Topology and chiral random matrix theory at nonzero imaginary chemical potential

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We study the effect of topology for a random matrix model of QCD at nonzero imaginary chemical potential or nonzero temperature. Nonuniversal fluctuations of Dirac eigenvalues lead to normalization factors that contribute to the θ dependence of the partition function. These normalization factors have to be canceled in order to reproduce the θ dependence of the QCD partition function. The reason for this behavior is that the topological domain of the Dirac spectrum (the region of the Dirac spectrum that is sensitive to the topological charge) extends beyond the microscopic domain at nonzero imaginary chemical potential or temperature. Such behavior could persist in certain lattice formulations of QCD.

I. INTRODUCTION

Chiral random matrix models [1,2] have been used with great success in the past 15 years to understand and compute universal features of the QCD Dirac spectrum; see Ref. [3] for a review. The effect of temperature or chemical potential can be included in these models in a schematic way to obtain qualitative, nonuniversal results for the QCD phase diagram. The main purpose of this paper is to point out and clarify a number of subtleties and nonuniversal effects that can arise when the effects of topology become important in such schematic random matrix models. In particular, we shall see that nontrivial normalization factors can occur which will be related to nonuniversal properties of the Dirac spectrum.

Let us first consider QCD at zero temperature with N_f quark flavors, which for simplicity we take to be of equal mass m. The QCD partition function, Z^{QCD}, can be considered at fixed θ angle or at fixed topological charge ν. In the former case, the θ angle can be introduced according to (see, e.g., [3,4])

$$m_R \rightarrow me^{i\theta/N_f}, \qquad m_L \rightarrow me^{-i\theta/N_f}, \qquad (1)$$

where m_R (m_L) is the mass that couples right-handed (left-handed) quarks with antiquarks of opposite chirality. We assume m to be real and positive.

If the number of right-handed and left-handed modes differs by ν, the product of the fermion determinants results in an overall factor $e^{i\nu\theta}$, and we have

$$Z^{\text{QCD}}(m, \theta) = \sum_{\nu=-\infty}^{\infty} e^{i\nu\theta} Z_\nu^{\text{QCD}}(m). \qquad (2)$$

This relation can be inverted to give the QCD partition function at fixed ν,

$$Z_\nu^{\text{QCD}}(m) = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-i\nu\theta} Z^{\text{QCD}}(m, \theta), \qquad (3)$$

which corresponds to a path integral restricted to gauge fields of topological charge ν.

It is generally assumed that the gauge field measure does not depend on the topological charge. When topological excitations can be considered as independent events, the central limit theorem dictates that the distribution of topological charge is given by

$$\mathcal{P}_\nu = \frac{1}{\sqrt{2\pi V\chi}} e^{-\nu^2/2V\chi}, \qquad (4)$$

where V is the space-time volume and χ is the topological susceptibility at $\theta = 0$. In the quenched theory (or, equivalently, for heavy quarks), $\chi = \chi_q$ is a mass-independent constant, whereas for light quarks, the topological charge is screened, resulting in a topological susceptibility at $\theta = 0$ given by [5]

$$\chi = m\Sigma, \qquad (5)$$

where Σ is the absolute value of the chiral condensate for $m = 0$ and $\theta = 0$.

In the microscopic domain of QCD, where the Compton wavelength of the pion is much larger than the size of the box, the mass and θ dependence of the QCD partition function is given by a random matrix theory (RMT) with the same global symmetries as those of QCD. Contrary to QCD, random matrix partition functions are defined in terms of integrals over the matrix elements of the Dirac operator at *fixed* topological charge rather than integrals over gauge fields at fixed θ angle, which contain the sum over topological charges. In this paper we will study random matrix theories that are deformed by an imaginary chemical potential or temperature. The deformation parameter will be denoted by u.

Given a random matrix partition function at fixed ν, the partition function at fixed θ is defined by

$$Z^{\text{RMT}}(m, \theta) = \sum_{\nu=-\infty}^{\infty} e^{i\nu\theta} \mathcal{P}_\nu \mathcal{N}_\nu Z_\nu^{\text{RMT}}(m), \qquad (6)$$

where we separated a potentially nontrivial normalization factor $\mathcal{N}_\nu$ and a weight factor $\mathcal{P}_\nu$ from $Z_\nu^{\mathrm{RMT}}(m)$. The factor $\mathcal{P}_\nu$ corresponds to the quenched distribution of topological charge given in Eq. (4) (with $\chi = \chi_q$). Other ν-dependent normalization factors that may arise in random matrix models of the QCD partition function are included in the factor $\mathcal{N}_\nu$. One of our objectives is to discuss the significance of these two factors. We shall see in Sec. III A that, contrary to QCD or chiral random matrix theories at $u = 0$ [1,6], $\mathcal{N}_\nu$ can become a nontrivial function of the deformation parameter. On the other hand, as will be shown in Sec. III B, for light quarks it makes no difference whether or not $\mathcal{P}_\nu$ is included in the sum over ν [6].

A related question we would like to address in this paper is which part of the Dirac spectrum is sensitive to the topological charge. The answer to this question could depend on the parameters of QCD or the chiral random matrix model, and we shall see below that this is actually the case. It also depends on the value of the quark mass, for which we distinguish the following scales: (i) The microscopic scale [1,2] where $mV\Sigma$ is kept fixed in the thermodynamic limit. This corresponds to the ε regime of chiral perturbation theory [7]. (ii) The chiral scale where $m\sqrt{V}$ is kept fixed in the thermodynamic limit. This corresponds to the p regime [8] of chiral perturbation theory. (iii) The macroscopic domain with $m \sim \Lambda_{\mathrm{QCD}}$. In the microscopic domain, the mass dependence of the QCD partition function is given by chiral random matrix theory. Actually, this domain extends beyond the microscopic domain all the way to the chiral scale. Therefore, it is appropriate to borrow the name "ergodic domain" from the theory of disordered systems [9] to distinguish the domain $m \ll 1/\Lambda_{\mathrm{QCD}}\sqrt{V}$ from the microscopic scaling domain. Note that we will sometimes consider the limit where $mV\Sigma$ approaches infinity with the understanding that the thermodynamic limit is taken first so that m is still in the microscopic domain.

The issues that will be addressed in this paper are already manifest for one quark flavor, and for simplicity we will only discuss this case. The one-flavor QCD partition function, given by the average fermion determinant, is a function of the quark mass and of the θ angle or the topological charge ν. If the eigenvalues of the (anti-Hermitian) Dirac operator at fixed ν are denoted by $i\lambda_k^\nu$, the QCD partition function at fixed ν can be expressed as

$$Z_\nu(m) = \left\langle \prod_k (i\lambda_k^\nu + m) \right\rangle, \qquad (7)$$

where the average is over gauge fields with fixed ν.

We know that in the microscopic domain (and in fact in the ergodic domain) the mass dependence of the one-flavor QCD partition function in the sector of topological charge ν is given by [4,7]

$$Z_\nu(m) \sim I_\nu(mV\Sigma). \qquad (8)$$

For large values of the argument the modified Bessel function I_ν becomes insensitive to its index ν, and thus Eq. (8) implies that the average fermion determinant does not depend on the topological charge when $mV\Sigma \gg 1$. In terms of Dirac eigenvalues one way to realize this is when only eigenvalues below this mass scale are affected by topology [see Eq. (7)]. However, more exotic scenarios are also possible. It could be that eigenvalues beyond the microscopic domain are sensitive to the topological charge. If m is in the microscopic domain, this might result in a ν-dependent overall factor $\mathcal{N}_\nu$ that could depend on the deformation parameter u and restores the ν independence of Z_ν for $mV\Sigma \gg 1$. To find out whether this scenario is realized, it makes sense to introduce the notion of the *topological domain of the Dirac spectrum*, which we define to be the part of the Dirac spectrum that is sensitive to the topological charge.

