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Multiparton correlations and "exclusive" cross sections

G. Calucci* and D. Treleani†

Dipartimento di Fisica Teorica dell'Università di Trieste and INFN, Sezione di Trieste, Strada Costiera 11, Miramare-Grignano, I-34014 Trieste, Italy

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In addition to the inclusive cross sections discussed within the QCD-parton model, in the regime of multiple parton interactions, different and more exclusive cross sections become experimentally viable and may be suitably measured. Indeed, in its study of double parton collisions, the quantity measured by the CDF was an "exclusive" rather than an inclusive cross section. The nonperturbative input to the "exclusive" cross sections is different with respect to the nonperturbative input of the inclusive cross sections and involves correlation terms of the hadron structure already at the level of single parton collisions. The matter is discussed in details keeping explicitly into account the effects of double and of triple parton collisions.

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I. INTRODUCTION

The growing importance of multiple parton interactions (MPI) at high energy has stimulated a lot of interest in the phenomenon, in view of the forthcoming results at the LHC [1,2]. MPI are essential to describe the features of the minimum bias and of the underlying event [3–9] and may represent an important background in many channels of interest at the LHC [1,2], not only in processes where the cross section is large, as the unitarity issue at the origin of the effect might suggest, but also in cases where the cross section is rather small, like for the search of the Higgs boson [10,11] or in the case of the production of equal sign W boson pairs [2,12,13]. On the other hand, MPI are by themselves an interesting topic of research, since by studying MPI one may obtain informations on the multiparton structure of the hadron [1,2].

Up until now, the direct observation of MPI has not been easy. The direct measurement of MPI requires in fact the identification of the final fragments of the multiple processes and the reduction of statistics, due to the request of large momentum exchange in each hard interaction, has restricted considerably the possibilities of a direct study of the phenomenon. To measure the MPI one needs moreover to separate the background due to hard radiation. A given multiparton's final state may in fact be produced either by a multiple or by a single parton collision. The separation between the two contributions has proven to be experimentally feasible in the case of double parton collisions [14–17]. The enhanced contribution of MPI at high energy will facilitate considerably direct studies of the phenomenon and one may reasonably expect that the separation of the two different contributions will be done more easily at the LHC, at least in the simplest cases of MPI.

In the regime where MPI may be observed directly, in addition to the inclusive cross sections usually considered in large momentum exchange processes, one has the possibility to measure diverse and more exclusive cross sections, computable in perturbation theory and linked differently to the hadron structure [18]. In pp interactions, MPI are dominated by independent collisions, initiated by different pairs of partons [19,20]. A direct consequence is that the multiparton inclusive cross sections are proportional to the moments of the distribution in the number of collisions [18]. On the other hand, a statistical distribution may be characterized either by its moments or by its different terms. While the moments of the distribution in the number of collisions are measured by the multiparton inclusive cross sections, the different terms of the distribution are measured by a different set of observables, which one may call "exclusive" cross sections. Inclusive and "exclusive" cross sections result from independent measurements and are linked in a different way to the hadron structure. The two sets of cross sections are however connected by sum rules. By testing the sum rules, namely, by looking at the number of terms needed to saturate the sum rules in a given phase space region, one measures the effects of unitarity corrections, which allows one to control the consistency of the analysis and provides an additional handle to obtain information on the multiparton correlations of the hadron structure.

Interestingly, in its study of MPI, the CDF experiment did not measure the inclusive cross section of double parton scattering. The events selected were in fact only those which contained just double parton collisions, while all events with triple scatterings (about 17% of the sample of all events with double parton scatterings) where removed [17]. The resulting quantity measured by CDF is hence different with respect to the inclusive cross sections usually discussed in large p_t physics. In fact, it represents precisely one of the "exclusive" cross sections recently discussed [18].

*giorgio.calucci@ts.infn.it
†daniele.treleani@ts.infn.it

While the inclusive cross sections are linked directly to the multiparton structure of the hadron, the link of the "exclusive" cross sections with the hadron structure is much more elaborate. The requirement of having only events with a given number of hard collisions implies that the corresponding cross section (being proportional to the probability of not having any further hard interaction) depends on the whole series of multiple hard collisions. The number of hard partonic collisions which can be observed directly is nevertheless limited, which allows one to discuss the "exclusive" cross sections by expanding in the number of elementary interactions. Simple and directly testable connections between inclusive and "exclusive" cross sections may hence be established by saturating the sum rules, which link "inclusive" and exclusive cross sections, with a finite number of terms.

The purpose of the present paper is to discuss the "exclusive" multiparton scattering cross sections, going up to the third order in the number of collisions and keeping two-body parton correlations explicitly into account, while the effects of the three-body parton correlations in the hadron structure will be neglected. Explicit expressions in terms of the two-body correlation parameters will be derived in a few simplest cases, where correlations will be assumed to depend only on the transverse coordinates.

II. "EXCLUSIVE" CROSS SECTIONS

In pp collisions, the inclusive cross sections are basically the moments of the distribution of the number of MPI [18]. The most basic information on the distribution in the number of collisions, the average number, is hence given by the single scattering inclusive cross section of the QCD-parton model. Analogously, the K-parton scattering inclusive cross section σ_K gives the Kth moment of the distribution in the number of collisions and is related directly to the K-body parton distribution of the hadron structure:

$$\begin{aligned}\sigma_K &= \frac{1}{K!}\int D_A(x_1\dots x_K;b_1\dots b_K)\hat\sigma(x_1x_1')\dots\hat\sigma(x_Kx_K')D_B(x_1'\dots x_K';b_1-\beta\dots b_K-\beta)dx_1dx_1'd^2b_1\dots dx_Kdx_K'd^2b_Kd^2\beta\\ &= \frac{\langle N(N-1)\dots(N-K+1)\rangle}{K!}\sigma_{\rm hard},\end{aligned} \tag{1}$$

where $\sigma_{\rm hard}$ represents the contribution to the total inelastic cross section due to all events with at least one hard interaction, while $D(x_1\dots x_K;b_1\dots b_K)$ is the K parton's density of the hadron structure, with transverse parton coordinates $b_1\dots b_K$ and fractional momenta $x_1\dots x_K$, β is the hadronic impact parameter and $\hat\sigma$ the parton-parton cross section.

