is very close to the constant 1. Therefore, the model is just the standard ΛCDM model $f = R - 6(1-\Omega_{m0})H_0^2$ and other cases with $m \neq 1$, including the model with $m = 2$ [17], should be ruled out.

B. Logarithmic Lagrangian

A model with a logarithmic Lagrangian in the Ricci scalar is induced by quantum effects in curved spacetime [16,18]. There are two kinds of such models we will discuss, respectively.

1. Theories of type $f = \alpha R^m[\ln(\beta R/H_0^2)]^n$

For these theories, Eqs. (16) and (18) give

$$\varepsilon = \frac{1}{K}\left|\frac{mn(m-1)[\frac{3\Omega_{m0}+K}{(2-m)K-3m\Omega_{m0}}]^2 + \frac{n(2m-1)(3\Omega_{m0}+K)}{(2-m)K-3m\Omega_{m0}} + n - 1}{mn(m-2)[\frac{3\Omega_{m0}+K}{(2-m)K-3m\Omega_{m0}}]^2 + \frac{2n(m-1)(3\Omega_{m0}+K)}{(2-m)K-3m\Omega_{m0}} + n - 1}\right|, \tag{24}$$

and

$$\beta = \frac{1}{K}\exp\left\{\frac{(3\Omega_{m0}+K)[2K-(3\Omega_{m0}+K)m]}{2K(K-3\Omega_{m0})}\right\}. \tag{25}$$

Because ε is very small, as an effective approximation, Eq. (24) reduces to

$$n = \frac{[(3\Omega_{m0}+K)m - 2K]^2}{2K(K-3\Omega_{m0})}. \tag{26}$$

Similar to the generalized ΛCDM model, the model pa-

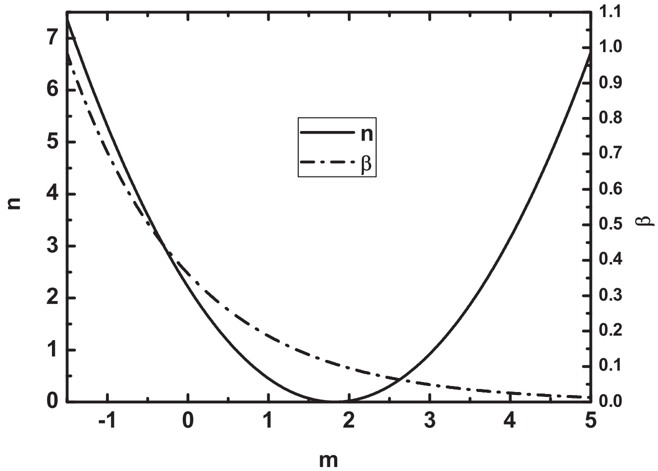

FIG. 2. Constraint relations among the dimensionless parameters of the model $f = \alpha R^m [\ln(\beta R/H_0^2)]^n$ with the density parameter $\Omega_{m0} = 0.3$.

rameters m, n, and β are correlative to each other with the constraint conditions Eqs. (25) and (26). The constraint relations among them are shown in Fig. 2. It indicates that this type of model with $n \leq 0$ is unacceptable, while in the case of $n > 0$, as an alternative of the dark energy, the model has the potential to explain the current acceleration.

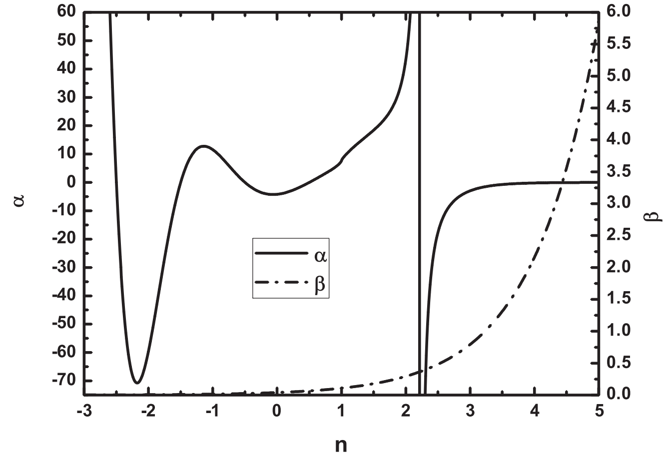

FIG. 3. Constraint relations among the dimensionless parameters of the model $f = R + \alpha H_0^2 [\ln(\beta R/H_0^2)]^n$ with the density parameter $\Omega_{m0} = 0.3$.

