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Hadronic light-by-light scattering in the muon $g - 2$: A new short-distance constraint on pion exchange

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Recently it was pointed out that for the evaluation of the numerically dominant pion-exchange contribution to the hadronic light-by-light scattering correction in the muon $g - 2$, a fully off-shell pion-photon-photon form factor should be used. Following this proposal, we first derive a new short-distance constraint on the off-shell form factor which enters at the external vertex for the muon $g - 2$ and show that it is related to the quark condensate magnetic susceptibility in QCD. We then evaluate the pion-exchange contribution in the framework of large-N_C QCD using an off-shell form factor which fulfills all short-distance constraints. With a value for the magnetic susceptibility as estimated in the same large-N_C framework, we obtain the result $a_\mu^{\mathrm{LbyL};\pi^0} = (72 \pm 12) \times 10^{-11}$. Updating our earlier results for the contributions from the exchanges of the η and η' using simple vector-meson dominance form factors, we obtain $a_\mu^{\mathrm{LbyL;PS}} = (99 \pm 16) \times 10^{-11}$ for the sum of all light pseudoscalars. Combined with available evaluations for the other contributions to hadronic light-by-light scattering this leads to the new estimate $a_\mu^{\mathrm{LbyL;had}} = (116 \pm 40) \times 10^{-11}$.

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I. INTRODUCTION

The hadronic light-by-light scattering contribution to the muon $g - 2$ has a long and troubled history. The relevant physics involves the nonperturbative regime of QCD below about 1–2 GeV. Furthermore, no direct experimental information is available, in contrast to the hadronic vacuum polarization contribution to the $g - 2$, which can be related to the cross section $e^+e^- \to$ hadrons. Therefore various models have been used over the years, starting first with a simple loop of some constituent quarks [1]. Later more realistic hadronic models with exchanges of light pseudoscalars, scalars and other resonances and loops with charged pions have been employed [2]. However, the coupling of hadrons to photons will, in general, involve some form factors which are very model-dependent [$\rho - \gamma$ mixing as in vector-meson dominance (VMD) models]. In the absence of any direct experimental checks, the size and even the sign of the light-by-light scattering contribution to the muon $g - 2$ was therefore uncertain for a long time. Actually, the sign has changed several times over the years due to some errors in the complicated calculations.

In Ref. [3] a systematic approach was proposed, based on the chiral expansion [4] and the large-N_C counting [5] of the various contributing diagrams. Soon afterwards, two very extensive evaluations appeared, Refs. [6–8], based on slightly different hadronic models. However, they both had a sign error in the numerically dominating pseudoscalar-exchange contribution as was pointed out a few years later in Refs. [9,10] and confirmed in Refs. [11–13].

Reference [9] mainly concentrated on the neutral pion-exchange contribution where the pion-photon-photon form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ enters. In general, low-energy hadronic models for form factors, e.g. based on some constituent quark model or on some resonance Lagrangian, do not satisfy all the large momentum asymptotics required by QCD. Using these form factors in loop diagrams thus leads to cutoff-dependent results. Even if the cutoff is varied in a reasonable range, e.g. ~1–2 GeV, the corresponding model uncertainty is completely uncontrollable. In order to eliminate (or at least reduce) this cutoff dependence, new models for $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ were proposed in Ref. [14] and then applied to hadronic light-by-light scattering in Ref. [9]. These models are based on the large-N_C picture of QCD [5], where, in leading order in N_C, an (infinite) tower of narrow resonances contributes in each channel of a particular Green's function. The low-energy and short-distance behavior of these Green's functions is then matched with results from QCD, using chiral perturbation theory [4] and the operator product expansion (OPE) [15], respectively. Based on the experience gained in many examples of low-energy hadronic physics, and from the use of dispersion relations and spectral representations for two-point functions, it is then assumed that with a minimal number of resonances in a given channel one can get a reasonable good description of the QCD Green's function in the real world (minimal hadronic Ansatz). Often only the lowest-lying resonance is considered [lowest-meson dominance (LMD)] [16–20], as a generalization of VMD.