An alternative way to the set of moments, to provide the whole information of the distribution, is represented by the set of the different terms of the probability distribution of multiple collisions. Correspondingly, in addition to the set of the inclusive cross sections σ_K, one may consider the set of the "exclusive" cross sections $\tilde\sigma_N$, where one selects the events where *only* N collisions are present. One hence has

$$\begin{aligned}\sigma_{\rm hard} &\equiv \sum_{N=1}^{\infty}\tilde\sigma_N,\\ \sigma_K &\equiv \sum_{N=K}^{\infty}\frac{N(N-1)\dots(N-K+1)}{K!}\tilde\sigma_N,\end{aligned} \tag{2}$$

which represents also a set of sum rules connecting the inclusive and the "exclusive" cross sections.

While the nonperturbative input to the inclusive cross section σ_K is given by the K-parton distributions of the hadron structure, as implicit in Eq. (2), the nonperturbative input to the "exclusive" cross sections is given by an infinite set of multiparton distributions. The request of being in a perturbative regime limits however the number of partonic collisions and the sum rules in Eq. (2) are saturated by a few terms, in such a way that the "exclusive" cross sections can be expressed by finite combinations of inclusive cross sections. A particular case is when all correlations C_n with $n > 2$ are negligible (higher order correlation terms may be introduced in the picture of the interaction as sketched in the Appendix). In that instance, all sums in Eq. (2) can be performed [21,22] and (neglecting all rescatterings) the hard cross section $\sigma_{\rm hard}$ can be expressed in the following functional form:

$$\begin{aligned}\sigma_{\rm hard}(\beta) = \Big[1-\exp\Big\{-\int du du'\partial_J\hat\sigma(u,u')\partial_{J'}\Big\}\Big]\\ \times Z_A[J]Z_B[J']|_{J=J'=1},\end{aligned} \tag{3}$$

where $u\equiv\{x,b\}$, $u'\equiv\{x',b'-\beta\}$,and $\hat\sigma(u,u')$ is the interaction probability of the two partons with coordinates u and u'. For simplicity, flavor indices are omitted and the integration limits are set by the limits of the phase space window where the MPI are observed. The two-body parton correlations are introduced through the functional

$$\mathcal{Z}[J+1] \equiv \exp\left\{\int D(u)J(u)du + \frac{1}{2}\int C(u,v)J(u)J(v)dudv\right\} = \sum_n \frac{1}{n!}\int J(u_1)\ldots J(u_n)D_n(u_1\ldots u_n)du_1\ldots du_n, \quad (4)$$

which generates the nonperturbative input of the n-parton inclusive cross sections, the inclusive n-body parton distributions $D_n(u_1\ldots u_n)$. Under these conditions, Eq. (3) can be worked out fully explicitly. One obtains [21,22]

$$\sigma_{\rm hard}(\beta) = 1 - \exp\left[-\frac{1}{2}\sum_n a_n - \frac{1}{2}\sum_n b_n/n\right], \quad (5)$$

where

$$a_n = (-1)^{n+1}\int D_A(u_1)\hat{\sigma}(u_1,u_1')C_B(u_1',u_2')\hat{\sigma}(u_2',u_2) \times C_A(u_2,u_3)\ldots\hat{\sigma}(u_n,u_n')D_B(u_n')\prod_{i=1}^n du_i du_i' \quad (6)$$

and the chain, which starts with A, may end either with A or with B, depending whether n is odd or even. For b_n one has

$$b_n = (-1)^{n+1}\int C_A(u_n,u_1)\hat{\sigma}(u_1,u_1')C_B(u_1',u_2')\ldots \times C_B(u_{n-1}',u_n')\hat{\sigma}(u_n',u_n)\prod_{i=1}^n du_i du_i' \quad (7)$$

and in this case only even values of n are possible.

The exponential in Eq. (5) represents the probability of no interaction at a given impact parameter β. All "exclusive" cross sections can be obtained from the argument of the exponential. One may start from the partonic interaction probability

$$1 - \prod_{i,j=1}^n (1-\hat{\sigma}_{ij}), \quad (8)$$

where in the product each index assumes a given value only once, in such a way that possible reinteractions are not included. The probability of having only a single interaction is expressed by

$$\left(-\frac{\partial}{\partial g}\right)\prod_{i,j=1}^n (1-g\hat{\sigma}_{ij})|_{g=1} = \sum_{kl}\hat{\sigma}_{kl}\prod_{ij\neq kl}^n (1-g\hat{\sigma}_{ij})|_{g=1}, \quad (9)$$

while the probabilities of a double and of a triple collision are

$$\begin{aligned}
&\frac{1}{2!}\left(-\frac{\partial}{\partial g}\right)^2\prod_{i,j=1}^n (1-g\hat{\sigma}_{ij})|_{g=1} = \frac{1}{2!}\sum_{kl}\sum_{rs}\hat{\sigma}_{kl}\hat{\sigma}_{rs}\prod_{ij\neq kl,rs}^n (1-g\hat{\sigma}_{ij})|_{g=1}\\
&\frac{1}{3!}\left(-\frac{\partial}{\partial g}\right)^3\prod_{i,j=1}^n (1-g\hat{\sigma}_{ij})|_{g=1} = \frac{1}{3!}\sum_{kl}\sum_{rs}\sum_{tu}\hat{\sigma}_{kl}\hat{\sigma}_{rs}\hat{\sigma}_{tu}\prod_{ij\neq kl,rs,tu}^n (1-g\hat{\sigma}_{ij})|_{g=1}
\end{aligned} \quad (10)$$

and the corresponding expressions for the "exclusive" cross sections are

$$\begin{aligned}
\left(-\frac{\partial}{\partial g}\right)e^{-X(g)}\Big|_{g=1} &= X'(g)e^{-X(g)}\Big|_{g=1}\\
\frac{1}{2!}\left(-\frac{\partial}{\partial g}\right)^2 e^{-X(g)}\Big|_{g=1} &= \frac{1}{2!}\{[X'(g)]^2 - X''(g)\}e^{-X(g)}\Big|_{g=1}\\
\frac{1}{3!}\left(-\frac{\partial}{\partial g}\right)^3 e^{-X(g)}\Big|_{g=1} &= \frac{1}{3!}\{X'''(g) + [X'(g)]^3 - 3X'(g)X''(g)\}e^{-X(g)}\Big|_{g=1}
\end{aligned} \quad (11)$$

where $X = \frac{1}{2}(\sum a_n + \sum b_n/n)$.