2. Theories of type $f = R + \alpha H_0^2 [\ln(\beta R/H_0^2)]^n$

From Eqs. (16) and (18) we can obtain

$$\varepsilon = \frac{1}{K} \left| \frac{n\alpha[n - 1 - \ln(\beta K)][\ln(\beta K)]^{n-2}}{K + 2n\alpha[\ln(\beta K)]^{n-1} - \alpha K n(n-1)[\ln(\beta K)]^{n-2}} \right|, \tag{27}$$

and

$$K(3\Omega_{m0} - K) = 2K\alpha[\ln(\beta K)]^n - n\alpha(K + 3\Omega_{m0}) \times [\ln(\beta K)]^{n-1}. \tag{28}$$

Because of the very small parameter ε, Eq. (27) can give

$$n\alpha[n - 1 - \ln(\beta K)][\ln(\beta K)]^{n-2} = 0. \tag{29}$$

From Eq. (29), we can get $n = 0$, $\ln(\beta K) = n - 1$, or $\ln(\beta K) = 0$. For the case of $n = 0$, we have $\alpha = -(K - 3\Omega_{m0})/2$; the Lagrangian reduces to the standard cosmological model $f = R - (K - 3\Omega_{m0})H_0^2/2$. For $\ln(\beta K) = n - 1$, we can obtain

$$\alpha = \frac{K(3\Omega_{m0} - K)}{[(K - 3\Omega_{m0})n - 2K](n-1)^{n-1}}, \tag{30}$$

and

$$\beta = \frac{1}{K} \exp(n - 1). \tag{31}$$

The constraint relations among model parameters are shown in Fig. 3. In the case of $n \geq 4$, the parameter $\alpha \to 0$, the corresponding Lagrangian reduces to $f = R$. It cannot explain the current acceleration, so, this case should be ruled out. Specifically, in the case of $n = 1$, we can get $\alpha = \pm K^2 \varepsilon$ and $\beta = 1/K$. Therefore the Lagrangian reduces to $f = R \pm \varepsilon K^2 H_0^2 \ln[R/(KH_0^2)]$. At present, due to $\ln[R/(KH_0^2)] \to 0$ with the curvature of the space-time $R \to KH_0^2$, so the model $f = R + \alpha H_0^2 \ln[R/(KH_0^2)]$ cannot explain the current acceleration of the Universe and should be ruled out. In the case of $\ln(\beta K) = 0$, the Lagrangian $f = R + \alpha H_0^2 \ln[R/(KH_0^2)]$ reduces to $f = R$ at present, and it cannot explain the cosmic acceleration.

C. The exponential gravity model

The so-called exponential $f(R)$ gravity models fit rather well with the supernovae Ia data, and its cosmological feasibility is also discussed in metric formulation [16,19]. There are two kinds of such models we will discuss, respectively.

1. Theories of type $f = \alpha R^m \exp[\beta R^n/(H_0^2)^n]$

In this type of model, Eqs. (16) and (18) can give

$$\varepsilon = \frac{1}{K} \left| \frac{\frac{2K(K - 3\Omega_{m0})}{(3\Omega_{m0} + K)^2} + n(\frac{2K}{3\Omega_{m0} + K} - m)}{-\frac{12\Omega_{m0} K}{(3\Omega_{m0} + K)^2} + n(\frac{2K}{3\Omega_{m0} + K} - m)} \right|, \tag{32}$$