Reference [9] obtained the result $a_\mu^{\mathrm{LbyL};\pi^0} = (58 \pm 10) \times 10^{-11}$ for the pion and $a_\mu^{\mathrm{LbyL;PS}} = (83 \pm 12) \times 10^{-11}$ for the sum of all light pseudoscalars

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π^0, η, and η'. These results are close to the (sign corrected) values $a_\mu^{\mathrm{LbyL};\pi^0} = (59 \pm 9) \times 10^{-11}$ $[a_\mu^{\mathrm{LbyL;PS}} = (85 \pm 13) \times 10^{-11}]$ obtained in Ref. [6] and $a_\mu^{\mathrm{LbyL};\pi^0} = (57 \pm 4) \times 10^{-11}$ $[a_\mu^{\mathrm{LbyL;PS}} = (82.7 \pm 6.4) \times 10^{-11}]$ in Refs. [7,8]. The results for the (corrected) full contributions at that time read $a_\mu^{\mathrm{LbyL;had}} = (83 \pm 32) \times 10^{-11}$ [6] and $a_\mu^{\mathrm{LbyL;had}} = (89.6 \pm 15.4) \times 10^{-11}$ [7,8].

Later Ref. [21] pointed out that some additional QCD short-distance constraint was not taken into account for the exchanges of pseudoscalars and axial-vector resonances in Refs. [6–9]. The authors of Ref. [21] argued that if one imposes this constraint, no momentum-dependent form factor can be present at the external vertex which couples to the soft photon relevant for the magnetic moment. In the absence of such a form factor, Ref. [21] then got enhanced results compared to the earlier evaluations $a_\mu^{\mathrm{LbyL};\pi^0} = (77 \pm 7) \times 10^{-11}$ $[a_\mu^{\mathrm{LbyL;PS}} = (114 \pm 10) \times 10^{-11}]$ and $a_\mu^{\mathrm{LbyL;had}} = (136 \pm 25) \times 10^{-11}$. As discussed in Ref. [22], a part of the enhancement of the result for $a_\mu^{\mathrm{LbyL;had}}$ in Ref. [21] is actually due to a different treatment of the axial vectors (ideal mixing instead of nonet symmetry) and the omission of the negative contributions from scalar exchanges and the charged pion loop. It is therefore not entirely related to the new short-distance constraint. Thus the slightly lower estimate $a_\mu^{\mathrm{LbyL;had}} = (110 \pm 40) \times 10^{-11}$ has been employed in the reviews [22,23]. Very recently, the value $a_\mu^{\mathrm{LbyL;had}} = (105 \pm 26) \times 10^{-11}$ has been proposed in Ref. [24].

However, recently Refs. [25,26] stressed the fact that one should actually use fully off-shell form factors for the evaluation of the light-by-light scattering contribution. This seems to have been overlooked in the recent literature, in particular, in Refs. [9,21,22,27]. The on-shell form factors as used in Refs. [9,27] actually violate four-momentum conservation at the external vertex. While Ref. [21] had already pointed out this violation of momentum conservation at the external vertex, they then only considered *on-shell* pion form factors, an approximation which yields the so-called *pion-pole* contribution and not the more general *pion-exchange* contribution with off-shell form factors. Putting the pion on-shell at the external vertex automatically leads to a constant form factor.

In the present paper we revisit the pion-exchange contribution in view of the observations made in Refs. [25,26]. We first derive a new QCD short-distance constraint for the off-shell form factor which enters at the external vertex and show that it is related to the quark condensate magnetic susceptibility in QCD. We also comment on how our short-distance constraint is connected with the one derived in Ref. [21]. In the second part we evaluate the pion-exchange contribution in the framework of large-N_C QCD with off-shell form factors both at the internal and the external vertex, taking into account the new short-distance constraint and an estimate for the magnetic susceptibility in QCD in the same large-N_C framework.

Strictly speaking, the identification of the pion-exchange contribution is only possible, if the pion is on-shell (or nearly on-shell). If one is (far) off the mass shell of the exchanged particle, it is not possible to separate different contributions to the $g - 2$, unless one uses some particular model where for instance elementary pions can propagate. In this sense, only the pion-pole contribution with on-shell form factors can be defined, at least in principle, in a model-independent way, although the numerical result will in general still depend on the model used for the on-shell form factors, unless one would know the "true" form factors. On the other hand, the pion-pole contribution is only a part of the full result, since in general, e.g. using some resonance Lagrangian, the form factors will enter the calculation with off-shell momenta. In this respect, we view our evaluation as being a part of a full calculation of hadronic light-by-light scattering using a resonance Lagrangian whose coefficients are tuned in such a way as to systematically reproduce the relevant QCD short-distance constraints, e.g. along the lines of the resonance chiral theory developed in Ref. [28].