It is convenient to expand X and its derivatives in the number of elementary collisions

$$X = X_1 + X_2 + X_3 + \ldots \quad (12)$$

where

$$\begin{aligned}
X_1 &= \int D_A(u)\hat{\sigma}(u,u')D_B(u')dudu'\\
X_2 &= -\frac{1}{2}\left[\int D_A(u_1)\hat{\sigma}(u_1,u_1')C_B(u_1',u_2')\hat{\sigma}(u_2',u_2)D_A(u_2)\prod_{i=1}^2 du_i du_i' + A\leftrightarrow B\right]\\
&\quad -\frac{1}{2}\int C_A(u_1,u_2)\hat{\sigma}(u_1,u_1')C_B(u_1',u_2')\hat{\sigma}(u_2',u_2)\prod_{i=1}^2 du_i du_i' \quad (13)\\
X_3 &= \int D_A(u_1)\hat{\sigma}(u_1,u_1')C_B(u_1',u_2')\hat{\sigma}(u_2',u_2)C_A(u_2,u_3)\hat{\sigma}(u_3,u_3')D_B(u_3')\prod_{i=1}^3 du_i du_i'.
\end{aligned}$$

The derivatives at $g = 1$ give

$$\begin{aligned}
X_1'(v, v') &= D_A(v)\hat{\sigma}(v, v')D_B(v') \\
X_2'(v, v') &= -\Big[D_A(v)\hat{\sigma}(v, v')\int C_B(v', u_1')\hat{\sigma}(u_1', u_1)D_A(u_1)du_1du_1' + A \leftrightarrow B\Big] \\
&\quad - \int C_A(u_1, v)\hat{\sigma}(v, v')C_B(v', u_1')\hat{\sigma}(u_1', u_1)du_1du_1' \\
X_3'(v, v') &= \Big[D_A(v)\hat{\sigma}(v, v')\int C_B(v', u_1')\hat{\sigma}(u_1', u_1)C_A(u_1, u_2)\hat{\sigma}(u_2, u_2')D_B(u_2')\prod_{i=1}^{2}du_idu_i' \\
&\quad + \int D_A(u_1)\hat{\sigma}(u_1, u_1')C_B(u_1', v')\hat{\sigma}(v', v)C_A(v, u_2)\hat{\sigma}(u_2, u_2')D_B(u_2')\prod_{i=1}^{2}du_idu_i' \\
&\quad + \int D_A(u_1)\hat{\sigma}(u_1, u_1')C_B(u_1', u_2')\hat{\sigma}(u_2', u_2)C_A(u_2, v)\hat{\sigma}(v, v')D_B(v')\prod_{i=1}^{2}du_idu_i'\Big]
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
X_2''(v_1, v_1'; v_2, v_2') &= -[D_A(v_1)\hat{\sigma}(v_1, v_1')C_B(v_1', v_2')\hat{\sigma}(v_2', v_2)D_A(v_2) + A \leftrightarrow B] \\
&\quad - C_A(v_2, v_1)\hat{\sigma}(v_1, v_1')C_B(v_1', v_2')\hat{\sigma}(v_2', v_2) \\
X_3''(v_1, v_1'; v_2, v_2') &= 2[D_A(v_1)\hat{\sigma}(v_1, v_1')C_B(v_1', v_2')\hat{\sigma}(v_2', v_2)\int C_A(v_2, u_1)\hat{\sigma}(u_1, u_1')D_B(u_1')du_1du_1' \\
&\quad + D_A(v_1)\hat{\sigma}(v_1, v_1')\int C_B(v_1', u_1')\hat{\sigma}(u_1', u_1)C_A(u_1, v_2)du_1du_1'\hat{\sigma}(v_2, v_2')D_B(v_2') \\
&\quad + \int D_A(u_1)\hat{\sigma}(u_1, u_1')C_B(u_1', v_1')du_1du_1'\hat{\sigma}(v_1', v_1)C_A(v_1, v_2)\hat{\sigma}(v_2, v_2')D_B(v_2')] \\
X_3'''(v_1, v_1'; v_2, v_2'; v_3, v_3') &= 6D_A(v_1)\hat{\sigma}(v_1, v_1')C_B(v_1', v_2')\hat{\sigma}(v_2', v_2)C_A(v_2, v_3)\hat{\sigma}(v_3, v_3')D_B(v_3').
\end{aligned} \tag{15}$$

By substituting the expansions in the number of elementary collisions in the expressions of the interaction probabilities and by expanding the exponential, one obtains the expressions:

$$\begin{aligned}
\tilde{\sigma}_1' &= (X_1' + X_2' + X_3')(1 - X_1 - X_2 + X_1 \cdot X_1/2) \\
2 \times \tilde{\sigma}_2'' &= (X_1' \cdot X_1' + 2X_1' \cdot X_2' - X_2'' - X_3'')(1 - X_1) \\
3 \times \tilde{\sigma}_3''' &= \tfrac{1}{2}(X_3''' + X_1' \cdot X_1' \cdot X_1' - 3X_1' \cdot X_2''),
\end{aligned} \tag{16}$$

where $\tilde{\sigma}_1'$, etc. are the "exclusive" cross sections, differentiated according with (14) and (15). The integrated "exclusive" cross sections hence are

$$\begin{aligned}
\tilde{\sigma}_1 &= X_1 - X_1^2 - X_1X_2 + X_1^3/2 + 2X_2 - 2X_2X_1 + 3X_3 \\
2 \times \tilde{\sigma}_2 &= X_1^2 + 4X_1X_2 - 2X_2 - 6X_3 - X_1^3 + 2X_1X_2 \\
3 \times \tilde{\sigma}_3 &= 3X_3 + (X_1)^3/2 - 3X_1X_2.
\end{aligned} \tag{17}$$

The sum rules of Eq. (2) are satisfied as follows:

$$\begin{aligned}
\tilde{\sigma}_1 + 2 \times \tilde{\sigma}_2 + 3 \times \tilde{\sigma}_3 &= X_1 - X_1^2 + 2X_2 + X_1^2 - 2X_2 - X_1X_2 + X_1^3/2 - 2X_2X_1 + 3X_3 + 4X_1X_2 - 6X_3 - X_1^3 + 2X_1X_2 \\
&\quad + 3X_3 + (X_1)^3/2 - 3X_1X_2 = X_1 \equiv \sigma_S \\
2 \times \tilde{\sigma}_2 + 6 \times \tilde{\sigma}_3 &= X_1^2 - 2X_2 + 4X_1X_2 - 6X_3 - X_1^3 + 2X_1X_2 + 6X_3 + (X_1)^3 - 6X_1X_2 = X_1^2 - 2X_2 \\
&\equiv 2 \times \sigma_D \\
6 \times \tilde{\sigma}_3 &= 6X_3 + (X_1)^3 - 6X_1X_2 \equiv 3! \times \sigma_T,
\end{aligned} \tag{18}$$

