Clarifying the covariant formalism for the Sunyaev-Zel'dovich effect due to relativistic nonthermal electrons

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We derive the covariant formalism associated with the relativistic Sunyaev-Zel'dovich effect due to a nonthermal population of high energy electrons in clusters of galaxies. More precisely, we show that the formalism proposed by Wright in 1979, based on an empirical approach to compute the inverse Compton scattering of a population of relativistic electrons on CMB photons, can actually be reinterpreted as a Boltzmann-like equation, in the single scattering approximation.

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I. INTRODUCTION

The purpose of this paper is (i) to revisit the formulation of the relativistic Sunyaev-Zel'dovich (rSZ) distortion of the cosmic microwave background (CMB) due to a population of nonthermal high energy electrons in clusters of galaxies, and (ii) to provide a unified and more comprehensive framework to compute the rSZ effect.

Inverse Compton scattering of nonrelativistic thermal electrons on cosmic microwave background (CMB) photons has been widely investigated in the literature since the 1960–1970's, notably with the seminal works by Sunyaev and Zel'dovich on the effect afterwards renamed SZ [1,2], and in early studies on relativistic electron cosmic rays in high energy astrophysics (e.g. [3–5]). The SZ effect generated by a population of hot (still nonrelativistic) electrons has also been addressed in detail since the 1980's (see e.g. [6–14]), while the SZ effect originating from relativistic electron cosmic rays has been discussed in e.g. Refs. [15–18]. So far, these works on hot thermal or relativistic nonthermal electrons have used different approaches.

For example, in Refs. [10–14], the authors have computed the relativistic corrections to the standard thermal SZ predictions due to the hot trail of intracluster thermal electrons (including the effect from multiple scattering). They used a Fokker-Planck expansion of the collisional Boltzmann equation in terms of the electron temperature-to-mass ratio $T/m \lesssim 0.1$, in which the full squared matrix amplitude of Compton scattering (that is given in any textbook of quantum field theory, e.g. [19]) was implemented for the relevant kinematics. Such an approach therefore not only encompasses but also extends the usual calculation of the standard thermal SZ effect, thus making this formalism very attractive. Nonetheless, it has not been applied yet to the case where there are two populations of electrons (one dominant semirelativistic population and a subdominant relativistic population).

The authors of e.g. Refs. [6–8,15,17,18] have considered the SZ contributions coming from a population of thermal electrons plus an extra source of nonthermal electron cosmic rays, as we want to do. However, they used a more empirical approach based on a classical radiative transfer method proposed by Chandrasekhar [20]. Indeed, they computed the spectral shift in the CMB photon intensity due to Compton scattering processes with electrons, scattering *off* some photon frequencies (from energy E to E') and scattering *in* some others (from energy E' to E). The squared matrix amplitude (that is the probability of frequency change) was expressed in the electron rest frame, leading to an expression close to its associated nonrelativistic limit, such as the one originally used by Chandrasekhar. Multiple scatterings were eventually implemented by further using Chandrasekhar's radiative approach.

In Sec. II, we demonstrate that Chandrasekhar's radiative transfer method and the Boltzmann-like approach mentioned above are in fact formally equivalent in the single scattering approximation. They thus give the same results, which we show by using the full relativistic squared matrix amplitude instead of the photon frequency redistribution probability function derived by Wright [6]. This therefore sketches a unified formalism for the computation of both hot thermal and relativistic nonthermal SZ effects, as subsequently found in [21]. For illustration, we derive the relativistic photon energy redistribution function in the CMB reference frame (therefore different from Wright's $P(s, \beta)$ function) that can be used to compute the relativistic SZ effect generated by an additional population of high energy electron cosmic rays in a cluster. For the sake of completeness, we revisit in Sec. III the full relativistic phase space and squared amplitude of the in-

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verse Compton process in the relativistic limit. Our conclusions are drawn in Sec. IV. Numerical applications of this framework will be given in a companion paper [22].

II. EQUIVALENCE BETWEEN RADIATIVE TRANSFER AND THE BOLTZMANN EQUATION

The purpose of this section is to demonstrate the equivalence between the approach based on the Boltzmann equation and Chandrasekhar's radiative transfer method to compute the rSZ effect, in the single scattering approximation.

A. Scattered intensities in the radiative transfer approach

Chandrasekhar's radiative transfer method has been widely used in the literature, notably to compute the SZ effect due to nonthermal electrons in clusters of galaxies (for a review, see [16]). The principle of this approach is recalled in the appendix (see Sec. II A), as well as the precise derivation of the CMB intensity deviation in the relativistic regime and in the single scattering approximation (see Sec. II B). Here we summarize the main results coming out of this approach.

The spectral shift in the CMB intensity due to scatterings with relativistic electrons can be expressed as the difference between two contributions. The first one is associated to the photons scattered *out* ($\hat{I}_\gamma^{\text{s-out}}(E_k)$) from the CMB spectrum, from energy E_k to energy $E'_k \neq E_k$, and the second is associated with the photons scattered *in* ($\hat{I}_\gamma^{\text{s-in}}(E_k)$) from energy E'_k to energy E_k (after averaging over the direction of the incoming photons):

$$\Delta I_\gamma(E_k) = \hat{I}_\gamma^{\text{s-in}}(E_k) - \hat{I}_\gamma^{\text{s-out}}(E_k), \tag{1}$$

The expressions of the *in* and *out* intensities can then be calculated. They are given in Sec. II B (see Eqs. (B2) and (B3)) as integrals over the line-of-sight dl:

$$\hat{I}^{\text{s-out}}(E_k) = \int dl \int \frac{d^3\vec{p}}{(2\pi)^3} f_e(E_p) \int \frac{d\Omega_{k'}}{32\pi^2} \frac{t^2|\mathcal{M}|^2}{(1-\beta\mu)} \times \frac{I_\gamma^0(E_k)}{E_p^2}$$

and

$$\hat{I}^{\text{s-in}}(E_k) = \int dl \int \frac{d^3 p}{(2\pi)^3} f_e(E_p) \int \frac{d\Omega_{k'}}{32\pi^2} \frac{|\tilde{\mathcal{M}}|^2}{E_p^2} \times \frac{I_\gamma^0(\tilde{t}E_k)}{\tilde{t}(1-\beta\mu)}$$

where I_γ^0 is the intensity associated with the photon blackbody spectrum (before interaction with the electrons) and f_e is the energy distribution function characterizing the relativistic electrons (of energy E_p). The term $|\mathcal{M}|^2$ is the squared matrix amplitude associated with the process (p, $k \to p'$, k') for the scattered-out part, and $|\tilde{\mathcal{M}}|^2$ describes the process (p, $k' \to p'$, k) for the scattered-in part. The terms t and $\tilde{t}$ feature the photon energy transfer for both processes, respectively, and are defined in Eqs. (25) and (B3). μ characterizes the angle between the incoming electron of momentum $\vec{p}$ and the photon of momentum $\vec{k}$ (see Eq. (27)).

As we will show in the next section, the photon frequency redistribution probability function considered by Wright in 1979 [6], which has been continuously used afterwards for rSZ predictions [8,15,17,18], will differ from our analysis in that it is derived in the electron rest frame, but both are consistent [21]. Anyway, the equivalence between the radiative transfer and covariant approaches—in the relativistic regime and single scattering approximation—can be formally demonstrated without the need for replacing the squared matrix amplitude by its expression.

B. Boltzmann-like formalism

1. Covariant collisional boltzmann equation

The theoretical framework developed in [10,11,13,14] to treat the semirelativistic corrections of the thermal SZ is the covariant collisional Boltzmann equation (see e.g. [23–26]), which expresses the momentum transfer between different interacting species.

When focusing on photons scattered by a gas of electrons in a covariant scheme, the spatial convective currents are irrelevant, and the time evolution of the photon phase space is given by the covariant collisional Boltzmann equation as:

$$\frac{d}{dt} f_\gamma(E_k) = I_{\text{coll}}(E_k). \tag{2}$$

The collision integral is defined by:

$$\begin{aligned} I_{\text{coll}}(E_k) &\equiv \frac{1}{2E_k} \int \prod_{X=p,p',k'} \frac{d^3\vec{X}}{(2\pi)^3 2E_X} \delta^4(p+k-p'-k') \\ &\quad \times |\mathcal{M}|^2 \{f_\gamma(E_k) f_e(E_p) - f_\gamma(E_{k'}) f_e(E_{p'})\} \\ &= -\int \frac{d^3\vec{p}}{(2\pi)^3} d\text{LIPS} \frac{|\mathcal{M}|^2}{4E_k E_p} \times \{f_\gamma(E_k) f_e(E_p) \\ &\quad - f_\gamma(E_{k'}) f_e(E_{p'})\}, \end{aligned} \tag{3}$$

where f_γ and f_e are the photon and electron distributions, respectively, and where we have neglected the Pauli blocking and stimulated emission factors (even if the latter were included, they would automatically disappear as soon as treating relativistic electrons, since in this case $f_e(E_p) \simeq f_e(E_{p'})$, as we will show in the following). This equation features the Lorentz invariant phase space dLIPS and the squared matrix amplitude $|\mathcal{M}|^2$ associated with the process (p, $k \leftrightarrow p'$, k'). These will be made explicit in Sec. III.