In QCD we have $\mathcal{N}_\nu = 1$ and, from Eqs. (2) and (8), the universal θ dependence of the partition function is given by

$$Z^{\mathrm{QCD}}(m, \theta) \sim e^{mV\Sigma \cos\theta}. \qquad (9)$$

It is plausible that the standard scenario discussed after Eq. (8) applies in this case, i.e., the topological domain of the Dirac spectrum does not extend beyond the microscopic domain. Exotic scenarios such as the one discussed above could occur in certain lattice formulations of QCD, and it would be interesting to test this directly. We shall further comment on this point in the conclusions.

The ergodic domain of QCD is given by random matrix theory, but since the average fermion determinant is sensitive to *all* eigenvalues, it could be that deformations of the random matrix model result in a topological domain that extends beyond the microscopic domain. In this paper we will see that this may happen in random matrix models at nonzero temperature/imaginary chemical potential.

The θ dependence of random matrix theories at nonzero temperature was discussed before in the literature [10]. In that work the temperature was introduced such that it only affects the eigenmodes corresponding to nonzero Dirac eigenvalues. This resulted in the same θ dependence as in the zero-temperature random matrix model [1]. Among others it was shown that the $O(m^2)$ term in the chiral Ward identity does not contribute in the chiral limit. This is not always the case. It was recently shown in the framework of chiral perturbation theory that in the superfluid phase of QCD at nonzero chemical potentials the $O(m^2)$ term cannot be neglected [11]. In this paper we will see that the $O(m^2)$ term in the chiral Ward identity contributes to the topological susceptibility for random matrix partition functions at nonzero temperature/imaginary chemical potential if the u-dependent normalization factor $\mathcal{N}_\nu$ is not included.

The structure of this paper is as follows. Chiral random matrix theories at zero and nonzero deformation parameter

will be introduced in Sec. II. The random matrix models are solved in Sec. III, where we also discuss the normalization factor $\mathcal{N}_\nu$ and the distribution of the topological charge $\mathcal{P}_\nu$. In Sec. IV we show that the chiral condensate for one flavor only has the correct behavior if the normalization factor $\mathcal{N}_\nu$ is included. The origin of $\mathcal{N}_\nu$ is studied in Secs. V and VI. In Sec. V we show that it is related to the extent of the topological domain, and in Sec. VI we find that the contribution of the pseudoscalar susceptibility does not vanish if $\mathcal{N}_\nu$ is not included. Concluding remarks are made in Sec. VII.

II. CHIRAL RANDOM MATRIX MODELS

A. Definition of the random matrix model

The random matrix model for $N_f = 1$ in the sector of topological charge ν is defined by [1]

$$Z_\nu(m) = \mathcal{C}_{N,\nu} \int \mathcal{D}W \det(D + m)e^{-(1/2)N\Sigma^2 \mathrm{tr}W^\dagger W} \quad (10)$$

with the random matrix Dirac operator defined by

$$D = \begin{pmatrix} 0 & iW \\ iW^\dagger & 0 \end{pmatrix}. \quad (11)$$

The integral $\mathcal{D}W$ is over the real and imaginary parts of the elements of the random matrix W, which has dimension $p \times q$. The Dirac operator (11) has $|p - q|$ exact zero modes. For this reason we interpret

$$\nu = p - q \quad (12)$$

as the topological charge. The total number of modes

$$N = p + q \quad (13)$$

will be interpreted as the volume. This corresponds to the choice of mode density

$$\frac{N}{V} = 1. \quad (14)$$

The normalization factor $\mathcal{C}_{N,\nu}$ is chosen such that the quenched partition function is normalized to unity, i.e.,

$$\mathcal{C}_{N,\nu} = \left(\frac{N\Sigma^2}{2\pi}\right)^{(1/4)(N^2 - \nu^2)}. \quad (15)$$

We will consider this random matrix model in the presence of an imaginary chemical potential iu. Using the chiral representation of the γ matrices, the u-deformed Dirac operator is given by [12–15]

$$D(u) = \begin{pmatrix} 0 & iW + iu\mathbb{1}_{p\times q} \\ iW^\dagger + iu\mathbb{1}_{q\times p} & 0 \end{pmatrix}, \quad (16)$$

where $(\mathbb{1}_{p\times q})_{k\ell} = \delta_{k\ell}$. Alternatively, u can be interpreted as a schematic temperature as was done in [12–14]. The argument goes as follows. The temperature enters in the Dirac operator through the matrix elements corresponding to ∂_0, with eigenvalues that are given by the Matsubara

frequencies. We include only the temperature dependence given by the lowest two Matsubara frequencies by adding the $p \times q$ temperature matrix $i\mathcal{T}$ to iW and $iW^\dagger$ in Eq. (11), where

$$\mathcal{T}_{kk} = \begin{cases} u & \text{for } k \leq \min\{p, q\}/2, \\ -u & \text{for } k > \min\{p, q\}/2, \end{cases} \quad (17)$$

and $\mathcal{T}_{k\ell} = 0$ for $k \neq \ell$. Using the invariance of the integration measure under unitary transformations $W \rightarrow UWV^{-1}$ with $U \in U(p)$ and $V \in U(q)$, the temperature matrix can be transformed into a diagonal matrix with all diagonal matrix elements equal to u, so that the Dirac operator is given by Eq. (16).

In the following, we shall refer to the model defined by Eq. (16) as model A.

B. Other random matrix models

Equation (16) is not the only way to introduce a nonzero temperature. Another possibility [10] is to first partition the N modes into $N_0 = p + q$ "zero" modes and a fixed number N_1 of "nonzero" modes, with $|\nu| = |p - q|$ actual zero modes of the Dirac operator. An $N_1 \times N_1$ temperature matrix is then added to the nonzero-mode component of the Dirac operator, while the zero-mode matrix elements remain temperature independent. In terms of the Dirac operator (16) this means that we add to an $(N_1/2 + p) \times (N_1/2 + q)$ random matrix W a diagonal matrix with $N_1/2$ elements equal to iu and $\min\{p, q\}$ elements equal to zero. (This is technically equivalent to the model considered in Ref. [13], although the physics background is different.) In the following, we shall refer to this model as model B.

A third possibility is to add to W a random matrix with matrix elements that are proportional to u. This model was introduced in Ref. [16] for imaginary u (i.e., real chemical potential) to describe the microscopic domain of QCD at nonzero baryon chemical potential. For real u, this results in a model that differs from the original model (11) simply by a rescaling of the parameter Σ according to $\Sigma \rightarrow \Sigma/\sqrt{1 + u^2}$. This model will be referred to as model C. Note that this model does not have a chiral phase transition. A less trivial model is obtained by introducing two or more different imaginary chemical potentials [17], but we will not discuss this possibility in this paper.

III. SOLUTION OF THE RANDOM MATRIX MODELS AND NORMALIZATION FACTORS

In this section we solve the random matrix models that were introduced in the previous section. We will find that the universal θ dependence is not recovered for model A at $u \neq 0$ unless additional normalization factors are included.