where σ_S, σ_D, and σ_T are, respectively, the single, double, and triple parton scattering inclusive cross sections. Explicitly,

$$\sigma_S = X_1 = \int D_A\hat{\sigma}D_B$$
$$\sigma_D = \frac{1}{2}[X_1^2 - 2X_2] = \frac{1}{2}\Big[\int D_A\hat{\sigma}D_B \cdot D_A\hat{\sigma}D_B + \int D_A\hat{\sigma}C_B\hat{\sigma}D_A + \int D_B\hat{\sigma}C_A\hat{\sigma}D_B + \int C_A\hat{\sigma}C_B\hat{\sigma}\Big] \tag{19}$$
$$= \frac{1}{2}\int [D_AD_A + C_A]\hat{\sigma}\,\hat{\sigma}[D_BD_B + C_B],$$

where $[DD + C] \equiv D_2$, the two-body parton distribution as defined in Eq. (4). An analogous expression may be written for σ_T.

The relations (18) may be inverted

$$\tilde{\sigma}_1 = \sigma_S - 2\sigma_D + 3\sigma_T \qquad \tilde{\sigma}_2 = \sigma_D - 3\sigma_T \qquad \tilde{\sigma}_3 = \sigma_T, \tag{20}$$

which allow one to express the scale parameters characterizing the double and triple parton collisions in terms of the single scattering inclusive cross section σ_S and of the single and double parton "exclusive" cross sections $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$:

$$\sigma_D = \sigma_S - \tilde{\sigma}_1 - \tilde{\sigma}_2 = \frac{1}{2}\frac{\sigma_S^2}{\sigma_{\rm eff}}$$
$$\sigma_T = \frac{1}{3}(\sigma_S - \tilde{\sigma}_1 - 2\tilde{\sigma}_2) = \frac{1}{6}\sigma_S^3\frac{1}{\tau\sigma_{\rm eff}^2}, \tag{21}$$

where the scale factor of the triple parton scattering cross section has been characterized by the dimensionless parameter τ.

III. CORRELATIONS IN TRANSVERSE SPACE

Multiple parton collisions are most important in the region of small fractional momenta, where the large population of partons may dilute correlations and, e.g., correlations due to energy conservation may not be of major importance. The simplest possibility is hence to neglect altogether correlations in fractional momenta and to work out some case where only correlations in transverse space are present and which allows an analytic treatment. To disentangle the effect of correlations in transverse space from other sources of correlation, we will consider the instance where all correlation terms give zero, when integrated on the transverse coordinates, in such a way that all other variables remain uncorrelated and, in particular, the distribution in the number of partons is Poissonian.

The actual dependence of the correlation on the transverse variables is not prescribed by general principles, in this section two choices are presented, a Gaussian shape and an exponential shape, both were used in a previous analysis [18]. The case of a multiparton distribution expressed by the superposition of Poissonians will also be discussed.

A. Gaussian density

Using Gaussian distributions for the parton densities in transverse space and for the correlations one obtains closed analytic expressions:

$$D(x, b) = G(x)f(b)$$
$$f(b) = g(b, R^2)$$
$$C(x_1, x_2; b_1, b_2) = G(x_1)G(x_2)h(b_1, b_2) \tag{22}$$
$$h(b_1, b_2) = c \cdot g(B, R^2/2)\bar{h}(b, \lambda^2)$$
$$\bar{h}(b, \lambda^2) = \frac{d}{d\gamma}\bar{g}(b, \lambda^2/\gamma)|_{\gamma=1},$$

where $G(x)$ represents the usual one-body parton distribution, $\bar{g}(b, \lambda^2/\gamma) \equiv \eta \cdot g(b, \lambda^2/\gamma)$,

$$g(b, R^2) = \frac{1}{\pi R^2}\exp(-b^2/R^2)$$

and

$$\mathbf{B} = [\mathbf{b}_1 + \mathbf{b}_2]/2 \qquad \mathbf{b} = [\mathbf{b}_1 - \mathbf{b}_2] \tag{23}$$

in such a way that the following relations hold

$$\int d^2b\, g(b, R^2) = 1,$$
$$\int d^2b_2\, g(\mathbf{b}_1 - \mathbf{b}_2, R_1^2)g(\mathbf{b}_2, R_2^2) = g(\mathbf{b}_1, R_1^2 + R_2^2) \tag{24}$$
$$\int h(b_1, b_2)d^2b = 0.$$

One may define the correlation length r_c as the value of b where the correlation term $\bar{h}(b, \lambda^2)$ changes sign. With our definition of $\bar{h}(b, \lambda^2)$, one has $r_c = \lambda$.

To define unambiguously the "correlation strength" c, one needs to normalize properly the correlation term $\bar{h}(b, \lambda^2)$ (which integrates to zero). Our choice is

$$\int_{|b|\le r_c}\bar{h}(b, \lambda^2)d^2b = 1,$$

which gives $\eta = e$, where e is the Euler's number.

The integrations on the transverse variables of the terms in Eqs. (16)–(18) give

$$D_A\hat{\sigma}D_B \to \int d^2b d^2\beta g(\mathbf{b}-\boldsymbol{\beta}, R_A^2) g(\mathbf{b}, R_B^2) = 1$$

$$D_A\hat{\sigma}D_B \cdot D_A\hat{\sigma}D_B \to \int d^2b_1 d^2b_2 d^2\beta g(\mathbf{b}_1-\boldsymbol{\beta}, R_A^2) g(\mathbf{b}_1, R_B^2) g(\mathbf{b}_2-\boldsymbol{\beta}, R_B^2) g(\mathbf{b}_2, R_B^2) = \frac{1}{2\pi(R_A^2+R_B^2)}$$

$$D_A\hat{\sigma}D_B \cdot D_A\hat{\sigma}D_B \cdot D_A\hat{\sigma}D_B \to \int d^2b_1 d^2b_2 d^2b_3 d^2\beta g(\mathbf{b}_1-\boldsymbol{\beta}, R_A^2) g(\mathbf{b}_1, R_B^2) g(\mathbf{b}_2-\boldsymbol{\beta}, R_A^2) g(\mathbf{b}_2, R_B^2) g(\mathbf{b}_3-\boldsymbol{\beta}, R_A^2) g(\mathbf{b}_3, R_B^2) = \frac{1}{3\pi^2(R_A^2+R_B^2)^2}$$