We should mention that recently another paper appeared [29] which evaluates the pion-exchange contribution using an off-shell form factor based on the nonlocal chiral quark model, obtaining the result $a_\mu^{\mathrm{LbyL};\pi^0} = (65 \pm 2) \times 10^{-11}$. We will comment on that paper below.

This paper is organized as follows. Section II contains the starting point for the calculation of the pion-exchange contribution to the muon $g - 2$, including the definition of the pion-photon-photon form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$. We also discuss the issue of using on-shell or off-shell form factors. In Sec. III we discuss several experimental and theoretical constraints on the form factor. In particular, we derive a new short-distance constraint on the off-shell form factor at the external vertex in hadronic light-by-light scattering. In Sec. IV we present a new evaluation of the pion-exchange contribution in the framework of large-N_C QCD and give some updated estimates for the η and η' exchange contributions using simple VMD form factors. We end with discussions and conclusions in Sec. V. In the appendix we give a parametrization of the numerical result for the pion-exchange contribution for arbitrary parameters of our model for the off-shell form factor.

II. THE PSEUDOSCALAR-EXCHANGE CONTRIBUTION

The numerically dominating contributions to hadronic light-by-light scattering are due to the neutral pseudoscalar-exchange diagrams shown in Fig. 1.

We first concentrate on the exchange of the neutral pion. The key object which enters the Feynman diagrams in Fig. 1 is the *off-shell* pion-photon-photon form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1 + q_2)^2, q_1^2, q_2^2)$ which is defined, up to small

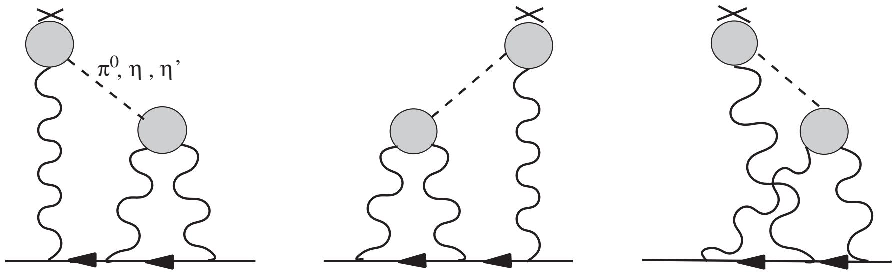

FIG. 1. The pseudoscalar-exchange contributions to hadronic light-by-light scattering. The shaded blobs represent the off-shell form factor $\mathcal{F}_{\mathrm{PS}^*\gamma^*\gamma^*}$ where PS $= \pi^0, \eta, \eta'$.

mixing effects with the states η and η', via the Green's function $\langle VVP\rangle$ in QCD

$$\int d^4x d^4y e^{i(q_1\cdot x+q_2\cdot y)}\langle 0|T\{j_\mu(x)j_\nu(y)P^3(0)\}|0\rangle = \varepsilon_{\mu\nu\alpha\beta}q_1^\alpha q_2^\beta \frac{i\langle\bar\psi\psi\rangle}{F_\pi}\frac{i}{(q_1+q_2)^2-m_\pi^2} \times \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2). \tag{1}$$

Here $j_\mu(x) = (\bar\psi\hat Q\gamma_\mu\psi)(x)$ [$\bar\psi \equiv (\bar u, \bar d, \bar s)$, $\hat Q =$ diag$(2,-1,-1)/3$ the charge matrix] is the light quark part of the electromagnetic current and $P^3 = \bar\psi i\gamma_5\frac{\lambda^3}{2}\psi = (\bar u i\gamma_5 u - \bar d i\gamma_5 d)/2$. Note that we denote by $\langle\bar\psi\psi\rangle$ the *single flavor* bilinear quark condensate. The form factor is of course Bose symmetric $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) = \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_2^2, q_1^2)$, as the two photons are indistinguishable.