and

$$\beta = \frac{2K}{(3\Omega_{m0} + K)nK^n} - \frac{m}{nK^n}. \tag{33}$$

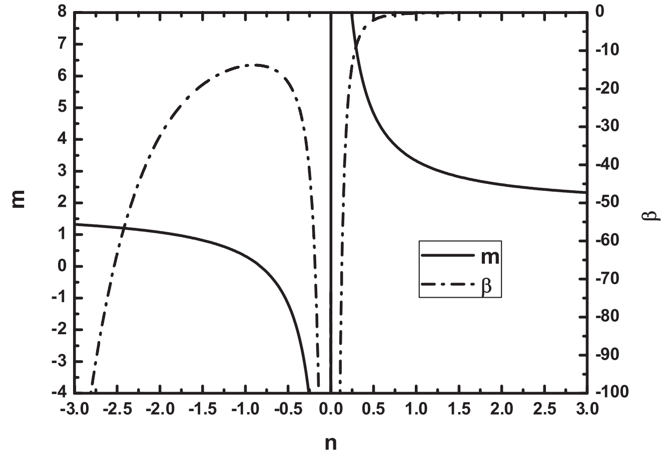

FIG. 4. Constraint relations among the dimensionless parameters of model $f = \alpha R^m \exp[\beta R^n/(H_0^2)^n]$ given at $n \neq 0$ and $\Omega_{m0} = 0.3$.

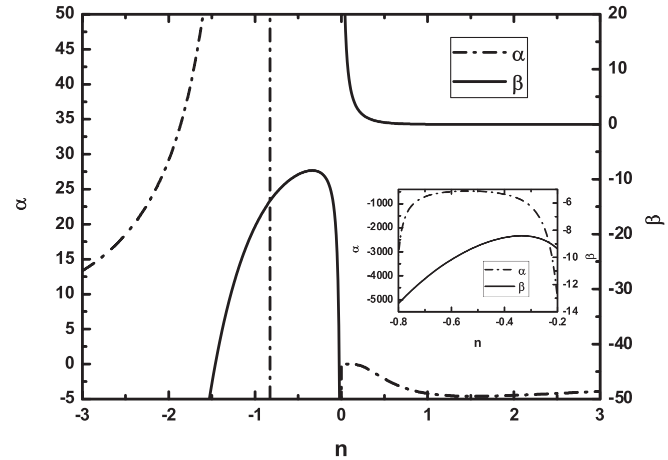

FIG. 5. Constraint relations among the dimensionless parameters of the model $f = R + \alpha H_0^2 \exp[\beta R^n/(H_0^2)^n]$ with the density parameter $\Omega_{m0} = 0.3$.

In the case of $n \neq 0$, we can take approximately $\varepsilon = 0$. So Eq. (32) gives

$$m = \frac{2K}{K + 3\Omega_{m0}}\left[\frac{K - 3\Omega_{m0}}{(K + 3\Omega_{m0})n} + 1\right]. \tag{34}$$

According to Eqs. (33) and (34), the constraint relations among the model parameters at $n \neq 0$ are shown in Fig. 4. The curve in Fig. 4 indicates that $n \geq 1.5$, $\beta \rightarrow 0$, and the Lagrangian reduces to $f = \mu R^m$ with $1.8 < m < 3$. The model with the Lagrangian $f = \mu R^m$ is required to have $|m - 1| < 7.2 \times 10^{-19}$ [6]. So, the theory of the type $f = \alpha R^m \exp[\beta R^n/(H_0^2)^n]$ is unacceptable when $n \geq 1.5$.

2. Theories of type $f = R + \alpha H_0^2 \exp[\beta R^n/(H_0^2)^n]$

For this type of model, Eqs. (16) and (18) give

$$\varepsilon = \frac{1}{K}\left|\frac{\alpha\beta n K^n \exp(\beta K^n)(\beta n K^n + n - 1)}{K + \alpha\beta n(2 - n)K^n \exp(\beta K^n) - \alpha n^2(\beta K^n)^2 \exp(\beta K^n)}\right|, \tag{35}$$

and

$$\alpha = \frac{K(K - 3\Omega_{m0})}{[(3\Omega_{m0} + K)\beta n K^n - 2K]\exp(\beta K^n)}. \tag{36}$$