$$D_A\hat{\sigma}C_B\hat{\sigma}D_A \to \int d^2b_1 d^2b_2 d^2\beta g(\mathbf{b}_1-\boldsymbol{\beta}, R_A^2) h_B(\mathbf{b}_1, \mathbf{b}_2) g(\mathbf{b}_2-\boldsymbol{\beta}, R_A^2) = \frac{c_B e}{\pi}\frac{\lambda_B^2}{(2R_A^2+\lambda_B^2)^2}$$

$$D_A\hat{\sigma}C_B\hat{\sigma}D_A \cdot D_A\hat{\sigma}D_B \to \int d^2b_1 d^2b_2 d^2b_3 d^2\beta g(\mathbf{b}_1-\boldsymbol{\beta}, R_A^2) h_B(\mathbf{b}_1, \mathbf{b}_2) g(\mathbf{b}_2-\boldsymbol{\beta}, R_A^2) g(\mathbf{b}_3-\boldsymbol{\beta}, R_A^2) g(\mathbf{b}_3, R_B^2) = \frac{c_B e}{3\pi^2}\frac{2\lambda_B^2}{(R_A^2+R_B^2)(2R_A^2+\lambda_B^2)^2}$$

$$C_A\hat{\sigma}C_B\hat{\sigma} \to \int d^2b_1 d^2b_2 d^2\beta h_A(\mathbf{b}_1-\boldsymbol{\beta}, \mathbf{b}_2-\boldsymbol{\beta}) h_B(\mathbf{b}_1, \mathbf{b}_2) = \frac{c_A c_B e^2}{\pi}\frac{2\lambda_A^2\lambda_B^2}{(\lambda_A^2+\lambda_B^2)^3}$$

$$C_A\hat{\sigma}C_B\hat{\sigma} \cdot D_A\hat{\sigma}D_B \to \int d^2b_1 d^2b_2 d^2b_3 d^2\beta h_A(\mathbf{b}_1-\boldsymbol{\beta}, \mathbf{b}_2-\boldsymbol{\beta}) h_B(\mathbf{b}_1, \mathbf{b}_2) g(\mathbf{b}_3-\boldsymbol{\beta}, R_A^2) g(\mathbf{b}_3, R_B^2) = \frac{c_A c_B e^2}{3\pi^2}\frac{4\lambda_A^2\lambda_B^2}{(R_A^2+R_B^2)(\lambda_A^2+\lambda_B^2)^3}$$

$$D_A\hat{\sigma}C_B\hat{\sigma}C_A\hat{\sigma}D_B \to \int d^2b_1 d^2b_2 d^2b_3 d^2\beta g(\mathbf{b}_1, R_A^2) h_B(\mathbf{b}_1-\boldsymbol{\beta}, \mathbf{b}_2-\boldsymbol{\beta}) h_A(\mathbf{b}_2, \mathbf{b}_3) g(\mathbf{b}_3-\boldsymbol{\beta}, R_B^2) = \frac{16 c_A c_B e^2}{3\pi^2 R_A^2 R_B^2} E(s_A^2, s_B^2, r^2) \tag{25}$$

where $s_{A,B} = (\lambda/R)_{A,B}$, $r = R_A/R_B$, and

$$E(s_A^2, s_B^2, r^2) \equiv \frac{s_A^2 s_B^2[32r^2 + 100 + 32r^{-2} + 6s_B^2(1+4r^{-2}) + 3s_A^2(3s_B^2 + 2(4r^2+1))]}{\{s_A^2[3s_B^2 + 2(4r^2+1)] + 2[6 + s_B^2(1+4r^{-2})]\}^3}.$$

Using Eqs. (20) and (21), all inclusive and "exclusive" cross sections, up to the triple order in the number of parton collisions, are expressed in terms of the single scattering inclusive cross section σ_S, of the "effective" cross section σ_{eff}, and of the parameter τ. In the case of collisions of two identical hadrons, the explicit expressions of σ_{eff} and τ are

$$\frac{1}{\sigma_{\text{eff}}} = \frac{3}{8\pi\bar{R}^2}\left\{1 + c\cdot e\frac{16\times 3\bar{s}^2}{(4+3\bar{s}^2)^2} + c^2\cdot e^2\frac{2}{3\bar{s}^2}\right\} \tag{26}$$

and

$$\frac{1}{\tau\sigma_{\text{eff}}^2} = \frac{3}{16\pi^2\bar{R}^4}\left\{1 + c\cdot e\frac{16\times 9\bar{s}^2}{(4+3\bar{s}^2)^2} + c^2\cdot e^2\left[\frac{2}{3\bar{s}^2} + 6\times 64 E\left(\frac{3\bar{s}^2}{2}, \frac{3\bar{s}^2}{2}, 1\right)\right]\right\}, \tag{27}$$

where $\bar{s} \equiv \lambda/\bar{R}$ ($\lambda = r_c$ is the correlation length) and $\bar{R}^2 \equiv \frac{3}{2}R^2$ is the square hadron radius measured in the generalized parton distributions [23,24].

B. Exponential density

The case where the parton density has an exponential shape is better displayed in Fourier-transform representation:

$$g(b, R^2) = \frac{1}{(2\pi)^2}\int e^{-i\mathbf{k}\cdot\mathbf{b}}\frac{1}{(1+k^2R^2)^2}d^2k, \tag{28}$$

while all other terms defined in Eq. (22) are redefined accordingly with this unique change. In particular, the correlation term is

$$h(b_1, b_2) = c\cdot g(B, R^2/2)\bar{h}(b, \lambda^2)$$

$$\bar{h}(b, \lambda^2) = \frac{d}{d\gamma}\bar{g}(b, \lambda^2/\gamma)\bigg|_{\gamma=1} = \frac{2\eta}{(2\pi)^2}\int e^{-i\mathbf{k}\cdot\mathbf{b}}\frac{(k\lambda)^2}{(1+k^2\lambda^2)^3}d^2k$$

and $\bar{g}(b, \lambda^2/\gamma) \equiv \eta\cdot g(b, \lambda^2/\gamma)$. As in the previous case, one defines the correlation length r_c as the value of the distance $|\mathbf{b}_1 - \mathbf{b}_2|$ where the correlation $\bar{h}(b, \lambda^2)$ changes sign. The relation with the parameter λ is $r_c = x_0\lambda$ with $x_0 \simeq 2.387$ [18]. The normalization parameter η, defined by the requirement

$$\int_{|b|\le r_c} \bar{h}(b,\lambda^2) d^2b = 1$$

is now $\eta \simeq 3.456$.

The integrations on the transverse variables cannot always be displayed in closed form, a relevant simplification is reached by taking the parameters R and λ to be equal in A and B.