A. Solution of model A

In this subsection we solve the random matrix model A given by Eq. (10) with Dirac operator (16). The procedure

is standard (see, e.g., [1,12]). We start by writing the determinant as a Grassmann integral,

$$\det(D(u) + m) = \int d\psi \, d\bar{\psi}$$
$$\times \exp\left[\left(\begin{array}{c} \bar{\psi}_L \\ \bar{\psi}_R \end{array}\right)^T (D(u) + m)\left(\begin{array}{c} \psi_R \\ \psi_L \end{array}\right)\right],$$
(18)

and perform the Gaussian average over the random matrix elements. After a Hubbard-Stratonovich transformation and integration over the Grassmann variables we obtain the following σ model:

$$Z_\nu^A(m) = \int d\sigma \, d\sigma^*(1 + u^2|\sigma + m|^{-2})^n$$
$$\times (\sigma + m)^p (\sigma^* + m)^q e^{-(1/2)N\Sigma^2 \sigma\sigma^*},$$
(19)

where $n = \min\{p, q\}$. Notice that the ν-dependent normalization constant introduced in Eq. (10) has canceled.

After changing variables $\sigma \to \sigma - m$ and $\sigma^* \to \sigma^* - m$ in Eq. (19) and then expressing the integral over (σ, σ^*) in polar coordinates (r, φ), the angular integral results in a modified Bessel function, and the partition function is given by the remaining integral over r,

$$Z_\nu^A(m) = 2\pi \int_0^\infty dr I_\nu(mN\Sigma^2 r) r^{|\nu|+1} (r^2 + u^2)^{(N-|\nu|)/2}$$
$$\times e^{-(1/2)N\Sigma^2(r^2+m^2)}.$$
(20)

For large N, this partition function can be evaluated by a saddle-point approximation. For m in the ergodic domain, the saddle point in the broken phase is at $\bar{r}^2 = 1/\Sigma^2 - u^2$. To leading order in $1/N$ the partition function is given by

$$Z_\nu^{A,\text{as}}(m) \sim I_\nu(mN\Sigma^A(u))\tau^{|\nu|},$$
(21)

where irrelevant prefactors have been ignored and

$$\Sigma^A(u) = \Sigma \tau(u) \quad \text{with} \quad \tau(u) = \sqrt{1 - \Sigma^2 u^2}.$$
(22)

A second-order phase transition to the chirally symmetric phase occurs at $u_c = 1/\Sigma$ [12].

The θ dependence of the partition function is obtained after performing the sum over ν according to Eq. (6). As will be explained in detail in the next subsection, for light quarks the sum is not affected by the distribution function $\mathcal{P}_\nu$ [6]. We will therefore set $\mathcal{P}_\nu = 1$.

Let us first consider the case $u = 0$ and take $\mathcal{N}_\nu = 1$. Using the identity for Bessel functions given by ([18], Eq. (9.6.33))

$$\sum_{\nu=-\infty}^\infty I_\nu(x)t^\nu = e^{(1/2)x(t+1/t)},$$
(23)

we find the universal result [4,6]

$$Z^A(m, \theta)|_{u=0} \sim e^{mN\Sigma \cos\theta}.$$
(24)

This shows that we do not need nontrivial normalization factors at $u = 0$.

Now consider the case $u \neq 0$. Because of the factor $\tau^{|\nu|}$, in this case Eq. (21) depends on ν for $mN\Sigma^A(u) \gg 1$. This is a nonuniversal result and would also lead to a nonuniversal θ dependence of Z^A after summing over ν. However, these problems can be fixed by introducing a u-dependent normalization factor

$$\mathcal{N}_\nu = \tau^{-|\nu|}.$$
(25)

Then with the replacement $\Sigma \to \Sigma^A(u)$ the sum over ν is the same as for $u = 0$. Again the sum is not affected by the distribution function $\mathcal{P}_\nu$, and we find the universal result

$$Z^A(m, \theta) \sim e^{mN\Sigma^A(u)\cos\theta}.$$
(26)

In QCD an imaginary chemical potential is equivalent to a constant vector field and can be gauged into the temporal boundary conditions of the fermion fields. This is not the case in random matrix theory, and therefore it should not come as a surprise that we need a ν-dependent normalization factor to recover the correct θ dependence. In agreement with universality properties of Dirac spectra at fixed ν [17,19–21] this normalization factor does not depend on the quark mass.

When u approaches $u_c = 1/\Sigma$, higher-order terms in the saddle-point approximation of Eq. (20) become important, and the integral has to be performed exactly. We will not further elaborate on this and only discuss the parameter domain where the leading-order saddle-point approximation is appropriate.

We will discuss further properties of model A in later sections but first turn to a discussion of the necessity of $\mathcal{P}_\nu$ and to a comparison with models B and C, where no u-dependent normalization factors will be needed.

B. On the necessity of $\mathcal{P}_\nu$

For large $|\nu|$ at fixed x the modified Bessel function can be approximated by ([18], Eq. (9.3.1))

$$I_\nu(x) \sim \frac{(x/2)^{|\nu|}}{|\nu|!}.$$
(27)

Therefore, if m is in the microscopic domain, the sum over ν in Eq. (6) is convergent without the Gaussian factor (4).

The sum over ν can be performed, up to exponentially suppressed contributions, using the approximation [4]

$$I_\nu(x) \sim \frac{1}{\sqrt{2\pi x}} e^{x - \nu^2/2x},$$
(28)

which follows from the uniform large-order expansion of the modified Bessel function and is valid for $1 \ll |\nu| \ll x$ ([18], Eq. (9.7.7)). It makes no difference whether or not we include the factor $\mathcal{P}_\nu$ in Eq. (6) since

$$e^{-(\nu^2/2N)((1/m\Sigma(u))+(1/\chi_q))} \sim e^{-(\nu^2/2mN\Sigma(u))}$$
(29)

for m in the ergodic domain. The topological susceptibility at $\theta = 0$ is therefore given by Eq. (5). From the approximation (28) we also see that all topological sectors with $\nu^2 \ll mN\Sigma(u)$ contribute equally to the partition function.

It was argued by Damgaard [6] that the factor $\mathcal{P}_\nu$ should be absent in the sum over ν in Eq. (6), although he also pointed out that the quenched limit could not be taken properly in this case. Our point of view is that the presence of $\mathcal{P}_\nu$ is immaterial for m in the microscopic domain, but that $\mathcal{P}_\nu$ becomes important at length scales below the inverse η' mass where it is believed to determine the local topological susceptibility and leads to the Witten-Veneziano formula for the η' mass [22–25]. Beyond this scale the topological susceptibility at $\theta = 0$ is given by Eq. (5).

