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# Hadronic quarkonium decays at order $v^7$

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We compute the complete imaginary part of the nonrelativistic QCD (NRQCD) Lagrangian at order $1/M^4$ in the heavy-quark mass expansion, which includes center of mass operators, and at order $\alpha_s^2$ in the matching coefficients. We also compute the imaginary part of the NRQCD Lagrangian at order $1/M^6$ and at order $\alpha_s^2$ that contributes to the S-wave and P-wave inclusive decay widths of heavy quarkonium into light hadrons at order $v^7$ in the heavy-quark velocity expansion. If we count $\alpha_s(M) \sim v^2$, the calculation provides the complete next-to-leading order corrections to the P-wave hadronic widths, and in the original NRQCD power counting, the complete next-to-leading order corrections to the vector S-wave widths, and part of the next-to-next-to leading order corrections to the pseudoscalar S-wave widths. In the S-wave case, we confirm previous findings and add new terms in a more conservative power counting. In the P-wave case, our results are in disagreement with previous ones. Constraints induced by Poincaré invariance on the NRQCD four-fermion sector are studied for the first time and provide an additional check of the calculation. Perspectives for phenomenological applications are discussed.

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## I. INTRODUCTION

Nonrelativistic effective field theories (NR EFT) of QCD [1,2] like nonrelativistic QCD (NRQCD) [3,4] offer a systematic framework to access heavy-quarkonium properties and, in particular, inclusive decay widths. Decay width formulas may be organized in a double expansion in the strong coupling constant $\alpha_s$, calculated at a large scale of the order of the heavy-quark mass $M$, and in the heavy-quark velocity $v$. Both expansion parameters are relatively small. In the bottomonium system, typical reference values are $\alpha_s(M_b) \approx 0.2$, $v_b^2 \approx 0.1$ and in the charmonium one, $\alpha_s(M_c) \approx 0.35$, $v_c^2 \approx 0.3$.

The increasing accuracy of the experimental measurements [1,5–7] calls for a corresponding accuracy in the theoretical predictions. The inclusive decay widths of $J/\psi$, $\psi(2S)$ and $\Upsilon(1S)$ into light hadrons are presently known within a few percent uncertainty, while the uncertainties in the inclusive decay widths of $\Upsilon(2S)$ and $\Upsilon(3S)$ are less than 10% [5]. Theoretical accuracies of about 5% both in the charmonium and in the bottomonium case require at least the calculation of $O(v^4, \alpha_s v^2, \alpha_s^2)$ corrections. The S-wave decay of $\eta_c$ into light hadrons is presently known within a 15% uncertainty, while the P-wave decays of $\chi_{cJ}$, with $J = 0, 1, 2$, are known within a 10% uncertainty [5]. In the P-wave case, the improvement of the experimental accuracy has been noticeable over the last few years and the data are now clearly sensitive to next-to-leading order (NLO) corrections [1,8]. Hence, for the decay of the charmonium P-wave states, theoretical accuracies matching the experimental ones require the calculation of $O(v^2, \alpha_s)$ corrections.

In this work, we consider relativistic corrections of order $v^2$ and $v^4$ to inclusive decays of P- and S-wave quarkonium into light hadrons, respectively. The leading order S-wave decay width is proportional to the square of the wave function in the origin and is therefore of order $v^3$. The leading order P-wave decay width is proportional to the square of the derivative of the wave function in the origin, and is therefore of order $v^5$. Then, corrections of order $v^4$ to S-wave decays and of order $v^2$ to P-wave decays provide in both cases decay widths at order $v^7$ in the relativistic expansion. We consider only processes where the quark and antiquark annihilate into two gluons. Hence, more precisely, the paper provides the $\alpha_s^2 v^7$ terms of the S-wave and P-wave inclusive decay widths.

In the S-wave case, corrections of order $v^2$ and $v^4$ were first considered in [4,9], respectively. We agree with their results if we use their power counting, but find additional contributions in the more conservative counting that we adopt. In the P-wave case, corrections of order $v^2$ were first calculated in [10]. Our results disagree with those results. In particular, we find different matching coefficients for the dimension 10 operators. Moreover, also adopting the power counting of [10], our decay widths appear to contain two matrix elements more.

The paper is organized in the following way. In Sec. II, we set up the formalism, discuss the power counting, introduce our basis of operators and give the general form of the decay widths at order $v^7$. In Sec. III, we

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calculate the short-distance imaginary parts of the NRQCD four-fermion operators by matching annihilation diagrams of order $\alpha_s^2$. Octet operators are calculated by matching diagrams with an external gluon. In Sec. IV, we show how Poincaré invariance is realized in the EFT in the form of exact relations among matching coefficients. Such relations provide an additional and independent check of some of the results. In Sec. V, we conclude by summarizing the present knowledge about inclusive decays and discussing phenomenological applications and future developments of this work. In Appendixes A and B, we list all the operators and the matching coefficients that have been employed through the paper.

## II. HADRONIC DECAY WIDTHS IN NRQCD

### A. NRQCD

The main mechanism for quarkonium to decay into light hadrons is quark-antiquark annihilation. It takes place at a scale which is twice the heavy-quark mass $M$. Since this scale is perturbative, quark-antiquark annihilation may be described within an expansion in the strong coupling constant $\alpha_s$. Experimentally, this is shown by the narrow widths of quarkonia below the open flavor threshold. The bound state dynamics, instead, is characterized by physical scales smaller than $M$, such that a perturbative expansion in $\alpha_s$ may not be allowed. It is however possible to take advantage of the nonrelativistic nature of the bound state and expand in the relative heavy-quark velocity $v$. In an EFT language, once the scale $M$ has been integrated out, the information on the decays is carried by contact terms (four-fermion operators) whose matching coefficients develop an imaginary part [4]. In NRQCD, the decay widths factorize in a high-energy contribution, encoded in the imaginary part of the four-fermion matching coefficients, and a low-energy contribution, encoded in the matrix elements of the four-fermion operators evaluated on the heavy-quarkonium states. The NRQCD factorization formula for the inclusive decay width of a quarkonium state $H$ into light hadrons (l.h.) is [4]

$$\Gamma(H \to \text{l.h.}) = 2\sum_n \frac{\text{Im}c^{(n)}}{M^{d_n-4}} \langle H|\mathcal{O}_{4\text{-f}}^{(n)}|H\rangle. \tag{1}$$

$|H\rangle$ is a mass dimension $-3/2$ normalized eigenstate of the NRQCD Hamiltonian with the quantum numbers of the quarkonium state $H$. The coefficients $c^{(n)}$ can be calculated in perturbation theory by matching Green functions or physical amplitudes in QCD and NRQCD. $\mathcal{O}_{4\text{-f}}^{(n)}$ stands for a generic four-fermion operator of dimension $d_n$, whose general form is $\psi^\dagger(\cdots)\chi\chi^\dagger(\cdots)\psi$, $\psi$ being the Pauli spinor that annihilates a quark and $\chi$ the one that creates an antiquark. The operators $(\cdots)$ may transform as singlets or octets under color SU(3) gauge transformations. In the first case, we denote the operator with the subscript 1, in the second with the subscript 8. A list of relevant four-fermion operators is provided in Appendix A.

It is the purpose of this work to calculate the order $\alpha_s^2$ contributions to the $c^{(n)}$ coefficients that multiply matrix elements up to order $v^7$. These involve operators up to dimension 10.

### B. Power counting

In the factorization formula (1), the matching coefficients $c^{(n)}$ are series in $\alpha_s$ while the matrix elements $\langle H|\mathcal{O}_{4\text{-f}}^{(n)}|H\rangle$ are series in $v$ and are, in general, nonperturbative objects. In NRQCD, several power countings are possible because of the several contributing energy scales. These are the relative momentum $Mv$, the binding energy $Mv^2$ and the typical hadronic scale $\Lambda_{\text{QCD}}$; additional scales may enter at higher orders in the calculation [11]. Whatever power counting one assumes, as long as $v \ll 1$, matrix elements of operators of higher dimensionality are suppressed by powers of $v$.

The NRQCD Lagrangian is constructed as an expansion in $1/M$ and hence it is independent of the power counting. We shall adopt a power counting, however, when assessing the size of the different matrix elements contributing to the decay widths. We will assume $Mv$ of the same order as $\Lambda_{\text{QCD}}$ and adopt the following rules. Matrix elements of the type $\langle H'|\mathcal{O}|H\rangle$, where $\mathcal{O}|H\rangle$ and $|H'\rangle$ have the same quantum numbers and color transformation properties in the dominant Fock state, scale (at leading order) like $(Mv)^{d-3}$, $d$ being the dimension of the operator $\mathcal{O}$. If $\mathcal{O}|H\rangle$ and $|H'\rangle$ do not have the dominant Fock state with the same quantum numbers, then the matrix element singles out a component of the quarkonium Fock state that is suppressed. The amount of suppression depends on the power counting and on the quantum numbers. As detailed in [12], the power counting we adopt implies that the octet components with quantum numbers $S$ and $L \pm 1$, $S$ and $L$, $S \pm 1$ and $L$ of a quarkonium state are suppressed by $v$ with respect to the singlet component with quantum numbers $S$ and $L$, while the components with $S$, $L \pm 2$ or $S \pm 1$, $L \pm 1$ are suppressed by $v^2$.

A different counting, which seems suitable for the situation $Mv^2 \sim \Lambda_{\text{QCD}}$ has been defined in [4] and used in [9,10]. Our power counting is more conservative than the one in [4], because we assume that all operators scale with the largest available scale, i.e. $Mv \sim \Lambda_{\text{QCD}}$, while in [4] this is not always the case and some operators have extra suppressions. As a consequence, one may recover the expressions in the power counting of [4] from our expressions simply by eliminating matrix elements that, in that counting, would be smaller than $v^7$: no new matrix element or matching coefficient needs to be added.

For a critical review and a discussion on the different power countings we refer to [2] and references therein.

### C. Four-fermion operators

The four-fermion sector of the NRQCD Lagrangian contains all four-fermion operators invariant under gauge transformations, rotations, translations, charge conjugation, parity and time inversion. They may be classified according to their dimensionality and color content. The analysis of the four-fermion operators involved in the hadronic decay widths at order $v^7$ closely parallels the one performed for electromagnetic decays in [12]. In the following, we focus on the main differences, that are mostly related to the contributions of color octet operators to the hadronic decay widths. The presence of color octet operators, acting on subleading components of the heavy-quarkonium Fock state, is one important and well-known characteristics of NRQCD [4].

We organize the four-fermion sector of the NRQCD Lagrangian according to the mass dimension and the color structures of the operators. In Sec. II C 2, we show how the number of (redundant) color singlet and octet operators may be reduced by using suitable field redefinitions. In Sec. II C 3, we introduce operators proportional to the total momentum of the heavy-quark–antiquark pair: at variance with the electromagnetic case, such operators contribute to the decay widths at order $v^7$. In Appendix A, we give some details on the construction of octet operators of higher dimension and the explicit list of all four-fermion operators that need to be considered at the order of accuracy we are working. Finally, in Sec. II C 4, we use the NRQCD power counting of Sec. II B to assess the importance of the different matrix elements, and in Sec. II D, we write the general form of the hadronic decay widths accurately up to order $v^7$.

#### 1. Operators from dimension 6 to dimension 10

For dimensional reasons, four-fermion operators of mass dimension 6 can only contain four-fermion fields, without any covariant derivative or gluon field. The only allowed color structures are $\mathbf{1}_c \otimes \mathbf{1}_c$ and $t^a \otimes t^a$. The color octet operator

$$\psi^\dagger t^a \chi \chi^\dagger t^a \psi \tag{2}$$

has non vanishing matrix element between the states $\langle (Q\bar{Q})_8 g| \ldots |(Q\bar{Q})_8 g\rangle$, which are subleading components of the heavy-quarkonium Fock state. Color octet matrix elements are particularly relevant for P-wave decays, where they contribute at leading order in the power counting.

Parity conservation forbids four-fermion operators of mass dimension 7. Four-fermion operators of dimension 8 can be built with two covariant derivatives or with a chromomagnetic field. For operators built with two derivatives, the possible color structures are $\mathbf{1}_c \otimes \mathbf{1}_c$ and $t^a \otimes t^a$. The construction of color singlet operators is straightforward, while some care has to be taken in the color octet case, because of the non-Abelian nature of the gauge group; see Appendix A. The covariant derivatives involved can be proportional either to the relative momentum of the quark and antiquark pair, for example, in an operator like

$$\psi^\dagger \overleftrightarrow{D} \chi \cdot \chi^\dagger \overleftrightarrow{D} \psi, \tag{3}$$

or to the total momentum of the pair, like in

$$\nabla(\psi^\dagger \chi) \cdot \nabla(\chi^\dagger \psi).$$

Also, operators containing both kind of derivatives can be built, like

$$\psi^\dagger \left(-\frac{i}{2}\overleftrightarrow{D}\right) \times \vec{\sigma} \chi \cdot \vec{\nabla}(\chi^\dagger \psi) + \text{H.c.}.$$

Operators containing the chromomagnetic field can appear with the different color structures $t^a \otimes \mathbf{1}_c$, $\mathbf{1}_c \otimes t^a$, $f^{abc} t^a \otimes t^b$ and $d^{abc} t^a \otimes t^b$:

$$\begin{aligned}
&\psi^\dagger g\vec{B} \cdot \vec{\sigma} \chi \chi^\dagger \psi + \text{H.c.}, \\
&\psi^\dagger g\vec{B}^a \cdot \vec{\sigma} \chi \chi^\dagger t^a \psi + \text{H.c.}, \\
&f^{abc} \psi^\dagger g\vec{B}^a \cdot \vec{\sigma} t^b \chi \chi^\dagger t^c \psi + \text{H.c.}, \\
&d^{abc} \psi^\dagger g\vec{B}^a \cdot \vec{\sigma} t^b \chi \chi^\dagger t^c \psi + \text{H.c.}.
\end{aligned} \tag{4}$$

Operators of dimension 9 can involve a covariant derivative and a chromoelectric field,

$$\psi^\dagger \chi \chi^\dagger (\overleftrightarrow{D} \cdot g\vec{E} + g\vec{E} \cdot \overleftrightarrow{D}) \psi + \text{H.c.}, \tag{5}$$

and again we have to consider all the possible color structures, as in Eq. (4). Finally, dimension 10 operators may involve four covariant derivatives or two covariant derivatives and a chromomagnetic field or two gluon fields. To clarify our terminology, we call "singlet operators" the ones in which both the ingoing and the outgoing $Q\bar{Q}$ pairs are singlets, as in (3). Although, any covariant derivative also contains an octet part, "octet operators" the ones in which both the ingoing and the outgoing $Q\bar{Q}$ pairs are octets, as in (2) or in the third and fourth lines of Eq. (4), and "singlet-octet transition operators" the ones in which one of the two pairs is an octet and the other is a singlet, as the first two operators of Eq. (4) or the one in Eq. (5). For details on the four-fermion operator definition and construction see Appendix A.

#### 2. Field redefinitions

The four-fermion basis built with all possible operators allowed by rotational and translational invariance, gauge invariance and invariance under the discrete symmetries of QCD is redundant since the number of four-fermion operators may be reduced by suitable field redefinitions. The analysis performed in [12] can be extended to hadronic singlet operators. Through the field redefinitions

$$\psi \to \psi + \frac{a}{M^5}\left[\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2, \chi\chi^\dagger\right]\psi,$$
$$\chi \to \chi - \frac{a}{M^5}\left[\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2, \psi\psi^\dagger\right]\chi \tag{6}$$

it is possible, for a suitable choice of the free parameter $a$, to trade the operator $\mathcal{T}_{1\text{–}8}(^1S_0, {}^1P_1)$, defined in Eq. (A25), for the linear combination of $\mathcal{Q}_1'(^1S_0) - \mathcal{Q}_1''(^1S_0)$, defined in Eq. (A26), while, through

$$\psi \underset{J}{\to} \psi + \frac{a}{M^5}\mathbf{T}^{(J)}_{ijlk}\sigma^l\left[\left(-\frac{i}{2}\overleftrightarrow{D}^i\right)\left(-\frac{i}{2}\overleftrightarrow{D}^j\right), \chi\chi^\dagger\right]\sigma^k\psi,$$
$$\chi \underset{J}{\to} \chi - \frac{a}{M^5}\mathbf{T}^{(J)}_{ijlk}\sigma^l\left[\left(-\frac{i}{2}\overleftrightarrow{D}^j\right)\left(-\frac{i}{2}\overleftrightarrow{D}^i\right), \psi\psi^\dagger\right]\sigma^k\chi, \tag{7}$$

where

$$\mathbf{T}^{(0)}_{ijlk} = \frac{\delta^{ij}\delta^{lk}}{3}, \tag{8}$$

$$\mathbf{T}^{(1)}_{ijlk} = \frac{\epsilon_{ijn}\epsilon_{kln}}{2}, \tag{9}$$

$$\mathbf{T}^{(2)}_{ijlk} = \frac{\delta^{il}\delta^{jk} + \delta^{jl}\delta^{ik}}{2} - \frac{\delta^{ij}\delta^{lk}}{3}, \tag{10}$$

the operators $\mathcal{T}^{(i)}_{1\text{–}8}(^3S_1, {}^3P)$, with $i = 0, 1, 2$, can be eliminated by a suitable choice of $a$ and by redefining the matching coefficients of $\mathcal{Q}_1'(^3S_1)$, $\mathcal{Q}_1''(^3S_1)$, $\mathcal{Q}_1'(^3S_1, {}^3D_1)$ and $\mathcal{Q}_1''(^3S_1, {}^3D_1)$ (see Eqs. (A25) and (A26) for the definition of these operators). As it was noted in [12], these field redefinitions do not change the sums of the coefficients $h_1'(^1S_0) + h_1''(^1S_0)$, $h_1'(^3S_1) + h_1''(^3S_1)$ and $h_1'(^3S_1, {}^3D_1) + h_1''(^3S_1, {}^3D_1)$.