Notice that this formalism rests on the microreversibility of the process.

Let us now discuss the meaning of the electron distribution written above. As we have already stressed, we are considering two different populations of electrons, one thermal and the other nonthermal. The former is assumed dominant, but we are interested in the subdominant population only. Formally, the above electron distribution f_e characterizes the sum of these two populations, that is $f_e = f_e^{n\text{-th}} + f_e^{\text{th}}$. Hence, the question of whether we can separate their respective contributions arises. When integrating over the electron momenta, we see that at low energy, say below 1 MeV typically, only the thermal part f_e^{th} will be relevant in the calculation, while at higher energy, only the nonthermal part will contribute. Since we restrict ourselves to the single scattering limit, we can therefore safely separate those two contributions since none of them will affect the other one. Consequently, in the following, we will only deal with the relativistic nonthermal part, and will set $f_e = f_e^{n\text{-th}}$ for convenience. Note that in the multiple scattering case, this would not hold anymore, since there would be a mixing of the SZ effects coming from both populations of electrons: for instance, a photon could first be scattered by a thermal electron, and then by a nonthermal one. On that account, we would have to treat both populations in a self-consistent manner. Nevertheless, such a complex analysis is beyond the scope of this paper, and probably too sophisticated regarding the small contribution to the SZ effect that is expected from relativistic electrons.

The collisional Boltzmann equation written above describes the time evolution of the photon phase space distribution due to elastic collisions with electrons. The electron distribution has therefore to be known at any time. In the following (Sec. II B 2), we will demonstrate that if one assumes that the electron phase space density does not change significantly over one (or even several) characteristic travel time of a photon through a cluster, then one can use this formalism to treat the SZ effect originating from nonthermal electrons in the single scattering approximation. The above hypothesis is justified since, if we take a typical scale of 10 Mpc for a cluster, then the mean time for a photon to cross it is ~30 Myr while the typical timescale for the relativistic electron energy loss by inverse Compton scattering on CMB photons is ~300 Myr.

2. Covariant formalism applied to the rSZ effect

Let us consider a cluster containing a tiny population of electrons, and track the photon distribution. The integration over the time, in Eq. (2), translates to an integration over a geodesic ($dl = cdt$):

$$\int df_\gamma(E_k) = -\int dl \int \frac{d^3\vec{p}}{(2\pi)^3} d\text{LIPS} \frac{|\mathcal{M}|^2}{4E_k E_p} \times \{f_\gamma(E_k)f_e(E_p) - f_\gamma(E_{k'})f_e(E_{p'})\}. \quad (4)$$

If we consider the electron density to be small enough, we can assume that a CMB photon has a very small probability to experience Compton scattering with electrons along its path through a cluster. In this case, the single scattering approximation is fully relevant and gives: $\int df_\gamma = f_\gamma^{\text{after}} - f_\gamma^{\text{before}}$, where f_γ^{before} is the original blackbody spectrum distribution before the CMB photons enter into the cluster, namely f_γ^0, and f_γ^{after} is the photon distribution after travelling through the cluster and experiencing one scattering. An observer looking in the direction of the cluster will therefore observe $f_\gamma^{\text{after}} = f_\gamma \neq f_\gamma^0$. Expressing now this equation in terms of the photon intensity, that is $I_\gamma(E) = E^3 f_\gamma(E)$, we obtain:

$$I_\gamma(E_k) = I_\gamma^0(E_k) - \int dl \int \frac{d^3\vec{p}}{(2\pi)^3} d\text{LIPS} \frac{|\mathcal{M}|^2}{4E_k E_p} \times \left\{I_\gamma(E_k)f_e(E_p, \vec{x}) - \frac{E_k^3}{E_{k'}^3} I_\gamma(E_{k'})f_e(E_{p'}, \vec{x})\right\}, \quad (5)$$

where I_γ^0 is the intensity of the pure blackbody spectrum of the CMB before entering in the cluster. The first term in the sum under the integral characterizes the frequencies shifted out from E_k to $E_{k'}$, while the second term characterizes the frequencies shifted in from $E_{k'}$ to E_k, both due to Compton scattering. Note that the electron distribution f_e is not ubiquitous anymore, and must now also carry information of the spatial distribution, which will be relevant for the line-of-sight integral along dl.

As we stated previously, if the number density of the relativistic electron population is very small, then the interactions between the CMB photons and this population is well approximated by the single scattering approximation. This hypothesis further enables us to treat several populations of electrons (e.g. a dominant, thermal and/or semirelativistic, population and a subdominant relativistic population) separately.

Let us now specifically focus on the relativistic population. As we are considering electrons with typical energies $E_p \gtrsim 1$ MeV and CMB photons with energies $E_k \lesssim 10^{-3}$ eV, we can safely make use of the limit $\alpha \equiv E_k/E_p \to 0$ in the above equation, which is fully valid in the whole range of the CMB photon energy distribution in the single scattering approximation. Within this limit, due to energy-momentum conservation in the Compton scattering, we will show in Sec. III that $E_{p'} \to E_p$, such that we can replace $f_e(E_{p'})$ by $f_e(E_p)$ (see Eqs. (33) and (35)). This is the key point to understand why the Boltzmann framework can be linked to the radiative transfer approach for the SZ effect associated with nonthermal electrons. Most importantly, in the single scattering approximation, the photon intensity I_γ under the integral of the right hand side can be replaced by that corresponding to the pure blackbody spectrum I_γ^0. Defining $\Delta I_\gamma \equiv I_\gamma - I_\gamma^0$, we obtain:

$$\Delta I_\gamma(E_k) = -2\int dl \int \frac{d^3\vec{p}}{(2\pi)^3} f_e(E_p, \vec{x}) d\mathrm{LIPS} \times \frac{|\mathcal{M}|^2}{4E_k E_p} \times \left\{ I^0_\gamma(E_k) - \frac{I^0_\gamma(E_{k'})}{t^3} \right\}, \tag{6}$$

where $t \equiv E_{k'}/E_k$, and where we have explicitly added a factor of 2 to account for the sum over the two photon polarizations (so that it should not appear in the definition of I^0_γ in the right hand side). This expression fully characterizes the intensity after removing the contribution of photons scattered-out (from E_k to $E_{k'}$) and after adding that of photons scattered-in (from $E_{k'}$ to E_k), exactly like à la Wright. The term proportional to $I^0_\gamma(E_k)$ is the scattered-out contribution, and the term proportional to $I^0_\gamma(E_{k'})$ is the scattered-in contribution. Hence Chandrasekhar's formalism, when applied to the relativistic SZ effect, as recalled in the previous section, is strictly equivalent to using the Boltzmann equation (in the single scattering approximation)—detailed expressions related to Chandrasekhar's formalism are derived in the appendix, see Sec. II B. We can rewrite the above equation in terms of the differential Compton cross section, after performing the integrals over $d^3\vec{p}\,'$ and $dE_{k'}$:

$$\Delta I_\gamma(E_k) = -2\int dl \int \frac{d^3\vec{p}}{(2\pi)^3} f_e(E_p, \vec{x}) \int d\Omega_{k'} \beta_{\mathrm{rel}} \frac{d\sigma}{d\Omega_{k'}} \times \left\{ I^0_\gamma(E_k) - \frac{I^0_\gamma(tE_k)}{t^3} \right\}, \tag{7}$$

where β_{rel} is the relative velocity between the incoming particles (see Eq. (51)), where $d\Omega_{k'}$ is the solid angle associated with photons of energy $E_{k'} \neq E_k$, and where the differential Compton cross section $d\sigma$ is defined in Eq. (52). We will sketch the full angular dependence in the next subsection.