C. Comparison with models B and C

For fixed topological charge ν the partition function of model B is given by

$$Z_\nu^B(m) = \int d\sigma d\sigma^* (|\sigma + m|^2 + u^2)^{N_1/2} (\sigma + m)^p$$
$$\times (\sigma^* + m)^q e^{-(1/2)N\Sigma^2 \sigma \sigma^*}, \qquad (30)$$

or, after introducing polar coordinates,

$$Z_\nu^B(m) = 2\pi \int_0^\infty dr I_\nu(mN\Sigma^2 r) r^{N_0+1}(r^2 + u^2)^{N_1/2}$$
$$\times e^{-(1/2)N\Sigma^2(r^2+m^2)}. \qquad (31)$$

Note that this partition function becomes independent of ν for large $mN\Sigma$. Since the correct θ dependence is obtained at $u = 0$ this model does not require additional normalization factors. The sum over ν with $\mathcal{P}_\nu = 1$ results in

$$Z^B(m, \theta) = 2\pi \int_0^\infty dr e^{mN\Sigma^2 r \cos\theta} r^{N_0+1}(r^2 + u^2)^{N_1/2}$$
$$\times e^{-(1/2)N\Sigma^2(r^2+m^2)}. \qquad (32)$$

Using a saddle-point approximation for large N, we find the universal θ dependence

$$Z^B(m, \theta) \sim e^{mN\Sigma^B(u)\cos\theta}, \qquad (33)$$

where [13]

$$\frac{\Sigma^B(u)}{\Sigma} = \left[\frac{1 - \Sigma^2 u^2 + \sqrt{(1 + \Sigma^2 u^2)^2 - 4\Sigma^2 u^2 N_1/N}}{2} \right]^{1/2}. \qquad (34)$$

The partition function of model C at deformation parameter u is equivalent to Eq. (19) at $u = 0$ with $\Sigma \to \Sigma^C(u) = \Sigma/\sqrt{1 + u^2}$, and we thus have the universal result

$$Z^C(m, \theta) \sim e^{mN\Sigma^C(u)\cos\theta}. \qquad (35)$$

Hence we see that in order to obtain the universal θ dependence of the partition function neither model B nor model C requires normalization factors $\mathcal{N}_\nu$ that depend on the deformation parameter u. However, let us emphasize again that the appearance of the $\mathcal{N}_\nu$ is a generic feature in RMT. In the remainder of this paper we will identify mechanisms that are responsible for this behavior.

IV. CHIRAL CONDENSATE AND TOPOLOGY

The case $N_f = 1$ we address in this paper is special since there is no $\text{SU}(N_f) \times \text{SU}(N_f)$ symmetry that could be spontaneously broken. Nevertheless, there could still be a chiral condensate, which can be calculated in the usual way,

$$|\langle \bar{\psi} \psi \rangle| = \frac{1}{V} \partial_m \log Z(m, \theta). \qquad (36)$$

The parameter Σ introduced earlier is defined to be equal to $|\langle \bar{\psi} \psi \rangle|$ at $\theta = 0$ for $m \to 0$ and $V \to \infty$. The functions $\Sigma(u)$ computed in Secs. III A and III C correspond to the u-dependent chiral condensate in the same limits. These limits can be taken in different orders [26], either

$$\Sigma^{(1)} = \lim_{V \to \infty} \lim_{m \to 0} \frac{1}{V} \partial_m \log Z(m, \theta = 0) \qquad (37)$$

or in the reverse order

$$\Sigma^{(2)} = \lim_{m \to 0} \lim_{V \to \infty} \frac{1}{V} \partial_m \log Z(m, \theta = 0). \qquad (38)$$

In Eq. (37), a nonzero chiral condensate implies the breaking of the $\text{U}_A(1)$ symmetry by instantons or the chiral anomaly [27], whereas in Eq. (38) a nonzero chiral condensate implies "spontaneous symmetry breaking" in the following sense. At *fixed* topology the QCD partition function has a $\text{U}_A(1)$ symmetry (in fact a covariance except at $\nu = 0$ where we have a symmetry). A nonzero chiral condensate spontaneously breaks this $\text{U}_A(1)$ symmetry at fixed topology.

From the universal expression (9) for the one-flavor partition function it is clear that the order of limits should not matter. We will now see that for model A this is only the case if the normalization factors $\mathcal{N}_\nu$ are included. Because in this section we only consider model A we omit the superscript A. Using Eq. (6) and the mass dependence of $Z_\nu(m)$ given by Eq. (21), we find that $\Sigma^{(1)}$ of model A is given by

$$\Sigma^{(1)} = \lim_{N \to \infty} \lim_{m \to 0} \frac{\partial_m [\mathcal{N}_1 Z_1(m) + \mathcal{N}_{-1} Z_{-1}(m)]}{N \mathcal{N}_0 Z_0(m)}, \qquad (39)$$

where the factor $\mathcal{P}_\nu$ has dropped out of numerator and denominator since it is essentially constant for $\nu \ll \sqrt{N}$. Using the result (21), we obtain

$$\Sigma^{(1)}(u) = (\mathcal{N}_1/\mathcal{N}_0)\Sigma\tau^2 = (\mathcal{N}_1/\mathcal{N}_0)\Sigma(1 - \Sigma^2 u^2). \qquad (40)$$

Next we calculate the chiral condensate using the reverse order of limits. Based on the discussion in Sec. III B we find that for $|\nu| \ll \sqrt{mN\Sigma}$ the condensate for fixed ν does not depend on ν. Its value is therefore equal to the value in the $\nu = 0$ sector. This was calculated in Ref. [12], resulting in

$$\Sigma^{(2)}(u) = \Sigma\tau = \Sigma\sqrt{1 - \Sigma^2 u^2}. \tag{41}$$

We thus see that the two condensates are only equal if the normalization factor $\mathcal{N}_1/\mathcal{N}_0 = 1/\sqrt{1 - \Sigma^2 u^2}$ from Eq. (25) is included. Therefore we have a nice consistency check of Eq. (25).

So far, we have assumed that we can choose p and q to have arbitrary $\nu = p - q$. Let us now fix the total number of states N. In this case the Dirac operator with ν zero modes has nonzero off-diagonal blocks of dimension $(N + \nu)/2 \times (N - \nu)/2$; see Eqs. (12) and (13). This implies that the parity of the topology is the same as the parity of N. In the following we assume that N, and therefore also ν, is even. Equation (9) shows that the chiral condensate can be extracted from

$$(\Sigma^{(1)})^2 = \lim_{N\to\infty} \lim_{m\to 0} \frac{1}{N^2} \frac{\partial_m^2 Z(m, \theta = 0)}{Z(m, \theta = 0)}. \tag{42}$$

For $m \to 0$, the numerator receives contributions from $\nu = 0$ and $\nu = \pm 2$, while only the $\nu = 0$ sector contributes to the denominator. For the $\nu = 0$ contribution we find [4]

$$(\Sigma^{(1)})^2_{\nu=0} = \lim_{N\to\infty} \lim_{m\to 0} \frac{1}{N^2} \frac{\partial_m^2 Z_0(m)}{Z_0(m)}$$
$$= \lim_{N\to\infty} \frac{2}{N^2} \left\langle \sum_{k=1}^{N/2} \frac{1}{(\lambda_k^{\nu=0})^2} \right\rangle_{N_f=1}, \tag{43}$$

where the average includes the fermion determinant. The right-hand side of Eq. (43) is dominated by the smallest eigenvalues. Note that this contribution is independent of the normalization of the partition function. The contributions of $\nu = \pm 2$ to the condensate are the same and can be written in terms of the Dirac eigenvalues as

$$(\Sigma^{(1)})^2_{\nu=\pm 2} = \lim_{N\to\infty} \frac{2}{N^2} \frac{\mathcal{N}_2}{\mathcal{N}_0} \frac{\left\langle \prod_{k=1}^{N/2-1} (\lambda_k^{\nu=2})^2 \right\rangle}{\left\langle \prod_{k=1}^{N/2} (\lambda_k^{\nu=0})^2 \right\rangle}, \tag{44}$$

where averages without subscript are with respect to the quenched partition function. This is essentially the ratio of the fermion determinants in the sectors $\nu = 2$ and $\nu = 0$. In the random matrix model A the expressions (43) and (44) evaluate to

$$(\Sigma^{(1)})^2_{\nu=0} = \frac{1}{2}\Sigma^2\tau^2, \tag{45}$$

$$(\Sigma^{(1)})^2_{\nu=\pm 2} = \frac{1}{4}\Sigma^2\tau^4 \frac{\mathcal{N}_2}{\mathcal{N}_0}, \tag{46}$$

so that the correct result for the chiral condensate is obtained with the normalization factors from Eq. (25).