It is also possible to exploit field redefinitions to reduce the number of octet operators. Consider the field redefinitions

$$\psi \to \psi + \frac{a}{M^5}\left[\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2, t^a\chi\chi^\dagger\right]t^a\psi,$$
$$\chi \to \chi - \frac{a}{M^5}\left[\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2, t^a\psi\psi^\dagger\right]t^a\chi, \tag{11}$$

where the definition of $\psi^\dagger\overleftrightarrow{D}^2 t^a\chi$ is given in Eq. (A13). Equation (11) induces the following transformation (8-8)

$$\begin{aligned}\psi^\dagger iD_0\psi + \chi^\dagger iD_0\chi \to{}& \psi^\dagger iD_0\psi + \chi^\dagger iD_0\chi \\ &- \frac{a}{M^5}\frac{1}{N_c}\mathcal{T}_{1\text{–}8}(^1P_1, {}^1S_0) \\ &- \frac{a}{2M^5}\mathcal{D}_{8\text{-}8}(^1S_0, {}^1P_1) \\ &+ \frac{a}{2M^5}\mathcal{F}_8(^1S_0),\end{aligned} \tag{12}$$

where $N_c = 3$ is the number of colors and the operators $\mathcal{D}_{8\text{-}8}(^1S_0, {}^1P_1)$ and $\mathcal{F}_8(^1S_0)$ are defined in Eq. (A25). The same field redefinitions induce the following transformation on the kinetic term

$$\psi^\dagger\frac{\vec{D}^2}{2M}\psi - \chi^\dagger\frac{\vec{D}^2}{2M}\chi \to \psi^\dagger\frac{\vec{D}^2}{2M}\psi - \chi^\dagger\frac{\vec{D}^2}{2M}\chi + \frac{2a}{M^6} \times (\mathcal{Q}_8'(^1S_0) - \mathcal{Q}_8''(^1S_0)), \tag{13}$$

where in the right-hand side we have neglected operators proportional to the center of mass momentum of the quark-antiquark pair. Eqs. (12) and (13) show that the operators $\mathcal{T}_{1\text{–}8}(^1P_1, {}^1S_0)$ and $\mathcal{Q}_8'(^1S_0) - \mathcal{Q}_8''(^1S_0)$ are not independent and that it is possible, for a suitable choice of the parameter $a$, to trade the one for a redefinition of the matching coefficient of the other and of $\mathcal{D}_{8\text{-}8}(^1S_0, {}^1P_1)$ and $\mathcal{F}_8(^1S_0)$.

With a closely related argument, introducing the field redefinitions

$$\psi \underset{J}{\to} \psi + \frac{a}{M^5}\mathbf{T}^{(J)}_{ijlk}\sigma^l\left[\left(-\frac{1}{4}\overleftrightarrow{D}^i\overleftrightarrow{D}^j\right), t^a\chi\chi^\dagger\right]t^a\sigma^k\psi,$$
$$\chi \underset{J}{\to} \chi - \frac{a}{M^5}\mathbf{T}^{(J)}_{ijlk}\sigma^l\left[\left(-\frac{1}{4}\overleftrightarrow{D}^j\overleftrightarrow{D}^i\right), t^a\psi\psi^\dagger\right]t^a\sigma^k\chi, \tag{14}$$

with $\mathbf{T}^{(J)}_{ijlk}$ given in Eqs. (8)–(10) and $\overleftrightarrow{D}^i\overleftrightarrow{D}^j t^a$ defined according to Eq. (A13), it is possible to set the parameter $a$ in such a way that the minimal basis of operators either contains the three operators $\mathcal{T}_{1\text{–}8}(^3P_J, {}^3S_1)$ or, with a different choice of $a$, the three operators $1/2(\mathcal{Q}_8'(^3S_1) - \mathcal{Q}_8''(^3S_1))$, $1/2(\mathcal{Q}_8'(^3S_1, {}^3D_1) - \mathcal{Q}_8''(^3S_1, {}^3D_1))$ and $\mathcal{T}^{(1)\prime}_{8-1}(^3S_1, {}^3P)$ defined in Eqs. (A25) and (A26). The first set of operators is more useful in dealing with P-wave decay widths and we will use it in the rest of the paper.

Note that the operator $\mathcal{T}^{(1)\prime}_{8-1}(^3S_1, {}^3P)$, as well as the operators $\mathcal{T}^{(i)}_{1\text{–}8}(^3S_1, {}^3P)$ previously introduced and $\mathcal{T}^{(1)\prime}_{1\text{–}8}(^3S_1, {}^3P)$, which is required by the matching, annihilate (create) a singlet $Q\bar{Q}$ pair with orbital angular momentum $L = 1$ but with no definite value of $J$. So, in our notation, we denote the annihilated pair just with its spin and orbital angular momentum quantum numbers, omitting the subscript $J$.

### *3. Operators proportional to the total momentum of the quark-antiquark pair*

The description of the hadronic decay widths up to order $v^7$ requires the inclusion of operators proportional to the total momentum of the quark-antiquark pair into the meson. By parity conservation these operators must contain at least two derivatives, so they have at least mass dimension 8. The two derivatives can act on the $Q\bar{Q}$ pair, like in

$$\mathcal{P}_{1a\,\mathrm{cm}} = \vec{\nabla}^i(\psi^\dagger\sigma^j\chi)\vec{\nabla}^i(\chi^\dagger\sigma^j\psi). \tag{15}$$

Since the $Q\bar{Q}$ pair is a color singlet, $\vec{\nabla}$ is an ordinary derivative. If the $Q\bar{Q}$ pair is a color octet, we can build an operator analogous to (15)

$$\mathcal{P}_{8a\,\mathrm{cm}} = \vec{D}^i_{ab}(\psi^\dagger t^b \sigma^j \chi)\vec{D}^i_{ac}(\chi^\dagger t^c \sigma^j \psi), \tag{16}$$

where $\vec{D}_{ab}$ is a covariant derivative in the adjoint representation.

Also operators containing a total derivative $\vec{\nabla}$ and a derivative $\overleftrightarrow{D}$, proportional to the relative momentum of the pair, can be built. In this case, since under charge conjugation $\vec{\nabla}(\psi^\dagger \chi) \to \vec{\nabla}(\psi^\dagger \chi)$ and $\psi^\dagger \overleftrightarrow{D}\chi \to -\psi^\dagger \overleftrightarrow{D}\chi$, the operators must contain a Pauli matrix in order to be charge conjugation invariant. An example is the operator

$$\mathcal{O}_{1\,\mathrm{cm}} = \psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\right) \times \vec{\sigma}\chi \cdot \vec{\nabla}(\chi^\dagger \psi) + \mathrm{H.c.}. \tag{17}$$

As explained in Sec. IV, the matching coefficients of the operators of mass dimension 8 proportional to the total momentum of the $Q\bar{Q}$ pair are completely determined by the coefficients of the dimension 6 operators. These relations are a manifestation of the Poincaré invariance of the effective field theory.

#### *4. Power counting of the four-fermion operators*

From the rules given in Sec. II B, it follows that

$$\langle H(^{2S+1}L_J)| \frac{1}{M^{d-4}} \mathcal{O}_1(^{2S+1}L_J)|H(^{2S+1}L_J)\rangle \sim Mv^{d-3}, \tag{18}$$

where $|H(^{2S+1}L_J)\rangle$ stands for a quarkonium state whose dominant Fock-space component is a $Q\bar{Q}$ pair with quantum numbers $S$, $L$ and $J$; $\mathcal{O}_1(^{2S+1}L_J)$ is a singlet four-fermion operator that acts on the $Q\bar{Q}$ pair with spin $S$, orbital angular momentum $L$ and total angular momentum $J$; and $d$ is the dimension of the operator.

The scaling of color octet matrix elements is affected by the suppression of the Fock state component they act on. For example, the power counting given in Sec. II B implies

$$\begin{aligned} &\langle H(^3P_0)| \frac{1}{M^2} \mathcal{O}_8(^3S_1)|H(^3P_0)\rangle \sim Mv^5, \\ &\langle H(^1S_0)| \frac{1}{M^2} \mathcal{O}_8(^3S_1)|H(^1S_0)\rangle \sim Mv^5. \end{aligned} \tag{19}$$

In the power counting that we adopt, the gluon field and the derivative that belong to a covariant derivative have the same scaling. If the gluon field selects a component of the quarkonium Fock state, which is suppressed, like in $\langle H(^3P_0)|\mathcal{O}_1(^3P_0)|H(^3P_0)\rangle$, then its contribution to the matrix element is subleading. If, however, the gluon field selects a component whose projection on the operator is not suppressed or the gluon is reabsorbed by other gluons in the operator, then it may happen that the gluon part in the covariant derivative gives to the matrix element a contribution that is larger than the one provided by the derivative part. For example, due to the gluons in the covariant derivatives, dimension 10 octet operators like $\mathcal{P}_8(^1P_1)$, $\mathcal{Q}'_8(^1S_0)$ and $\mathcal{Q}_8(^1D_2)$, as well as the singlet operator $\mathcal{Q}_1(^1D_2)$, contribute to the decay width of the quarkonium state $H(^1S_0)$ at order $v^7$. Similar operators contribute at order $v^7$ also to the decay width of the quarkonium states $H(^3S_1)$ and $H(^3P_J)$.

Concerning the scaling of the singlet-octet matrix elements, in the power counting of Sec. II B both the chromoelectric and chromomagnetic fields scale as their mass dimension, $(Mv)^2$, so the scaling of a matrix element is $Mv^{d-3}v^s$, where $v^s$ takes into account the suppression of the Fock state the operator acts on. For example, consider the matrix elements of the dimension 8 operators defined in Eq. (A24):

$$\langle H(^1S_0)| \frac{1}{M^4} \mathcal{S}_{1-8}(^1S_0, {}^3S_1)|H(^1S_0)\rangle, \tag{20}$$

and

$$\langle H(^3S_1)| \frac{1}{M^4} \mathcal{S}_{1-8}(^3S_1, {}^1S_0)|H(^3S_1)\rangle. \tag{21}$$

The operator $\mathcal{S}_{1-8}(^1S_0, {}^3S_1)$ destroys a singlet $Q\bar{Q}$ pair with quantum numbers $^1S_0$ and creates an octet $Q\bar{Q}$ pair with quantum numbers $^3S_1$ and a gluon (and vice versa), the operator $\mathcal{S}_{1-8}(^3S_1, {}^1S_0)$ destroys a singlet $Q\bar{Q}$ pair with quantum numbers $^3S_1$ and creates an octet $Q\bar{Q}$ pair with quantum numbers $^1S_0$ and a gluon (and vice versa). Hence, both matrix elements scale like $Mv^6$.

Equations (A25) define octet operators of dimension 9, and since the octet Fock-space component is suppressed by $v$, we have

$$\langle H(^3S_1)| \frac{1}{M^5} \mathcal{T}^{(1)\prime}_{1-8}(^3S_1, {}^3P)|H(^3S_1)\rangle \sim Mv^7, \tag{22}$$

and

$$\langle H(^3P_J)| \frac{1}{M^5} \mathcal{T}_{1-8}(^3P_J, {}^3S_1)|H(^3P_J)\rangle \sim Mv^7. \tag{23}$$

For the reasons discussed above, in our power counting, matrix elements of octet operators of dimension 9, like $\mathcal{D}_{8\text{-}8}(^1S_0, {}^1P_1)$, are not necessarily negligible at order $v^7$ because of the gluons in the covariant derivatives, which may couple to other gluons in the operator and in the quarkonium Fock state. For instance, we have

$$\langle H(^1S_0)| \frac{1}{M^5} \mathcal{D}_{8\text{-}8}(^1S_0, {}^1P_1)|H(^1S_0)\rangle \sim Mv^7. \tag{24}$$

Matrix elements of the operator $\mathcal{F}_8(^1S_0)$ are smaller than $v^7$ because of the suppression induced by the Gauss law. Note that also the matrix element of the following dimension 10 operator is negligible at order $v^7$:

$$\langle H(^3P_0)| \frac{1}{M^6} \psi^\dagger \vec{B} \cdot \overleftrightarrow{D}\chi\chi^\dagger \overleftrightarrow{D} \cdot \vec{\sigma}\psi|H(^3P_0)\rangle \sim Mv^8. \tag{25}$$

Finally, we discuss the scaling of matrix elements of operators proportional to the total momentum of the $Q\bar{Q}$ pair. We work in a frame in which the heavy quarkonium is at rest. In this frame, operators proportional to the total momentum of the pair have nonvanishing matrix elements only between subleading components of the heavy-quarkonium Fock state, containing at least one gluon. Lattice data indicate that higher gluonic excitations between the $Q\bar{Q}$ pair are separated from the lowest quarkonium state by a mass gap of order $\Lambda_{\rm QCD}$ (for a detailed discussion, see [2] and references therein). Therefore, gluons in subleading components of the Fock space must be counted as soft $(q^0, \vec{q}) \sim (Mv, Mv)$, where $Mv \sim \Lambda_{\rm QCD}$. The emission of a soft gluon leaves the $Q\bar{Q}$ pair with a total momentum of order $Mv$, hence, the scaling of the operators $\vec{\nabla}$ and $\vec{D}_{ab}$ acting on the $Q\bar{Q}$ pair is $\sim Mv$. Consider, for example, the matrix element of the operator $\mathcal{O}_{8\,\rm cm}$ between $^3S_1$ states

$$\langle H(^3S_1)|\mathcal{O}_{8\,\rm cm}|H(^3S_1)\rangle = \langle ^3S_1|\psi^\dagger t^a\left(-\frac{i}{2}\overleftrightarrow{D}\right)\times\vec{\sigma}\chi \cdot \vec{D}_{ab}(\chi^\dagger t^b\psi) + {\rm H.c.}|(^1S_0)_8 g\rangle + \ldots . \tag{26}$$

The leading order contribution to the left-hand side of Eq. (26) comes from the matrix element between the components $|^3S_1\rangle$ and $|(^1S_0)_8 g\rangle$ of $|H(^3S_1)\rangle$, the gluon in the incoming state being annihilated by the gluon field in $\overleftrightarrow{D}$. The matrix element in Eq. (26) gets a $v$ suppression from each derivative, and a further $v$ suppression from the $|(^1S_0)_8 g\rangle$ state. Therefore it scales like $v^6$ and is suppressed by $v^3$ with respect to the leading contribution to the decay width. The operator $\mathcal{P}_{8a\,\rm cm}$ has a nonvanishing matrix element if both the incoming and outgoing states contain a gluon. For example, it contributes to the decay width of $H(^3P_0)$:

$$\langle H(^3P_0)|\mathcal{P}_{8a\,\rm cm}|H(^3P_0)\rangle = \langle (^3S_1)_8 g|\vec{D}^i_{ab}(\psi^\dagger t^b\sigma^j\chi)\vec{D}^i_{ac} \cdot (\chi^\dagger t^c\sigma^j\psi) + {\rm H.c.}|(^3S_1)_8 g\rangle + \ldots . \tag{27}$$

The matrix element in Eq. (27) gets two powers of $v$ from the derivatives and two from the states, so it scales like $v^7$, and contributes to the P-wave decay width at the order we are interested in.

We note that for electromagnetic decays, operators proportional to the total momentum of the $Q\bar{Q}$ pair do not contribute to decay widths calculated in the quarkonium center of mass rest frame. The reason is the following: Electromagnetic operators are obtained by inserting the vacuum projector $|0\rangle\langle 0|$ in hadronic operators. As a consequence, any matrix element involving derivatives acting on both the quark-antiquark fields may be reduced by integration by parts either to a matrix element that does not involve an operator with derivatives acting on the quark-antiquark fields or to a global derivative of a matrix element of the type $\langle 0|(\ldots)|H\rangle$. The first one is a standard matrix element that does not involve the center of mass momentum, the last one vanishes in the quarkonium center of mass rest frame.

### D. Hadronic decay widths

Having assumed a power counting and having chosen a basis of operators, we are in the position to provide explicit factorization formulas for S-wave and P-wave inclusive decays. The S-wave decay widths at order $v^7$ are

$$\begin{aligned}
\Gamma(^1S_0 \to {\rm l.h.}) = {}& \frac{2\,{\rm Im}f_1(^1S_0)}{M^2}\langle H(^1S_0)|\mathcal{O}_1(^1S_0)|H(^1S_0)\rangle + \frac{2\,{\rm Im}g_1(^1S_0)}{M^4}\langle H(^1S_0)|\mathcal{P}_1(^1S_0)|H(^1S_0)\rangle \\
&+ \frac{2\,{\rm Im}f_8(^3S_1)}{M^2}\langle H(^1S_0)|\mathcal{O}_8(^3S_1)|H(^1S_0)\rangle + \frac{2\,{\rm Im}f_8(^1S_0)}{M^2}\langle H(^1S_0)|\mathcal{O}_8(^1S_0)|H(^1S_0)\rangle \\
&+ \frac{2\,{\rm Im}f_8(^1P_1)}{M^4}\langle H(^1S_0)|\mathcal{O}_8(^1P_1)|H(^1S_0)\rangle + \frac{2\,{\rm Im}s_{1-8}(^1S_0,{}^3S_1)}{M^4}\langle H(^1S_0)|\mathcal{S}_{1-8}(^1S_0,{}^3S_1)|H(^1S_0)\rangle \\
&+ \frac{2\,{\rm Im}f'_{8\,\rm cm}}{M^4}\langle H(^1S_0)|\mathcal{O}'_{8\,\rm cm}|H(^1S_0)\rangle + \frac{2\,{\rm Im}g_{8a\,\rm cm}}{M^4}\langle H(^1S_0)|\mathcal{P}_{8a\,\rm cm}|H(^1S_0)\rangle \\
&+ \frac{2\,{\rm Im}f_{1\,\rm cm}}{M^4}\langle H(^1S_0)|\mathcal{O}_{1\,\rm cm}|H(^1S_0)\rangle + \frac{2\,{\rm Im}h'_1(^1S_0)}{M^6}\langle H(^1S_0)|\mathcal{Q}'_1(^1S_0)|H(^1S_0)\rangle \\
&+ \frac{2\,{\rm Im}h''_1(^1S_0)}{M^6}\langle H(^1S_0)|\mathcal{Q}''_1(^1S_0)|H(^1S_0)\rangle + \frac{2\,{\rm Im}g_8(^3S_1)}{M^4}\langle H(^1S_0)|\mathcal{P}_8(^3S_1)|H(^1S_0)\rangle \\
&+ \frac{2\,{\rm Im}g_8(^1S_0)}{M^4}\langle H(^1S_0)|\mathcal{P}_8(^1S_0)|H(^1S_0)\rangle + \frac{2\,{\rm Im}g_8(^1P_1)}{M^6}\langle H(^1S_0)|\mathcal{P}_8(^1P_1)|H(^1S_0)\rangle \\
&+ \frac{2\,{\rm Im}h'_8(^1S_0)}{M^6}\langle H(^1S_0)|\mathcal{Q}'_8(^1S_0)|H(^1S_0)\rangle + \frac{2\,{\rm Im}h_8(^1D_2)}{M^6}\langle H(^1S_0)|\mathcal{Q}_8(^1D_2)|H(^1S_0)\rangle \\
&+ \frac{2\,{\rm Im}h_1(^1D_2)}{M^6}\langle H(^1S_0)|\mathcal{Q}_1(^1D_2)|H(^1S_0)\rangle + \frac{2\,{\rm Im}d_8(^1S_0,{}^1P_1)}{M^5}\langle H(^1S_0)|\mathcal{D}_{8\text{-}8}(^1S_0,{}^1P_1)|H(^1S_0)\rangle,
\end{aligned} \tag{28}$$