The corresponding contribution to the muon $g-2$ may be worked out with the result [9][1]

$$a_\mu^{\mathrm{LbyL};\pi^0} = -e^6\int\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{1}{q_1^2q_2^2(q_1+q_2)^2[(p+q_1)^2-m_\mu^2][(p-q_2)^2-m_\mu^2]} \times\Bigg[\frac{\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(q_2^2, q_1^2, (q_1+q_2)^2)\mathcal{F}_{\pi^{0*}\gamma^*\gamma}(q_2^2, q_2^2, 0)}{q_2^2-m_\pi^2}T_1(q_1,q_2;p) + \frac{\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2)\mathcal{F}_{\pi^{0*}\gamma^*\gamma}((q_1+q_2)^2, (q_1+q_2)^2, 0)}{(q_1+q_2)^2-m_\pi^2}T_2(q_1,q_2;p)\Bigg], \tag{2}$$

with

$$T_1(q_1,q_2;p) = \tfrac{16}{3}(p\cdot q_1)(p\cdot q_2)(q_1\cdot q_2) - \tfrac{16}{3}(p\cdot q_2)^2q_1^2 - \tfrac{8}{3}(p\cdot q_1)(q_1\cdot q_2)q_2^2 + 8(p\cdot q_2)q_1^2q_2^2 - \tfrac{16}{3}(p\cdot q_2)(q_1\cdot q_2)^2 + \tfrac{16}{3}m_\mu^2q_1^2q_2^2 - \tfrac{16}{3}m_\mu^2(q_1\cdot q_2)^2, \tag{3}$$

$$T_2(q_1,q_2;p) = \tfrac{16}{3}(p\cdot q_1)(p\cdot q_2)(q_1\cdot q_2) - \tfrac{16}{3}(p\cdot q_1)^2q_2^2 + \tfrac{8}{3}(p\cdot q_1)(q_1\cdot q_2)q_2^2 + \tfrac{8}{3}(p\cdot q_1)q_1^2q_2^2 + \tfrac{8}{3}m_\mu^2q_1^2q_2^2 - \tfrac{8}{3}m_\mu^2(q_1\cdot q_2)^2, \tag{4}$$

where $p^2 = m_\mu^2$ and the external photon has now zero four-momentum. The first and the second graphs in Fig. 1 give rise to identical contributions, leading to the term with T_1, whereas the third graph gives the contribution involving T_2. The factor T_2 has been symmetrized with respect to the exchange $q_1 \leftrightarrow -q_2$.

Instead of the expressions in Eq. (2), Refs. [9,27] and maybe also earlier works, considered on-shell form factors, e.g. for the term involving T_2, one would write [26]

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2)\mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, (q_1+q_2)^2, 0). \tag{5}$$

Often the first argument of the on-shell form factor is omitted in the literature, i.e. the form factor is written as $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) \equiv \mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2)$. Although pole dominance might be expected to give a reasonable approximation, it is not correct as it was used in those references, as stressed in Refs. [21,25,26]. The point is that the form factor sitting at the external photon vertex in the pole approximation $\mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, (q_1+q_2)^2, 0)$ for $(q_1+q_2)^2 \neq m_\pi^2$ violates four-momentum conservation, since the momentum of the external (soft) photon vanishes. The latter requires $\mathcal{F}_{\pi^{0*}\gamma^*\gamma}((q_1+q_2)^2, (q_1+q_2)^2, 0)$. In order to avoid this inconsistency, Ref. [21] proposed to use instead

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2)\mathcal{F}_{\pi^0\gamma\gamma}(m_\pi^2, m_\pi^2, 0), \tag{6}$$

i.e. a constant form factor at the external vertex, which is given by the Wess-Zumino-Witten (WZW) anomaly [30]. The absence of a form factor at the external vertex in the

[1]To be precise, the corresponding expression with *on-shell* form factors is given in Ref. [9].

pion-pole approximation follows automatically, if one carefully considers the momentum dependence of the form factor. This procedure is also consistent with any quantum field theoretical framework for hadronic light-by-light scattering, for instance, if one uses a (resonance) Lagrangian to derive the form factors, and where a different treatment of the internal and external vertex, apart from the kinematics, is not possible. On the other hand, taking the diagram more literally, would require

$$\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) \times \mathcal{F}_{\pi^{0*}\gamma^*\gamma}((q_1+q_2)^2, (q_1+q_2)^2, 0), \quad (7)$$

as the more appropriate amplitude; see Eq. (2). References [25,26] advocate the use of fully off-shell form factors at both vertices and we will follow this approach in the rest of this paper. The difference to the procedure adopted in Ref. [21] will be important when we discuss their short-distance constraint.