The question we wish to address in the sections below is why model A requires the u-dependent normalization factors (25). We will relate this question to the properties of the Dirac eigenvalues. As we have already discussed in the introduction, the requirement that $Z_\nu(m)$ be independent of ν for $mV\Sigma \gg 1$ can explain why a normalization factor $\mathcal{N}_\nu \neq 1$ is needed if the topological domain of the Dirac spectrum extends beyond the microscopic domain. Equation (44) shows that the consistency relation $\Sigma^{(1)} = \Sigma^{(2)}$ should also be related to the properties of the Dirac eigenvalues, to which we turn now.

V. EIGENVALUE FLUCTUATIONS AND MICROSCOPIC UNIVERSALITY

In the numerical calculation of this section we keep N fixed as discussed at the end of the previous section. Motivated by Eq. (44), we consider the ratio R_n of the products of eigenvalues for $\nu = 2$ and $\nu = 0$ as a function of the number of eigenvalues included in the product,

$$R_n \equiv \frac{1}{N^2} \frac{\left\langle \prod_{k=1}^{n-1} (\lambda_k^{\nu=2})^2 \right\rangle}{\left\langle \prod_{k=1}^{n} (\lambda_k^{\nu=0})^2 \right\rangle}. \tag{47}$$

For $n = N/2$ all eigenvalues are included in the product, and for model A the value of this ratio follows from Eqs. (44) and (46),

$$R_\infty = \lim_{N\to\infty} R_{N/2} = \frac{1}{8}\Sigma^2\tau^4. \tag{48}$$

We have evaluated the ratio R_n numerically for model A, using an ensemble of 10^6 random matrices (16) of dimension $N = 400$ distributed according to the Gaussian factor in Eq. (10). The mass has been set to zero. In Fig. 1 we plot the ratio R_n/R_∞ versus n for $u = 0$, $u = 0.5$, and $u = 0.8$. We observe that for $u = 0$ the ratio of determinants saturates in the ergodic domain ($n \lesssim \sqrt{N} = 20$). This is not the case for $u = 0.5$ and $u = 0.8$, where *all* eigenvalues contribute to the ratio of the two determinants.

This is further illustrated in Fig. 2, where we plot the ratio

$$\Delta\lambda_n \equiv \frac{\langle\lambda_n^{\nu=2}\rangle - \langle\lambda_{n+1}^{\nu=0}\rangle}{\langle\lambda_n^{\nu=0}\rangle - \langle\lambda_{n+1}^{\nu=0}\rangle} \tag{49}$$

versus n. The motivation for constructing this particular ratio is as follows. The microscopic eigenvalues are expected to behave universally after rescaling with the chiral condensate and the volume. The universal result for the spectral density of microscopic eigenvalues in the
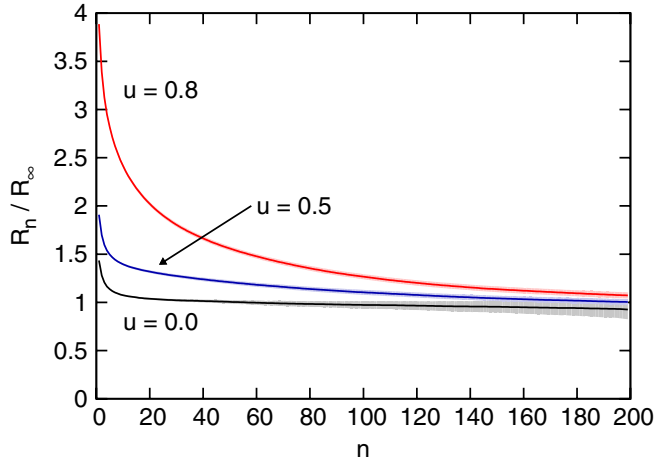
FIG. 1 (color online). Convergence of the ratio R_n of determinants for $\nu = 2$ and $\nu = 0$ as a function of the number n of eigenvalues included for an ensemble of 10^6 400×400 matrices. Results are shown for $u = 0.0$, $u = 0.5$, and $u = 0.8$. The shaded areas correspond to the statistical errors.

quenched case and in the topological sector ν is [28]

$$\rho_s(\xi) = \frac{\xi}{2}[J_\nu(\xi)^2 - J_{\nu+1}(\xi)J_{\nu-1}(\xi)], \qquad (50)$$

where J_ν is a Bessel function and $\xi \equiv \lambda N \Sigma$. Its large-$\xi$ behavior is given by

$$\rho_s(\xi) = \frac{1}{\pi} - \frac{\cos(\nu\pi - 2\xi)}{2\pi\xi} \qquad (51)$$

so that for $\lambda_n N \Sigma \gg 1$ we have $\langle \lambda_n^{\nu=2} \rangle \approx \langle \lambda_{n+1}^{\nu=0} \rangle$ and therefore $\Delta\lambda_n \to 0$. This is indeed what we find in Fig. 2 for $u = 0$. Notice that Eq. (50) has been obtained by taking the microscopic limit and is only valid for eigenvalues well below the chiral scale. For $u = 0$ we find that $\Delta\lambda_n = 0$ also beyond the microscopic domain and conclude that in

this case the topological domain does not extend beyond the microscopic domain. For $u \neq 0$, however, the situation is completely different. All eigenvalues are in the topological domain and only the first few eigenvalues show universal behavior. Comparing the results for $N = 400$ and $N = 800$ in Fig. 2, we observe that the universal domain, i.e., the domain where the eigenvalue ratio $\Delta\lambda_n$ does not depend on u, increases with N proportional to $\sqrt{N}$. This is in agreement with microscopic universality for $u < u_c = 1/\Sigma$, which states that the distribution of low-lying eigenvalues is universal after rescaling them by the chiral condensate. If we consider the Dirac spectrum around x, the correction terms to this universal behavior are of the order Nx^2. This implies that the number of eigenvalues with universal fluctuations around $\lambda = 0$ scales with $\sqrt{N}$.

Based on Fig. 2, a plausible explanation for the behavior of the ratio of the determinants seen in Fig. 1 can be given in terms of the u dependence of the average position of the eigenvalues. For this reason we plot in Fig. 3 the same ratios as in Fig. 1, but normalized with respect to the average positions of the eigenvalues. The ratio $\tilde{R}_n$ defined by

$$\tilde{R}_n \equiv \frac{\left\langle \prod_{k=1}^{n-1}(\lambda_k^{\nu=2}/\langle \lambda_k^{\nu=2} \rangle)^2 \right\rangle}{\left\langle \prod_{k=1}^{n}(\lambda_k^{\nu=0}/\langle \lambda_k^{\nu=0} \rangle)^2 \right\rangle} \qquad (52)$$

is shown for $u = 0.0$, $u = 0.5$, and $u = 0.8$.