$$\begin{aligned}\Gamma({}^3S_1 \to \text{l.h.}) = {} & \frac{2\,\mathrm{Im} f_1({}^3S_1)}{M^2}\langle H({}^3S_1)|\mathcal{O}_1({}^3S_1)|H({}^3S_1)\rangle + \frac{2\,\mathrm{Im}\, g_1({}^3S_1)}{M^4}\langle H({}^3S_1)|\mathcal{P}_1({}^3S_1)|H({}^3S_1)\rangle \\
& + \frac{2\,\mathrm{Im} f_8({}^1S_0)}{M^2}\langle H({}^3S_1)|\mathcal{O}_8({}^1S_0)|H({}^3S_1)\rangle + \frac{2\,\mathrm{Im}\, f_8({}^3S_1)}{M^2}\langle H({}^3S_1)|\mathcal{O}_8({}^3S_1)|H({}^3S_1)\rangle \\
& + \sum_{J=0}^{2}\frac{2\,\mathrm{Im} f_8({}^3P_J)}{M^4}\langle H({}^3S_1)|\mathcal{O}_8({}^3P_J)|H({}^3S_1)\rangle + \frac{2\,\mathrm{Im}\, s_{1-8}({}^3S_1, {}^1S_0)}{M^4}\langle H({}^3S_1)|\mathcal{S}_{1-8}({}^3S_1, {}^1S_0)|H({}^3S_1)\rangle \\
& + \frac{2\,\mathrm{Im}\, f_{8\,\mathrm{cm}}}{M^4}\langle H({}^3S_1)|\mathcal{O}_{8\,\mathrm{cm}}|H({}^3S_1)\rangle + \frac{2\,\mathrm{Im}\, g_{8c\,\mathrm{cm}}}{M^4}\langle H({}^3S_1)|\mathcal{P}_{8c\,\mathrm{cm}}|H({}^3S_1)\rangle \\
& + \frac{2\,\mathrm{Im}\, f'_{1\,\mathrm{cm}}}{M^4}\langle H({}^3S_1)|\mathcal{O}'_{1\,\mathrm{cm}}|H({}^3S_1)\rangle + \frac{2\,\mathrm{Im} h'_1({}^3S_1)}{M^6}\langle H({}^3S_1)|\mathcal{Q}'_1({}^3S_1)|H({}^3S_1)\rangle \\
& + \frac{2\,\mathrm{Im} h''_1({}^3S_1)}{M^6}\langle H({}^3S_1)|\mathcal{Q}''_1({}^3S_1)|H({}^3S_1)\rangle + \frac{2\,\mathrm{Im} g_1({}^3S_1, {}^3D_1)}{M^4}\langle H({}^3S_1)|\mathcal{P}_1({}^3S_1, {}^3D_1)|H({}^3S_1)\rangle \\
& + \frac{2\,\mathrm{Im} g_8({}^1S_0)}{M^4}\langle H({}^3S_1)|\mathcal{P}_8({}^1S_0)|H({}^3S_1)\rangle + \frac{2\,\mathrm{Im}\, g_8({}^3S_1)}{M^4}\langle H({}^3S_1)|\mathcal{P}_8({}^3S_1)|H({}^3S_1)\rangle \\
& + \frac{2\,\mathrm{Im} t^{(1)\prime}_{1-8}({}^3S_1, {}^3P)}{M^5}\langle H({}^3S_1)|\mathcal{T}^{(1)\prime}_{1-8}({}^3S_1, {}^3P)|H({}^3S_1)\rangle + \sum_{J=0}^{2}\frac{2\,\mathrm{Im} g_8({}^3P_J)}{M^6}\langle H({}^3S_1)|\mathcal{P}_8({}^3P_J)|H({}^3S_1)\rangle \\
& + \frac{2\,\mathrm{Im}\, h'_8({}^3S_1)}{M^6}\langle H({}^3S_1)|\mathcal{Q}'_8({}^3S_1)|H({}^3S_1)\rangle + \sum_{J=0}^{2}\Bigg[\frac{2\,\mathrm{Im} h_8({}^3D_J)}{M^6}\langle H({}^3S_1)|\mathcal{Q}_8({}^3D_J)|H({}^3S_1)\rangle \\
& + \frac{2\,\mathrm{Im} h_1({}^3D_J)}{M^6}\langle H({}^3S_1)|\mathcal{Q}_1({}^3D_J)|H({}^3S_1)\rangle\Bigg] + \sum_{k=0,2}\frac{2\,\mathrm{Im}\, d^{(k)}_8({}^3S_1, {}^3P)}{M^5}\langle H({}^3S_1)|\mathcal{D}^{(k)}_{8\text{-}8}({}^3S_1, {}^3P)|H({}^3S_1)\rangle \\
& + \frac{2\,\mathrm{Im} g_8({}^3P_2, {}^3F_2)}{M^6}\langle H({}^3S_1)|\mathcal{P}_8({}^3P_2, {}^3F_2)|H({}^3S_1)\rangle. \end{aligned} \tag{29}$$

In Eqs. (28) and (29), the first matrix element scales like $v^3$, the following four in the second and third line like $v^5$, the following two like $v^6$ and the others like $v^7$. S-wave decay widths at order $v^7$ were computed in [9]. For $\Gamma({}^1S_0 \to \text{l.h.})$, the decay width in [9] does not include the matrix elements of the operators proportional to the total momentum of the $Q\bar{Q}$ pair, the matrix element of $\mathcal{Q}_1({}^1D_2)$ and any other matrix element of octet operators with the exception of $\mathcal{O}_8({}^3S_1)$, $\mathcal{O}_8({}^1S_0)$ and $\mathcal{O}_8({}^1P_1)$. In the power counting adopted in [9], which is described in [4], all these matrix elements are suppressed by further powers of $v$ and they can be neglected at this order of the expansion. For the same reason, the expression for $\Gamma({}^3S_1 \to \text{l.h.})$ in [9] does not include all the matrix elements of operators proportional to the total momentum of the $Q\bar{Q}$ pair, the matrix elements of $\mathcal{Q}_1({}^3D_J)$ and $\mathcal{P}_1({}^3S_1, {}^3D_1)$, and any other matrix element of octet operators with the exception of $\mathcal{O}_8({}^3S_1)$, $\mathcal{O}_8({}^1S_0)$ and $\mathcal{O}_8({}^3P_J)$.

The P-wave decay widths at order $v^7$ are

$$\begin{aligned}\Gamma(^3P_J \to \text{l.h.}) &= \frac{2\,\text{Im}f_1(^3P_J)}{M^4}\langle H(^3P_J)|\mathcal{O}_1(^3P_J)|H(^3P_J)\rangle + \frac{2\,\text{Im}f_8(^3S_1)}{M^2}\langle H(^3P_J)|\mathcal{O}_8(^3S_1)|H(^3P_J)\rangle \\ &+ \frac{2\,\text{Im}g_1(^3P_J)}{M^6}\langle H(^3P_J)|\mathcal{P}_1(^3P_J)|H(^3P_J)\rangle + \frac{2\,\text{Im}g_8(^3S_1)}{M^4}\langle H(^3P_J)|\mathcal{P}_8(^3S_1)|H(^3P_J)\rangle \\ &+ \frac{2\,\text{Im}g_8(^3S_1,{}^3D_1)}{M^4}\langle H(^3P_J)|\mathcal{P}_8(^3S_1,{}^3D_1)|H(^3P_J)\rangle + \frac{2\,\text{Im}g_{8a\,\text{cm}}}{M^4}\langle H(^3P_J)|\mathcal{P}_{8a\,\text{cm}}|H(^3P_J)\rangle \\ &+ \frac{2\,\text{Im}t_{1-8}(^3P_J,{}^3S_1)}{M^5}\langle H(^3P_J)|\mathcal{T}_{1-8}(^3P_J,{}^3S_1)|H(^3P_J)\rangle + \frac{2\,\text{Im}f_8(^1P_1)}{M^4}\langle H(^3P_J)|\mathcal{O}_8(^1P_1)|H(^3P_J)\rangle \\ &+ \frac{2\,\text{Im}f_8(^1S_0)}{M^2}\langle H(^3P_J)|\mathcal{O}_8(^1S_0)|H(^3P_J)\rangle + \frac{2\,\text{Im}f_1(^1S_0)}{M^2}\langle H(^3P_J)|\mathcal{O}_1(^1S_0)|H(^3P_J)\rangle \\ &+ \frac{2\,\text{Im}f_8(^3P_J)}{M^4}\langle H(^3P_J)|\mathcal{O}_8(^3P_J)|H(^3P_J)\rangle + \frac{2\,\text{Im}h_8'(^3S_1)}{M^6}\langle H(^3P_J)|\mathcal{Q}_8'(^3S_1)|H(^3P_J)\rangle \\ &+ \frac{2\,\text{Im}h_8'(^3S_1,{}^3D_1)}{M^6}\langle H(^3P_J)|\mathcal{Q}_8'(^3S_1,{}^3D_1)|H(^3P_J)\rangle + \sum_{k=1}^{J+1}\frac{2\,\text{Im}h_8(^3D_k)}{M^6}\langle H(^3P_J)|\mathcal{Q}_8(^3D_k)|H(^3P_J)\rangle \\ &+ \frac{2\,\text{Im}f_1(^3S_1)}{M^2}\langle H(^3P_J)|\mathcal{O}_1(^3S_1)|H(^3P_J)\rangle + \sum_{i=1,8}\delta_{J2}\frac{2\,\text{Im}g_i(^3P_2,{}^3F_2)}{M^6}\langle H(^3P_2)|\mathcal{P}_i(^3P_2,{}^3F_2)|H(^3P_2)\rangle, \quad (30)\end{aligned}$$

where $J = 0, 1, 2$.

In Eq. (30), the first two matrix elements scale like $v^5$, the remaining ones like $v^7$. P-wave decay widths at order $v^7$ were computed in [10], where the power counting of [4] was used: they appear to contain only the first four terms of Eq. (30). It seems, however, that also by adopting the power counting of [4] at least the matrix elements of the operators $\mathcal{P}_8(^3S_1, {}^3D_1)$ and $\mathcal{P}_{8a\,\text{cm}}$ should be added.

## III. MATCHING

In this section, we calculate the order $\alpha_s^2$ contributions to the imaginary parts of the matching coefficients that appear in Eqs. (28)–(30). The method consists in equating (matching) the imaginary parts of scattering amplitudes in QCD and NRQCD along the lines of [4].

In the QCD part of the matching, the ingoing quark and the outgoing antiquark are represented by the Dirac spinors $u(\vec{p})$ and $v(\vec{p})$, respectively, whose explicit expressions are

$$u(\vec{p}) = \sqrt{\frac{E_p + M}{2E_p}}\begin{pmatrix}\xi \\ \frac{\vec{p}\cdot\vec{\sigma}}{E_p+M}\xi\end{pmatrix}, \qquad v(\vec{p}) = \sqrt{\frac{E_p + M}{2E_p}}\begin{pmatrix}\frac{\vec{p}\cdot\vec{\sigma}}{E_p+M}\eta \\ \eta\end{pmatrix}, \quad (31)$$

where $E_p = \sqrt{\vec{p}^2 + M^2}$, and $\xi$ and $\eta$ are Pauli spinors. In the NRQCD part of the matching, the ingoing quark and the outgoing antiquark are represented by the Pauli spinors $\xi$ and $\eta$, respectively.

We will match singlet, octet and singlet-octet transition operators at order $\alpha_s^2$; to this purpose we will consider both the scattering amplitudes $Q\bar{Q} \to Q\bar{Q}$ and $Q\bar{Q}g \to Q\bar{Q}$, with no more than two gluons in the intermediate states.

In the center of mass rest frame, the energy and momentum conservation imposes the following kinematical constraints on the scattering $Q\bar{Q} \to Q\bar{Q}$,

$$|\vec{p}| = |\vec{k}|, \qquad \vec{p} + \vec{p}\,' = 0, \qquad \vec{k} + \vec{k}' = 0, \quad (32)$$

and on the scattering $Q\bar{Q}g \to Q\bar{Q}$,

$$E_p + E_{p'} + |\vec{q}| = 2E_k, \qquad \vec{p} + \vec{p}\,' + \vec{q} = 0, \qquad \vec{k} + \vec{k}' = 0, \quad (33)$$

where $\vec{p}$, $\vec{p}\,'$ are the ingoing and $\vec{k}$, $\vec{k}'$ the outgoing quark and antiquark momenta, while $\vec{q}$ is the momentum of the ingoing gluon, which is on mass shell.

The matching does not rely on any specific power counting and can be performed order by order in $1/M$ [13]. We will perform the matching up to order $1/M^6$, which is the highest power in $1/M$ appearing in Eqs. (28)–(30). In practice, we expand the QCD amplitude with respect to all external three-momenta. Note that, in the relativistic expansion, the gluon momentum $|\vec{q}|$ is proportional to (three-momenta)$^2/M$. In the matching calculation, therefore, the gluon three-momentum appears with an extra $1/M$ suppression with respect to the quark and antiquark three-momenta. In the case of the $Q\bar{Q}g \to Q\bar{Q}$ scattering, the expansion in the gluon momentum may develop infrared singularities, i.e. terms proportional to $1/|\vec{q}|$. These terms cancel in the matching, as expected, having QCD and NRQCD the same infrared structure. For a detailed discussion see [12]. In the hadronic calculation, individual diagrams that contribute to the imaginary part of the $Q\bar{Q}g \to Q\bar{Q}$ scattering amplitude containing interactions between the gluon in the initial state and gluon propagators develop also collinear singularities, i.e. terms proportional to $1/(1 \pm \cos\theta)$, $\theta$ being the angle between the incoming gluon momentum and the momentum flowing in one of the gluon propagators put on shell to get the imaginary contribution. These singular terms cancel in the sum of all diagrams. Finally, we expect that, since the matching does not rely on a power counting and scattering amplitudes do not have a definite angular momentum, the matching will determine more coefficients than needed in Eqs. (28)–(30).

### A. $Q\bar{Q}$ to light hadrons: singlet matching

The matching of the $Q\bar{Q} \to gg(q\bar{q}) \to Q\bar{Q}$ amplitude is performed by equating the sum of the imaginary parts of the QCD diagrams shown in Fig. 1 [taken by cutting the gluon propagators or the light quark propagators according to $1/k^2 \to -2\pi i\delta(k^2)\theta(k^0)$] to the sum of all the NRQCD diagrams of the type shown in Fig. 2. The first two diagrams in Fig. 1 contain both a color singlet and a color octet part, coming from the decompositions

$$\begin{aligned} t^a t^b \otimes t^b t^a &= \frac{C_F}{2N_c}\mathbf{1}_c \otimes \mathbf{1}_c + \frac{N_c^2-2}{2N_c} t^a \otimes t^a, \\ t^a t^b \otimes t^a t^b &= \frac{C_F}{2N_c}\mathbf{1}_c \otimes \mathbf{1}_c - \frac{1}{N_c} t^a \otimes t^a, \end{aligned} \tag{34}$$

while the other five Feynman diagrams contribute only to the octet part.

The calculation of the box diagrams in Fig. 1 gives the matching coefficients of the dimension 6, 8 and 10 singlet operators proportional to the relative momentum of the $Q\bar{Q}$ pair, listed in Eqs. (A16)–(A19) and (A26)–(A28). We quote the coefficients of the dimension 6 and dimension 8 operators in Appendix B. They agree with those calculated in [4]. We refer to [14] and references therein for an updated list of imaginary parts of matching coefficients of dimension 6 and 8 four-fermion operators; some of them are known at next-to-leading order. For dimension 10 operators we find

$$\mathrm{Im}\, h_1(^1D_2) = \frac{2}{15}\frac{\alpha_s^2\pi C_F}{2N_c}, \tag{35}$$

$$\mathrm{Im}\, h_1'(^1S_0) + \mathrm{Im} h_1''(^1S_0) = \frac{68}{45}\frac{\alpha_s^2\pi C_F}{2N_c}, \tag{36}$$

$$\mathrm{Im}\, g_1(^3P_0) = -7\frac{\alpha_s^2\pi C_F}{2N_c}, \tag{37}$$

$$\mathrm{Im}\, g_1(^3P_2) = -\frac{8}{5}\frac{\alpha_s^2\pi C_F}{2N_c}, \tag{38}$$

$$\mathrm{Im}\, g_1(^3P_2, {}^3F_2) = -\frac{20}{21}\frac{\alpha_s^2\pi C_F}{2N_c}. \tag{39}$$

The four-fermion operators to which the matching coefficients refer are listed in Appendix A.

The coefficients relevant for P-wave decay widths at order $v^7$ are (37) and (38). They were first computed in [10], but our results disagree with the ones reported there. Note that while Eqs. (37) and (38) agree in the QED limit with the results of [12], the QED limit of the results in [10] is in disagreement both with [12,15].

Equation (36) agrees with the one found in [9]. By matching the diagrams of Fig. 1 we cannot resolve $\mathrm{Im}h_1'(^1S_0)$ and $\mathrm{Im}h_1''(^1S_0)$ separately. These coefficients multiply operators that contribute to the $v^4$ corrections of the S-wave decay widths. Equation (35) contributes to the leading order decay width of the singlet state of the $D$ multiplet, which for charmonium and bottomonium has not yet been observed; it agrees with the result of [16].

### B. $Q\bar{Q}$ to light hadrons: octet matching

The calculation of the diagrams in Fig. 1 provides also the coefficients of dimension 6, 8 and 10 color octet operators. Again, since we work in the center of mass

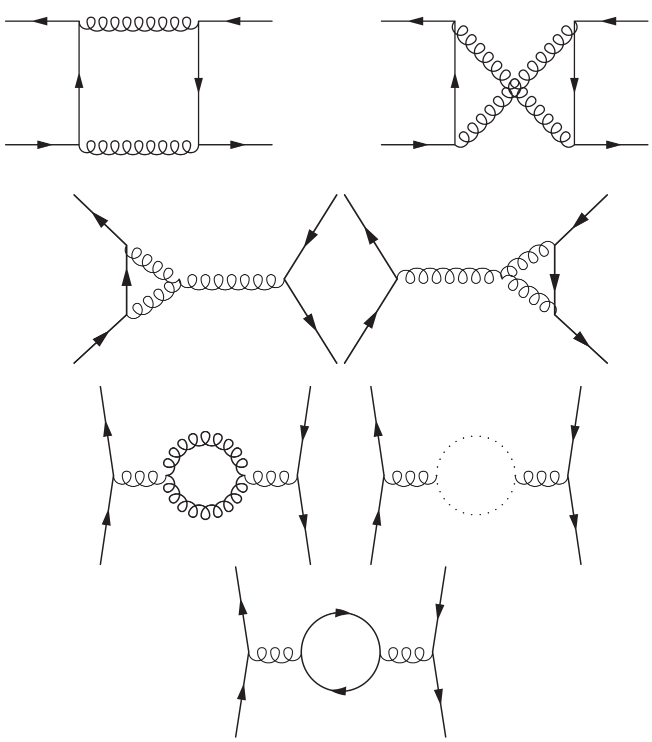

FIG. 1. QCD Feynman diagrams describing the amplitude $Q\bar{Q} \to Q\bar{Q}$ at order $\alpha_s^2$.