We take $\varepsilon = 0$ approximately, and Eq. (35) gives

$$\beta n K^n \exp(\beta K^n)(\beta n K^n + n - 1) = 0. \tag{37}$$

Equation (37) requires $n = 0$, $\beta = 0$, or $\beta n K^n = 1 - n$. For the case of $n = 0$, from Eq. (36), we can get $\alpha = (3\Omega_{m0} - K)/(2e^{\beta K})$, therefore the Lagrangian reduces to $f = R + (3\Omega_{m0} - K)e^{\beta}H_0^2/(2e^{\beta K})$. For $\beta = 0$, we can obtain $\alpha = -(K - 3\Omega_{m0})/2$, and the Lagrangian is $f = R - (K - 3\Omega_{m0})H_0^2/2$. In the two cases, the standard ΛCDM paradigm is recovered. In the case of $\beta n K^n = 1 - n$, the constraint conditions are

$$\alpha = \frac{K(K - 3\Omega_{m0})}{[3\Omega_{m0} - K - (3\Omega_{m0} + K)n]\exp(\frac{1-n}{n})}, \tag{38}$$

and

$$\beta = \frac{1 - n}{nK^n}. \tag{39}$$

The constraint relations among the model parameters are shown in Fig. 5. When $0 < n < 0.2$, the parameter $\alpha \rightarrow 0$, the Lagrangian reduces to $f = R$, and it cannot explain the current acceleration. For the case of $n \geq 0.5$, the parameter $\beta \rightarrow 0$. This means that the model has no meaningful difference with the standard cosmological model. For the case of $n \leq -1$, as an alternative of the dark energy, the model has the potential to explain the cosmic acceleration at present.

IV. CONCLUSION AND DISCUSSION

The $f(R)$ models can provide an explanation for the late-time acceleration of the Universe. With the modified general relativity, it results in a time-varying gravitational constant. In this paper, we clarified how the time variation of the effective gravitational constant constrains the parameters of $f(R)$ models in Palatini formalism. The necessary conditions for $f(R)$ models which can be reconcilable with the value $|\dot{\bar{G}}/\bar{G}|_0 < 10^{-13}\ \mathrm{yr}^{-1}$ are obtained. It provides an extremely simple method to investigate the feasibility of the $f(R)$ model. We also studied some concrete models and gave the relations among model parameters and the restrictions on values of model parameters.

For the model $f = \sqrt[m]{R^m + \Lambda(H_0^2)^m}$, the constraint conditions require $|m - 1| < 6 \times 10^{-3}$. It is very close to the standard ΛCDM model. Other cases in which m deviates very far from 1 should be ruled out. The parameter n in theories of type $f = \alpha R^m[\ln(\beta R/H_0^2)]^n$ must be larger than zero. The case of $n \leq 0$ for this type of theories is unacceptable. Theories with $f = R + \alpha H_0^2[\ln(\beta R/H_0^2)]^n$ are reasonable for $n < 4$ and $n \neq 1$. When $n = 1$ or $n \geq 4$, the relation between parameters n and α requires $\alpha \to 0$. In these cases, the model should be ruled out because its Lagrangian reduces to $f = R$ and cannot explain the accelerated expansion of the Universe at present time. As a possible alternative of the dark energy, the model $f = \alpha R^m \exp[\beta R^n/(H_0^2)^n]$ has the potential to explain the cosmic acceleration in the case of $n < 0$. In the model $f = R + \alpha H_0^2 \exp[\beta R^n/(H_0^2)^n]$, the parameter $\alpha \to 0$ when $0 < n < 0.2$ that the Lagrangian reduces to the unacceptable form $f = R$. For the case of $n \geq 0.5$, the parameter $\beta \to 0$, the model reduces to the standard cosmological model. For the case of $n \leq -1$, the model provides a reasonable explanation of the cosmic acceleration.