We conclude that the u dependence of the ratio of the determinants is almost exclusively due to the effect of u on the average position of the eigenvalues.

In the theory of disordered systems, a frequently used measure to test the breakdown of universality is the number variance [9]. This is the variance of the number of levels in an interval containing $\bar{n}$ eigenvalues on average. In Fig. 4 we display the number variance Σ^2 versus the average
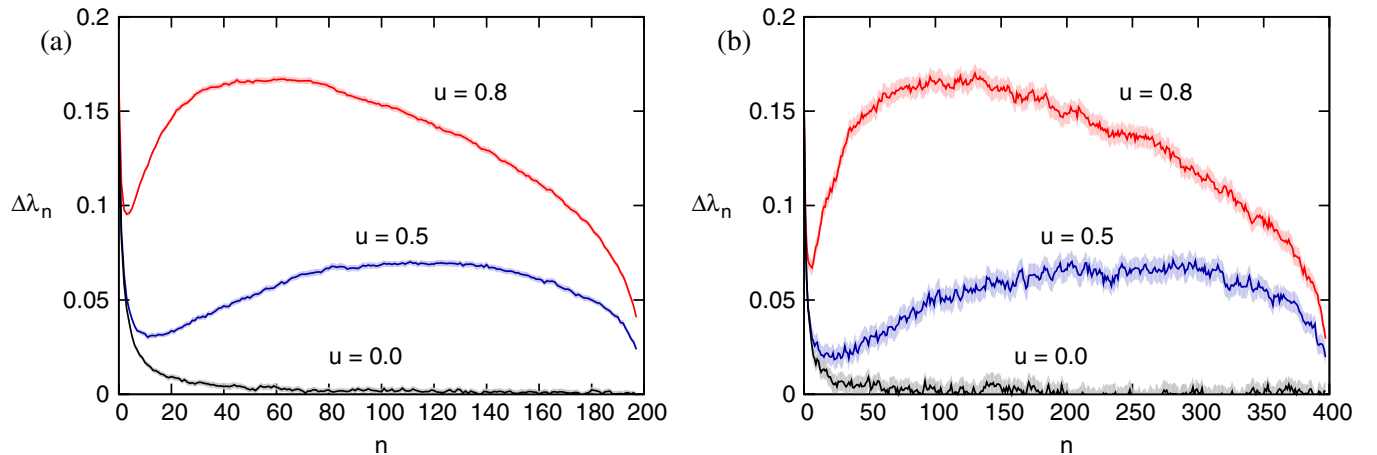

FIG. 2 (color online). Topological shift $\Delta\lambda_n$ of the eigenvalues for (a) an ensemble of 10^6 400×400 matrices and (b) an ensemble of 10^5 800×800 matrices. The shaded areas correspond to the statistical errors.
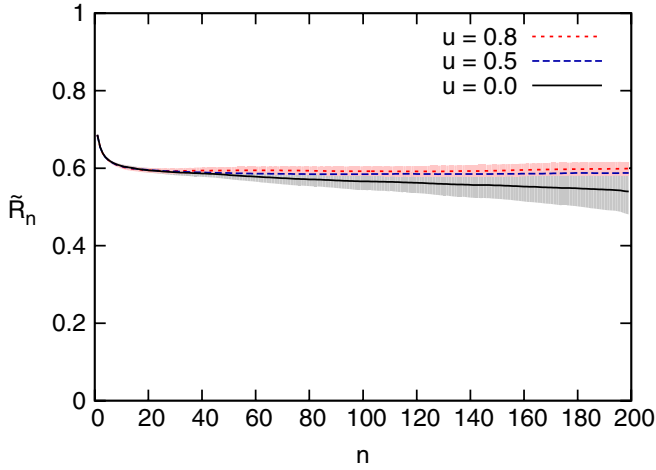
FIG. 3 (color online). Convergence of the ratio $\tilde{R}_n$ of determinants for $\nu = 2$ and $\nu = 0$ as a function of the number n of eigenvalues included. Results are for an ensemble of 10^6 400×400 matrices. The shaded areas correspond to the statistical errors.
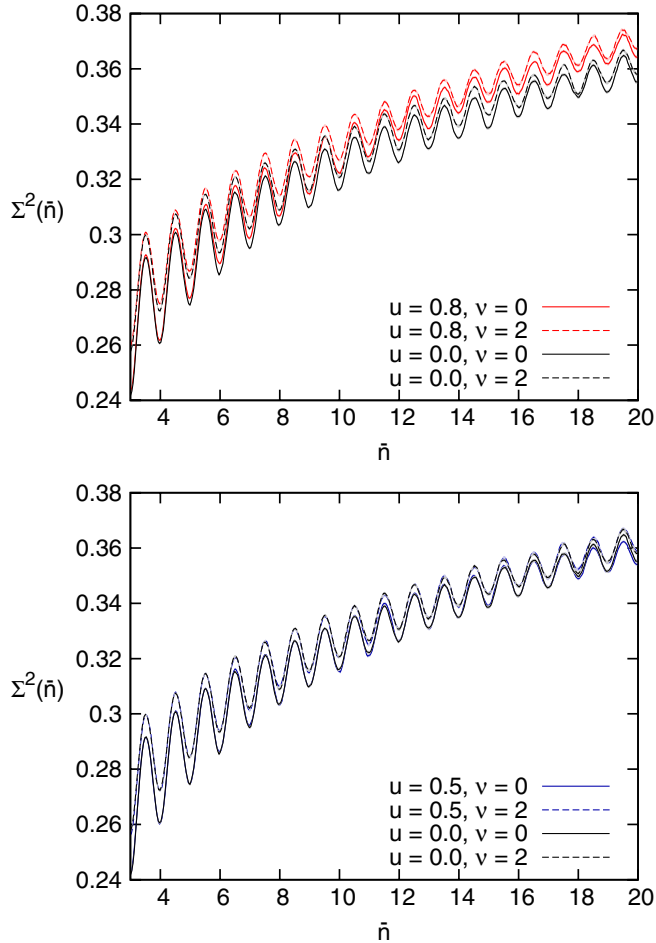

FIG. 4 (color online). Number variance versus the average number $\bar{n}$ of levels in the interval $[0, n]$ for an ensemble of 10^6 400×400 matrices. The curves for $u = 0.0$ and $u = 0.5$ only start to deviate from each other at $\bar{n} \geq 15$.
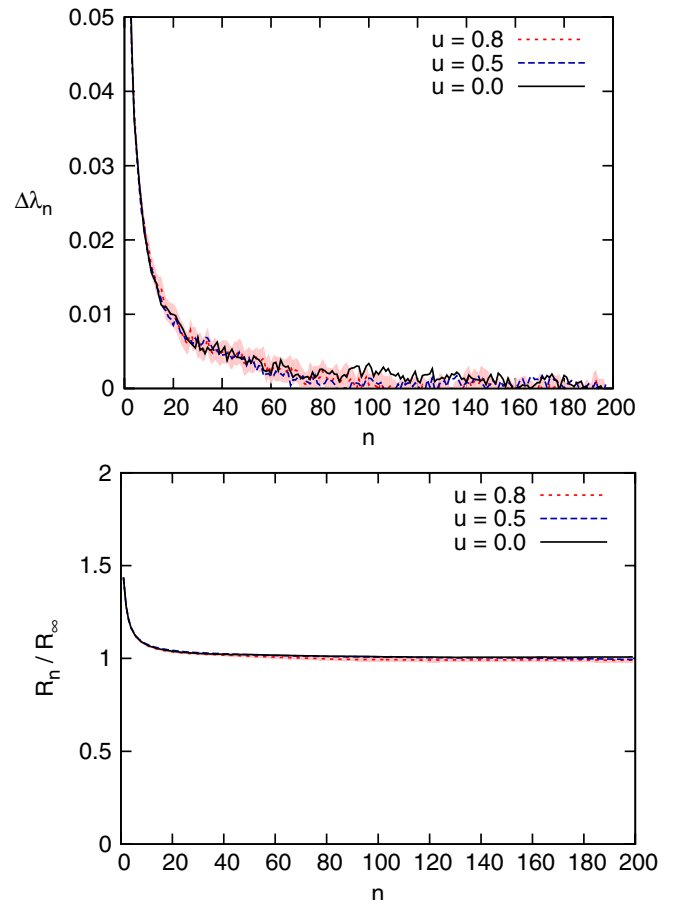
FIG. 5 (color online). Topological shift $\Delta\lambda_n$ and the ratio R_n/R_∞ for an ensemble of 10^6 400×400 matrices with $N_1/N = 0.75$ for model B [10].