In this case the integrations on the transverse variables of the terms in Eqs. (16)–(18) give

$$\begin{aligned}
D_A\hat{\sigma}D_B &\to \int d^2bd^2\beta g(\mathbf{b}-\boldsymbol{\beta})g(\mathbf{b}) = 1 \\
D_A\hat{\sigma}D_B\cdot D_A\hat{\sigma}D_B &\to \int d^2b_1d^2b_2d^2\beta g(\mathbf{b}_1-\boldsymbol{\beta})g(\mathbf{b}_1)g(\mathbf{b}_2-\beta)g(\mathbf{b}_2) = \frac{1}{4\pi R^2}\frac{1}{7} \\
D_A\hat{\sigma}C_B\hat{\sigma}D_A &\to \int d^2b_1d^2b_2d^2\beta g(\mathbf{b}_1-\boldsymbol{\beta})h(\mathbf{b}_1,\mathbf{b}_2)g(\mathbf{b}_2-\boldsymbol{\beta}) = \frac{c}{4\pi R^2}\eta F\left[\frac{1}{s^2}\right] \\
F(a) &\equiv \frac{3+44a-36a^2-12a^3+a^4+12a(2+3a)\ln(a)}{3(a-1)^6a} \\
C_A\hat{\sigma}C_B\hat{\sigma} &\to \int d^2b_1d^2b_2d^2\beta h(\mathbf{b}_1-\boldsymbol{\beta},\mathbf{b}_2-\boldsymbol{\beta})h(\mathbf{b}_1,\mathbf{b}_2) = \frac{c^2}{4\pi R^2}\frac{2\eta^2}{15s^2} \\
D_A\hat{\sigma}D_B\cdot D_A\hat{\sigma}D_B\cdot D_A\hat{\sigma}D_B &\to \frac{1}{(4\pi)^2}\frac{1}{R^4}\int_0^\infty \frac{(1+x+y)[(1+x+y)^2+6xy]dxdy}{(1+x)^4(1+y)^4[(1+x+y)^2-4xy]^{7/2}} \simeq \frac{1}{(4\pi R^2)^2}\times 0.030 \\
D_A\hat{\sigma}D_B\cdot D_A\hat{\sigma}C_B\hat{\sigma}D_A &\to c\frac{\eta}{(4\pi)^2}\frac{1}{R^4}\int_0^\infty 2\frac{(1+x/4+y)^2-xy/2}{(1+x/2+y)^3[(1+x+2y+(x/2-y)^2]^{3/2}}\frac{s^2y}{(1+s^2y)^3} \\
&\quad\times\frac{dxdy}{(1+x)^4(1+x/2)^2} \equiv \frac{c}{(4\pi R^2)^2}\eta H\left[\frac{1}{s^2}\right] \\
C_A\hat{\sigma}C_B\hat{\sigma}\cdot D_A\hat{\sigma}D_B &\to c^2\eta^2\frac{1}{4\pi^2}\frac{1}{R^4}\frac{1}{s^2}\int_0^\infty\frac{dx}{(1+x)^4(1+x/2)^4}\frac{y^2dy}{(1+y)^6} \simeq \frac{c^2}{(4\pi R^2)^2}\frac{\eta^2}{s^2}\times 0.0256 \\
D_A\hat{\sigma}C_B\hat{\sigma}C_A\hat{\sigma}D_B &\to c^2\eta^2\frac{1}{(4\pi R^2)^2}\int_0^\pi 4\frac{d\phi}{\pi}\int_0^\infty\frac{dx}{(1+x)^2(1+x/2)^2}\frac{dy}{(1+y)^2(1+y/2)^2}\frac{w}{(1+w)^3}\frac{w'}{(1+w')^3} \\
&\equiv \frac{c^2}{(4\pi R^2)^2}\eta^2L\left[\frac{1}{s^2}\right],
\end{aligned} \tag{29}$$

where $s = \lambda/R$, $w = s^2[x+y/4-\sqrt{xy}\cos\phi]$, and $w' = s^2[x/4+y-\sqrt{xy}\cos\phi]$.

The functions F, H, and L defined above, are given in Table I for four different values of the parameter s^2.

The root mean square radius $\bar{R}$ in now given by $\bar{R}^2 = 12R^2$, so a more meaningful reference parameter is $\bar{s} \equiv r_c/\bar{R} = x_0/\sqrt{12}\cdot\lambda/R$. As a function of $\bar{R}$, $\bar{s}$, and of the correlation strength c, the expressions of the effective cross section and of the parameter τ are

TABLE I. The functions F, H, and L in Eqs. (29) for three different values of s^2.

s^2	0.5	1	2	3
$\bar{s}^2$	0.2372	0.4744	0.9488	1.4232
$F[1/s^2]$	0.0140	0.0667	0.2735	0.5834
$H[1/s^2]$	0.0104	0.0120	0.0119	0.0110
$L[1/s^2]$	0.0089	0.0094	0.0078	0.0064

$$\frac{1}{\sigma_{\rm eff}} = \frac{3}{7\pi\bar{R}^2}\left\{1+c\cdot 14\eta F\left(\frac{1}{12}\frac{x_0^2}{\bar{s}^2}\right) + c^2\cdot\frac{14}{15}\eta^2\left(\frac{1}{12}\frac{x_0^2}{\bar{s}^2}\right)\right\} \tag{30}$$

and

$$\frac{1}{\tau\sigma_{\rm eff}^2} \simeq \frac{0,27}{\pi^2\bar{R}^4}\left\{1+c\cdot 400\eta H\left(\frac{1}{12}\frac{x_0^2}{\bar{s}^2}\right) + c^2\cdot\eta^2\left[\frac{x_0^2}{\bar{s}^2}\frac{0,64}{3}+800L\left(\frac{1}{12}\frac{x_0^2}{\bar{s}^2}\right)\right]\right\}. \tag{31}$$