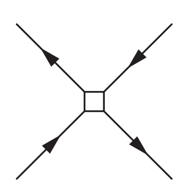

FIG. 2. Generic NRQCD four-fermion Feynman diagram. The empty box stands for one of the four-fermion vertices induced by the operators listed in Appendix A, Eqs. (A16)–(A21) and (A26)–(A31).

rest frame, we cannot obtain the matching coefficients of the dimension 8 and 10 operators proportional to the center of mass momentum.

The coefficients of the dimension 6 and 8 operators are quoted in Appendix B and agree with those obtained in [4,17]. The coefficients of the dimension 10 operators are new results of this work. We find

$$\mathrm{Im}\, h_8'({}^3S_1) + \mathrm{Im} h_8''({}^3S_1) = \frac{29}{108}\alpha_s^2 \pi n_f + \frac{1}{108}\alpha_s^2\pi N_c, \tag{40}$$

$$\mathrm{Im}\, h_8'({}^3S_1, {}^3D_1) + \mathrm{Im} h_8''({}^3S_1, {}^3D_1) = \frac{23}{72}\alpha_s^2 \pi n_f + \frac{1}{18}\alpha_s^2\pi N_c, \tag{41}$$

$$\mathrm{Im}\, h_8({}^3D_1) = \frac{1}{24}\alpha_s^2 \pi n_f + \frac{1}{12}\alpha_s^2\pi N_c, \tag{42}$$

$$\mathrm{Im}\, h_8({}^3D_2) = \frac{1}{30}\alpha_s^2\pi N_c, \tag{43}$$

$$\mathrm{Im}\, h_8({}^3D_3) = \frac{1}{21}\alpha_s^2\pi N_c, \tag{44}$$

$$\mathrm{Im}\, h_8({}^1D_2) = \frac{2}{15}\alpha_s^2\pi \frac{N_c^2-4}{4N_c}, \tag{45}$$

$$\mathrm{Im}\, h_8'({}^1S_0) + \mathrm{Im} h_8''({}^1S_0) = \frac{68}{45}\alpha_s^2\pi \frac{N_c^2-4}{4N_c}, \tag{46}$$

$$\mathrm{Im}\, g_8({}^1P_1) = -\frac{3}{20}\alpha_s^2\pi N_c, \tag{47}$$

$$\mathrm{Im}\, g_8({}^3P_0) = -7\alpha_s^2\pi \frac{N_c^2-4}{4N_c}, \tag{48}$$

$$\mathrm{Im}\, g_8({}^3P_2) = -\frac{8}{5}\alpha_s^2\pi \frac{N_c^2-4}{4N_c}, \tag{49}$$

$$\mathrm{Im}\, g_8({}^3P_2, {}^3F_2) = -\frac{20}{21}\alpha_s^2\pi \frac{N_c^2-4}{4N_c}. \tag{50}$$

The four-fermion operators to which the matching coefficients refer are listed in Appendix A.

### C. $Q\bar{Q}g$ to light hadrons

We show in Figs. 3–8 the diagrams that contribute to the $Q\bar{Q}g \to Q\bar{Q}$ scattering amplitude with terms with color content $t^a \otimes \mathbf{1}$ or $\mathbf{1} \otimes t^a$. The imaginary part of $Q\bar{Q}g \to Q\bar{Q}$ is computed considering all possible cuts of the gluon propagators. Diagrams in Fig. 4 and 7 develop collinear singularities that cancel when all possible cuts are taken into account. In these figures, the cuts are explicitly indicated.

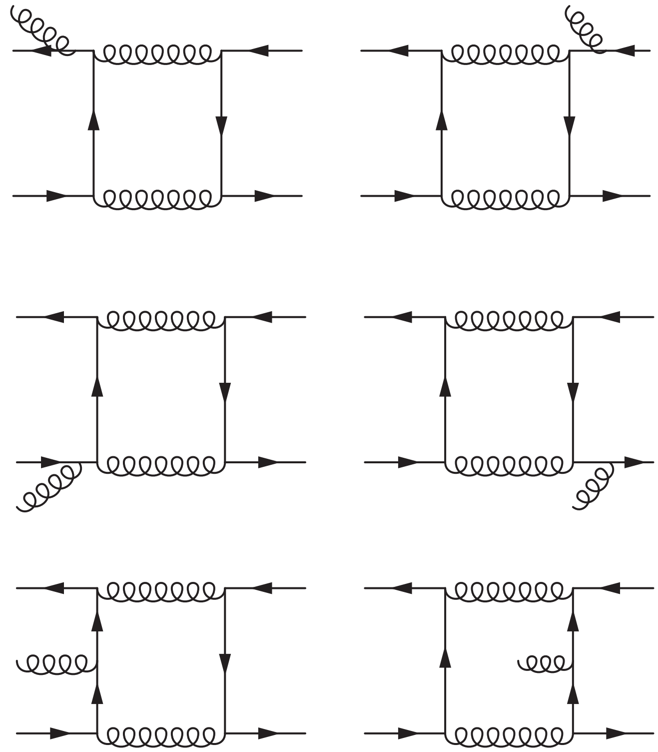

FIG. 3. Box diagrams: the gluon in the initial state interacts with a fermion leg. The other six diagrams, in which the two gluon propagators cross, have not been displayed.

In the matching procedure, the QCD amplitude is equated to the sum of all NRQCD diagrams of the type shown in Fig. 9. These are all diagrams of NRQCD with an ingoing $Q\bar{Q}$ pair and a gluon and an outgoing $Q\bar{Q}$ pair. They can involve four-fermion operators and a gluon coupled to the

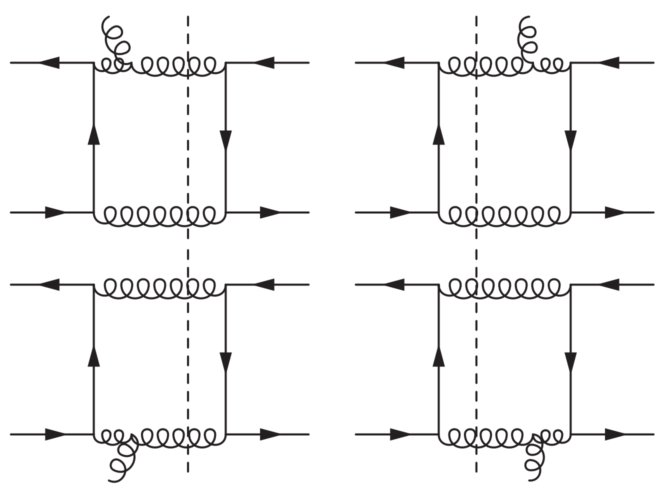

FIG. 4. Box diagrams: the gluon in the initial state interacts with a gluon propagator. The other four diagrams, in which the two gluon propagators cross, have not been displayed.

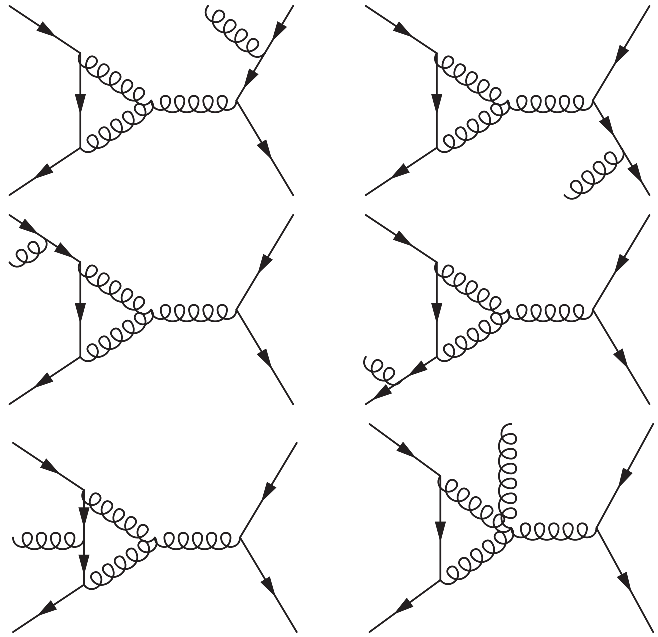

FIG. 5. Vertex corrections: the gluon in the initial state couples to a fermion leg or to a three-gluon vertex.

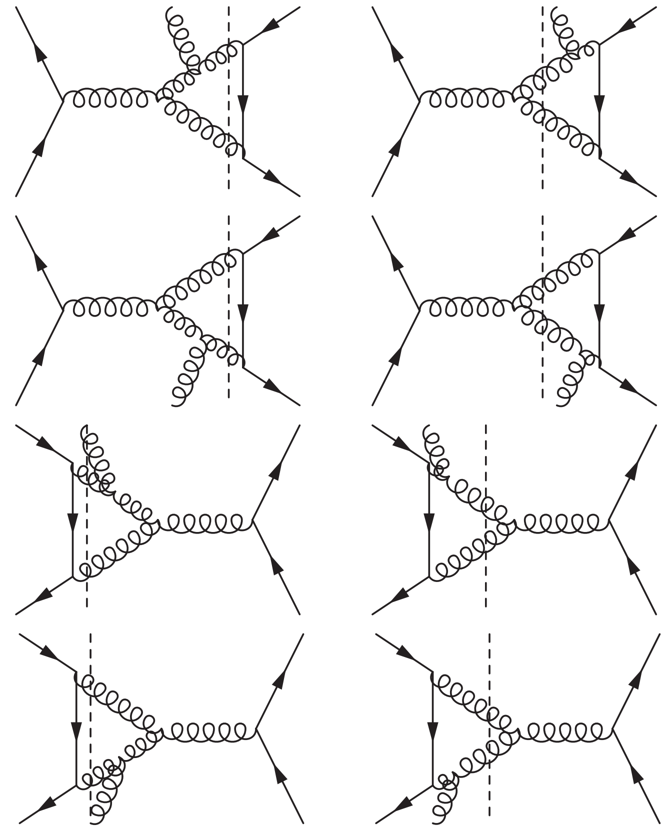

FIG. 7. Vertex corrections: the gluon in the initial state interacts with a gluon propagator.

quark or the antiquark line, but also four-fermion operators that couple to gluons. Four-fermion operators that induce octet to singlet transitions on the $Q\bar{Q}$ pair may be one of the operators listed in Eqs. (A24) and (A25), but also one of the four-fermion operators involving only covariant derivatives, which, despite being usually denoted as singlet (or octet) operators, couple to the gluon field through the term $-it^a g\vec{A}^a$ in the covariant derivative and therefore have a singlet-octet component.

The calculation of the imaginary part of the $Q\bar{Q}g \to Q\bar{Q}$ scattering amplitude allows us to find the matching coefficients of dimension 8 and dimension 9 singlet-octet transition operators and dimension 8 operators proportional to

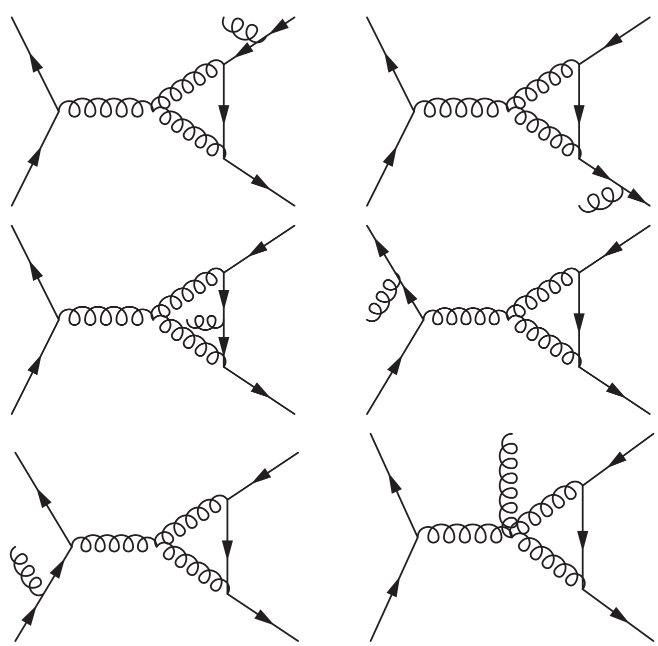

FIG. 6. Vertex corrections: the gluon in the initial state couples to a fermion leg or to a three-gluon vertex.

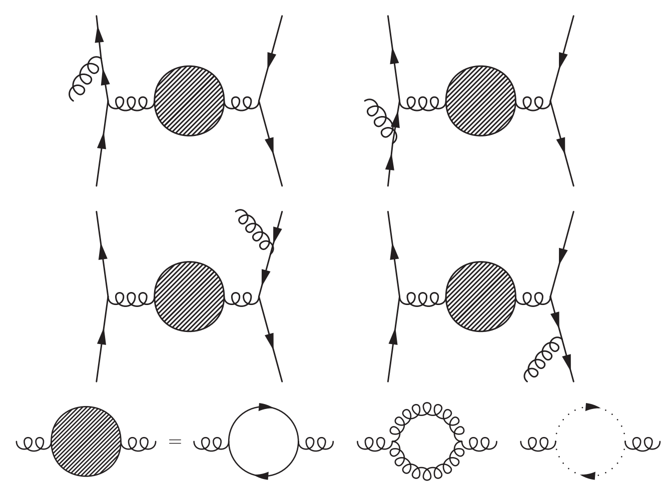

FIG. 8. Vacuum polarization: the gluon in the initial state interacts with a fermion leg.

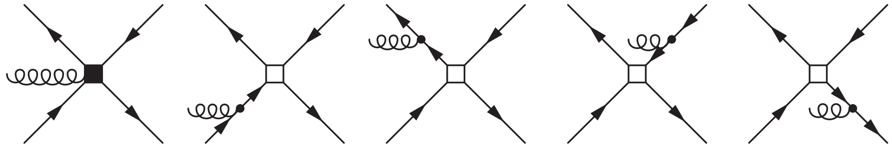

FIG. 9. Generic NRQCD four-fermion Feynman diagrams involving an ingoing $Q\bar{Q}$ pair and a gluon and an outgoing $Q\bar{Q}$ pair. The black box with a gluon attached to it and the empty box stand, respectively, for one of the four-fermion–one-gluon vertices and for one of the four-fermion vertices induced by the operators listed in Appendix A, Eqs. (A16)–(A31). The black dot with a gluon attached to it stands for one of the quark-gluon vertices induced by the bilinear part of the NRQCD Lagrangian given in Eq. (A1).

the total momentum of the $Q\bar{Q}$ pair. It also allows us to fix the individual coefficients appearing in Eqs. (36) and (46). As discussed in Sec. II C 2, the basis of operators that we chose contains as independent operators $\mathcal{T}_{1-8}(^3P_J, {}^3S_1)$, with $J = 0, 1, 2$. In Appendix B, we give for completeness also the matching coefficients computed with the other possible choice of independent operators, $1/2(\mathcal{Q}_8'(^3S_1) - \mathcal{Q}_8''(^3S_1))$, $1/2(\mathcal{Q}_8'(^3S_1, {}^3D_1) - \mathcal{Q}_8''(^3S_1, {}^3D_1))$ and $\mathcal{T}_{8-1}^{(1)\prime}(^3S_1, {}^3P)$. This second set allows us to establish the individual coefficients of the operators appearing in Eqs. (40) and (41), but it is less useful for the discussion of the P-wave decay widths.

The matching coefficients are

$$\mathrm{Im}\, s_{1-8}(^1S_0, {}^3S_1) = -\frac{1}{4}\alpha_s^2\pi + \frac{1}{12}\frac{\alpha_s^2\pi n_f}{N_c}, \tag{51}$$

$$\mathrm{Im}\, s_{1-8}(^3S_1, {}^1S_0) = 0, \tag{52}$$

$$\mathrm{Im}\, h_1'(^1S_0) = \frac{10}{9}\alpha_s^2\pi\frac{C_F}{2N_c} - \frac{1}{48}\alpha_s^2\pi, \tag{53}$$

$$\mathrm{Im}\, h_1''(^1S_0) = \frac{2}{5}\alpha_s^2\pi\frac{C_F}{2N_c} + \frac{1}{48}\alpha_s^2\pi, \tag{54}$$

$$\mathrm{Im}\, h_8'(^1S_0) = \frac{10}{9}\alpha_s^2\pi\frac{N_c^2-4}{4N_c}, \tag{55}$$

$$\mathrm{Im}\, h_8''(^1S_0) = \frac{2}{5}\alpha_s^2\pi\frac{N_c^2-4}{4N_c}, \tag{56}$$

$$\mathrm{Im}\, t_{1-8}^{(1)\prime}(^3S_1, {}^3P) = -\frac{1}{8}\alpha_s^2\pi\frac{N_c^2-4}{4N_c^2}, \tag{57}$$

$$\mathrm{Im}\, h_1'(^3S_1) = \frac{1}{12}\alpha_s^2\pi\frac{N_c^2-4}{4N_c^2}, \tag{58}$$

$$\mathrm{Im}\, h_1''(^3S_1) = -\frac{1}{12}\alpha_s^2\pi\frac{N_c^2-4}{4N_c^2}, \tag{59}$$

$$\mathrm{Im}\, h_1'(^3S_1, {}^3D_1) = \frac{1}{4}\alpha_s^2\pi\frac{N_c^2-4}{4N_c^2}, \tag{60}$$

$$\mathrm{Im}\, h_1''(^3S_1, {}^3D_1) = -\frac{1}{4}\alpha_s^2\pi\frac{N_c^2-4}{4N_c^2}, \tag{61}$$

$$\mathrm{Im} t_{1-8}(^3P_0, {}^3S_1) = -\frac{3}{2}\alpha_s^2\pi\frac{C_F}{2N_c} + \left(\frac{61}{240} + \frac{7}{192}\frac{n_f}{N_c}\right)\alpha_s^2\pi, \tag{62}$$

$$\mathrm{Im}\, t_{1-8}(^3P_1, {}^3S_1) = \left(\frac{1}{72} + \frac{107}{576}\frac{n_f}{N_c}\right)\alpha_s^2\pi, \tag{63}$$

$$\mathrm{Im}\, t_{1-8}(^3P_2, {}^3S_1) = \left(\frac{1}{10} + \frac{25}{576}\frac{n_f}{N_c}\right)\alpha_s^2\pi. \tag{64}$$

The four-fermion operators to which the matching coefficients refer are listed in Appendix A. The total momentum of the ingoing $Q\bar{Q}$ being different from 0, the matching calculation for $Q\bar{Q}g \to Q\bar{Q}$ also provides the coefficients for the operators defined in (A22) and (A23):

$$\mathrm{Im}\, f_{1\,\mathrm{cm}} = \frac{1}{4}\alpha_s^2\pi\frac{C_F}{2N_c}, \tag{65}$$

$$\mathrm{Im}\, f_{1\,\mathrm{cm}}' = 0, \tag{66}$$

$$\mathrm{Im}\, f_{8\,\mathrm{cm}} = \frac{1}{4}\alpha_s^2\pi\frac{N_c^2-4}{4N_c}, \tag{67}$$

$$\mathrm{Im}\, f_{8\,\mathrm{cm}}' = \frac{1}{24}\alpha_s^2\pi n_f, \tag{68}$$

$$\mathrm{Im}\, g_{1a\,\mathrm{cm}} = 0, \tag{69}$$

$$\mathrm{Im}\, g_{1b\,\mathrm{cm}} = 0, \tag{70}$$

$$\mathrm{Im}\, g_{1c\,\mathrm{cm}} = -\frac{1}{4}\alpha_s^2\pi\frac{C_F}{2N_c}, \tag{71}$$

$$\mathrm{Im}\, g_{8a\,\mathrm{cm}} = -\frac{1}{24}\alpha_s^2\pi n_f, \tag{72}$$

$$\mathrm{Im}\, g_{8b\,\mathrm{cm}} = 0, \tag{73}$$

$$\mathrm{Im}\, g_{8c\,\mathrm{cm}} = -\frac{1}{4}\alpha_s^2\pi\frac{N_c^2-4}{4N_c}. \tag{74}$$

We have checked the matching coefficients (65)–(74) by repeating the calculation of the diagrams in Fig. 1 up to order $1/M^4$ in the general frame

$$\vec{p} = \frac{1}{2}\vec{q} + \vec{p}_r, \qquad \vec{k} = \frac{1}{2}\vec{q} + \vec{k}_r,$$
$$\vec{p}\,' = \frac{1}{2}\vec{q} - \vec{p}_r, \qquad \vec{k}' = \frac{1}{2}\vec{q} - \vec{k}_r.$$

Equations (51)–(74) are original results of this work.