number $\bar{n}$ of eigenvalues in an interval starting at zero. The curves for $u = 0.0$ and $u = 0.5$ coincide for $\bar{n} \leq 15$, while the deviations between $u = 0.0$ and $u = 0.8$ are already significant for $\bar{n} \geq 5$. This is in agreement with the discussion of Fig. 2.

In Fig. 5 we show the behavior of the Dirac eigenvalues in model B. We observe that in this model the topological domain does not extend beyond the microscopic domain even for $u \neq 0$. This is also the case for model C, which at $u \neq 0$ is equivalent to model A at $u = 0$ after rescaling the chiral condensate $\Sigma \to \Sigma^C(u)$. The results for model C are therefore identical to the $u = 0$ results in Figs. 1–4. We thus have a further piece of evidence that nontrivial normalization factors $\mathcal{N}_\nu$ only appear if the topological domain extends beyond the microscopic domain.

VI. TOPOLOGICAL AND PSEUDOSCALAR SUSCEPTIBILITY

As mentioned in the introduction, the θ dependence of the QCD partition function is obtained by introducing left-handed and right-handed quark masses according to $z = me^{i\theta}$ and $z^* = me^{-i\theta}$, respectively; see Eq. (1). Denoting the left-hand side of Eq. (2) by $Z(z, z^*)$, with the superscript

QCD omitted for simplicity, the topological susceptibility at arbitrary θ angle is given by

$$\chi_t = \frac{1}{V}(\langle \nu^2 \rangle - \langle \nu \rangle^2) = -\frac{1}{V}\partial_\theta^2 \log Z(z, z^*)$$

$$= \frac{1}{V}(z\partial_z + z^*\partial_{z^*})\log Z(z, z^*)$$

$$+ \frac{1}{V}[z^2\partial_z^2 + z^{*2}\partial_{z^*}^2 - 2zz^*\partial_z\partial_{z^*}]\log Z(z, z^*). \quad (53)$$

Because $m\partial_m = z\partial_z + z^*\partial_{z^*}$, the first term on the right-hand side of this equation is equal to $m|\langle \bar\psi \psi \rangle|$; see Eq. (36). The second term on the right-hand side of Eq. (53) is equal to m^2 times the pseudoscalar (PS) susceptibility given by

$$m^2\chi_{PS} = V\langle (z\bar\psi_L\psi_R - z^*\bar\psi_R\psi_L)^2 \rangle_{N_f=1}$$

$$- V\langle z\bar\psi_L\psi_R - z^*\bar\psi_R\psi_L \rangle_{N_f=1}^2. \quad (54)$$
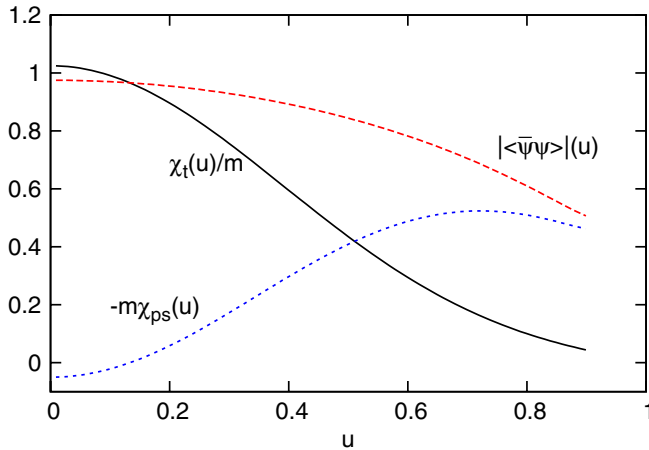
Thus Eq. (53) becomes

$$\chi_t = m|\langle \bar\psi \psi \rangle| + m^2\chi_{PS}. \quad (55)$$

This is the well-known chiral Ward identity relating χ_t to the chiral condensate and the pseudoscalar susceptibility [5]. Note that $|\langle \bar\psi \psi \rangle| = \Sigma \cos\theta + O(m)$.

The random matrix partition function $Z^A(m, \theta)$ with m in the ergodic domain can be calculated explicitly from Eq. (6), setting $\mathcal{P}_\nu = 1$ according to the discussion in Sec. III B. We will set $\mathcal{N}_\nu = \tau^{-|\nu|(1-\varepsilon)}$, where setting ε to 0 or 1 allows us to switch between including or not including $\mathcal{N}_\nu$.
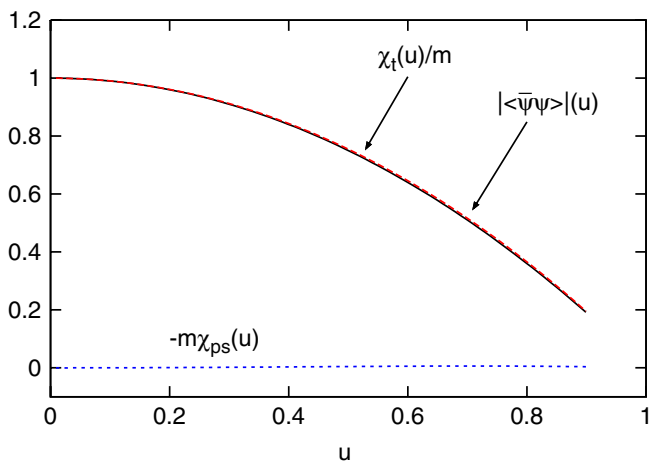
We first replace the Bessel function I_ν in Eq. (20) by the integral representation