III. EXPERIMENTAL AND THEORETICAL CONSTRAINTS ON THE PION-PHOTON-PHOTON FORM FACTOR

The form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2)$ defined in Eq. (1) is determined by nonperturbative physics of QCD and cannot (yet) be calculated from first principles. Therefore, various hadronic models have been used in the literature, sometimes combined with short-distance constraints from perturbative QCD at high momenta. At low energies, the form factor is normalized by the decay amplitude, $\mathcal{A}(\pi^0 \to \gamma\gamma) \equiv e^2 \mathcal{F}_{\pi^0\gamma\gamma}(m_\pi^2, 0, 0)$ in our conventions. In the chiral limit, $m_q \to 0$, $q = u, d, s$, this amplitude is fixed by the WZW anomaly

$$\mathcal{A}^{(0)}(\pi^0 \to \gamma\gamma) = -\frac{e^2 N_C}{12\pi^2 F_0}. \quad (8)$$

For massive light quarks, this expression receives corrections. In particular, the pion decay constant in the chiral limit F_0 is replaced by its physical counterpart $F_\pi = F_0[1 + \mathcal{O}(m_q)]$:

$$\mathcal{A}(\pi^0 \to \gamma\gamma) = -\frac{e^2 N_C}{12\pi^2 F_\pi}[1 + \mathcal{O}(m_q)]. \quad (9)$$

It turns out that the additional $\mathcal{O}(m_q)$ corrections in this relation are numerically small [31], so that one may drop them to a good approximation. The measured decay width $\Gamma(\pi^0 \to \gamma\gamma) = (7.74 \pm 0.6)$ eV [32] is then well reproduced for $F_\pi = 92.4$ MeV. Therefore, all hadronic models for the form factor have to satisfy the low-energy constraint

$$\mathcal{F}_{\pi^0\gamma\gamma}(m_\pi^2, 0, 0) = -\frac{N_C}{12\pi^2 F_\pi}. \quad (10)$$

Sometimes this normalizing value is used to define a constant "WZW form factor" $\mathcal{F}^{\rm WZW}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) \equiv -N_C/(12\pi^2 F_\pi)$. This notion of a constant form factor is, however, very misleading. For off-shell momenta away from the physical point in Eq. (10) the value of this WZW form factor has no physical meaning. Recall that the WZW effective Lagrangian only yields the first term in the low-energy and chiral expansion of the corresponding $\langle VVP \rangle$ Green's function.

For an on-shell pion, there is also experimental data available for one on-shell and one off-shell photon, from the process $e^+e^- \to e^+e^-\pi^0$. Several experiments [33,34] thereby fairly well confirm the Brodsky-Lepage [35] behavior for large Euclidean momentum

$$\lim_{Q^2\to\infty} \mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, -Q^2, 0) \sim -\frac{2F_\pi}{Q^2} \quad (11)$$

and any satisfactory model should reproduce this behavior.

Apart from these experimental constraints, any consistent hadronic model for the off-shell form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2)$ should match at large momentum with short-distance constraints from QCD that can be calculated using the OPE. In Ref. [14] the short-distance properties for the three-point function $\langle VVP \rangle$ in Eq. (1) in the chiral limit and assuming octet symmetry have been worked out in detail (see also Refs. [17,20] for earlier partial results). At least for the pion the chiral limit should be a not too bad approximation[2]; however, for the η and, in particular, for the non-Goldstone boson η' further analysis will be necessary.

It is important to notice that the Green's function $\langle VVP \rangle$ is an order parameter of chiral symmetry. Therefore, it vanishes to all orders in perturbative QCD in the chiral limit, so that the behavior at short distances is smoother than expected from naive power counting arguments. Two limits are of interest. In the first case, the two momenta become simultaneously large, which in position space describes the situation where the space-time arguments of all three operators tend towards the same point at the same rate. To leading order and up to corrections of order $\mathcal{O}(\alpha_s)$ one obtains the following behavior for the form factor[3]:

$$\lim_{\lambda\to\infty} \mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((\lambda q_1 + \lambda q_2)^2, (\lambda q_1)^2, (\lambda q_2)^2) = \frac{F_0}{3}\frac{1}{\lambda^2}\frac{q_1^2 + q_2^2 + (q_1+q_2)^2}{q_1^2 q_2^2} + \mathcal{O}\left(\frac{1}{\lambda^4}\right). \quad (12)$$

[2] As pointed out in Ref. [36], the integrals in Eq. (2) are infrared safe for $m_\pi \to 0$. This can also be seen within the effective field theory approach to light-by-light scattering proposed in Refs. [10,13].

[3] In the chiral limit, the relation between the off-shell form factor and the single invariant function $\mathcal{H}_V$ which appears in $\langle VVP \rangle$ is given by $\mathcal{F}_{\pi^{0*}_2\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) = -(2/3)(F_0/\langle\bar\psi\psi\rangle_0)(q_1+q_2)^2\mathcal{H}_V(q_1^2, q_2^2, (q_1+q_2)^2)$; see Ref. [14] for details.