C. Superposition of Poissonians

A particular case, where all n-body correlations are important and which can be worked out explicitly, is when the parton distribution is given by the superposition of different Poissonians. The superposition of Poissonians is naturally obtained when introducing diffraction in a multichannel eikonal model of high energy hadronic interactions [25–27]. To have some indication on this case, we consider the simplest possibility where the probability P_n, to find n partons within a given kinematical range, is given by the sum of two Poissonians with average numbers n_1 and n_2

$$P_n = \left[\gamma \frac{n_1^n}{n!} e^{-n_1} + (1-\gamma)\frac{n_2^n}{n!} e^{-n_2}\right]$$
$$n_{1,2} = \int n_{1,2}(x, b) dx d^2 b, \tag{32}$$
$$n_{1,2}(x, b) = G(x) g(b, R_{1,2}^2),$$

where γ gives the relative weight of the two Poissonians and the integration limits in x are defined by the kinematical range relevant to the case of interest. Notice that, as we want to disentangle the effect of correlations in the transverse coordinates, while $n_1(x, b) \neq n_2(x, b)$, the integrated values n_1 and n_2 are equal. The average density of partons with fractional momentum x and transverse coordinate b is

$$\langle n \rangle = \sum_{n=1}^{\infty} n P_n = [\gamma n_1 + (1-\gamma) n_2] = \int D(x, b) dx d^2 b; \tag{33}$$
$$D(x, b) \equiv G(x)[\gamma g(b, R_1^2) + (1-\gamma) g(b, R_2^2)]$$

while for the average density of pairs of partons with coordinates x_1, b_1 and x_2, b_2 one obtains

$$\langle n(n-1) \rangle = \sum_{n=1}^{\infty} n(n-1) P_n = [\gamma n_1^2 + (1-\gamma) n_2^2] = \int D_2(x_1, b_1; x_2, b_2) dx_1 dx_2 d^2 b_1 d^2 b_2;$$
$$D_2(x_1, b_1; x_2, b_2) \equiv G(x_1)G(x_2)[\gamma g(b_1, R_1^2) g(b_2, R_1^2) + (1-\gamma) g(b_1, R_2^2) g(b_2, R_2^2)] \tag{34}$$

and analogously

$$D_n(x_1, b_1 \ldots x_n, b_n) \equiv G(x_1) \ldots G(x_n)[\gamma g(b_1, R_1^2) \ldots g(b_n, R_1^2) + (1-\gamma) g(b_1, R_2^2) \ldots g(b_n, R_2^2)]. \tag{35}$$

The expression of the inclusive cross section of N independent parton collisions σ_N hence is

$$\sigma_N = \frac{1}{N!} \sigma_S^N \times \left\{\gamma^2 \int [g(\beta, 2R_1^2)]^N d^2\beta + 2\gamma(1-\gamma) \times \int [g(\beta, R_1^2 + R_2^2)]^N d^2\beta + (1-\gamma)^2 \int [g(\beta, 2R_2^2)]^N d^2\beta\right\}. \tag{36}$$

The actual calculations are carried out for the case of the Gaussian parton density, where the effective cross section and the parameter τ of the triple scattering inclusive cross section are given by

$$\frac{1}{\sigma_{\rm eff}} = \frac{3}{4\pi \bar{R}^2}\left\{\gamma^2 \cdot \frac{1}{2\alpha} + 2\gamma(1-\gamma) \cdot \frac{1}{2} + (1-\gamma)^2 \cdot \frac{1}{2(2-\alpha)}\right\} \tag{37}$$

and

$$\frac{1}{\tau \sigma_{\rm eff}^2} = \frac{3}{4\pi \bar{R}^4}\left\{\gamma^2 \cdot \left(\frac{1}{2\alpha}\right)^2 + 2\gamma(1-\gamma) \cdot \left(\frac{1}{2}\right)^2 + (1-\gamma)^2 \cdot \left(\frac{1}{2(2-\alpha)}\right)^2\right\}, \tag{38}$$

where we made the positions

$$R_1^2 = \alpha R^2, \qquad R_2^2 = (2-\alpha) R^2 \quad \text{and} \quad R^2 = \frac{2}{3}\bar{R}^2$$

with $\bar{R}^2$ the mean square hadron radius measured in the generalized parton distributions.

One recognizes that the three different terms in the curly brackets are the contributions to the scale factors due to all possible combinations of the different sizes of the two interacting hadrons ($R_1 - R_1$, $R_1 - R_2 + R_2 - R_1$ and $R_2 - R_2$)[26].

To make contact with the general formalism previously discussed, one may identify the correlation term by the relation

$$D_2(x_1, b_1; x_2, b_2) = [D(x_1, b_1) D(x_2, b_2) + C(x_1, b_1; x_2, b_2)], \tag{39}$$

which gives

$$\begin{aligned} C(x_1, b_1; x_2, b_2) &= G(x_1)G(x_2)\{[\gamma g(b_1, R_1^2) g(b_2, R_1^2) + (1-\gamma) g(b_1, R_2^2) g(b_2, R_2^2)] \\ &\quad - [\gamma g(b_1, R_1^2) + (1-\gamma) g(b_1, R_2^2)] \times [\gamma g(b_2, R_1^2) + (1-\gamma) g(b_2, R_2^2)]\} \\ &= \gamma(1-\gamma) G(x_1) G(x_2) \times \{[g(b_1, R_1^2) - g(b_1, R_2^2)] \times [g(b_2, R_1^2) - g(b_2, R_2^2)]\}. \end{aligned} \tag{40}$$

The correlation strength c is hence expressed as a function of the relative weight of the two Poissonians γ by the relation

$$c = \gamma(1-\gamma), \tag{41}$$

while the correlation length r_c, now defined by the change of sign of the correlation terms for b_1 or $b_2 = r_c$, is given below as a function of the mean square root hadron radius $\bar{R}$ and of the parameter α, which controls the relative value of the two transverse radii R_1 and R_2:

$$r_c = \bar{R}\left[\frac{1}{3}\alpha(2-\alpha)\ln\left(\frac{2-\alpha}{\alpha}\right)\right]^{1/2}. \quad (42)$$

The case discussed in [27] corresponds to $c = \frac{1}{4}$ and $r_c = \frac{\bar{R}}{2}\sqrt{\ln 3} \simeq 0.52\bar{R}$. With $\bar{R}2 = 0.42\ \mathrm{fm}^2$ [28] one obtains $\sigma_{\mathrm{eff}} \simeq 30$ mb, too large to explain the value of σ_{eff} observed by CDF [16,17].

Notice that in the two-channel eikonal model discussed in [25], one obtains a value of σ_{eff} in agreement with the experimental indication. The reason is that, in the model, compact hadronic configurations are characterized by a stronger Pomeron coupling, which corresponds associating a higher partonic population to the compact configurations. In the present case, the distribution in the number of partons is, on the contrary, the same in the two configurations with transverse distances R_1 and R_2 (namely, $n_1 = n_2$, after integrating on b). One may hence conclude that the experiment indicates that the fluctuation of the whole hadron structure in its transverse size, to the extent suggested by diffraction, is not enough to explain the value of σ_{eff}, which may, on the contrary, require the introduction of correlation terms of the kind discussed above in subsecs. III A and III B, or/and of correlations in fractional momenta. In this last instance the multiparton distribution of the hadron structure is different from a Poissonian, also after integrating on the parton's kinematical variables.