The key point of the paper is that our discussions are based on the supposition $|\dot{\bar{G}}/\bar{G}| < 10^{-13}\ \mathrm{yr}^{-1}$ at the redshift $z = 0$. It is noted that Eqs. (16) and (18) depend on the current curvature of space-time R_0. As long as $|\dot{\bar{G}}/\bar{G}|_{z=0} \ll 2H_0$, we obtain the relation $R_0 = KH_0^2$ from Eq. (14), which is independent of the form of the function $f(R)$. Accordingly, the ratio $|\dot{\bar{G}}/\bar{G}|_{z=0}$ can be relaxed to $|\dot{\bar{G}}/\bar{G}|_{z=0} < 10^{-12}\ \mathrm{yr}^{-1}$ with $H_0 \sim 0.7 \times 10^{-10}\ \mathrm{yr}^{-1}$, which is far larger than the above adopted $|\dot{\bar{G}}/\bar{G}|_{z=0} < 10^{-13}\ \mathrm{yr}^{-1}$. The only influence is that it widens the range of the dimensionless number ε to $\varepsilon < 5 \times 10^{-3}$. This relaxation does not alter our results, and the obtained conclusions are always valid. If the relaxed supposition $|\dot{\bar{G}}/\bar{G}|_{z=0} < 10^{-12}\ \mathrm{yr}^{-1}$ is not true, the constraints on the model parameters will not be obtained using this method because we cannot get the relation $R_0 = KH_0^2$. We emphasize finally that the adopted assumption, even being quite reasonable, is still a hypothesis which requires a further experimental verification.

ACKNOWLEDGMENTS

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[1] A.G. Riess *et al.*, Astron. J. **116**, 1009 (1998); S. Perlmutter *et al.*, Nature (London) **391**, 51 (1998); J.L. Tonry *et al.*, Astrophys. J. **594**, 1 (2003); R. Knop *et al.*, Astrophys. J. **598**, 102 (2003); B. Barris *et al.*, Astrophys. J. **602**, 571 (2004).
[2] S. Capozziello *et al.*, Int. J. Mod. Phys. D **12**, 1969 (2003); S.M. Carroll, V. Duvvuri, M. Trodden, and M.S. Turner, Phys. Rev. D **70**, 043528 (2004).
[3] T.P. Sotiriou and V. Faraoni, arXiv:0805.1726 [Rev. Mod. Phys. (to be published)].
[4] D. Barraco and V.H. Hamity, Phys. Rev. D **62**, 044027 (2000).
[5] J.A.R. Cembranos, Phys. Rev. D **73**, 064029 (2006); V. Faraoni, Phys. Rev. D **74**, 104017 (2006); X.T.P. Sotiriou, Phys. Lett. B **645**, 389 (2007).
[6] S. Capozziello, V.F. Cardone, and A. Troisi, J. Cosmol. Astropart. Phys. 08 (2006) 001; Y. Sobouti, Astron. Astrophys. **464**, 921 (2007); S. Capozziello, V.F. Cardone, and A. Troisi, Mon. Not. R. Astron. Soc. **375**, 1423 (2007); S. Carloni, P. Dunsby, S. Capozziello, and A. Troisi, Classical Quantum Gravity **22**, 4839 (2005); A. Borowiec, W. Godlowski, and M. Szydlowski, Phys. Rev. D **74**, 043502 (2006); J. Santos, J.S. Alcaniz, M.J. Rebouc, and F.C. Carvalho, Phys. Rev. D **76**, 083513 (2007); S. Tsujikawa, Phys. Rev. D **76**, 023514 (2007); T. Clifton and J.D. Barrow, Phys. Rev. D **72**, 103005 (2005).
[7] G. Cognola, E. Elizalde, S. Nojiri, S.D. Odintsov, and S. Zerbini, J. Cosmol. Astropart. Phys. 02 (2005) 010; M. Fairbairn and S. Rydbeck, J. Cosmol. Astropart. Phys. 12 (2007) 005; R. Bean, D. Bernat, L. Pogosian, A. Silvestri, and M. Trodden, Phys. Rev. D **75**, 064020 (2007); P. Zhang, Phys. Rev. D **76**, 024007 (2007); K. Kainulainen, V. Reijonen, and D. Sunhede, Phys. Rev. D **76**, 043503 (2007); C.G. Bohmer, L. Hollenstein, and F.S.N. Lobo, Phys. Rev. D **76**, 084005 (2007); J.D. Evans, L.M.H. Hall, and P. Caillol, Phys. Rev. D **77**, 083514 (2008); J. Santos, J.S. Alcaniz, F.C. Carvalho, and N. Pires, Phys. Lett. B **669**, 14 (2008); F.C. Carvalho, E.M. Santos, J.S. Alcaniz, and J. Santos, J. Cosmol. Astropart. Phys. 09 (2008) 008.
[8] S. Fay, R. Tavakol, and S. Tsujikawa, Phys. Rev. D **75**, 063509 (2007).
[9] T.P. Sotiriou, Classical Quantum Gravity **23**, 5117 (2006); V. Faraoni, Phys. Rev. D **75**, 067302 (2007); A. Iglesias, N. Kaloper, A. Padilla, and M. Park, Phys. Rev. D **76**, 104001 (2007); H. Motavali, S. Capozziello, and M.R.A. Jog, Phys. Lett. B **666**, 10 (2008); S. Tsujikawa and T. Tatekawa, Phys. Lett. B **665**, 325 (2008); V. Faraoni, Phys. Lett. B **665**, 135 (2008).
[10] C.J. Copi, A.N. Davis, and L.M. Krauss, Phys. Rev. Lett. **92**, 171301 (2004); A.S. Al-Rawaf, Astrophys. Space Sci. **310**, 173 (2007); R. Nagata, T. Chiba, and N. Sugiyama, Phys. Rev. D **69**, 083512 (2004); K.C. Chan and M.C. Chu, Phys. Rev. D **75**, 083521 (2007).
[11] E.V. Pitjeva, Astron. Lett. **31**, 340 (2005); Solar System