C. Equivalence between the radiative transfer and covariant approaches

Equations (6) and (7) contain two contributions, which can conveniently be expressed as:

$$\Delta I_\gamma(E_k) = I^{\mathrm{in}}_\gamma(E_k) - I^{\mathrm{out}}_\gamma(E_k), \tag{8}$$

where the upper script *out* characterizes the part of the spectrum which is shifted out from energy E_k to other energies, while *in* stands for other frequencies which are shifted in to energy E_k because of scattering. According to Eq. (7), the term $I^{\mathrm{out}}_\gamma(E_k)$ is given by:

$$I^{\mathrm{out}}_\gamma(E_k) = 2\int dl \int \frac{d^3\vec{p}}{(2\pi)^3} f_e(E_p, \vec{x}) \times \int d\Omega_{k'} \beta_{\mathrm{rel}} \frac{d\sigma}{d\Omega_{k'}} I^0_\gamma(E_k), \tag{9}$$

where we recall that the factor of 2 in front of the right hand side accounts for the sum over the two photon polarizations (see Eq. (6)). This can be rewritten as:

$$I^{\mathrm{out}}_\gamma(E_k) = 2\tau I^0_\gamma(E_k), \tag{10}$$

where we define:

$$\tau \equiv \int dl \int \frac{d^3\vec{p}}{(2\pi)^3} f_e(E_p, \vec{x}) \int d\Omega_{k'} \beta_{\mathrm{rel}} \frac{d\sigma}{d\Omega_{k'}} = \int dl \int \frac{d^3\vec{p}}{(2\pi)^3} f_e(E_p, \vec{x}) \int \frac{d\Omega_{k'}}{64\pi^2} \frac{t^2|\mathcal{M}|^2}{E_p^2(1-\beta\mu)}. \tag{11}$$

We have used the definition of the differential cross section from Eq. (53), and that of the relative velocity $\beta_{\mathrm{rel}} = (1 - \beta\mu)$. Now we need to average over the solid angle $d\Omega_k = d\phi d\mu$ associated with the incoming photons, so that the actual measured shifted intensity will be:

$$\hat{I}^{\mathrm{out}}_\gamma(E_k) = 2\hat{\tau} I^0_\gamma(E_k) \qquad \hat{\tau} \equiv \int \frac{d\Omega_k}{4\pi}\tau. \tag{12}$$

If we perform the integral over $d\Omega_{k'} = d\phi' d\mu'$ appearing in the definition of $\hat{\tau}$, we obtain the full relativistic (inverse) Compton cross section given in Eq. (54), which still depends on μ, E_k and E_p. Then, we can perform the average over the ingoing photon solid angle $d\Omega_k = 2\pi d\mu$. The result is found to depend on powers of $\alpha\gamma^2 = \gamma E_k/m$. Though for CMB photons $\alpha\gamma^2$ is negligible up to ultrarelativistic electron energies, that is as long as $\gamma \lesssim 10^8$, we still give the result at next-to-leading order (see Eq. (55)):

$$\hat{\tau} = \int dl \int \frac{d^3\vec{p}}{(2\pi)^3} f_e(E_p, \vec{x}) \sigma_{\mathrm{T}} \left(1 - \frac{8}{3}\alpha\gamma^2 + \frac{2\alpha}{3}\right), \tag{13}$$

where σ_{T} is the Thomson scattering cross section. Note that it is quite consistent to find a result in which the cross section is close to the Thomson cross section for the diffused *out* term, since we are in the limit $\gamma E_k \ll m$. Indeed, this corresponds exactly to seeing a low energy photon in the electron energy rest frame: in this sense, as very well known, we are in the Thomson regime, while not dealing with a Thomson diffusion at all. If we had found other dependences on angles under the integral, we could not have integrated the cross section out, as it will be shown to be the case for the scattered *in* a bit further.

We recall that the spatial dependence of the electron population has to be included in the distribution f_e. τ can be, as displayed above, interpreted as the optical depth associated with the electron population. To provide an expression closer to the standard lores, we can assume that the energy and the spatial distributions of electrons can be factorized, such that $f_e(E_p, \vec{x}) = n_e(\vec{x})\tilde{f}_e(E_p)$, where $\tilde{f}_e$ is the normalized momentum distribution ($\int d^3\vec{p}/(2\pi)^3 \tilde{f}_e(E_p) = 1$). This naturally gives $\int d^3\vec{p}/(2\pi)^3 f_e(E_p, \vec{x}) = n_e(\vec{x})$, to which one can stick instead of the previous trick, depending on one's convenience. We can therefore rewrite the previous equation as:

$$\hat{\tau} = \mathcal{K}\tau_{\rm nr} \tag{14}$$

where:

$$\mathcal{K} \equiv \int \frac{d^3\vec{p}}{(2\pi)^3}\tilde{f}_e(E_p)\Big(1 - \frac{8}{3}\alpha\gamma^2 - \frac{2\alpha}{3}\Big), \quad \text{and} \qquad \tau_{\rm nr} \equiv \int dl\sigma_T n_e(\vec{x}). \tag{15}$$

We thus recover the standard definition of the optical depth $\tau_{\rm nr}$ for nonrelativistic thermal electrons when neglecting the terms proportional to α and $\alpha\gamma^2$. The factor $\mathcal{K} \lesssim 1$ has to be considered as a relativistic correction to apply to $\tau_{\rm nr}$. Finally, we remind that this optical depth will have to be averaged over the spatial resolution of the telescope for quantitative prediction purposes [see Eq. (23)].

The scattered *in* term of Eq. (8), $I_\gamma^{\rm in}(E_k)$, is given, according to Eq. (7), by:

$$\begin{aligned} I_\gamma^{\rm in}(E_k) &= 2\int dl \int \frac{d^3\vec{p}}{(2\pi)^3} f_e(E_p, \vec{x}) \times \int d\Omega_{k'}\beta_{\rm rel}\frac{d\sigma}{d\Omega_{k'}}\frac{I_\gamma^0(tE_k)}{t^3} \\ &= 2\int dl \int \frac{d^3\vec{p}}{(2\pi)^3} f_e(E_p, \vec{x}) \times \int \frac{d\Omega_{k'}}{64\pi^2}\frac{|\mathcal{M}|^2}{E_p^2(1-\beta\mu)}\frac{I_\gamma^0(tE_k)}{t}. \end{aligned} \tag{16}$$

We recall that $t \equiv E_{k'}/E_k$ is the ratio of the diffused to initial photon energies for the scattered-out process (and conversely for the scattered-in process), and that the factor of 2 in front of the right hand side accounts for the sum over two photon polarizations (see Eq. (6)). In the second equality, we have used the expression of the differential cross section given in Eq. (53), where appears the squared amplitude $|\mathcal{M}|^2$ of the inverse Compton process. We further need to average to above expression, as previously done for the *out* part, over the solid angle $d\Omega = d\phi d\mu$, and the final expression for the scattered-in term is thus:

$$\hat{I}_\gamma^{\rm in}(E_k) = \int \frac{d\Omega_k}{4\pi} I_\gamma^{\rm in}(E_k). \tag{17}$$

As one can see, Eqs. (10) and (16) are similar to Eqs. (B2) and (B3). While the scattered-out processes are strictly equivalent, a slight difference appears in the scattered-in, coming from the squared amplitude ($|\mathcal{M}|^2$ versus $|\tilde{\mathcal{M}}|^2$) and the energy transfer (t versus $\tilde{t}$). Nevertheless, this difference is completely removed in the relativistic limit, since we have in this case $|\mathcal{M}|^2 = |\tilde{\mathcal{M}}|^2$ and $t = \tilde{t}$. Hence the two methods are indeed equivalent.

D. Redistribution function for the rSZ effect

In Sec. III D, we will derive the full expression of the differential inverse Compton cross section for relativistic electrons and soft photons, and also provide some results in the corresponding limit $\alpha = E_k/E_p \to 0$ (see Eq. (53)). Anticipating these results, and using Eqs. (16) and (17), this implies that the angular average of $I_\gamma^{\rm in}(E_k)$ can actually be written at leading order in α as:

$$\hat{I}_\gamma^{\rm in}(E_k) = 2\int dl \int dp f_e(E_p, \vec{x}) \times \int d\mu' \int d\mu \mathcal{F}(\beta, \mu, \mu') I_\gamma^0(tE_k), \tag{18}$$

where we define:

$$\mathcal{F}(\beta, \mu, \mu') \equiv \frac{\beta^2 m^2}{(2\pi)^3}\frac{3\sigma_{\rm T}}{16}\frac{(1-\beta\mu')}{(1-\beta\mu)^2}\Big\{2 - 2K(1-\mu\mu') + K^2\Big[(1-\mu\mu')^2 + \frac{1}{2}(1-\mu^2)(1-\mu'^2)\Big]\Big\}. \tag{19}$$

K is defined in Eq. (49). The function $\mathcal{F}(\beta, \mu, \mu')$ is meant to be related, while not strictly, to the product $p(\mu)\cdot q(\mu, \mu')$ (where $q(\mu, \mu')$ is in fact the squared matrix amplitude) which was displayed in Wright's paper [6]. However, it is different from Wright's expression, notably because this latter is derived in the electron rest frame, but they have been shown to be consistent [21]. Moreover, we see that in Eq. (18), we cannot factorize out the optical depth τ as defined by Eq. (12): it is mixed with the definition of $\mathcal{F}$, and it cannot be taken out of the integral over the angles. So by reexpressing the photon zenith angles μ and μ' in the electron rest frame, it is possible to derive a much more suitable expression for numerical computations [21] consistent with the results obtained by Wright.