The second situation of interest corresponds to the case where the relative distance between only two of the three operators in $\langle VVP\rangle$ becomes small. It so happens that the corresponding behaviors in momentum space involve, apart from the correlator $\langle AP\rangle$ which, in the chiral limit, is saturated by the single-pion intermediate state

$$\int d^4x e^{ip\cdot x}\langle 0|T\{A_\mu^a(x)P^b(0)\}|0\rangle = \delta^{ab}\langle\bar\psi\psi\rangle_0\frac{p_\mu}{p^2} \quad (13)$$

(we denote by $\langle\bar\psi\psi\rangle_0$ the single flavor bilinear quark condensate in the chiral limit), the two-point function $\langle VT\rangle$ of the vector current and the antisymmetric tensor density

$$\delta^{ab}(\Pi_{\rm VT})_{\mu\rho\sigma}(p) = \int d^4x e^{ip\cdot x}\langle 0|T\left\{V_\mu^a(x)\left(\bar\psi\sigma_{\rho\sigma}\frac{\lambda^b}{2}\psi\right)(0)\right\}|0\rangle, \quad (14)$$

with $\sigma_{\rho\sigma}=\frac{i}{2}[\gamma_\rho,\gamma_\sigma]$ (the similar correlator between the axial current and the tensor density vanishes as a consequence of invariance under charge conjugation). Conservation of the vector current and invariance under parity then give

$$(\Pi_{\rm VT})_{\mu\rho\sigma}(p) = (p_\rho\eta_{\mu\sigma}-p_\sigma\eta_{\mu\rho})\Pi_{\rm VT}(p^2). \quad (15)$$

When the space-time arguments of the two vector currents in $\langle VVP\rangle$ approach each other, the leading term in the OPE leads to the Green's function $\langle AP\rangle$ and the short-distance behavior of the form factor reads

$$\lim_{\lambda\to\infty}\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(q_2^2,(\lambda q_1)^2,(q_2-\lambda q_1)^2) = \frac{2F_0}{3}\frac{1}{\lambda^2}\frac{1}{q_1^2}+\mathcal{O}\left(\frac{1}{\lambda^3}\right). \quad (16)$$

Further important information on the on-shell pion form factor has been obtained in Ref. [37] based on higher-twist terms in the OPE and worked out in [38]. In the chiral limit one obtains the behavior

$$\lim_{\lambda\to\infty}\frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,(\lambda q_1)^2,(\lambda q_1)^2)}{\mathcal{F}_{\pi^0\gamma\gamma}(0,0,0)} = -\frac{8}{3}\pi^2F_0^2\left\{\frac{1}{\lambda^2q_1^2}+\frac{8}{9}\frac{\delta^2}{\lambda^4q_1^4}+\mathcal{O}\left(\frac{1}{\lambda^6}\right)\right\}, \quad (17)$$

where δ^2 parametrizes the relevant higher-twist matrix element. The sum rule estimate performed in [38] yields the value $\delta^2=(0.2\pm0.02)$ GeV^2.

On the other hand, when the space-time argument of one of the vector currents in $\langle VVP\rangle$ approaches the argument of the pseudoscalar density one obtains the relation [14]

$$\lim_{\lambda\to\infty}\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((\lambda q_1+q_2)^2,(\lambda q_1)^2,q_2^2) = -\frac{2}{3}\frac{F_0}{\langle\bar\psi\psi\rangle_0}\Pi_{\rm VT}(q_2^2)+\mathcal{O}\left(\frac{1}{\lambda}\right). \quad (18)$$

In particular, at the external vertex in light-by-light scattering in Eq. (2), the following limit is relevant:

$$\lim_{\lambda\to\infty}\mathcal{F}_{\pi^{0*}\gamma^*\gamma}((\lambda q_1)^2,(\lambda q_1)^2,0) = -\frac{2}{3}\frac{F_0}{\langle\bar\psi\psi\rangle_0}\Pi_{\rm VT}(0)+\mathcal{O}\left(\frac{1}{\lambda}\right). \quad (19)$$

Note that there is no falloff in this limit, unless $\Pi_{\rm VT}(0)$ vanishes.