Research **39**, 176 (2005).

[12] S. Capozziello and A. Troisi, Phys. Rev. D **72**, 044022 (2005); G. J. Olmo, Phys. Rev. D **72**, 083505 (2005).

[13] C. G. Shao, J. Shen, B. Wang, and R. K. Su, Classical Quantum Gravity **23**, 3707 (2006).

[14] D. F. Torres, Phys. Rev. D **66**, 043522 (2002); R. Gannouji, D. Polarski, A. Ranquet, and A. A. Starobinsky, J. Cosmol. Astropart. Phys. 09 (2006) 016.

[15] A. G. Riess *et al.*, Astrophys. J. **607**, 665 (2004); H. K. Jassal, J. S. Bagla, and T. Padmanabhan, Phys. Rev. D **72**, 103503 (2005); S. Capozziello, V. F. Cardone, E. Elizalde, S. Nojiri, and S. D. Odintsov, Phys. Rev. D **73**, 043512 (2006); J. Q. Xia, G. B. Zhao, B. Feng, H. Li, and X. Zhang, Phys. Rev. D **73**, 063521 (2006); P. Astier *et al.*, Astron. Astrophys. **447**, 31 (2006); D. N. Spergel *et al.*, Astrophys. J. **170**, 377 (2007); C. L. MacLeod and C. J. Hogan, Phys. Rev. D **77**, 043512 (2008).

[16] L. Amendola, R. Gannouji, D. Polarski, and S. Tsujikawa, Phys. Rev. D **75**, 083504 (2007).

[17] S. Baghram, M. Farhang, and S. Rahvar, Phys. Rev. D **75**, 044024 (2007); M. S. Movahed, S. Baghram, and S. Rahvar, Phys. Rev. D **76**, 044008 (2007).

[18] S. Nojiri and S. D. Odintsov, Int. J. Geom. Methods Mod. Phys. **4**, 115 (2007); S. E. P. Bergliaffa, Phys. Lett. B **642**, 311 (2006); X. H. Meng and P. Wang, Phys. Lett. B **584**, 1 (2004).

[19] S. Nojiri and S. D. Odintsov, Phys. Lett. B **652**, 343 (2007); B. Li and M. C. Chu, Phys. Rev. D **74**, 104010 (2006); Y. S. Song, W. Hu, and I. Sawicki, Phys. Rev. D **75**, 044004 (2007); B. Li and J. D. Barrow, Phys. Rev. D **75**, 084010 (2007).