The trick to recover a formulation of the *in* process similar to the classical radiative transfer method used by Wright is the same as employed for the *out* process. Indeed, we can still describe the electron phase space density as $f_e(E_p, \vec{x}) = n_e(\vec{x})\tilde{f}_e(E_p)$, where $\tilde{f}_e$ is normalized to unity, and utilize the expression of the nonrelativistic optical depth $\tau_{\rm nr}$ given in Eq. (16). Armed with this, we can express *in* contribution à la Wright as:

$$\hat{I}_\gamma^{\rm in}(E_k) = 2\tau_{\rm nr}\int dp \int d\mu \int d\mu' \hat{\mathcal{F}}(\beta, \mu, \mu') I_\gamma^0(tE_k), \tag{20}$$

where we have defined:

$$\hat{\mathcal{F}}(\beta, \mu, \mu') \equiv \frac{\mathcal{F}(\beta, \mu, \mu')}{\sigma_{\rm T}}\tilde{f}_e(E_p). \tag{21}$$

These expressions are formally encoding the same physical information as those derived by Wright. The function $\hat{\mathcal{F}}$

has to be interpreted as the photon frequency redistribution function for a given normalized energy distribution of relativistic electrons such that $\int d^3\vec{p}/(2\pi)^3\tilde{f}_e(E_p) = 1$.

To summarize, the Boltzmann approach allows to derive contributions for the scattered *out* and *in* processes which are similar to the radiative transfer method when regarding the relativistic SZ effect. These contributions are recalled here:

$$\begin{aligned}\hat{I}^{\rm out}_\gamma(E_k) &= 2\mathcal{K}\tau_{\rm nr}I^0_\gamma(E_k)\\ \hat{I}^{\rm in}_\gamma(E_k) &= 2\tau_{\rm nr}\int dp\int d\mu\int d\mu'\hat{\mathcal{F}}(\beta,\mu,\mu')I^0_\gamma(tE_k).\end{aligned} \tag{22}$$

We recall that the factors of two appearing above explicitly account for the sum over the two polarization states of the CMB photons, so that I^0_γ must be defined for 1° of freedom only. Note that for numerical computations, the photon frequency redistribution function $\mathcal{F}$ is more suitably expressed when photon zenith angles are reexpressed in the electron rest frame, as shown by [21] to which we refer the reader for computational purposes.

E. Experimental signature of the deviation

Finally, since an experiment has a finite angular resolution, the relevant quantity to use when doing predictions for specific instruments is the average value of $\Delta I_\gamma(E_k)$ over the solid angle $\Delta\Omega_{\rm res}$ carried by the experimental resolution, that is:

$$\Delta I^{\rm obs}_\gamma(E_k) = \frac{1}{4\pi}\int d\Omega_{\rm res}\Delta I_\gamma(E_k), \tag{23}$$

where the dependence on $\Omega_{\rm res}$ of the intensity is hidden in the spatial dependence of the electron population characterized by $f_e(p,\vec{x})$. More precisely, if the electron density is spherical in the cluster, then it only depends on the distance $r = |\vec{x}-\vec{x}_{\rm cc}|$ to the cluster center. This radius r can be related to the angle $\psi_{\rm res}$ scanning the resolution range through the relation:

$$r = \sqrt{l^2+d^2-2dl\cos\psi_{\rm res}},$$

where d is the distance from the observer to the cluster center, and l is the distance as measured along the line-of-sight.

III. (INVERSE) COMPTON SCATTERING

In this section, we rederive the famous results for the squared matrix amplitude and the phase space factors associated with the electron-photon elastic scattering process in the special kinematics of inverse Compton scattering. The number of references associated with the Compton scattering is obviously very large, but one can find some more details in e.g. [19,27,28].

A. Kinematics

The kinematics of the process we are interested in is the scattering of an electron of four-momentum p off a photon of four-momentum k, resulting in final states p' and k' for the electron and the photon, respectively. We deal with an elastic collision, in which the conservation of four-momentum holds, so that:

$$E_p + E_k = E_{p'} + E_{k'}, \qquad \vec{p}+\vec{k} = \vec{p}'+\vec{k}'. \tag{24}$$

We can now derive almost everything we need from these basic equations of classical collision theory and from the very simple geometry of the problem [20]. Note that the coming results will hold in any Lorentz frame, unless we specify it.

1. The scattered photon energy

The first quantity that we can calculate is the energy $E_{k'}$ of the outgoing photon after the collision. Starting from Eq. (24), and substituting $E_{p'} = \sqrt{m^2+p'^2}$, and $\vec{p}' = \vec{p}+\vec{k}-\vec{k}'$, we find:

$$\begin{aligned}E^2_{p'} &= (E_p+E_k-E_{k'})^2 \Leftrightarrow m^2 + (\vec{p}+\vec{k}-\vec{k}')^2\\ &= (E_p+E_k-E_{k'})^2 \Leftrightarrow E_{k'} = tE_k \quad \text{with}\\ t &\equiv \frac{(1-\beta\mu)}{(1-\beta\mu')+\alpha(1-\Delta)},\end{aligned} \tag{25}$$

where $\beta \equiv p/E_p$ is the electron velocity, where

$$\alpha \equiv \frac{E_k}{E_p}, \tag{26}$$

which will be a quantity very useful to study the different astrophysical regimes; and where we have defined the cosine of the angle θ (θ') between the incident electron and the initial (outgoing) photon by μ (μ', respectively), and that of the angle Θ between the initial and final photon directions by Δ:

$$\begin{aligned}\mu \equiv \cos\theta = \frac{\vec{p}\cdot\vec{k}}{|\vec{p}|\cdot|\vec{k}|} \qquad \mu' \equiv \cos\theta' = \frac{\vec{p}\cdot\vec{k}'}{|\vec{p}|\cdot|\vec{k}'|}\\ \Delta \equiv \cos\Theta = \frac{\vec{k}\cdot\vec{k}'}{|\vec{k}|\cdot|\vec{k}'|}.\end{aligned} \tag{27}$$

We can already have a look to the limit value of the scattered energy when the incoming electron energy is much larger than that of the incoming photon $E_p \gg E_k \Leftrightarrow \alpha \to 0$, which is typically the case in any collision between a CMB photon and an electron (indeed, $E_k \ll m$):

$$t = \frac{E_{k'}}{E_k} \overset{\alpha\to 0}{\longrightarrow} \frac{(1-\beta\mu)}{(1-\beta\mu')}. \tag{28}$$

We see that, as $\beta \to 1$, $E_{k'}$ is maximized for frontal collisions scattering a photon of initial direction $\mu = -1$ to the opposite direction $\mu' = 1$. In this case, $E_{k'} \to 4\gamma^2 E_k$, obtained after a Taylor expansion of $\beta \simeq 1 - 1/(2\gamma^2)$ up to the second order in $\gamma \to \infty$.

2. Geometry

It will be found convenient to express Δ as a function of the other cosines μ and μ'. Without loss of generality, we can choose the incident electron direction along the z axis of any frame $Oxyz$. Therefore, we can define the coordinates of the quadri-momenta as follows:

$$\begin{aligned} p &= E_p(1, 0, 0, \beta) \\ k &= E_k(1, \sin\theta\cos\phi, \sin\phi\sin\theta, \cos\theta) \\ k' &= E_{k'}(1, \sin\theta'\cos\phi', \sin\phi'\sin\theta', \cos\theta'), \end{aligned} \tag{29}$$

where we have used $\beta = |\vec{p}|/E_p$ and the fact that $|\vec{k}| = E_k$ for photons. The coordinates of p' are readily deduced from the above expressions by invoking energy-momentum conservation. With these coordinates, we recover the previous definitions of μ and μ' (see Eq. (27)), and we can now express Δ, the cosine of the angle between the incoming and outgoing photons, as:

$$\begin{aligned} \Delta = \cos\Theta = \mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos\Psi, \\ \text{where } \Psi \equiv \phi - \phi'. \end{aligned} \tag{30}$$

Armed with these definitions and quantities, we can now survey the full relativistic phase space as well as the squared amplitude of the process.