## IV. POINCARÉ INVARIANCE CONSTRAINTS

We can use Poincaré symmetry to obtain independent checks on some of the matching coefficients derived in the previous sections. Here we outline the procedure, following the method of Ref. [18].

NRQCD is constructed by expanding (and matching) QCD in the nonrelativistic limit. As a consequence, while translations and rotations are still explicit symmetries of the NRQCD action, the explicit invariance of the QCD action under boost is lost in the nonrelativistic regime. However, the boost invariance of QCD manifests itself in a nonlinear realization, constraining the form of the NRQCD Hamiltonian.

The constraints posed by Poincaré invariance on the bilinear sector of the NRQCD Lagrangian have been studied extensively in [13,18]. The computation of the matching coefficients (65)–(74) completes our knowledge of the imaginary part of the NRQCD Lagrangian at order $1/M^4$, including four-fermion operators proportional to the total momentum of the $Q\bar{Q}$ pair, which, due to their suppression in $v$ have not been considered before. Therefore, we can now study the constraints induced by Poincaré invariance in the four-fermion sector of the NRQCD Lagrangian. We adopt the method described in [18] by constructing the generators of time translation $H$, space translations $\vec{P}$, rotations $\vec{J}$ and boosts $\vec{K}$ inside the effective theory and by imposing that the commutation relations of the Poincaré algebra are respected. Since rotation and translation invariance are manifestly maintained in NRQCD, the commutation relations involving only $H$, $\vec{P}$ and $\vec{J}$ are trivially satisfied while the commutation relations involving the boost generators $\vec{K}$ impose restrictions among the matching coefficients:

$$[P^i, K^j] = -i\delta^{ij}H, \tag{75}$$

$$[H, K^i] = -iP^i, \tag{76}$$

$$[J^i, K^j] = i\varepsilon^{ijk}K^k, \tag{77}$$

$$[K^i, K^j] = -i\varepsilon^{ijk}J^k. \tag{78}$$

The construction of the generators proceeds in the following way: $\vec{P}$ and $\vec{J}$ can be obtained from the symmetric energy-momentum tensor [18,19]:

$$\vec{P} = \int d^3x\psi^\dagger(-i\vec{D})\psi + \chi^\dagger(-i\vec{D})\chi + \frac{1}{2}[\vec{\Pi}^a\times, \vec{B}^a], \tag{79}$$

$$\vec{J} = \int d^3x\psi^\dagger\Big(\vec{x}\times(-i\vec{D}) + \frac{\vec{\sigma}}{2}\Big)\psi + \chi^\dagger\Big(\vec{x}\times(-i\vec{D}) + \frac{\vec{\sigma}}{2}\Big)\chi + \frac{1}{2}\vec{x}\times[\vec{\Pi}^a\times, \vec{B}^a], \tag{80}$$

where $(\psi, i\psi^\dagger)$, $(\chi, i\chi^\dagger)$ and, in the $A^0 = 0$ gauge, $[A_i, \Pi_a^i = \partial\mathcal{L}_{\mathrm{NRQCD}}/\partial(\partial_0 A_i^a)]$, are the pairs of canonical variables. The NRQCD Hamiltonian density $h_{\mathrm{NRQCD}}$ can be obtained from a Legendre transformation of the Lagrangian density:

$$\begin{aligned} H_{\mathrm{NRQCD}} &= \int d^3x h_{\mathrm{NRQCD}} \\ &= \int d^3x\psi^\dagger\Big(M - c_1\frac{\vec{D}^2}{2M} - c_F\frac{\vec{\sigma}\cdot g\vec{B}}{2M}\Big)\psi \\ &\quad + \chi^\dagger\Big(-M + c_1\frac{\vec{D}^2}{2M} + c_F\frac{\vec{\sigma}\cdot g\vec{B}}{2M}\Big)\chi \\ &\quad + \frac{1}{2}(\vec{\Pi}^a\cdot\vec{\Pi}^a + \vec{B}^a\cdot\vec{B}^a) - \sum_{i=1,8}\frac{1}{M^2}(f_i(^3S_1)\mathcal{O}_i \\ &\quad \times(^3S_1) + f_i(^1S_0)\mathcal{O}_i(^1S_0)) - \sum_{i=1,8}\frac{1}{M^4}(g_i(^3S_1) \\ &\quad \times \mathcal{P}_i(^3S_1) + g_i(^1S_0)\mathcal{P}_i(^1S_0) + \ldots) + \ldots. \end{aligned} \tag{81}$$

The coefficient $c_1$ is equal to 1 at all orders in $\alpha_s$; see [13,18].

A way to construct $\vec{K}$ is to write down the most general expression consistent with the NRQCD symmetries and to match it to the QCD boost generator, $\vec{K} = -t\vec{P} + \int d^3x\frac{1}{2}\{\vec{x}, h_{\mathrm{QCD}}\}$. This procedure is analogous to the one followed in the construction of the NRQCD Lagrangian: new matching coefficients, typical of $\vec{K}$, appear. The form of $\vec{K}$ in NRQCD is

$$\vec{K} = -t\vec{P} + \int d^3x\frac{1}{2}\{\vec{x}, h_{\mathrm{NRQCD}}\} - \sum_{l=1}^{\infty}\int d^3x\frac{k_l}{M^l}\vec{K}^{(l)}. \tag{82}$$

This form is chosen in analogy to the QCD boost generator and satisfies (75). $\vec{K}^{(l)}$ contains all the possible operators with mass dimension $M^l$ that are vectors under rotation, are odd under parity and are invariant under $C$ and $T$ transformations.

We now compute the imaginary, four-fermion part of the commutator (76) at order $1/M^3$. To this aim, we need the bilinear NRQCD Hamiltonian at order $1/M$, the four-fermion part of the NRQCD Hamiltonian at order $1/M^4$, the operator $\vec{K}^{(1)}$ and four-fermion operators in the boost generator, which first appear in $\vec{K}^{(4)}$. The form of $\vec{K}^{(1)}$

$$\vec{K}^{(1)} = \frac{1}{2}\psi^\dagger \frac{\vec{\sigma}}{2} \times (-i\vec{D})\psi - \frac{1}{2}\chi^\dagger \frac{\vec{\sigma}}{2} \times (-i\vec{D})\chi, \quad (83)$$

and its coefficient $k_1$ were obtained in [18], where it was shown that $k_1 = 1$ to all orders in $\alpha_s$. In $\vec{K}^{(4)}$, four-fermion operators like

$$\vec{K}^{(4)} = \frac{1}{2}\psi^\dagger \frac{\vec{\sigma}}{2} \times (-i\overleftrightarrow{D})\chi\chi^\dagger\psi$$

appear. We do not give the detailed form of $\vec{K}^{(4)}$ since an explicit calculation shows that $1/M^4 \int d^3x[\vec{K}^{(4)}(x), H] = \mathcal{O}(1/M^5)$.

Using the canonical commutation relations we find for singlet operators at order $1/M^3$:

$$\begin{aligned}[H, K^j] = \frac{1}{M^3}\int d^3x\Big[&(\partial^j(\psi^\dagger\chi)\chi^\dagger\psi - \psi^\dagger\chi\partial^j(\chi^\dagger\psi))\Big(\frac{1}{2}\,\mathrm{Im} f_1(^1S_0) + 2\,\mathrm{Im} g_{1c\,\mathrm{cm}}\Big) + (\partial^j(\psi^\dagger\sigma^i\chi)\chi^\dagger\sigma^i\psi \\ &- \psi^\dagger\sigma^i\chi\partial^j(\chi^\dagger\sigma^i\psi))\Big(\frac{1}{2}\,\mathrm{Im} f_1(^3S_1) + 2\,\mathrm{Im} g_{1a\,\mathrm{cm}}\Big) + (\partial^i(\psi^\dagger\sigma^i\chi)\chi^\dagger\sigma^j\psi - \psi^\dagger\sigma^j\chi\partial^i(\chi^\dagger\sigma^i\psi)) \\ &\times(2\,\mathrm{Im} g_{1b\,\mathrm{cm}}) - i\epsilon^{jlm}(\psi^\dagger\sigma^l\overleftrightarrow{\partial}^m\chi\chi^\dagger\psi - \psi^\dagger\chi\chi^\dagger\sigma^l\overleftrightarrow{\partial}^m\psi)\Big(\frac{1}{4}\,\mathrm{Im} f_1(^1S_0) - \mathrm{Im} f_{1\,\mathrm{cm}}\Big) \\ &- i\epsilon^{jlm}(\psi^\dagger\sigma^l\chi\chi^\dagger\overleftrightarrow{\partial}^m\psi - \psi^\dagger\overleftrightarrow{\partial}^m\chi\chi^\dagger\sigma^l\psi)\Big(\frac{1}{4}\,\mathrm{Im} f_1(^3S_1) - \mathrm{Im} f'_{1\,\mathrm{cm}}\Big)\Big] = 0, \end{aligned} \quad (84)$$

and for octet operators:

$$\begin{aligned}[H, K^j] = \frac{1}{M^3}\int d^3x\Big[&(\partial^j(\psi^\dagger t^a\chi)\chi^\dagger t^a\psi - \psi^\dagger t^a\chi\partial^j(\chi^\dagger t^a\psi))\Big(\frac{1}{2}\,\mathrm{Im} f_8(^1S_0) + 2\,\mathrm{Im} g_{8c\,\mathrm{cm}}\Big) + (\partial^j\psi^\dagger t^a\sigma^i\chi)\chi^\dagger t^a\sigma^i\psi \\ &- \psi^\dagger t^a\sigma^i\chi\partial^j(\chi^\dagger t^a\sigma^i\psi))\Big(\frac{1}{2}\,\mathrm{Im} f_8(^3S_1) + 2\,\mathrm{Im} g_{8a\,\mathrm{cm}}\Big) + (\partial^i(\psi^\dagger\sigma^i t^a\chi)\chi^\dagger\sigma^j\psi - \psi^\dagger\sigma^j\chi\partial^i(\chi^\dagger t^a\sigma^i\psi)) \\ &\times(2\,\mathrm{Im} g_{8b\,\mathrm{cm}}) - i\epsilon^{jlm}(\psi^\dagger t^a\sigma^l\overleftrightarrow{\partial}^m\chi\chi^\dagger t^a\psi - \psi^\dagger t^a\chi\chi^\dagger t^a\sigma^l\overleftrightarrow{\partial}^m\psi)\Big(\frac{1}{4}\,\mathrm{Im} f_8(^1S_0) - \mathrm{Im} f_{8\,\mathrm{cm}}\Big) \\ &- i\epsilon^{jlm}(\psi^\dagger t^a\sigma^l\chi\chi^\dagger t^a\overleftrightarrow{\partial}^m\psi - \psi^\dagger t^a\overleftrightarrow{\partial}^m\chi\chi^\dagger t^a\sigma^l\psi)\Big(\frac{1}{4}\,\mathrm{Im} f_8(^3S_1) - \mathrm{Im} f'_{8\,\mathrm{cm}}\Big)\Big] = 0. \end{aligned} \quad (85)$$

Equations (84) and (85) imply that

$$\begin{aligned} \mathrm{Im}\, g_{1c\,\mathrm{cm}} &= -\frac{1}{4}\,\mathrm{Im} f_1(^1S_0), \\ \mathrm{Im} g_{8c\,\mathrm{cm}} &= -\frac{1}{4}\,\mathrm{Im} f_8(^1S_0), \\ \mathrm{Im} g_{1a\,\mathrm{cm}} &= -\frac{1}{4}\,\mathrm{Im} f_1(^3S_1), \\ \mathrm{Im} g_{8a\,\mathrm{cm}} &= -\frac{1}{4}\,\mathrm{Im} f_8(^3S_1), \\ \mathrm{Im} g_{1b\,\mathrm{cm}} &= \mathrm{Im} g_{8b\,\mathrm{cm}} = 0, \end{aligned} \quad (86)$$

$$\begin{aligned} \mathrm{Im}\, f_{1\,\mathrm{cm}} &= \frac{1}{4}\,\mathrm{Im} f_1(^1S_0), \qquad \mathrm{Im} f'_{1\,\mathrm{cm}} = \frac{1}{4}\,\mathrm{Im} f_1(^3S_1), \\ \mathrm{Im} f_{8\,\mathrm{cm}} &= \frac{1}{4}\,\mathrm{Im} f_8(^1S_0), \qquad \mathrm{Im} f'_{8\,\mathrm{cm}} = \frac{1}{4}\,\mathrm{Im} f_8(^3S_1). \end{aligned} \quad (87)$$

Relations of the same form as Eqs. (86) and (87) hold also for the matching coefficients of the electromagnetic operators. Equations (86) and (87) imply that the knowledge of the imaginary part of matching coefficients of the dimension 6 operators completely determines the imaginary part of the coefficients of the operators defined in Eqs. (A22) and (A23), proportional to the total momentum of the $Q\bar{Q}$ pair. The coefficients (65)–(74), obtained in the previous section, satisfy Eqs. (86) and (87).

## V. SUMMARY AND OUTLOOK

In the paper, we have calculated the hadronic inclusive quarkonium decay widths in NRQCD at order $v^7$ in the relativistic expansion and at order $\alpha_s^2$. The electromagnetic S- and P-wave decay widths have been previously calculated at order $v^7$ in [9,12,15]. If we count $\alpha_s(M) \sim v^2$, terms of order $\alpha_s(M)^2 v^7$ are part of the next-to-next-to leading order (NNLO) corrections to the pseudoscalar S-wave decays and part of the NLO corrections to the vector S-wave and the P-wave hadronic decays.

The results for the S-wave hadronic decay widths are given in Eqs. (28) and (29) with the coefficients at order $\alpha_s^2$ listed in Appendix B. Let us first consider S-wave vector decays. In the power counting of [4], those coefficients together with previous results, including contributions to the decay width coming from three-gluon decays and loop corrections [4,17,20–23], provide us with the full NLO expression of the hadronic inclusive decay widths, i.e. with the full expression up to order $\alpha_s^4 v^3$, $\alpha_s^3 v^5$ and $\alpha_s^2 v^7$. In the more conservative power counting adopted here, the octet terms $\sum_{k=0,2} \frac{2\,\mathrm{Im} d_8^{(k)}(^3S_1, ^3P)}{M^5} \times \langle H(^3S_1)|\mathcal{D}^{(k)}_{8\text{-}8}(^3S_1, ^3P)|H(^3S_1)\rangle$ need to be included. The matching coefficients $d_8^{(k)}(^3S_1, ^3P)$ are however unknown. In the case of S-wave pseudoscalar decays, the largest uncertainties in the decay width come from the NNLO correction in $\alpha_s$ to the matching coefficient $\mathrm{Im} f_1(^1S_0)$, from the NLO correction in $\alpha_s$ to the coefficient $\mathrm{Im} g_1(^1S_0)$, and from the $\alpha_s^2$ expression of $\mathrm{Im} d_8(^1S_0, ^1P_1)$, which are all unknown. If we count $\alpha_s(M) \sim v^2$, these are the only missing ingredients to complete the NNLO corrections to the pseudoscalar decay widths. Note that to complete the NNLO corrections to the pseudoscalar width, the NNLO expression of $\mathrm{Im} f_1(^1S_0)$ and the NLO expression of $\mathrm{Im} g_1(^1S_0)$ would be necessary also in the power counting of [4]. We recall that matching amplitudes with loops, like those required for calculating $\mathrm{Im} f_1(^1S_0)$ and $\mathrm{Im} g_1(^1S_0)$ at NNLO and NLO, respectively, and with two external gluons, like those required for calculating the $\mathrm{Im} d_8$ coefficients, have been beyond the scope of this work.