$$I_\nu(x) = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, e^{i\nu\varphi + x\cos\varphi}, \quad (56)$$
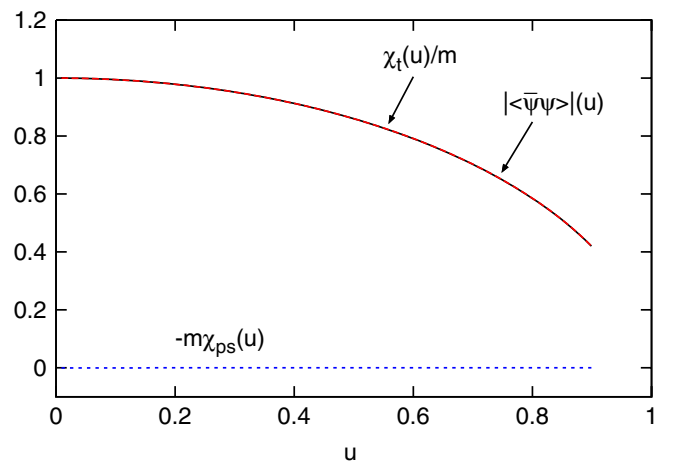
sum the resulting geometric series in ν, and perform a
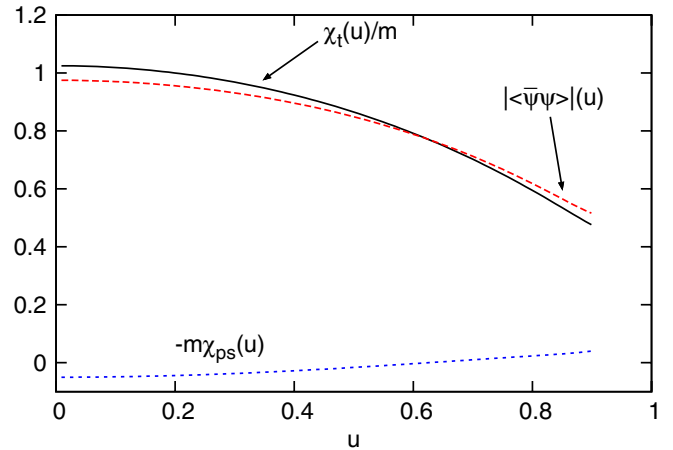
(a) $mN\Sigma \gg 1$, $\mathcal{N}_\nu = 1$.

(b) $mN\Sigma \gg 1$, $\mathcal{N}_\nu$ from Eq. (25).

(c) $mN\Sigma \ll 1$, $\mathcal{N}_\nu = 1$.

(d) $mN\Sigma \ll 1$, $\mathcal{N}_\nu$ from Eq. (25).

FIG. 6 (color online).   Contributions to χ_t^A for $N = 10^3$ and $m = 5 \cdot 10^{-2}$ (a) and (b), and $m = 5 \cdot 10^{-5}$ (c) and (d), with or without the normalization factor $\mathcal{N}_\nu$. We set $\theta = 0$ and $\Sigma = 1$. The curves were obtained by numerical evaluation of Eq. (57) in connection with Eq. (53).

saddle-point approximation of the radial integral including next-to-leading order corrections in m to find

$$Z^A(m, \theta) \sim \int_0^{2\pi} d\varphi \, \frac{1 - \tau^{2\varepsilon}}{1 - 2\tau^\varepsilon \cos\varphi + \tau^{2\varepsilon}}$$
$$\times \exp[mN\Sigma\tau\cos(\varphi - \theta)]$$
$$\times \exp\left[\frac{1}{4\tau^2} Nm^2\Sigma^2\cos^2(\varphi - \theta)\right]. \quad (57)$$

Note that

$$\lim_{u \to 0} \frac{1 - \tau^{2\varepsilon}}{2\pi(1 - 2\tau^\varepsilon \cos\varphi + \tau^{2\varepsilon})} = \delta(\varphi) \qquad (58)$$

but also

$$\lim_{\varepsilon \to 0} \frac{1 - \tau^{2\varepsilon}}{2\pi(1 - 2\tau^\varepsilon \cos\varphi + \tau^{2\varepsilon})} = \delta(\varphi). \qquad (59)$$

Therefore for $\varepsilon \to 0$ or $u \to 0$ we find

$$Z^A(m, \theta) \sim \exp\left[mN\Sigma\tau\cos\theta + \frac{1}{4\tau^2} Nm^2\Sigma^2\cos^2\theta\right] \quad (60)$$

and thus by Eq. (53)

$$\chi_t^A(u) = m\Sigma^A(u)\cos\theta + O(m^2), \qquad (61)$$

which is consistent with results obtained by Crewther [5]. We conclude that for $u = 0$ or if we include the normalization factor (25) the contribution of the pseudoscalar susceptibility vanishes in the chiral limit.

The situation is different, however, if we do not include the $\mathcal{N}_\nu$. For $mN\Sigma \gg 1$ the contribution of the pseudoscalar susceptibility to the topological susceptibility becomes comparable to that of the chiral condensate but with opposite sign and thus leads to a significant suppression of the topological susceptibility (see Fig. 6). Because the saddle-point approximation breaks down close to $u = 1$ we do not plot the curves of Fig. 6 in this region. For $mN\Sigma \ll 1$ the exponent in Eq. (57) can be expanded, and after evaluating the integral analytically we find

$$Z^A(m, \theta) \sim 1 + mN\Sigma\tau^{1+\varepsilon}\cos\theta. \qquad (62)$$

This result agrees with Fig. 6 and shows that in this limit the contribution of the pseudoscalar susceptibility at $u \neq 0$ is small also without $\mathcal{N}_\nu$.

Metlitski and Zhitnitsky have recently found another situation in which the $O(m^2)$ term in Eq. (55) becomes important, i.e., the superfluid phase of QCD with two or three colors [11]. Of course, if we include the $\mathcal{N}_\nu$ in model A (as we should) we do not see this effect. Nevertheless, our observation may potentially be of importance; see the conclusions.

For models B and C no normalization factors $\mathcal{N}_\nu$ are needed to ensure a vanishing contribution of the pseudoscalar susceptibility.

The vanishing of the contribution of the pseudoscalar susceptibility also imposes constraints on the ν dependence of pseudoscalar correlators and can be used as a check of results that were recently derived for the ε domain [29,30].

VII. CONCLUSIONS

It is well-known that random matrix models for QCD at *zero* imaginary chemical potential (or temperature) u have the correct θ dependence. In this paper we have shown that this is not automatically the case for $u \neq 0$. We obtain the correct θ dependence only after introducing ν-dependent normalization factors $\mathcal{N}_\nu$ in the sum over topologies.

To explain this we have introduced the topological domain of the Dirac spectrum, which is defined as the part of the Dirac spectrum that is sensitive to the topological charge. We have shown that for $u = 0$ the topological domain coincides with the microscopic domain. This is also the case at $u \neq 0$ for models for which no ν-dependent normalization factors are needed to obtain the correct θ dependence. However, for the model we analyzed that requires nontrivial normalization factors, the complete Dirac spectrum is inside the topological domain. This results in a partition function that gives universal behavior for small Dirac eigenvalues, but has bulk spectral correlations that depend both on u and on the topological charge. In the thermodynamic limit this leads to an additional u-dependent factor in the partition function at fixed topological charge which results in an incorrect θ dependence of the partition function. To obtain a partition function with the usual behavior in the chiral limit, one has to introduce additional ν-dependent normalization factors in the sum over topologies.

Our observations are of potential importance for lattice QCD at nonzero imaginary chemical potential or temperature. Depending on, e.g., the fermion formulation or the algorithm used, it could be that nontrivial normalization factors are needed in the sum over topological sectors, and these could even persist in the continuum limit. To find out whether such normalization factors might be necessary, it would be interesting to determine the topological domain as a function of the deformation parameters. This is feasible with current lattice technology. To be consistent with the general properties of QCD, the topological domain should not extend beyond the microscopic domain. Future work will tell us if this interesting picture prevails.

ACKNOWLEDGMENTS

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