B. Phase space

The first relevant quantity when dealing with the probability of any quantum scattering process is the Lorentz invariant phase space dLIPS, namely, the possible final states to which the process can lead. With the same conventions as previously, we can write:

$$d\text{LIPS} = \frac{d^3\vec{p}'}{(2\pi)^3}\frac{d^3\vec{k}'}{(2\pi)^3}\frac{(2\pi)^4\delta^4(p+k-p'-k')}{4E_{p'}E_{k'}}. \tag{31}$$

By performing the integral over $d^3\vec{p}'$, we get:

$$\int_{\vec{p}'} d\text{LIPS} = \frac{d^3\vec{k}'}{4(2\pi)^2}\frac{\delta(E_p + E_k - E_r - E_{k'})}{E_r E_{k'}}, \tag{32}$$

where, accounting for the momentum conservation $E_{p'} \xrightarrow{\delta^3(P_i - P_f)} E_r$, we define:

$$\begin{aligned} E_r &\equiv \sqrt{m^2 + (\vec{p} + \vec{k} - \vec{k}')^2} \\ &= \sqrt{E_p^2 + E_k^2 + E_{k'}^2 + 2E_p\beta(E_k\mu - E_{k'}\mu') - 2E_kE_{k'}\Delta} \end{aligned} \tag{33}$$

where the last line is obtained after having expressed μ as a function of t. Therefore, we see that the argument in the δ Dirac function of Eq. (31) is a non trivial function of $E_{k'}$, $f(E_{k'}) = E_p + E_k - E_r - E_{k'}$. By means of the properties of the Dirac functions, we can write:

$$\delta(f(E_{k'})) = \sum_i \frac{\delta(E_k^i - E_{k'})}{|f'(E_k^i)|},$$

where E_k^i are the zeros of f, and f' stands for the derivative with respect to $E_{k'}$. There is only one zero, $E_{k'} = E_k^0 = tE_k$, given by Eq. (25), so that we find:

$$\begin{aligned} f'(E_k^0) &= -\frac{E_p}{E_r^0}(\alpha(t-\Delta) - \beta\mu' + E_r^0/E_p) \quad \text{with} \\ E_r^0 &\equiv E_r(E_{k'} = E_k^0) \\ &= E_p\sqrt{1 + 2\alpha\beta(\mu - t\mu') + \alpha^2(1 - 2t\Delta + t^2)} \\ &= E_p|1 + \alpha(1-t)|. \end{aligned} \tag{34}$$

Therefore, the Dirac function becomes:

$$\begin{aligned} \delta(f(E_{k'})) &= \frac{E_r^0\delta(E_k^0 - E_{k'})}{E_p|\mathcal{B}|}, \\ \text{where } \mathcal{B} &\equiv \alpha(t-\Delta) - \beta\mu' + E_r^0/E_p \\ &= (1-\beta\mu') + \alpha(1-\Delta). \end{aligned} \tag{35}$$

We can go ahead with the derivation of the phase space factor, which can now be written as:

$$\begin{aligned} \int_{E_{k'},\vec{p}'} d\text{LIPS} &= \int_{E_{k'}} \frac{d\Omega_{k'}}{4(2\pi)^2}\frac{k'^2dk'}{E_rE_{k'}}\frac{E_r^0\delta(E_k^0 - E_{k'})}{E_p|\mathcal{B}|} \\ &= \frac{d\Omega_{k'}}{4(2\pi)^2}\frac{E_k}{E_p}\frac{t}{|\mathcal{B}|} = \frac{d\Omega_{k'}}{4(2\pi)^2}\frac{\alpha t^2}{(1-\beta\mu)}. \end{aligned} \tag{36}$$

The second line is obtained after integration over $k'^2dk' = E_{k'}^2dE_{k'}$.

As previously, it is interesting to consider the limit $\alpha \to 0$ (see Eq. (26)), which is valid in the case of CMB photons scattered by hot or relativistic electrons ($E_p \gg E_k$). In this limit, $t \to (1-\beta\mu)/(1-\beta\mu')$ and we finally get:

$$\int_{E_{k'},\vec{p}'} d\text{LIPS} \xrightarrow{\alpha\to 0} \frac{d\Omega_{k'}}{4(2\pi)^2}\frac{\alpha(1-\beta\mu)}{(1-\beta\mu')^2} \tag{37}$$

This result describes the whole phase space function for electron interactions with CMB photons. It can be expressed in terms of dt or $d\xi$ (where $\xi \equiv 1/t$), if we make the change of variable $t \leftrightarrow \mu'$ or equivalently $t \leftrightarrow \mu$. With the definition of t given in Eq. (25), and in the limit $\alpha \to 0$ we have:

$$d\mu' = \frac{(1-\beta\mu)}{\beta}\frac{dt}{t^2} \qquad d\mu = -\frac{(1-\beta\mu')}{\beta}dt. \tag{38}$$

Thus, in terms of t or ξ, the phase space is merely:

$$\int_{E_{k'},\vec{p}'} d\mu d\text{LIPS} \overset{\alpha\to 0}{\to} \frac{d\phi'}{4(2\pi)^2}\frac{\alpha}{\beta} d\mu dt,$$
$$\left(\overset{\alpha\to 0}{\to} -\frac{d\phi'}{4(2\pi)^2}\frac{\alpha t}{\beta} d\mu' dt\right), \quad \text{or}$$
$$\int_{E_{k'},\vec{p}'} d\mu d\text{LIPS} \overset{\alpha\to 0}{\to} -\frac{d\phi'}{4(2\pi)^2}\frac{\alpha}{\beta} d\mu \frac{d\xi}{\xi^2},$$
$$\left(\overset{\alpha\to 0}{\to} \frac{d\phi'}{4(2\pi)^2}\frac{\alpha}{\beta} d\mu' \frac{d\xi}{\xi^3}\right). \tag{39}$$

C. Squared matrix amplitude

Only two Feynman diagrams have to be considered for the (inverse) Compton diffusion at leading order. The amplitude of the process $p + k \to p' + k'$ is to be found in any textbook introducing quantum field theory (see e.g. [19]), and can be written as:

$$\mathcal{M} = -ie^2 \bar{u}_f(p')\mathcal{A} u_i(p), \quad \text{with}$$
$$\mathcal{A} = \not\varepsilon_f^\star \frac{(\not p + \not k + m)}{s - m^2}\not\varepsilon_i + \not\varepsilon_i \frac{(\not p - \not k' + m)}{u - m^2}\not\varepsilon_f^\star, \tag{40}$$

where i and f flag the initial and final states, p and k (prime) are the four-momenta of the incoming (outgoing) electron and photon, respectively, m is the electron mass and $u_{i/f}$ is the electron four-spinor, while $\varepsilon_{i/f}$ is the space-like photon four-polarization. The Mandelstam variables are:

$$s = (p + k)^2 = m^2 + 2p.k = m^2 + 2E_p E_k(1 - \beta\mu)$$
$$u = (p - k')^2 = m^2 - 2p.k' = m^2 - 2E_p E_{k'}(1 - \beta\mu'). \tag{41}$$

If we define $D_s \equiv s - m^2 = 2p.k$, $D_u \equiv u - m^2 = -2p.k'$, $q_s \equiv p + k$ and $q_u = p - k'$, then the squared value of the amplitude in Eq. (40), once averaged over the incoming electron spin, is given by:

$$\frac{1}{4}\sum_{\text{spin,pol}} |\mathcal{M}|^2 = \frac{1}{2}\sum_{\text{pol}} \frac{e^4}{2D_s D_u}\text{Tr}\{D_s^2 \mathcal{T}_1 + D_u^2 \mathcal{T}_2 + D_s D_u \mathcal{T}_3\}$$
$$\text{with } \mathcal{T}_1 = (\not p' + m)\not\varepsilon_i(\not q_u + m)\not\varepsilon_f^\star(\not p + m)\not\varepsilon_f(\not q_u + m)\not\varepsilon_i^\star$$
$$\mathcal{T}_2 = (\not p' + m)\not\varepsilon_f^\star(\not q_s + m)\not\varepsilon_i(\not p + m)\not\varepsilon_i^\star(\not q_s + m)\not\varepsilon_f$$
$$\mathcal{T}_3 = (\not p' + m)\not\varepsilon_f^\star(\not q_s + m)\not\varepsilon_i(\not p + m)\not\varepsilon_f(\not q_u + m)\not\varepsilon_i^\star$$
$$+ (\varepsilon_i \leftrightarrow \varepsilon_f \&\& q_s \leftrightarrow q_u). \tag{42}$$

After some Dirac algebra calculation, and after averaging over the photon polarization states, one can find the very well known result for the squared Compton amplitude (see e.g. [19]):

$$|\mathcal{M}|^2 = 2e^4\left\{\frac{p\cdot k}{p\cdot k'} + \frac{p\cdot k'}{p\cdot k} + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) + m^4\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right)^2\right\}. \tag{43}$$

Using the conventions displayed in Sec. III A 2, we have the four-products:

$$p\cdot k = m\gamma E_k(1 - \beta\mu) \tag{44}$$

and

$$p\cdot k' = m\gamma E_{k'}(1 - \beta\mu') \tag{45}$$

so Eq. (43) finally reduces to:

$$|\mathcal{M}|^2 = 2e^4\left\{1 + \left(1 - \frac{(1-\Delta)}{\gamma^2(1-\beta\mu)(1-\beta\mu')}\right)^2 + \frac{\varepsilon^2(1-\Delta)^2}{\gamma^2(1-\beta\mu')[(1-\beta\mu') + \alpha(1-\Delta)]}\right\}$$
$$= 2e^4\left\{2 + \frac{g}{\tilde{\gamma}^2}\left(\frac{g}{\tilde{\gamma}^2} - 2\right) + \frac{\alpha^2 g^2}{1 + \alpha g}\right\}, \tag{46}$$

where we define:

$$g \equiv (1-\Delta)/(1-\beta\mu') \quad \text{and} \quad \tilde{\gamma}^2 \equiv \gamma^2(1-\beta\mu). \tag{47}$$