The result for the P-wave hadronic decay width, calculated up to order $\alpha_s(M)^2 v^7$, is given in Eq. (30) with the coefficients at order $\alpha_s^2$ given in Appendix B. In the case of P-wave vector decays, the present calculation together with previous results, including contributions to the decay width coming from three-gluon decays and loop corrections [17,24], provides us with the full expression of the hadronic inclusive decay widths up to order $\alpha_s^3 v^5$ and $\alpha_s^2 v^7$. Explicitly we have

$$\begin{aligned}
\Gamma(^3P_0 \to \text{l.h.}) = {}& \frac{4}{3}\frac{\alpha_s^2(2M)\pi}{M^4}\left[1 + \frac{\alpha_s}{\pi}\left(\frac{343}{27} + \frac{5}{16}\pi - \frac{58}{81}n_f\right)\right]\langle H(^3P_0)|\mathcal{O}_1(^3P_0)|H(^3P_0)\rangle \\
& + \frac{\alpha_s^2(2M)\pi n_f}{3M^2}\left[1 + \frac{\alpha_s}{\pi}\left(\frac{107}{6} + 2\log 2 - \frac{3}{4}\pi^2 - \frac{5}{9}n_f + \left(-\frac{73}{4} + \frac{67}{36}\pi^2\right)\frac{5}{n_f}\right)\right]\langle H(^3P_0)|\mathcal{O}_8(^3S_1)|H(^3P_0)\rangle \\
& - \frac{28}{9}\frac{\alpha_s^2\pi}{M^6}\langle H(^3P_0)|\mathcal{P}_1(^3P_0)|H(^3P_0)\rangle - \frac{4}{9}\frac{\alpha_s^2\pi n_f}{M^4}\langle H(^3P_0)|\mathcal{P}_8(^3S_1)|H(^3P_0)\rangle \\
& - \frac{\alpha_s^2\pi n_f}{3M^4}\langle H(^3P_0)|\mathcal{P}_8(^3S_1, ^3D_1)|H(^3P_0)\rangle - \frac{\alpha_s^2\pi n_f}{12M^4}\langle H(^3P_0)|\mathcal{P}_{8a\,\mathrm{cm}}|H(^3P_0)\rangle \\
& + \frac{\alpha_s^2\pi}{M^5}\left(-\frac{19}{120} + \frac{7}{288}n_f\right)\langle H(^3P_0)|\mathcal{T}_{1-8}(^3P_0, ^3S_1)|H(^3P_0)\rangle + \frac{1}{2}\frac{\alpha_s^2\pi}{M^4}\langle H(^3P_0)|\mathcal{O}_8(^1P_1)|H(^3P_0)\rangle \\
& + \frac{5}{6}\frac{\alpha_s^2\pi}{M^2}\langle H(^3P_0)|\mathcal{O}_8(^1S_0)|H(^3P_0)\rangle + \frac{4}{9}\frac{\alpha_s^2\pi}{M^2}\langle H(^3P_0)|\mathcal{O}_1(^1S_0)|H(^3P_0)\rangle + \frac{5}{2}\frac{\alpha_s^2\pi}{M^4}\langle H(^3P_0)|\mathcal{O}_8(^3P_0)|H(^3P_0)\rangle \\
& + \frac{\alpha_s^2\pi}{M^6}\left(-\frac{2}{3} + \frac{23}{54}n_f\right)\langle H(^3P_0)|\mathcal{Q}'_8(^3S_1)|H(^3P_0)\rangle + \frac{\alpha_s^2\pi}{M^6}\left(-\frac{1}{30} + \frac{5}{9}n_f\right)\langle H(^3P_0)|\mathcal{Q}'_8(^3S_1, ^3D_1)|H(^3P_0)\rangle \\
& + \frac{\alpha_s^2\pi}{M^6}\left(\frac{1}{2} + \frac{1}{12}n_f\right)\langle H(^3P_0)|\mathcal{Q}_8(^3D_1)|H(^3P_0)\rangle,
\end{aligned} \tag{88}$$

$$\begin{aligned}
\Gamma(^3P_1 \to \text{l.h.}) = {}& \frac{2\alpha_s^3(2M)}{M^4}\left[\left(\frac{587}{81} - \frac{317}{432}\pi^2\right) - \frac{32}{243}n_f\right]\langle H(^3P_1)|\mathcal{O}(^3P_1)|H(^3P_1)\rangle \\
& + \frac{\alpha_s^2(2M)\pi n_f}{3M^2}\left[1 + \frac{\alpha_s}{\pi}\left(\frac{107}{6} + 2\log 2 - \frac{3}{4}\pi^2 - \frac{5}{9}n_f + \left(-\frac{73}{4} + \frac{67}{36}\pi^2\right)\frac{5}{n_f}\right)\right]\langle H(^3P_1)|\mathcal{O}_8(^3S_1)|H(^3P_1)\rangle \\
& - \frac{4}{9}\frac{\alpha_s^2\pi n_f}{M^4}\langle H(^3P_1)|\mathcal{P}_8(^3S_1)|H(^3P_1)\rangle - \frac{\alpha_s^2\pi n_f}{3M^4}\langle H(^3P_1)|\mathcal{P}_8(^3S_1, ^3D_1)|H(^3P_1)\rangle \\
& - \frac{\alpha_s^2\pi n_f}{12M^4}\langle H(^3P_1)|\mathcal{P}_{8a\,\mathrm{cm}}|H(^3P_1)\rangle + \frac{\alpha_s^2\pi}{M^5}\left(\frac{1}{36} + \frac{107}{864}n_f\right)\langle H(^3P_1)|\mathcal{T}_{1-8}(^3P_1, ^3S_1)|H(^3P_1)\rangle \\
& + \frac{1}{2}\frac{\alpha_s^2\pi}{M^4}\langle H(^3P_1)|\mathcal{O}_8(^1P_1)|H(^3P_1)\rangle + \frac{5}{6}\frac{\alpha_s^2\pi}{M^2}\langle H(^3P_1)|\mathcal{O}_8(^1S_0)|H(^3P_1)\rangle + \frac{4}{9}\frac{\alpha_s^2\pi}{M^2}\langle H(^3P_1)|\mathcal{O}_1(^1S_0)|H(^3P_1)\rangle \\
& + \frac{\alpha_s^2\pi}{M^6}\left(-\frac{2}{3} + \frac{23}{54}n_f\right)\langle H(^3P_1)|\mathcal{Q}'_8(^3S_1)|H(^3P_1)\rangle + \frac{\alpha_s^2\pi}{M^6}\left(-\frac{1}{30} + \frac{5}{9}n_f\right)\langle H(^3P_1)|\mathcal{Q}'_8(^3S_1, ^3D_1)|H(^3P_1)\rangle \\
& + \frac{\alpha_s^2\pi}{M^6}\left(\frac{1}{2} + \frac{1}{12}n_f\right)\langle H(^3P_1)|\mathcal{Q}_8(^3D_1)|H(^3P_1)\rangle + \frac{1}{5}\frac{\alpha_s^2\pi}{M^6}\langle H(^3P_1)|\mathcal{Q}_8(^3D_2)|H(^3P_1)\rangle,
\end{aligned} \tag{89}$$

$$\begin{aligned}
\Gamma(^3P_2\to \mathrm{l.h.}) = {}& \frac{16}{45}\frac{\alpha_s^2(2M)\pi}{M^4}\left[1+\frac{\alpha_s}{\pi}\left(\frac{1801}{72}-\frac{337}{128}\pi^2+5\log 2-\frac{29}{27}n_f\right)\right]\langle H(^3P_2)|\mathcal{O}_1(^3P_2)|H(^3P_2)\rangle \\
&+\frac{\alpha_s^2\pi n_f}{3M^2}\left[1+\frac{\alpha_s(2M)}{\pi}\left(\frac{107}{6}+2\log 2-\frac{3}{4}\pi^2-\frac{5}{9}n_f+\left(-\frac{73}{4}+\frac{67}{36}\pi^2\right)\frac{5}{n_f}\right)\right]\langle H(^3P_2)|\mathcal{O}_8(^3S_1)|H(^3P_2)\rangle \\
&-\frac{32}{45}\frac{\alpha_s^2\pi}{M^6}\langle H(^3P_2)|\mathcal{P}_1(^3P_2)|H(^3P_2)\rangle-\frac{4}{9}\frac{\alpha_s^2\pi n_f}{M^4}\langle H(^3P_2)|\mathcal{P}_8(^3S_1)|H(^3P_2)\rangle \\
&-\frac{\alpha_s^2\pi n_f}{3M^4}\langle H(^3P_2)|\mathcal{P}_8(^3S_1,{}^3D_1)|H(^3P_2)\rangle-\frac{\alpha_s^2\pi n_f}{12M^4}\langle H(^3P_2)|\mathcal{P}_{8a\,\mathrm{cm}}|H(^3P_2)\rangle \\
&+\frac{\alpha_s^2\pi}{M^5}\left(\frac{1}{5}+\frac{25}{864}n_f\right)\langle H(^3P_2)|\mathcal{T}_{1-8}(^3P_2,{}^3S_1)|H(^3P_2)\rangle+\frac{1}{2}\frac{\alpha_s^2\pi}{M^4}\langle H(^3P_2)|\mathcal{O}_8(^1P_1)|H(^3P_2)\rangle \\
&+\frac{5}{6}\frac{\alpha_s^2\pi}{M^2}\langle H(^3P_2)|\mathcal{O}_8(^2S_0)|H(^3P_2)\rangle+\frac{4}{9}\frac{\alpha_s^2\pi}{M^2}\langle H(^3P_2)|\mathcal{O}_1(^1S_0)|H(^3P_2)\rangle+\frac{2}{3}\frac{\alpha_s^2\pi}{M^4}\langle H(^3P_2)|\mathcal{O}_8(^3P_2)|H(^3P_2)\rangle \\
&+\frac{\alpha_s^2\pi}{M^6}\left(-\frac{2}{3}+\frac{23}{54}n_f\right)\langle H(^3P_2)|\mathcal{Q}_8'(^3S_1)|H(^3P_2)\rangle+\frac{\alpha_s^2\pi}{M^6}\left(-\frac{1}{30}+\frac{5}{9}n_f\right)\langle H(^3P_2)|\mathcal{Q}_8'(^3S_1,{}^3D_1)|H(^3P_2)\rangle \\
&+\frac{\alpha_s^2\pi}{M^6}\left(\frac{1}{2}+\frac{1}{12}n_f\right)\langle H(^3P_2)|\mathcal{Q}_8(^3D_1)|H(^3P_2)\rangle+\frac{1}{5}\frac{\alpha_s^2\pi}{M^6}\langle H(^3P_2)|\mathcal{Q}_8(^3D_2)|H(^3P_2)\rangle \\
&+\frac{2}{7}\frac{\alpha_s^2\pi}{M^6}\langle H(^3P_2)|\mathcal{Q}_8(^3D_3)|H(^3P_2)\rangle-\frac{80}{189}\frac{\alpha_s^2\pi}{M^6}\langle H(^3P_2)|\mathcal{P}_1(^3P_2,{}^3F_2)|H(^3P_2)\rangle \\
&-\frac{50}{63}\frac{\alpha_s^2\pi}{M^6}\langle H(^3P_2)|\mathcal{P}_8(^3P_2,{}^3F_2)|H(^3P_2)\rangle. \qquad (90)
\end{aligned}$$

A general source of concern is the proliferation of matrix elements with the increasing order of the expansion in $v$. Spin symmetry and vacuum saturation [4] may help to reduce the number of matrix elements by relating different spin states and hadronic with electromagnetic matrix elements. The actual number of independent matrix elements depends on the power counting.

In the power counting of [4], only the first six matrix elements of Eqs. (88) and (90) and the first five of (89) contribute. In [10], it was assumed that only the first four matrix elements of Eqs. (88) and (90) and the first three of (89) contribute.

The conservative power counting adopted here has been suggested in [25] to be appropriate when $\Lambda_{\mathrm{QCD}} \gg mv^2$. Under this condition, matrix elements are nonperturbative quantities and should be evaluated on the lattice. One can also take advantage of the factorization provided by potential NRQCD [11,25,26]. According to it, the matrix elements can be factorized into the product of the quarkonium wave function in the origin squared (or derivatives of it) and few universal nonperturbative correlation functions, eventually achieving a reduction in the number and a simplification of the nonperturbative operators needed.

We also note that the convergence of the perturbative series of the matching coefficient is typically poor. For a discussion and references we refer for instance to [27].

Phenomenological applications of the expressions of the decay widths will therefore entail work in two complimentary directions: (1) improving the knowledge of the NRQCD matrix elements either by direct evaluation, for example, by fitting the experimental data, by lattice calculations, and by models, or by exploiting the hierarchy of scales still entangled in NRQCD using EFTs of lower energy, like potential NRQCD; (2) improving the convergence of the perturbative series of the matching coefficients by resumming large contributions either related to large logarithms, or of the type discussed, for instance, in [28].

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## APPENDIX A: SUMMARY AND DEFINITION OF THE NRQCD OPERATORS

The two-fermion sector of the NRQCD Lagrangian relevant for the matching discussed in Sec. III is

$$\mathcal{L}_{2\text{-f}} = \psi^\dagger\Big(iD_0 + \frac{\vec{D}^2}{2M} + \frac{\vec{\sigma}\cdot g\vec{B}}{2M} + \frac{(\vec{D}\cdot g\vec{E})}{8M^2} - \frac{\vec{\sigma}\cdot[-i\vec{D}\times, g\vec{E}]}{8M^2} + \frac{(\vec{D}^2)^2}{8M^3} + \frac{\{\vec{D}^2, \vec{\sigma}\cdot g\vec{B}\}}{8M^3} - \frac{3}{32M^4}\{\vec{D}^2, \vec{\sigma}\cdot[-i\vec{D}\times, g\vec{E}]\} + \frac{3}{32M^4}\{\vec{D}^2, (\vec{D}\cdot g\vec{E})\} + \frac{\vec{D}^6}{16M^5}\Big)\psi + \text{c.c.}, \tag{A1}$$

where $\sigma^i$ are the Pauli matrices, $iD_0 = i\partial_0 - t^a gA_0^a$, $i\vec{D} = i\vec{\nabla} + t^a g\vec{A}^a$, $[\vec{D}\times, \vec{E}] = \vec{D}\times\vec{E} - \vec{E}\times\vec{D}$, $E^i = F^{i0}$ and $B^i = -\epsilon_{ijk}F^{jk}/2(\epsilon_{123} = 1)$. We have not displayed terms of order $1/M^6$ or smaller and matching coefficients of $O(\alpha_s)$ or smaller. The general structure of the four-fermion sector of the NRQCD Lagrangian is

$$\mathcal{L}_{4\text{-f}} = \sum_n \frac{c^{(n)}}{M^{d_n-4}}\mathcal{O}^{(n)}_{4\text{-f}}. \tag{A2}$$

Here, we list the operators relevant for the matching performed in Sec. III ordered by dimension. We use $\overleftrightarrow{D} \equiv \vec{D} - \overleftarrow{D}$.

For the octet operators defined in Eqs. (A20)–(A22) and (A29)–(A31), since the covariant derivative $\overleftrightarrow{D}$ does not commute with the color matrix $t^a$ we need to specify the ordering between the two and verify that the resulting operator is gauge invariant. Let us consider, for example, the operator $\mathcal{O}_8(^1P_1)$,

$$\mathcal{O}_8(^1P_1) = \psi^\dagger\overleftrightarrow{D}t^a\chi\chi^\dagger\overleftrightarrow{D}t^a\psi, \tag{A3}$$

and the three different orderings:

$$[\psi^\dagger\overleftrightarrow{D}t^a\chi]^{(1)} \equiv -(\vec{D}\psi)^\dagger t^a\chi + \psi^\dagger\vec{D}t^a\chi, \tag{A4}$$

$$[\psi^\dagger\overleftrightarrow{D}t^a\chi]^{(2)} \equiv -(\vec{D}t^a\psi)^\dagger\chi + \psi^\dagger t^a\vec{D}\chi, \tag{A5}$$

$$[\psi^\dagger\overleftrightarrow{D}t^a\chi]^{(3)} \equiv -(\vec{D}\psi)^\dagger t^a\chi + \psi^\dagger t^a\vec{D}\chi. \tag{A6}$$

Under the gauge transformation

$$\psi \to (1 + i\omega^a t^a)\psi, \qquad \chi \to (1 + i\omega^a t^a)\chi, \qquad A^{a\mu} \to A^{a\mu} - \frac{1}{g}\partial^\mu\omega^a + f^{abc}A^{b\mu}\omega^c, \tag{A7}$$

(A4)–(A6) transform, respectively, as

$$\delta[\psi^\dagger\overleftrightarrow{D}t^a\chi]^{(1)} = f^{abc}\omega^c[\psi^\dagger\overleftrightarrow{D}t^b\chi]^{(1)} + f^{abc}\vec{\partial}\omega^c\psi^\dagger t^b\chi, \tag{A8}$$

$$\delta[\psi^\dagger\overleftrightarrow{D}t^a\chi]^{(2)} = f^{abc}\omega^c[\psi^\dagger\overleftrightarrow{D}t^b\chi]^{(2)} - f^{abc}\vec{\partial}\omega^c\psi^\dagger t^b\chi, \tag{A9}$$

$$\delta[\psi^\dagger\overleftrightarrow{D}t^a\chi]^{(3)} = f^{abc}\omega^c[\psi^\dagger\overleftrightarrow{D}t^b\chi]^{(3)}. \tag{A10}$$

Only the last ordering leads to a gauge invariant definition of $\mathcal{O}_8(^1P_1)$:

$$\delta\mathcal{O}_8(^1P_1) = f^{abc}\omega^c([\psi^\dagger\overleftrightarrow{D}t^b\chi]^{(3)}[\chi^\dagger\overleftrightarrow{D}t^a\psi]^{(3)} + [\psi^\dagger\overleftrightarrow{D}t^a\chi]^{(3)}[\chi^\dagger\overleftrightarrow{D}t^b\psi]^{(3)}) = 0. \tag{A11}$$

Therefore, we define

$$\psi^\dagger\overleftrightarrow{D}t^a\chi \equiv -(\vec{D}\psi)^\dagger t^a\chi + \psi^\dagger t^a\vec{D}\chi. \tag{A12}$$

Generalizing to operators containing more than one covariant derivative, we define

$$\psi^\dagger\overleftrightarrow{D}^{i_1}\ldots\overleftrightarrow{D}^{i_n}t^a\chi \equiv (-1)^n(\vec{D}^{i_1}\ldots\vec{D}^{i_n}\psi)^\dagger t^a\chi + (-1)^{n-1}(\vec{D}^{i_2}\ldots\vec{D}^{i_n}\psi)^\dagger t^a\vec{D}^{i_1}\chi + \ldots + \psi^\dagger t^a\vec{D}^{i_1}\ldots\vec{D}^{i_n}\chi. \tag{A13}$$

The singlet-octet transition operators are denoted by $\mathcal{O}_{1-8}(^{2S+1}L_J, {}^{2S'+1}L'_{J'})$ or $\mathcal{O}_{8-1}(^{2S+1}L_J, {}^{2S'+1}L'_{J'})$. In the first case the first set of quantum numbers refers to the $Q\bar{Q}$ pair in a color singlet state, the second to the $Q\bar{Q}$ pair in the octet state, while in the second case the first set of quantum numbers refers to the $Q\bar{Q}$ pair in a color octet state and the second one to the $Q\bar{Q}$ pair in the singlet state. In some cases, it has been found convenient to introduce singlet-octet transition operators that annihilate (create) states containing a $Q\bar{Q}$ pair and a gluon in which the total angular momentum $J$ of the quark-antiquark pair does not have a definite value (a definite value could be attributed by further decomposing these operators in irreducible spherical tensors). This is the case of the operators $\mathcal{T}^{(i)}_{1-8}(^3S_1, {}^3P)$ and $\mathcal{T}^{(i)}_{8-1}(^3S_1, {}^3P)$. In these cases, we cannot use the quantum number $J$ and we have to denote the state just by the orbital angular momentum and spin quantum numbers.

The symbols $A^{(i}B^{j)}$ and $S^{((ij)}A^{k)}$, used in the definitions of some four-fermion operators denote symmetric and traceless two and three indices tensors, according to

$$A^{(i}B^{j)} = \frac{A^iB^j + A^jB^i}{2} - \frac{\delta^{ij}}{3}\vec{A}\cdot\vec{B}, \quad \text{(A14)}$$

$$S^{((ij)}A^{k)} = \frac{1}{3}(S^{(ij)}A^k + S^{(ik)}A^j + S^{(jk)}A^i) - \frac{2}{15}(\delta^{ij}\delta^{lk} + \delta^{ik}\delta^{lj} + \delta^{jk}\delta^{li})S^{(ml)}A^m. \quad \text{(A15)}$$

For some details on the decomposition of Cartesian tensors in terms of irreducible spherical tensors see [12].