IV. CONCLUDING REMARKS

Multiple parton interactions are going to play an important role at the LHC, both in the description of the properties of the minimum bias and of the underlying event and as a background to various channels of interest for the search of new physics. The study of MPI represents moreover the basic handle to obtain information on unknown nonperturbative features of the hadron structure, namely, the correlations between partons.

In the present paper, we have tried to identify the quantities which are most suitable to obtain the information on the nonperturbative features associated to the presence of MPI, in the case where the MPI are dominated by independent collisions initiated by different pairs of partons. To this purpose, in addition to the inclusive cross sections considered until now in hard processes, we made use also of the information provided by the "exclusive" cross sections. Following a previous article where the "exclusive" cross sections were introduced [18], we have hence analyzed MPI considering systematically all terms up to triple scatterings. The inclusive and the exclusive MPI cross sections are linked by the sum rules of Eq. (2) and, by checking the number of terms needed to saturate the sum rules, one has a direct control on the importance of the unitarity corrections. The case where no more that three parton collisions give significant contributions leads to very simple relations between inclusive and "exclusive" cross sections, the relations (18) and (20) which, being a consequence of the definitions of the cross sections, hold rather in general. One may thus obtain the values of the relevant parameters, σ_{eff} for the double parton collisions and the dimensionless parameter τ for the triple (having defined in that case the scale factor as $\tau\sigma_{\mathrm{eff}}^2$).

A convenient way to measure the scale factors may be through Eqs. (21), which make use only of single and double collisions terms. Notice that if the sum rules of Eq. (2) are saturated with three terms in a given phase space window (and hence in a given interval of x values) Eqs. (21) must hold. By measuring σ_S, $\tilde{\sigma}_1$, and $\tilde{\sigma}_2$ as a function of x in the given interval, one may hence obtain reliable information on the dependence of the correlation terms on x.

In our approach, correlations are introduced in the most general way, as deviations of the multiparton distributions from the Poissonian. Rather than trying to propose definite correlation models, our philosophy is hence to identify the observable quantities which are most suitable to obtain information on the correlation terms of the hadron structure. To have an idea of where the correlation parameters (correlation length and correlation strength) may be most relevant, we have considered a few simplest cases. Of course correlations will depend on all variables and, in particular, on fractional momenta, because of conservation laws. Nevertheless, conservation laws may not play a very important role when the parton population is large, namely, at small x. In Sec. III, we have worked out in full detail three simplified instances, where the dependence of correlations on x may be neglected and which allows a full (or almost full) analytic treatment (Gaussian and exponential parton densities and correlations, multiparton distribtuions given by a superposition of Poissonian). The relevant nonperturbative information, namely the quantities σ_{eff} and τ, are given as a function of the correlation parameters in Eqs. (26) and (27), in Eqs. (30) and (31), and in Eqs. (37) and (38) in the three different cases. Notice that the hypothesis of a negligible dependence of correlations on fractional momenta is easily tested experimentally by looking at the dependence of σ_{eff} and τ on x.

In our discussion, we did not allow for the differences between partons (gluons and quarks, different flavors, valence and sea). When considering definite reaction channels, the relations obtained have hence to be adapted, taking into account that the information on correlations will have to be related to the different kinds of initial state partons involved in the interactions.

APPENDIX

When looking at the effects of the correlation terms of the original multiparton distribution in processes where three or more partons undergo hard scattering, a natural question is how to deal with higher order correlation terms. In this Appendix we sketch the procedure to deal with this problem. When taking in consideration three-body correlations the starting point is

$$\mathcal{Z}[J+1] \equiv \exp\left\{\int D(u)J(u)du + \frac{1}{2}\int C(u,v)J(u)J(v)dudv + \frac{1}{6}\int T(u,v,w)J(u)J(v)J(w)dudvdw\right\}$$
$$= \sum_n \frac{1}{n!}\int J(u_1)\ldots J(u_n)D_n(u_1\ldots u_n)du_1\ldots du_n.$$

Although it is not possible any more to obtain a closed form for $\sigma_{\mathrm{hard}}(\beta)$, a general expansion procedure is available and well known [29]: every functional admitting a formal series expansion can be expressed as

$$\Phi[J] = \Phi[\delta/\delta\chi]\exp\int J(u)\chi(u)du\Big|_{\chi=0};$$

so the previous expression may be rewritten in the following form:

$$\mathcal{Z}[J+1] \equiv \exp\left[\frac{1}{6}\int T(u,v,w)\frac{\delta}{\delta\chi(u)}\frac{\delta}{\delta\chi(v)}\frac{\delta}{\delta\chi(w)}dudvdw\right]$$
$$\times \exp\left\{\int [D(u)+\chi(u)]J(u)du + \frac{1}{2}\int C(u,v)J(u)J(v)dudv\right\}\Big|_{\chi=0}.$$

The second exponential will give rise to terms similar to the ones in Eqs. (6) and (7), the only difference is in the terms a_n, which contain $D+\chi$ instead of D. The whole expression is suitable for an expansion in T. When acting with the derivatives one finds that every term T is connected either with three terms $M(u,v)$, defined by the relation below, ending in turn on a density D or with two of them, one of which ends on one D and the other is closed on the same term T.

$$a_n = (-1)^{n+1}\int D_A(u_1)\hat{\sigma}(u_1,u_1')C_B(u_1',u_2')\hat{\sigma}(u_2',u_2)C_A(u_2,u_3)\ldots\cdots\hat{\sigma}(u_n,u_n')D_B(u_n')\prod_{i=1}^n du_i du_i'$$
$$\equiv (-1)^{n+1}\int D_A(u_1)\hat{\sigma}(u_1,u_1')M_n(u_1',u_n')\hat{\sigma}(u_n,u_n')D_B(u_n')\prod_{i=1}^n du_i du_i'.$$

In particular, the first two terms are

$$\int T(u,v,w)\hat{\sigma}(u,u')D(u')\hat{\sigma}(v,v')D(v')\hat{\sigma}(w,w')D(w')dudvdwdu'dv'dw' \quad \text{and}$$
$$\int T(u,v,w)\hat{\sigma}(u,u')\hat{\sigma}(v,v')C(u',v')\hat{\sigma}(w,w')D(w')dudvdwdu'dv'dw';$$

and one expects the first to be the most important.

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