$\gamma \equiv E_p/m$ is the usual Lorentz factor for the incoming electron, and $\varepsilon \equiv E_k/m = \alpha\gamma$ is the incoming photon energy in units of the electron mass. The second equality given in Eq. (46) is very convenient because the result of the limit $E_p \gg E_k \Leftrightarrow \alpha \to 0$ appears explicitly. Within this limit, the squared amplitude is merely:

$$|\mathcal{M}|^2 \overset{\alpha\to 0}{\to} 2e^4\left\{2 + \frac{g}{\tilde{\gamma}^2}\left(\frac{g}{\tilde{\gamma}^2} - 2\right)\right\}. \tag{48}$$

Should we implement the nonrelativistic limit $\beta \to 0$, we would immediately recover the well known angular dependence of the Thomson scattering, that is $|\mathcal{M}|^2 \propto 1 + \Delta^2$. Anyway, the above expression depends on the azimuthal angle Ψ through Δ (see Eq. (30)), and it is interesting to perform the azimuthal integral for deeper insights. It is clear that only the even powers of $\cos\Psi$ will contribute to this integral. We find:

$$\langle|\mathcal{M}|^2\rangle_\Psi \overset{\alpha\to 0}{\to} 2e^4\left\{2 - 2K(1-\mu\mu') + K^2\left[[(1-\mu\mu')^2 + \frac{1}{2}(1-\mu^2)(1-\mu'^2)]\right]\right\}$$
$$\text{with} \quad K \equiv [\tilde{\gamma}^2(1-\beta\mu')]^{-1}. \tag{49}$$

From this expression, it is easy to calculate the limit $\beta \to 0$, that is the expression corresponding to the scattering of a photon of energy $E_k \ll m$ off an electron at rest. Indeed, within this limit, $K \to 1$, and the squared amplitude becomes:

$$\langle|\mathcal{M}|^2\rangle_\Psi \overset{\beta\to 0}{\to} 2e^4\left\{1+\frac{1}{2}(1-\mu^2)(1-\mu'^2)+\mu^2\mu'^2\right\}. \tag{50}$$

This is the famous nonrelativistic limit derived by Chandrasekhar in his textbook [20]. This expression cannot be used in relativistic SZ calculations, but it turns out that a form very similar can be derived when expressing the photon zenith angles μ and μ' in the electron rest frame, that is $\mu = (\beta - \mu_0)/(1-\beta\mu_0)$ and $\mu' = (\beta - \mu'_0)/(1-\beta\mu'_0)$, as nicely detailed in [21]. This Lorentz transformation of the angles is what Wright actually used to derive his energy transfer function [6] .

D. Differential (inverse) Compton cross section

The (inverse) Compton differential cross section is defined as usual for two body elastic scattering (p, $k \to p'$, k'):

$$d^2\sigma = \frac{(2\pi)^4}{4p\cdot k}\left\{\prod_{X=p',k'}\frac{d^3\vec{X}}{(2\pi)^3 2E_X}\right\}\delta^4(p+k-p'-k')|\mathcal{M}|^2, \tag{51}$$

where $|\mathcal{M}|^2$ already includes the sum and average over polarizations, and where the 4-product $p\cdot k = E_pE_k\beta_{\rm rel}$ in terms of the energies of the incoming particles and their associated relative velocity. After integration over $\vec{p}'$ and $E_{k'}$, by virtue of the energy-momentum conservation, the angular differential cross section reads:

$$\frac{d\sigma}{d\Omega_{k'}} = \frac{1}{4E_kE_p\beta_{\rm rel}}\int_{\vec{p}',E_{k'}} d{\rm LIPS}|\mathcal{M}|^2. \tag{52}$$

After substituting $\beta_{\rm rel} = (1-\beta\mu)$ (see Eq. (44)):

$$\begin{aligned}\frac{d\sigma}{d\Omega_{k'}} &= \frac{1}{64\pi^2}\frac{t^2}{E_p^2(1-\beta\mu)^2}|\mathcal{M}|^2\\ &= \frac{3\sigma_{\rm T}}{16\pi}\frac{t^2}{\gamma^2(1-\beta\mu)^2}\frac{|\mathcal{M}|^2}{2e^4}\\ &\overset{\alpha\to 0}{\to}\frac{3\sigma_{\rm T}}{16\pi\gamma^2(1-\beta\mu')^2}\left\{2-2K(1-\mu\mu')\right.\\ &\quad\left.+K^2\left[[(1-\mu\mu')^2+\frac{1}{2}(1-\mu^2)(1-\mu'^2)]\right]\right\},\end{aligned} \tag{53}$$

where we have used the standard definition of the Thomson cross section $\sigma_{\rm T} = 8\pi r_0^2/3$, with the classical electron radius $r_0 \equiv e^2/(4\pi m) = \alpha_{\rm fs}/m$. The first two lines give the exact and general expression of the (inverse) Compton scattering differential cross section. The full relativistic expressions of t and $|\mathcal{M}|^2$ have been derived in Eqs. (25) and (46), respectively. The third line shows the relativistic limit at leading order, when $\alpha = E_k/E_p \to 0$, including the average over the photon azimuthal angle, according to Eq. (49) where K is also defined. If we further integrate [29] the full differential cross section over $d\Omega_{k'}$, we find (see e.g. [27,28]):

$$\begin{aligned}\sigma(E_p,E_k) &= \frac{3\sigma_{\rm T}}{4}\left\{\frac{(\chi^2-2\chi-2)}{2\chi^3}\ln(1+2\chi)\right.\\ &\quad\left.+\frac{(1+2\chi)^2+(1+\chi)^2(1+2\chi)-\chi^3}{\chi^2(1+2\chi)^2}\right\}\\ &\overset{\alpha\gamma^2\to 0}{\to}\sigma_{\rm T},\end{aligned} \tag{54}$$

where $\chi \equiv \alpha\gamma^2(1-\beta\mu)$. We have featured the limit when $\alpha\gamma^2 = \gamma E_k/m \to 0$. It makes sense to recover the Thomson cross section, since if we transpose the process in the electron rest frame, the photon would have an energy of γE_k, still much lower than m as long as $\gamma \lesssim 10^8$, which corresponds to the classical Thomson regime. While not describing a Thomson scattering at all, the corresponding *integrated* cross section is recovered as long as $\gamma E_k \ll m$: this is a mere consequence of special relativity, and very well know for decades in astrophysics [3–5].

Another useful quantity to derive is the integral over the incoming and outgoing photons angles of the Taylor expansion of $\beta_{\rm rel}d\sigma/d\Omega_{k'}$ in the limit $\alpha \to 0$. Up to the second order in α, we get:

$$\begin{aligned}\int d\mu\int d\mu'\int d\phi'\beta_{\rm rel}\frac{d\sigma}{d\Omega_{k'}} &\overset{\alpha\to 0}{\to} 2\sigma_{\rm T}\left\{1+\frac{2\alpha}{3}(1-4\gamma^2)\right.\\ &\quad\left.-\frac{26\alpha^2}{5}(\gamma^2-2\gamma^4)\right\},\end{aligned} \tag{55}$$

where we have expressed the cosine Δ as a function of μ, μ' and ϕ' according to Eq. (27) before performing the integral.

IV. CONCLUSION

We have shown that the approach that Wright developed in Ref. [6] was equivalent to using a Boltzmann-like equation in the single scattering approximation, already very well described in the literature (e.g. [10,11,13,14]). The same result has been recovered in a recent subsequent work by Nozawa and Kohyama [21]. Here, we have established the relativistic expressions valid in the CMB frame to employ within Wright's formalism, summarized in Eqs. (22). They differ from Wright's expressions which are instead derived in the electron rest frame [21], and which are probably more suited for numerical computations. Finally, we have recovered that the SZ shift at energy E_k due to a relativistic population of electrons is given by an integral over a function which is still proportional to $I_\gamma^0(E_k) - I_\gamma^0(tE_k)/t^3$, where I_γ^0 is the intensity of the pure blackbody CMB spectrum and $t = E_{k'}/E_k$ is the ratio of scattered-to-focused frequencies.

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APPENDIX A: RECALL OF THE RADIATIVE TRANSFER APPROACH

In this appendix, we would like to demonstrate that the radiative transfer method used by Wright in [6] can be deduced from the Boltzmann approach presented in this paper, but by starting from the radiative approach instead. This will provide more insights on the correct frequency redistribution function to be used for the CMB photons.