(i) Operators of dimension 6

$$\mathcal{O}_1(^1S_0) = \psi^\dagger\chi\chi^\dagger\psi, \quad \mathcal{O}_1(^3S_1) = \psi^\dagger\vec{\sigma}\chi\cdot\chi^\dagger\vec{\sigma}\psi. \quad \text{(A16)}$$

$$\mathcal{O}_8(^1S_0) = \psi^\dagger t^a\chi\chi^\dagger t^a\psi, \quad \mathcal{O}_8(^3S_1) = \psi^\dagger\vec{\sigma}t^a\chi\cdot\chi^\dagger\vec{\sigma}t^a\psi. \quad \text{(A17)}$$

(ii) Operators of dimension 8

$$\mathcal{P}_1(^1S_0) = \frac{1}{2}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2\chi\chi^\dagger\psi + \text{H.c.},$$
$$\mathcal{P}_1(^3S_1) = \frac{1}{2}\psi^\dagger\vec{\sigma}\chi\cdot\chi^\dagger\vec{\sigma}\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2\psi + \text{H.c.},$$
$$\mathcal{P}_1(^3S_1, {}^3D_1) = \frac{1}{2}\psi^\dagger\sigma^i\chi\chi^\dagger\sigma^j\left(-\frac{i}{2}\right)^2\overleftrightarrow{D}^{(i}\overleftrightarrow{D}^{j)}\psi + \text{H.c.}. \quad \text{(A18)}$$

$$\mathcal{O}_1(^1P_1) = \psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\right)\chi\cdot\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\right)\psi,$$
$$\mathcal{O}_1(^3P_0) = \frac{1}{3}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\cdot\vec{\sigma}\right)\chi\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\cdot\vec{\sigma}\right)\psi,$$
$$\mathcal{O}_1(^3P_1) = \frac{1}{2}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\times\vec{\sigma}\right)\chi\cdot\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\times\vec{\sigma}\right)\psi,$$
$$\mathcal{O}_1(^3P_2) = \psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^{(i}\sigma^{j)}\right)\chi\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^{(i}\sigma^{j)}\right)\psi. \quad \text{(A19)}$$

$$\mathcal{P}_8(^1S_0) = \frac{1}{2}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2 t^a\chi\chi^\dagger t^a\psi + \text{H.c.},$$
$$\mathcal{P}_8(^3S_1) = \frac{1}{2}\psi^\dagger\vec{\sigma}t^a\chi\cdot\chi^\dagger\vec{\sigma}\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2 t^a\psi + \text{H.c.},$$
$$\mathcal{P}_8(^3S_1, {}^3D_1) = \frac{1}{2}\psi^\dagger\sigma^i t^a\chi\chi^\dagger\sigma^j\left(-\frac{i}{2}\right)^2\overleftrightarrow{D}^{(i}\overleftrightarrow{D}^{j)} t^a\psi + \text{H.c.}. \quad \text{(A20)}$$

$$\mathcal{O}_8(^1P_1) = \psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\right)t^a\chi\cdot\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\right)t^a\psi,$$
$$\mathcal{O}_8(^3P_0) = \frac{1}{3}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\cdot\vec{\sigma}\right)t^a\chi\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\cdot\vec{\sigma}\right)t^a\psi,$$
$$\mathcal{O}_8(^3P_1) = \frac{1}{2}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\times\vec{\sigma}\right)t^a\chi\cdot\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\times\vec{\sigma}\right)t^a\psi,$$
$$\mathcal{O}_8(^3P_2) = \psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^{(i}\sigma^{j)}\right)t^a\chi\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^{(i}\sigma^{j)}\right)t^a\psi. \quad \text{(A21)}$$

$$\mathcal{O}_{1\,\text{cm}} = \psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\right)\times\vec{\sigma}\chi\cdot\vec{\nabla}(\chi^\dagger\psi) + \text{H.c.},$$
$$\mathcal{O}'_{1\,\text{cm}} = -\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\right)\chi\cdot\vec{\nabla}\times(\chi^\dagger\vec{\sigma}\psi) + \text{H.c.},$$
$$\mathcal{O}_{8\,\text{cm}} = \psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\right)\times\vec{\sigma}t^a\chi\cdot\vec{D}_{ab}(\chi^\dagger t^b\psi) + \text{H.c.},$$
$$\mathcal{O}'_{8\,\text{cm}} = -\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\right)t^a\chi\cdot\vec{D}_{ab}\times(\chi^\dagger t^b\vec{\sigma}\psi) + \text{H.c.}. \quad \text{(A22)}$$

$$\mathcal{P}_{1a\,\text{cm}} = \nabla^i(\psi^\dagger\sigma^j\chi)\nabla^i(\chi^\dagger\sigma^j\psi),$$
$$\mathcal{P}_{1b\,\text{cm}} = \vec{\nabla}\cdot(\psi^\dagger\vec{\sigma}\chi)\vec{\nabla}\cdot(\chi^\dagger\vec{\sigma}\psi),$$
$$\mathcal{P}_{1c\,\text{cm}} = \vec{\nabla}(\psi^\dagger\chi)\cdot\vec{\nabla}(\chi^\dagger\psi),$$
$$\mathcal{P}_{8a\,\text{cm}} = D^i_{ab}(\psi^\dagger t^a\sigma^j\chi)D^i_{ac}(\chi^\dagger t^c\sigma^j\psi),$$
$$\mathcal{P}_{8b\,\text{cm}} = \vec{D}_{ab}\cdot(\psi^\dagger t^b\vec{\sigma}\chi)\vec{D}_{ac}\cdot(\chi^\dagger t^c\vec{\sigma}\psi),$$
$$\mathcal{P}_{8c\,\text{cm}} = \vec{D}_{ab}(\psi^\dagger t^b\chi)\cdot\vec{D}_{ac}(\chi^\dagger t^c\psi). \quad \text{(A23)}$$

$$\mathcal{S}_{1-8}(^1S_0, {}^3S_1) = \frac{1}{2}\psi^\dagger g\vec{B}\cdot\vec{\sigma}\chi\chi^\dagger\psi + \text{H.c.},$$
$$\mathcal{S}_{1-8}(^3S_1, {}^1S_0) = \frac{1}{2}\psi^\dagger g\vec{B}\chi\cdot\chi^\dagger\vec{\sigma}\psi + \text{H.c.}. \quad \text{(A24)}$$

(iii) Operators of dimension 9

$$\begin{aligned}
\mathcal{T}_{1-8}({}^1S_0, {}^1P_1) &= \frac{1}{2}\psi^\dagger\chi\chi^\dagger(\overleftrightarrow{D}\cdot g\vec{E} + g\vec{E}\cdot\overleftrightarrow{D})\psi + \text{H.c.},\\
\mathcal{F}_{8\text{-}8}({}^1S_0, {}^1P_1) &= \frac{1}{2}f^{abc}\psi^\dagger t^a\chi\chi^\dagger t^b(\overleftrightarrow{D}\cdot g\vec{E}^c + g\vec{E}^c\cdot\overleftrightarrow{D})\psi + \text{H.c.},\\
\mathcal{D}_{8\text{-}8}({}^1S_0, {}^1P_1) &= \frac{1}{2}d^{abc}\psi^\dagger t^a\chi\chi^\dagger t^b(\overleftrightarrow{D}\cdot g\vec{E}^c + g\vec{E}^c\cdot\overleftrightarrow{D})\psi + \text{H.c.},\\
\mathcal{D}^{(0)}_{8\text{-}8}({}^3S_1, {}^3P) &= \frac{1}{6}d^{abc}\psi^\dagger t^a\vec{\sigma}\chi\cdot\chi^\dagger\vec{\sigma}(\overleftrightarrow{D}\cdot g\vec{E}^b + g\vec{E}^b\cdot\overleftrightarrow{D})t^c\psi + \text{H.c.},\\
\mathcal{D}^{(2)}_{8-8}({}^3S_1, {}^3P) &= \frac{1}{2}d^{abc}\psi^\dagger t^a\sigma^i\chi\chi^\dagger\sigma^j(\overleftrightarrow{D}^{(i}g\vec{E}^{bj)} + g\vec{E}^{(bi}\overleftrightarrow{D}^{j)})t^c\psi + \text{H.c.},\\
\mathcal{F}_8({}^1S_0) &= \frac{i}{2}f^{abc}\psi^\dagger(\vec{D}\cdot\vec{E})^b t^c\chi\chi^\dagger t^a\psi + \text{H.c.},\\
\mathcal{T}_{1-8}({}^1P_1, {}^1S_0) &= \frac{1}{2}\psi^\dagger g\vec{E}\chi\cdot\chi^\dagger D\psi + \text{H.c.},\\
\mathcal{T}^{(0)}_{1-8}({}^3S_1, {}^3P) &= \frac{1}{6}\psi^\dagger\vec{\sigma}\chi\cdot\chi^\dagger\vec{\sigma}(\overleftrightarrow{D}\cdot g\vec{E} + g\vec{E}\cdot\overleftrightarrow{D})\psi + \text{H.c.},\\
\mathcal{T}^{(1)}_{1-8}({}^3S_1, {}^3P) &= \frac{1}{4}\psi^\dagger\vec{\sigma}\chi\cdot\chi^\dagger\vec{\sigma}\times(-\overleftrightarrow{D}\times g\vec{E} - g\vec{E}\times\overleftrightarrow{D})\psi + \text{H.c.},\\
\mathcal{T}^{(1)\prime}_{1-8}({}^3S_1, {}^3P) &= \frac{1}{4}\psi^\dagger\vec{\sigma}\chi\cdot\chi^\dagger\vec{\sigma}\times(\overleftrightarrow{D}\times g\vec{E} - g\vec{E}\times\overleftrightarrow{D})\psi + \text{H.c.},\\
\mathcal{T}^{(1)\prime}_{8-1}({}^3S_1, {}^3P) &= \frac{1}{4}\psi^\dagger t^a\vec{\sigma}\chi\cdot\chi^\dagger\vec{\sigma}\times(\overleftrightarrow{D}\times g\vec{E}^a - g\vec{E}^a\times\overleftrightarrow{D})\psi + \text{H.c.},\\
\mathcal{T}^{(2)}_{1-8}({}^3S_1, {}^3P) &= \frac{1}{2}\psi^\dagger\sigma^i\chi\chi^\dagger\sigma^j(\overleftrightarrow{D}^{(i}g\vec{E}^{j)} + g\vec{E}^{(i}\overleftrightarrow{D}^{j)})\psi + \text{H.c.},\\
\mathcal{T}_{1-8}({}^3P_0, {}^3S_1) &= \frac{1}{6}\psi^\dagger(\overleftrightarrow{D}\cdot\vec{\sigma})\chi\chi^\dagger\vec{\sigma}\cdot g\vec{E}\psi + \text{H.c.},\\
\mathcal{T}_{1-8}({}^3P_1, {}^3S_1) &= \frac{1}{4}\psi^\dagger(\overleftrightarrow{D}\times\vec{\sigma})\chi\cdot\chi^\dagger\vec{\sigma}\times g\vec{E}\psi + \text{H.c.},\\
\mathcal{T}_{1-8}({}^3P_2, {}^3S_1) &= \frac{1}{2}\psi^\dagger(\overleftrightarrow{D}^{(i}\sigma^{j)})\chi\chi^\dagger\sigma^{(i}gE^{j)}\psi + \text{H.c.}.
\end{aligned} \tag{A25}$$

(iv) Operators of dimension 10

$$\begin{aligned}
\mathcal{Q}'_1({}^1S_0) &= \psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2\chi\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2\psi,\\
\mathcal{Q}''_1({}^1S_0) &= \frac{1}{2}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\right)^4\chi\chi^\dagger\psi + \text{H.c.},\\
\mathcal{Q}'_1({}^3S_1) &= \psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2\vec{\sigma}\chi\cdot\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2\vec{\sigma}\psi,\\
\mathcal{Q}''_1({}^3S_1) &= \frac{1}{2}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\right)^4\vec{\sigma}\chi\cdot\chi^\dagger\vec{\sigma}\psi + \text{H.c.},\\
\mathcal{Q}'_1({}^3S_1, {}^3D_1) &= \frac{1}{2}\psi^\dagger\left(-\frac{i}{2}\right)^2\overleftrightarrow{D}^{(i}\overleftrightarrow{D}^{j)}\sigma^i\chi\chi^\dagger\sigma^j\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2\psi + \text{H.c.},\\
\mathcal{Q}''_1({}^3S_1, {}^3D_1) &= \frac{1}{2}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2\left(-\frac{i}{2}\right)^2\overleftrightarrow{D}^{(i}\overleftrightarrow{D}^{j)}\sigma^i\chi\chi^\dagger\sigma^j\psi + \text{H.c.}.
\end{aligned} \tag{A26}$$

$$\begin{aligned}
\mathcal{P}_1({}^1P_1) &= \frac{1}{2}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2\left(-\frac{i}{2}\overleftrightarrow{D}^i\right)\chi\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^i\right)\psi + \text{H.c.},\\
\mathcal{P}_1({}^3P_0) &= \frac{1}{6}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\cdot\vec{\sigma}\right)\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2\chi\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\cdot\vec{\sigma}\right)\psi + \text{H.c.},\\
\mathcal{P}_1({}^3P_1) &= \frac{1}{4}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\times\vec{\sigma}\right)\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2\chi\cdot\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\times\vec{\sigma}\right)\psi + \text{H.c.},\\
\mathcal{P}_1({}^3P_2) &= \frac{1}{2}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^{(i}\sigma^{j)}\right)\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2\chi\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^{(i}\sigma^{j)}\right)\psi + \text{H.c.},\\
\mathcal{P}_1({}^3P_2,{}^3F_2) &= \frac{1}{2}\psi^\dagger\left(-\frac{i}{2}\right)^2\overleftrightarrow{D}^{(i}\overleftrightarrow{D}^{j)}\left(-\frac{i}{2}\overleftrightarrow{D}\cdot\vec{\sigma}\right)\chi\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^{(i}\sigma^{j)}\right)\psi\\
&\quad-\frac{1}{5}\psi^\dagger\left(-\frac{i}{2}\right)\overleftrightarrow{D}^{(i}\sigma^{j)}\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2\chi\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^{(i}\sigma^{j)}\right)\psi + \text{H.c.}.
\end{aligned}\tag{A27}$$

$$\begin{aligned}
\mathcal{Q}_1({}^1D_2) &= \psi^\dagger\left(-\frac{i}{2}\right)^2\overleftrightarrow{D}^{(i}\overleftrightarrow{D}^{j)}\chi\chi^\dagger\left(-\frac{i}{2}\right)^2\overleftrightarrow{D}^{(i}\overleftrightarrow{D}^{j)}\psi,\\
\mathcal{Q}_1({}^3D_3) &= \psi^\dagger\left(-\frac{i}{2}\right)^2\overleftrightarrow{D}^{((i}\overleftrightarrow{D}^{j)}\sigma^{l)}\chi\chi^\dagger\left(-\frac{i}{2}\right)^2\overleftrightarrow{D}^{((i}\overleftrightarrow{D}^{j)}\sigma^{l))}\psi,\\
\mathcal{Q}_1({}^3D_2) &= \frac{2}{3}\psi^\dagger\left(-\frac{i}{2}\right)^2\left(\varepsilon^{ilm}\overleftrightarrow{D}^{(j}\overleftrightarrow{D}^{l)}\sigma^m+\frac{1}{2}\varepsilon^{ijl}\overleftrightarrow{D}^{(m}\overleftrightarrow{D}^{l)}\sigma^m\right)\chi\chi^\dagger\left(-\frac{i}{2}\right)^2\left(\varepsilon^{inp}\overleftrightarrow{D}^{(j}\overleftrightarrow{D}^{n)}\sigma^p+\frac{1}{2}\varepsilon^{ijn}\overleftrightarrow{D}^{(p}\overleftrightarrow{D}^{n}\sigma^p\right)\psi,\\
\mathcal{Q}_1({}^3D_1) &= \psi^\dagger\left(-\frac{i}{2}\right)^2\overleftrightarrow{D}^{(i}\overleftrightarrow{D}^{j)}\sigma^i\chi\chi^\dagger\left(-\frac{i}{2}\right)^2\overleftrightarrow{D}^{(l}\overleftrightarrow{D}^{j)}\sigma^l\psi.
\end{aligned}\tag{A28}$$

$$\begin{aligned}
\mathcal{Q}'_8({}^1S_0) &= \psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2 t^a\chi\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2 t^a\psi,\\
\mathcal{Q}''_8({}^1S_0) &= \frac{1}{2}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\right)^4 t^a\chi\chi^\dagger t^a\psi + \text{H.c.},\\
\mathcal{Q}'_8({}^3S_1) &= \psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2\vec{\sigma}t^a\chi\cdot\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2\vec{\sigma}t^a\psi,\\
\mathcal{Q}''_8({}^3S_1) &= \frac{1}{2}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\right)^4\vec{\sigma}t^a\chi\cdot\chi^\dagger\vec{\sigma}t^a\psi + \text{H.c.},\\
\mathcal{Q}'_8({}^3S_1,{}^3D_1) &= \frac{1}{2}\psi^\dagger\left(-\frac{i}{2}\right)^2\overleftrightarrow{D}^{(i}\overleftrightarrow{D}^{j)}\sigma^i t^a\chi\chi^\dagger\sigma^j\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2 t^a\psi + \text{H.c.},\\
\mathcal{Q}''_8({}^3S_1,{}^3D_1) &= \frac{1}{2}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2\left(-\frac{i}{2}\right)^2\overleftrightarrow{D}^{(i}\overleftrightarrow{D}^{j)}\sigma^i t^a\chi\chi^\dagger\sigma^j t^a\psi + \text{H.c.}.
\end{aligned}\tag{A29}$$

$$\begin{aligned}
\mathcal{P}_8({}^1P_1) &= \frac{1}{2}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2\left(-\frac{i}{2}\overleftrightarrow{D}^i\right)t^a\chi\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^i\right)t^a\psi + \text{H.c.},\\
\mathcal{P}_8({}^3P_0) &= \frac{1}{6}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\cdot\vec{\sigma}\right)\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2 t^a\chi\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\cdot\vec{\sigma}\right)t^a\psi + \text{H.c.},\\
\mathcal{P}_8({}^3P_1) &= \frac{1}{4}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\times\vec{\sigma}\right)\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2 t^a\chi\cdot\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}\times\vec{\sigma}\right)t^a\psi + \text{H.c.},\\
\mathcal{P}_8({}^3P_2) &= \frac{1}{2}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^{(i}\sigma^{j)}\right)\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2 t^a\chi\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^{(i}\sigma^{j)}\right)t^a\psi + \text{H.c.},\\
\mathcal{P}_8({}^3P_2,{}^3F_2) &= \frac{1}{2}\psi^\dagger\left(-\frac{i}{2}\right)^2\overleftrightarrow{D}^{(i}\overleftrightarrow{D}^{j)}\left(-\frac{i}{2}\overleftrightarrow{D}\cdot\vec{\sigma}\right)t^a\chi\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^{(i}\sigma^{j)}\right)t^a\psi\\
&\quad-\frac{1}{5}\psi^\dagger\left(-\frac{i}{2}\right)\overleftrightarrow{D}^{(i}\sigma^{j)}\left(-\frac{i}{2}\overleftrightarrow{D}\right)^2 t^a\chi\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^{(i}\sigma^{j)}\right)t^a\psi + \text{H.c.}.
\end{aligned}\tag{A30}$$

$$\mathcal{Q}_8(^1D_2) = \psi^\dagger\left(-\frac{i}{2}\right)^2 \overleftrightarrow{D}^{(i}\overleftrightarrow{D}^{j)} t^a \chi\chi^\dagger\left(-\frac{i}{2}\right)^2 \overleftrightarrow{D}^{(i}\overleftrightarrow{D}^{j)} t^a\psi,$$

$$\mathcal{Q}_8(^3D_3) = \psi^\dagger\left(-\frac{i}{2}\right)^2 \overleftrightarrow{D}^{((i}\overleftrightarrow{D}^{j)}\sigma^{l)} t^a \chi\chi^\dagger\left(-\frac{i}{2}\right)^2 \overleftrightarrow{D}^{((i}\overleftrightarrow{D}^{j)}\sigma^{l))} t^a\psi,$$

$$\mathcal{Q}_8(^3D_2) = \frac{2}{3}\psi^\dagger\left(-\frac{i}{2}\right)^2\left(\varepsilon^{ilm}\overleftrightarrow{D}^{(j}\overleftrightarrow{D}^{l)}\sigma^m + \frac{1}{2}\varepsilon^{ijl}\overleftrightarrow{D}^{(m}\overleftrightarrow{D}^{l)}\sigma^m\right)t^a\chi\chi^\dagger\left(-\frac{i}{2}\right)^2\left(\varepsilon^{inp}\overleftrightarrow{D}^{(j}\overleftrightarrow{D}^{n)}\sigma^p + \frac{1}{2}\varepsilon^{ijn}\overleftrightarrow{D}^{(p}\overleftrightarrow{D}^{n)}\sigma^p\right)t^a\psi,$$

$$\mathcal{Q}_8(^3D_1) = \psi^\dagger\left(-\frac{i}{2}\right)^2 \overleftrightarrow{D}^{(i}\overleftrightarrow{D}^{j)}\sigma^i t^a \chi\chi^\dagger\left(-\frac{i}{2}\right)^2 \overleftrightarrow{D}^{(l}\overleftrightarrow{D}^{j)}\sigma^l t^a\psi. \qquad \text{(A31)}$$

## APPENDIX B: SUMMARY OF MATCHING COEFFICIENTS

In the following, we list all the imaginary parts of the matching coefficients of the four-fermion operators up to dimension 10, calculated at $O(\alpha_s^2)$ in the strong coupling constant in Sec. III.