As pointed out in Ref. [39], the value of $\Pi_{\rm VT}(p^2)$ at zero momentum is related to the quark condensate magnetic susceptibility χ in QCD in the presence of a constant external electromagnetic field, introduced in Ref. [40]:

$$\langle 0|\bar q\sigma_{\mu\nu}q|0\rangle_F = ee_q\chi\langle\bar\psi\psi\rangle_0F_{\mu\nu}, \quad (20)$$

with $e_u=2/3$ and $e_d=-1/3$. With our definition of $\Pi_{\rm VT}$ in Eq. (14) one then obtains the relation (see also Ref. [41])

$$\Pi_{\rm VT}(0) = -\frac{\langle\bar\psi\psi\rangle_0}{2}\chi, \quad (21)$$

and therefore the behavior at the external vertex from Eq. (19) can be rewritten as

$$\lim_{\lambda\to\infty}\mathcal{F}_{\pi^{0*}\gamma^*\gamma}((\lambda q_1)^2,(\lambda q_1)^2,0) = \frac{F_0}{3}\chi+\mathcal{O}\left(\frac{1}{\lambda}\right). \quad (22)$$

Unfortunately there is no agreement in the literature what the actual value of χ should be. In comparing different results one has to keep in mind that χ actually depends on the renormalization scale μ. In Ref. [40] the estimate $\chi(\mu=0.5\text{ GeV})=-(8.16^{+2.95}_{-1.91})$ GeV^{-2} was given in a QCD sum rule evaluation of nucleon magnetic moments. This value was confirmed by the recent reanalysis [42] which yields $\chi=-(8.5\pm1.0)$ GeV^{-2}, although no scale μ has been specified. A similar value $\chi=-N_C/(4\pi^2F_\pi^2)=-8.9$ GeV^{-2} was obtained in Ref. [43]. From the explicit expression of χ it is not immediately clear what should be the relevant scale μ. Since pion dominance was used in the matching with the OPE below some higher states, it was argued in Ref. [43] that the normalization point is probably rather low, $\mu\sim$ 0.5 GeV. Calculations within the instanton liquid model yield $\chi^{\rm ILM}(\mu\sim0.5\text{–}0.6\text{ GeV})=-4.32$ GeV^{-2} [44], where the scale is set by the inverse average instanton size ρ^{-1}. The value of $\chi\langle\bar\psi\psi\rangle_0=42$ MeV at the same scale obtained in Ref. [44] agrees roughly with the result 35–40 MeV from Ref. [45] derived in the same model.

The leading short-distance behavior of the two-point function $\Pi_{\rm VT}$ in Eq. (15) is given by [14] (see also Ref. [46])

$$\lim_{\lambda\to\infty}\Pi_{\rm VT}((\lambda p)^2) = -\frac{1}{\lambda^2}\frac{\langle\bar\psi\psi\rangle_0}{p^2}+\mathcal{O}\left(\frac{1}{\lambda^4}\right). \quad (23)$$

Assuming that $\Pi_{\rm VT}(p^2)$ is well described by the multiplet of the lowest-lying vector mesons (LMD) and satisfies the OPE constraint from Eq. (23) leads to the Ansatz

[14,39,47]

$$\Pi_{\rm VT}^{\rm LMD}(p^2) = -\langle\bar\psi\psi\rangle_0 \frac{1}{p^2 - M_V^2}. \tag{24}$$

Using Eq. (21) then leads to the estimate $\chi^{\rm LMD} = -2/M_V^2 = -3.3\ {\rm GeV}^{-2}$ [47]. Again, it is not obvious at which scale this relation holds. In analogy to estimates of low-energy constants in chiral Lagrangians [28], it might be at $\mu = M_V$, although in principle the renormalization scale of χ is not related to the one of low-energy constants; see the discussion in Ref. [48]. This LMD estimate was soon afterwards improved by taking into account higher resonance states (ρ', ρ'') in the framework of QCD sum rules, with the results $\chi(0.5\ {\rm GeV}) = -(5.7 \pm 0.6)\ {\rm GeV}^{-2}$ [39] and $\chi(1\ {\rm GeV}) = -(4.4 \pm 0.4)\ {\rm GeV}^{-2}$ [49]. A more recent analysis [50] yields, however, a smaller absolute value $\chi(1\ {\rm GeV}) = -(3.15 \pm 0.30)\ {\rm GeV}^{-2}$, close to the original LMD estimate. Further arguments for the latter value are also given in Ref. [41] and references therein, by studying the coupling of the tensor current to the ρ meson. For a quantitative comparison of all these estimates for χ we would have to run them to a common scale, for instance, 1 GeV, which can obviously not be done within perturbation theory starting from such low scales as $\mu = 0.5$ GeV.[4] Finally, even if the renormalization-group running could be performed nonperturbatively, it is not clear what would be the relevant scale μ in the context of hadronic light-by-light scattering.