We are looking for a redistribution function that expresses the probability $P(t)$ for a CMB photon to have its energy E_k shifted to $E_{k'}$ by a factor of $t \equiv E_{k'}/E_k$—we could equivalently deal with $P(s)$ with $s \equiv \ln(t)$—in order to be able to write (see e.g. [16]) the modified intensity as:

$$I_\gamma(E_k) = \int dt P(t) \times I_\gamma^0(tE_k). \tag{A1}$$

The integral is performed from the minimal value of the scattered energy $t_{\rm min} = 1/(4\gamma_{\rm max}^2 - 1)$ up to its maximal value $t_{\rm max} = (4\gamma_{\rm max}^2 - 1)$, as allowed by the kinematics.

According to the radiative transfer method, such a probability depends on the electron optical depth τ and on the frequency redistribution function $P_1(t)$ for a single scattering. In the limit of single scattering, it is merely given by:

$$\begin{aligned} P(t) &= e^{-\tau}\{\delta(t-1) + \tau P_1(t)\} \stackrel{\tau \ll 1}{\to} (1-\tau)\delta(t-1) \\ &\quad + \tau P_1(t) + O(\tau^2), \end{aligned} \tag{A2}$$

where, in the second line, we have taken the limit of very small optical depth, which is the case in the problem of interest here, but also more generally for electrons in clusters.

If we report the previous equation in Eq. (A1), we find:

$$\Delta I_\gamma(E_k) = \tau \int dt (P_1(t) - \delta(t-1)) \times I_\gamma^0(tE_k). \tag{A3}$$

Two terms appear in the sum in the integral, which are readily interpreted as the positive contribution of CMB photons scattered *in* from energy $E_{k'} = tE_k$ to energy E_k for the former, and as the negative contribution of photons which have been shifted *out* from energy E_k to other energies. More explicitly, those contributions are given by:

$$I_\gamma^{\rm out}(E_k) \equiv \tau \times I_\gamma^0(E_k), \tag{A4}$$

and:

$$I_\gamma^{\rm in}(E_k) \equiv \tau \int dt P_1(t) \times I_\gamma^0(tE_k). \tag{A5}$$

If we scrutinize Eqs. (9) and (20), we can find some convincing similarities, while not straightforwardly. With the definition of τ given in Eq. (12), the former is perfectly consistent with Eq. (A4). Nevertheless, it is a bit more tricky for the latter. Indeed, given our definition, because τ already includes the integral over t, it would have to be considered as an operator acting on I_γ. We have actually determined the full relativistic formula, summarized in Eqs. (22).

APPENDIX B: SCATTERED *IN* AND *OUT* PHOTONS TREATED SEPARATELY

1. Definitions

Let us start with some basic definitions. The energy density of the blackbody spectrum, of average temperature T_0, before traveling through the cluster is given by:

$$\rho_\gamma^0 = \int d^3\vec{k} E_k f_\gamma^0(E_k),$$

with $d^3k = E_k^2 dE_k d\Omega_k$ and

$$f_\gamma^0(E_k) = \frac{1}{e^{E_k/T_0} - 1}.$$

From this equations, we deduce that the associated intensity is:

$$I_\gamma^0(E_k) = E_k^3 f_\gamma^0(E_k) = \frac{E_k^3}{e^{E_k/T_0} - 1}. \tag{B1}$$

2. Intensity associated with the relativistic effect

When interacting with the relativistic electrons, some CMB photons initially at energy E_k can be scattered-out at an energy $E_{k'}$. This effect generates a deficit in the number of CMB photons that is expected at an energy E_k. However, other CMB photons with an energy $E_{k'} \neq E_k$ will be scattered-in the cluster to a final energy E_k. The photons associated with the former process (responsible for the deficit) will be referred to as ''scattered-out'' photons (and will be denoted hereafter ''s-out'') while those associated with the second process will be referred to as ''scattered-in photons'' (and will be denoted ''s-in''). Let us now examine these two processes separately.

A. Intensity associated with the photons scattered-out

After traveling through the cluster and interacting with the population of relativistic electrons, a CMB photon (with an initial energy E_k) can be shifted to a new energy $E_{k'} \neq E_k$, according to the reaction $k + p \to k' + p'$. The CMB energy spectrum is modified accordingly. For an observer who aims at measuring the resulting intensity of the CMB spectrum, such interactions with a population of relativistic electrons translates into a deficit in the number of photons at an energy E_k with respect to the blackbody distribution.

If we disregard the thermal and kinetic SZ effect for a moment, the number of photons which remain at this energy E_k after traveling through the cluster is the number of photons which had this energy before entering the cluster minus the number of photons $n_\gamma^{\text{s-out}}$ which have changed of energy after traveling through the cluster.

This number density $n_\gamma^{\text{s-out}}$ of scattered photons is basically the convolution over the 3-momentum $d^3\vec{k}$ between the initial blackbody occupation number f_γ^0 and the interaction rate with the electrons (that is $\Gamma_\gamma = n_e\sigma\beta_{\text{rel}}$ if $\sigma\beta_{\text{rel}}$ is independent of the ingoing electron energy E_p). For relativistic electrons, the dependence of the cross section on the energy E_p is not trivial and the interaction rate is rather given by:

$$\Gamma_\gamma(E_k) = \int \frac{d^3\vec{p}}{(2\pi)^3} f_e(E_p)[\sigma\beta_{\text{rel}}]_{pk\to p'k'}$$

where

$$[\sigma\beta_{\text{rel}}]_{pk\to p'k'} = \frac{1}{4E_pE_k}\int_{p',k'} d\text{LIPS}|\mathcal{M}|^2$$

with β_{rel} the relative velocity and where the Lorentz invariant phase space $d\text{LIPS} = d\text{LIPS}_{p',k'}$ is defined in Eq. (31), and further developed in Eq. (36). $|\mathcal{M}|^2 = |\mathcal{M}|^2_{pk\to p'k'}$ is given in Eq. (46). To derive the full distribution function of the photons which have been scattered out from energy E_k to energy $E_{k'}$, we have to multiply (or weight) the blackbody distribution by the interaction rate integrated over the time:

$$\begin{aligned} f_\gamma^{\text{s-out}}(E_k) &= f_\gamma^0(E_k)\int dt\Gamma_\gamma(E_k) \\ &= f_\gamma^0(E_k)\int dt\int \frac{d^3p}{(2\pi)^3} f_e(E_p)[\sigma\beta_{\text{rel}}]_{kp\to k'p'}, \end{aligned}$$

which provides the full number density of photons scattered-out from any energy E_k to any $E_{k'}$, that is the number density of photons having interacted:

$$n_\gamma^{\text{s-out}} = \int \frac{d^3k}{(2\pi)^3} f_\gamma^{\text{s-out}}(E_k).$$

Now, we are interested in the specific density removed from a given energy E_k and shifted to any energy $E_{k'}$, so we have to come back to the distribution function. Hence, the intensity associated with the scattered-out photons reads:

$$I_\gamma^{\text{s-out}}(E_k) \equiv E_k^3 f_\gamma^{\text{s-out}}(E_k),$$

where $f_\gamma^{\text{s-out}}(E_k)$ represents the distribution function associated with the photons which have experienced elastic scatterings with electrons in the cluster. Since an observer will detect a modification of the photon distribution along the line-of-sight, the deficit in the CMB intensity at an energy E_k that could be observed is merely obtained by using the geodesic relation $dt = dl/c$ ($= dl$ in natural units):

$$\begin{aligned} \frac{\hat{I}^{\text{s-out}}(E_k)}{E_k^3} &= \int dl \int \frac{d^3\vec{p}}{(2\pi)^3} f_e(E_p) \\ &\quad\times \int_{p',k'} d\text{LIPS}\frac{|\mathcal{M}|^2}{4E_pE_k} f_\gamma^0(E_k) \\ &= \int dl\int\frac{d^3\vec{p}}{(2\pi)^3} f_e(E_p)\int\frac{d\Omega_{k'}}{64\pi^2}\frac{t^2|\mathcal{M}|^2}{(1-\beta\mu)} \\ &\quad\times \frac{f_\gamma^0(E_k)}{E_p^2}, \end{aligned}$$

where the Lorentz invariant phase space has been made explicit. We have finally:

$$\begin{aligned} \hat{I}^{\text{s-out}}(E_k) &= \int dl\int\frac{d^3\vec{p}}{(2\pi)^3} f_e(E_p)\int\frac{d\Omega_{k'}}{32\pi^2}\frac{t^2|\mathcal{M}|^2}{(1-\beta\mu)} \\ &\quad\times\frac{I_\gamma^0(E_k)}{E_p^2} \end{aligned} \tag{B2}$$

where we used the fact that $I_\gamma^0(E_k) = 2E_k^3 f_\gamma^0(E_k)$, making explicit the factor of 2 coming from the sum over the photon polarization states, and where the squared amplitude $|\mathcal{M}|^2 = |\mathcal{M}|^2_{p,k\to p',k'}$. We exactly recover the scattered-out term as derived in the Boltzmann-like formalism (see Eqs. (10) and (12)). Of course, one will have to average over the solid angle of the incoming photons $d\Omega_k = d\phi d\mu$.