In the presentation of the results we give for completeness also the matching coefficients obtained by using a basis of operators that includes $1/2(\mathcal{Q}_8'(^3S_1) - \mathcal{Q}_8''(^3S_1))$, $1/2(\mathcal{Q}_8'(^3S_1, {}^3D_1) - \mathcal{Q}_8''(^3S_1, {}^3D_1))$ and $\mathcal{T}^{(1)\prime}_{8-1}(^3S_1, {}^3P)$ instead of $\mathcal{T}_{1-8}(^3P_0, {}^3S_1)$, $\mathcal{T}_{1-8}(^3P_1, {}^3S_1)$, $\mathcal{T}_{1-8}(^3P_2, {}^3S_1)$. It is understood that when this basis is used, the coefficients $\mathrm{Im}t_{1-8}(^3P_J, {}^3S_1)$, with $J = 0, 1, 2$ are set to 0. Vice versa if our basis contains the operators $\mathcal{T}_{1-8}(^3P_J, {}^3S_1)$, with $J = 0, 1, 2$ the coefficients $\mathrm{Im}h_8'(^3S_1) - \mathrm{Im}h_8''(^3S_1)$, $\mathrm{Im}h_8'(^3S_1, {}^3D_1) - \mathrm{Im}h_8''(^3S_1, {}^3D_1)$ and $\mathrm{Im}t^{(1)\prime}_{8-1}(^3S_1, {}^3P)$ are set to 0.

| Operator of dim. 6 | Matching coefficient | Im (Value) |
|---|---|---|
| $\mathcal{O}_1(^1S_0)$ | $\mathrm{Im}f_1(^1S_0)$ | $\alpha_s^2\pi\frac{C_F}{2N_c}$ [4] |
| $\mathcal{O}_1(^3S_1)$ | $\mathrm{Im}f_1(^3S_1)$ | 0 |
| $\mathcal{O}_8(^1S_0)$ | $\mathrm{Im}f_8(^1S_0)$ | $\alpha_s^2\pi\frac{N_c^2-4}{4N_c}$ [4] |
| $\mathcal{O}_8(^3S_1)$ | $\mathrm{Im}f_8(^3S_1)$ | $\frac{1}{6}\alpha_s^2\pi n_f$ [4] |

| Operator of dim. 8 | Matching coefficient | Im (Value) |
|---|---|---|
| $\mathcal{P}_1(^1S_0)$ | $\mathrm{Im}g_1(^1S_0)$ | $-\frac{4}{3}\alpha_s^2\pi\frac{C_F}{2N_c}$ [4] |
| $\mathcal{P}_1(^3S_1)$ | $\mathrm{Im}g_1(^3S_1)$ | 0 |
| $\mathcal{P}_1(^3S_1, {}^3D_1)$ | $\mathrm{Im}g_1(^3S_1, {}^3D_1)$ | 0 |
| $\mathcal{O}_1(^1P_1)$ | $\mathrm{Im}f_1(^1P_1)$ | 0 |
| $\mathcal{O}_1(^3P_0)$ | $\mathrm{Im}f_1(^3P_0)$ | $3\alpha_s^2\pi\frac{C_F}{2N_c}$ [4] |
| $\mathcal{O}_1(^3P_1)$ | $\mathrm{Im}f_1(^3P_1)$ | 0 |
| $\mathcal{O}_1(^3P_2)$ | $\mathrm{Im}f_1(^3P_2)$ | $\frac{4}{5}\alpha_s^2\pi\frac{C_F}{2N_c}$ [4] |
| $\mathcal{S}_{1-8}(^1S_0, {}^3S_1)$ | $\mathrm{Im}s_{1-8}(^1S_0, {}^3S_1)$ | $\frac{\alpha_s^2\pi}{4N_c}(\frac{1}{3}nf - N_c)$ |
| $\mathcal{S}_{1-8}(^3S_1, {}^1S_0)$ | $\mathrm{Im}s_{1-8}(^3S_1, {}^3S_0)$ | 0 |
| $\mathcal{P}_8(^1S_0)$ | $\mathrm{Im}g_8(^1S_0)$ | $-\frac{4}{3}\alpha_s^2\pi\frac{N_c^2-4}{4N_c}$ [4,17] |
| $\mathcal{P}_8(^3S_1)$ | $\mathrm{Im}g_8(^3S_1)$ | $-\frac{2}{9}\alpha_s^2\pi n_f$ [4,17] |
| $\mathcal{P}_8(^3S_1, {}^3D_1)$ | $\mathrm{Im}g_8(^3S_1, {}^3D_1)$ | $-\frac{1}{6}\alpha_s^2\pi n_f$ [4,17] |
| $\mathcal{O}_8(^1P_1)$ | $\mathrm{Im}f_8(^1P_1)$ | $\frac{\alpha_s^2\pi N_c}{12}$ [4,17] |
| $\mathcal{O}_8(^3P_0)$ | $\mathrm{Im}f_8(^3P_0)$ | $3\alpha_s^2\pi\frac{N_c^2-4}{4N_c}$ [4,17] |
| $\mathcal{O}_8(^3P_1)$ | $\mathrm{Im}f_8(^3P_1)$ | 0 [4,17] |
| $\mathcal{O}_8(^3P_2)$ | $\mathrm{Im}f_8(^3P_2)$ | $\frac{4}{5}\alpha_s^2\pi\frac{N_c^2-4}{4N_c}$ [4,17] |
| $\mathcal{O}_{1\,\mathrm{cm}}$ | $\mathrm{Im}f_{1\,\mathrm{cm}}$ | $\frac{1}{4}\alpha_s^2\pi\frac{C_F}{2N_c}$ |
| $\mathcal{O}'_{1\,\mathrm{cm}}$ | $\mathrm{Im}f'_{1\,\mathrm{cm}}$ | 0 |
| $\mathcal{O}_{8\,\mathrm{cm}}$ | $\mathrm{Im}f_{8\,\mathrm{cm}}$ | $\frac{1}{4}\alpha_s^2\pi\frac{N_c^2-4}{4N_c}$ |
| $\mathcal{O}'_{8\,\mathrm{cm}}$ | $\mathrm{Im}f'_{8\,\mathrm{cm}}$ | $\frac{1}{24}\alpha_s^2\pi n_f$ |
| $\mathcal{P}_{1a\,\mathrm{cm}}$ | $\mathrm{Im}g_{1a\,\mathrm{cm}}$ | 0 |
| $\mathcal{P}_{1b\,\mathrm{cm}}$ | $\mathrm{Im}g_{1b\,\mathrm{cm}}$ | 0 |
| $\mathcal{P}_{1c\,\mathrm{cm}}$ | $\mathrm{Im}g_{1c\,\mathrm{cm}}$ | $-\frac{1}{4}\alpha_s^2\pi\frac{C_F}{2N_c}$ |
| $\mathcal{P}_{8a\,\mathrm{cm}}$ | $\mathrm{Im}g_{8a\,\mathrm{cm}}$ | $-\frac{1}{24}\alpha_s^2\pi n_f$ |
| $\mathcal{P}_{8b\,\mathrm{cm}}$ | $\mathrm{Im}g_{8b\,\mathrm{cm}}$ | 0 |
| $\mathcal{P}_{8c\,\mathrm{cm}}$ | $\mathrm{Im}g_{8c\,\mathrm{cm}}$ | $-\frac{1}{4}\alpha_s^2\pi\frac{N_c^2-4}{4N_c}$ |

| Operator of dim. 9 | Matching coefficient | Im (Value) |
|---|---|---|
| $T^{(1)\prime}_{1-8}(^3S_1, {}^3P)$ | $\mathrm{Im}t^{(1)\prime}_{1-8}(^3S_1, {}^3P)$ | $-\frac{1}{8}\alpha_s^2\pi\frac{N_c^2-4}{4N_c^2}$ |
| $\mathcal{T}^{(1)\prime}_{8-1}(^3S_1, {}^3P)$ | $\mathrm{Im}t^{(1)\prime}_{8-1}(^3S_1, {}^3P)$ | $\frac{1}{24}\alpha_s^2\pi\frac{n_f}{N_c} + \frac{1}{48}\alpha_s^2\pi - \frac{1}{8}\alpha_s^2\pi\frac{C_F}{N_c}$ |
| $\mathcal{T}_{1-8}(^3P_0, {}^3S_1)$ | $\mathrm{Im}t_{1-8}(^3P_0, {}^3S_1)$ | $-\frac{3}{2}\alpha_s^2\pi\frac{C_F}{2N_c} + (\frac{61}{240} + \frac{7}{192}\frac{n_f}{N_c})\alpha_s^2\pi$ |
| $\mathcal{T}_{1-8}(^3P_1, {}^3S_1)$ | $\mathrm{Im}t_{1-8}(^3P_1, {}^3S_1)$ | $(\frac{1}{72} + \frac{107}{576}\frac{n_f}{N_c})\alpha_s^2\pi$ |
| $\mathcal{T}_{1-8}(^3P_2, {}^3S_1)$ | $\mathrm{Im}t_{1-8}(^3P_2, {}^3S_1)$ | $(\frac{1}{10} + \frac{25}{576}\frac{n_f}{N_c})\alpha_s^2\pi$ |

| Operator of dim. 10 | Matching coefficient | Im (Value) |
|---|---|---|
| $\mathcal{Q}_1'(^1S_0)$ | $\mathrm{Im}h_1'(^1S_0)$ | $\frac{10}{9}\alpha_s^2\pi\frac{C_F}{2N_c} - \frac{1}{48}\alpha_s^2\pi$ |
| $\mathcal{Q}_1''(^1S_0)$ | $\mathrm{Im}h_1''(^1S_0)$ | $\frac{2}{5}\alpha_s^2\pi\frac{C_F}{2N_c} + \frac{1}{48}\alpha_s^2\pi$ |
| $\mathcal{Q}_1'(^3S_1)$ | $\mathrm{Im}h_1'(^3S_1)$ | $\frac{1}{12}\alpha_s^2\pi\frac{N_c^2-4}{4N_c^2}$ |
| $\mathcal{Q}_1''(^3S_1)$ | $\mathrm{Im}h_1''(^3S_1)$ | $-\frac{1}{12}\alpha_s^2\pi\frac{N_c^2-4}{4N_c^2}$ |
| $\mathcal{Q}_1'(^3S_1, {}^3D_1)$ | $\mathrm{Im}h_1'(^3S_1, {}^3D_1)$ | $\frac{1}{4}\alpha_s^2\pi\frac{N_c^2-4}{4N_c^2}$ |
| $\mathcal{Q}_1''(^3S_1, {}^3D_1)$ | $\mathrm{Im}h_1''(^3S_1, {}^3D_1)$ | $-\frac{1}{4}\alpha_s^2\pi\frac{N_c^2-4}{4N_c^2}$ [9] |

| Operator of dim. 10 | Matching coefficient | Im (Value) |
|---|---|---|
| $\mathcal{P}_1(^1P_1)$ | $\mathrm{Im}g_1(^1P_1)$ | 0 |
| $\mathcal{P}_1(^3P_0)$ | $\mathrm{Im}g_1(^3P_0)$ | $-7\alpha_s^2\pi\frac{C_F}{2N_c}$ |
| $\mathcal{P}_1(^3P_1)$ | $\mathrm{Im}g_1(^3P_1)$ | 0 |
| $\mathcal{P}_1(^3P_2)$ | $\mathrm{Im}g_1(^3P_2)$ | $-\frac{8}{5}\alpha_s^2\pi\frac{C_F}{2N_c}$ |
| $\mathcal{P}_1(^3P_2,{}^3F_2)$ | $\mathrm{Im}g_1(^3P_2,{}^3F_2)$ | $-\frac{20}{21}\alpha_s^2\pi\frac{C_F}{2N_c}$ |
| $\mathcal{Q}_1(^1D_2)$ | $\mathrm{Im}h_1(^1D_2)$ | $\frac{2}{15}\alpha_s^2\pi\frac{C_F}{2N_c}$ [16] |
| $\mathcal{Q}_1(^3D_1)$ | $\mathrm{Im}h_1(^3D_1)$ | 0 |
| $\mathcal{Q}_1(^3D_2)$ | $\mathrm{Im}h_1(^3D_2)$ | 0 |
| $\mathcal{Q}_1(^3D_3)$ | $\mathrm{Im}h_1(^3D_3)$ | 0 |
| $\mathcal{Q}_8'(^1S_0)$ | $\mathrm{Im}h_8'(^1S_0)$ | $\frac{10}{9}\alpha_s^2\pi\frac{N_c^2-4}{4N_c}$ |
| $\mathcal{Q}_8''(^1S_0)$ | $\mathrm{Im}h_8''(^1S_0)$ | $\frac{2}{5}\alpha_s^2\pi\frac{N_c^2-4}{4N_c}$ |
| $\frac{\mathcal{Q}_8'(^3S_1)+\mathcal{Q}_8''(^3S_1)}{2}$ | $\mathrm{Im}h_8'(^3S_1)+\mathrm{Im}h_8''(^3S_1)$ | $\frac{29}{108}\alpha_s^2\pi n_f+\frac{1}{108}\alpha_s^2\pi N_c$ |
| $\frac{\mathcal{Q}_8'(^3S_1,{}^3D_1)+\mathcal{Q}_8''(^3S_1,{}^3D_1)}{2}$ | $\mathrm{Im}h_8'(^3S_1,{}^3D_1)+\mathrm{Im}h_8''(^3S_1,{}^3D_1)$ | $\frac{23}{72}\alpha_s^2\pi n_f+\frac{1}{18}\alpha_s^2\pi N_c$ |
| $\frac{\mathcal{Q}_8'(^3S_1)-\mathcal{Q}_8''(^3S_1)}{2}$ | $\mathrm{Im}h_8'(^3S_1)-\mathrm{Im}h_8''(^3S_1)$ | $\frac{17}{108}\alpha_s^2\pi n_f-\frac{41}{108}\alpha_s^2\pi N_c+\frac{1}{3}\alpha_s^2\pi C_F$ |
| $\frac{\mathcal{Q}_8'(^3S_1,{}^3D_1)-\mathcal{Q}_8''(^3S_1,{}^3D_1)}{2}$ | $\mathrm{Im}h_8'(^3S_1,{}^3D_1)-\mathrm{Im}h_8''(^3S_1,{}^3D_1)$ | $\frac{17}{72}\alpha_s^2\pi n_f-\frac{23}{45}\alpha_s^2\pi N_c+\alpha_s^2\pi C_F$ |
| $\mathcal{P}_8(^1P_1)$ | $\mathrm{Im}g_8(^1P_1)$ | $-\frac{3}{20}\alpha_s^2\pi N_c$ |
| $\mathcal{P}_8(^3P_0)$ | $\mathrm{Im}g_8(^3P_0)$ | $-7\alpha_s^2\pi\frac{N_c^2-4}{4N_c}$ |
| $\mathcal{P}_8(^3P_1)$ | $\mathrm{Im}g_8(^3P_1)$ | 0 |
| $\mathcal{P}_8(^3P_2)$ | $\mathrm{Im}g_8(^3P_2)$ | $-\frac{8}{5}\alpha_s^2\pi\frac{N_c^2-4}{4N_c}$ |
| $\mathcal{P}_8(^3P_2,{}^3F_2)$ | $\mathrm{Im}g_8(^3P_2,{}^3F_2)$ | $-\frac{20}{21}\alpha_s^2\pi\frac{N_c^2-4}{4N_c}$ |
| $\mathcal{Q}_8(^1D_2)$ | $\mathrm{Im}h_8(^1D_2)$ | $\frac{2}{15}\alpha_s^2\pi\frac{N_c^2-4}{4N_c}$ |
| $\mathcal{Q}_8(^3D_1)$ | $\mathrm{Im}h_8(^3D_1)$ | $\frac{1}{24}\alpha_s^2\pi n_f+\frac{1}{12}\alpha_s^2\pi N_c$ |
| $\mathcal{Q}_8(^3D_2)$ | $\mathrm{Im}h_8(^3D_2)$ | $\frac{1}{30}\alpha_s^2\pi N_c$ |
| $\mathcal{Q}_8(^3D_3)$ | $\mathrm{Im}h_8(^3D_3)$ | $\frac{1}{21}\alpha_s^2\pi N_c$ |


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