B. Intensity associated with the photons scattered-in

Let us now estimate the number of photons with an arbitrary energy $E_{k'}$ which have actually been scattered to an energy E_k by a population of electrons of energy E_p. A first approach consists in doing as in the previous subsection, by defining the interaction rate of scattering shifting photons of energy $E_{k'}$ to any energy $E_k \neq E_{k'}$;

$$\Gamma_\gamma(E_{k'}) = \int\frac{d^3\vec{p}}{(2\pi)^3} f_e(E_p)[\sigma\beta_{\text{rel}}]_{pk'\to p'k}.$$

The number density of scattered-in photons from any energy $E_{k'}$ to any energy E_k is therefore given, in terms of the associated distribution function, by:

$$n_\gamma^{\text{s-in}} = \int\frac{d^3\vec{k}'}{(2\pi)^3} f_\gamma^{\text{s}}(E_{k'}) \quad f_\gamma^{\text{s}}(E_{k'}) \equiv f_\gamma^0(E_{k'})\int dt\Gamma_\gamma(E_{k'}).$$

Nevertheless, this is the number density of photons scattered to *any* energy $E_k \neq E_{k'}$, that is the total number density of photons having interacted, while we are only interested in the scattered photons from *any* energy $E_{k'}$ to a *given* energy E_k. Therefore, we have to modify a bit the above picture, and use differential quantities. The differential interaction rate of shifting photons from energy $E_{k'}$ to energy E_k reads:

$$\frac{d\tilde{\Gamma}_\gamma(E_{k'}, E_k)}{d^3\vec{k}} = \int \frac{d^3\vec{p}}{(2\pi)^3} f_e(E_p)\beta_{\rm rel}\frac{d\sigma_{pk'\to p'k}}{d^3\vec{k}}.$$

To derive the distribution function associated with all photons scattered from energy $E_{k'}$ to energy E_k, we need to convolve the above differential rate with the initial blackbody function and integrate over time and over all energies $E_{k'}$:

$$f_\gamma^{\text{s-in}}(E_k) = \int \frac{d^3\vec{k}'}{(2\pi)^3} f_\gamma^0(E_{k'}) \int dt \frac{d\tilde{\Gamma}_\gamma(E_{k'}, E_k)}{d^3\vec{k}}$$
$$= \int \frac{d^3\vec{k}'}{(2\pi)^3} f_\gamma^0(E_{k'})$$
$$\times \int \frac{d^3p}{(2\pi)^3} f_e(E_p)\beta_{\rm rel}\frac{d\sigma_{pk'\to p'k}}{d^3\vec{k}}.$$

We have:

$$\beta_{\rm rel}\frac{d\sigma_{pk'\to p'k}}{d^3\vec{k}} = \frac{(2\pi)^4}{4E_{k'}E_p}\int \frac{d^3\vec{p}'}{(2\pi)^3}\delta^4(p + k' - p' - k) \times \frac{|\tilde{\mathcal{M}}|^2}{4E_kE_{p'}}$$

where $|\tilde{\mathcal{M}}|^2 = |\mathcal{M}|^2_{pk'\to p'k}$. We see that the phase space is more subtle than in the scattered-out term. We can actually perform the integral over $d^3\vec{k}'$ and $d^3\vec{p}'$, by means of the conservation of energy-momentum. This will differ a bit from the scattered-out case, but we will apply the same treatment as in Sec. III B, to which we refer the reader for more details. We define:

$$d\widetilde{\rm LIPS} = \frac{(2\pi)^4}{4E_kE_{p'}}\frac{d^3\vec{p}'}{(2\pi)^3}\frac{d^3\vec{k}'}{(2\pi)^3}\delta^4(p + k' - p' - k).$$

This is not the common Lorentz invariant phase space for the process $pk' \to p'k$, which should instead feature integrals over p' and k, but this is a useful quantity in the present calculation. If we perform the integral over $d^3\vec{p}'$, we get:

$$\int_{\vec{p}'} d\widetilde{\rm LIPS} = \frac{d^3\vec{k}'}{16\pi^2E_k\tilde{E}_r}\delta(E_p + E_{k'} - \tilde{E}_r - E_k)$$

with:

$$\tilde{E}_r \equiv E_p\sqrt{1 + \alpha'^2 + \alpha^2 + 2\beta(\alpha'\mu' - \alpha\mu) - 2\alpha\alpha'\Delta},$$

where $\alpha \equiv E_k/E_p$, $\alpha' \equiv E_{k'}/E_p$, and the other angular variables are already defined in Eq. (27). We can further transpose the Dirac δ function according to:

$$\delta(E_p + E_{k'} - \tilde{E}_r - E_k) = \frac{\tilde{E}_r^0\delta(E_{k'} - E_k^0)}{E_p|\tilde{\mathcal{B}}|},$$

with:

$$E_k^0 \equiv \tilde{t}E_k \qquad \tilde{t} \equiv \frac{(1 - \beta\mu)}{(1 - \beta\mu') + \alpha(\Delta - 1)}$$
$$\tilde{\mathcal{B}} \equiv (1 - \beta\mu') + \alpha(\Delta - 1)$$
$$\tilde{E}_r^0 \equiv \tilde{E}_r(E_k^0) = E_p|1 - \alpha(1 - \tilde{t})|.$$

Hence, performing the integral over $d^3\vec{k}'$, we can rewrite the phase space as:

$$\int_{\vec{p}', E_{k'}} d\widetilde{\rm LIPS} = \frac{d\Omega_{k'}}{16\pi^2}\frac{\alpha\tilde{t}^3}{(1 - \beta\mu)},$$

such that the scattered-in distribution function can be written as:

$$f_\gamma^{\text{s-in}}(E_k) = \int dt \int \frac{d^3p}{(2\pi)^3} f_e(E_p)\int \frac{d\Omega_{k'}}{64\pi^2}\frac{|\tilde{\mathcal{M}}|^2}{E_p^2}\frac{\tilde{t}^2 f_\gamma^0(\tilde{t}E_k)}{(1 - \beta\mu)}$$

We can now derive the intensity of the scattered-in photons at energy E_k:

$$\hat{I}^{\text{s-in}}(E_k \leftarrow E_{k'}) = E_k^3 f_\gamma^{\text{s-in}}(E_k).$$

If we transform the time integral into a line-of-sight integral, we finally get:

$$\hat{I}^{\text{s-in}}(E_k) = \int dl \int \frac{d^3p}{(2\pi)^3} f_e(E_p)\int \frac{d\Omega_{k'}}{32\pi^2}\frac{|\tilde{\mathcal{M}}|^2}{E_p^2} \times \frac{I_\gamma^0(\tilde{t}E_k)}{\tilde{t}(1 - \beta\mu)} \qquad \text{(B3)}$$

where we have used $I_\gamma^0(\tilde{t}E_k) = 2(\tilde{t}E_k)^3 f_\gamma^0(\tilde{t}E_k)$—making explicit the factor of 2 coming from the sum over the photon polarization states—and where the squared amplitude $|\tilde{\mathcal{M}}|^2 = |\mathcal{M}|^2_{pk'\to p'k}$. Of course, we will further need to perform the averaging over the solid angle $d\Omega_k = d\phi d\mu$ associated with the detected photon. Nevertheless, it is already interesting to note that we recover an expression which is similar to Eq. (16), which is the result for the *in* process obtained in the Boltzmann-like formalism, but with slight differences. Indeed, in the Boltzmann approach, the squared amplitude is $|\mathcal{M}|^2_{pk\to p'k'}$, and the expression of $t = E_{k'}/E_k$ slightly differs from that of $\tilde{t}$. As regards the energy ratios, the difference comes from the denominators: there is a factor of $\alpha(1 - \Delta)$ in this of the former, and of $-\alpha(1 - \Delta)$ in that the latter. The physical interpretation of this slight difference is rather simple. Indeed, $\alpha = E_k/E_p$ in both cases, but though E_k is the energy of the incoming photon in the former case, it is that of the outgoing photon in the latter case. Nevertheless, since $\alpha \to 0$ for CMB photons as scattered by relativistic electrons, this has no effect. Concerning the squared amplitudes, we can also verify that $|\mathcal{M}|^2$ and $|\tilde{\mathcal{M}}|^2$ are equivalent within this limit. There is therefore no difference between the scattered-in contribution as computed in the approach developed in this appendix and that computed within the Boltzmann-like formalism in the limit $\alpha \longrightarrow 0$, which fully applies for relativistic electrons.

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[29] We temporarily slightly change our convention which defines the cosine Δ with respect to μ and μ'—see Eq. (27). We now set the main axis of the problem along $\vec{k}'$ instead of $\vec{p}$, so that it is now μ which depends on Δ and μ' and so that we can perform the integral over $d\mu'$ without accounting for Δ. Such a trick is aimed at comparing our results with the standard calculation of the integrated Compton cross section.