A short-distance constraint on the pion-exchange contribution to the hadronic light-by-light scattering correction in the muon $g-2$ itself was derived in Ref. [21]. The relevant kinematical configuration for the s-channel exchange of the pion is shown in Fig. 2.

In general one has $q_1 + q_2 + q_3 + q_4 = 0$, but for the muon $g-2$ the soft photon limit $q_4 \to 0$ will be relevant. The authors of Ref. [21] then consider the limit $q_1^2 \sim q_2^2 \gg q_3^2$, where $q_3 = -(q_1 + q_2)$. Since in this limit the leading term in the OPE of the two vector currents associated with the momenta q_1 and q_2 yields the axial-vector current, they can relate the matrix element $\langle VVV|\gamma\rangle$ which enters for the muon $g-2$ to the famous anomalous triangle diagram $\langle AV|\gamma\rangle$ [52], which is highly constrained; see Refs. [43,53]. From this they deduce that no momentum-dependent form factor should be used at the external vertex, but only a constant factor. They thus obtain the following intermediate expression for the light-by-light scattering amplitude[5]:

[4]A further complication arises in comparisons with papers from the early 1980s because not only $\mu = 0.5$ GeV was frequently used, but also 1-loop running with a low $\Lambda_{\rm QCD}^{n_f=3} = $ 100–150 MeV, whereas more recent estimates yield $\Lambda_{\overline{\rm MS}}^{n_f=3} = $ 346 MeV (at 4-loop) [51].

[5]We have rescaled the form factor in Eq. (18) in Ref. [21] to agree with our normalization in Eq. (10) and used Minkowski space notation.

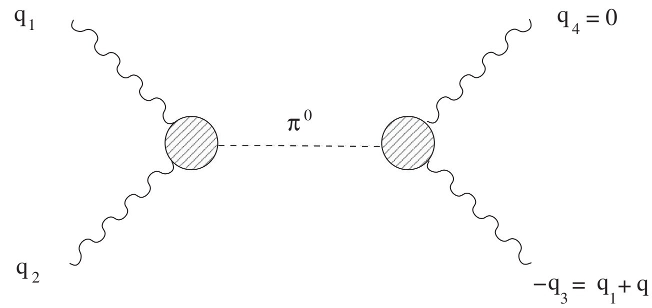

FIG. 2. Pion exchange in the s channel in hadronic light-by-light scattering. The photon with zero momentum $q_4 = 0$ represents the external soft photon for the corresponding contribution to the muon $g-2$.

$$\mathcal{A}_{\pi^0} = \frac{3}{2F_\pi} \frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)}{q_3^2 - m_\pi^2} (f_{2;\mu\nu}\tilde f_1^{\nu\mu})(\tilde f_{\rho\sigma} f_3^{\sigma\rho}) + \text{permutations}, \tag{25}$$

where $f_i^{\mu\nu} = q_i^\mu \epsilon_i^\nu - q_i^\nu \epsilon_i^\mu$ and $\tilde f_{i;\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} f_i^{\rho\sigma}$ for $i = 1, 2, 3$ denote the field strength tensors of the internal photons with polarization vectors ϵ_i. The field strength tensor of the external soft photon is defined similarly by $f^{\mu\nu} = q_4^\mu \epsilon_4^\nu - q_4^\nu \epsilon_4^\mu$. Except in $\tilde f_{\rho\sigma}$ the limit $q_4 \to 0$ is understood in Eq. (25), in particular, in $f_3^{\sigma\rho}$ and in the pion propagator.

Note the absence of a second form factor $\mathcal{F}_{\pi^0\gamma^*\gamma}(q_3^2, 0)$ in Eq. (25) at the external vertex. The authors of Ref. [21] rightly point out that such a momentum-dependent form factor at the external vertex would violate momentum conservation and criticize the procedure adopted in earlier works [6–9]. However, it is obvious from their expressions [Eq. (18) in Ref. [21]], reproduced here in Eq. (25), that they only consider the on-shell pion form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) \equiv \mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2)$ at the internal vertex and not the off-shell pion form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}(q_3^2, q_1^2, q_2^2)$. Note that the expression in Eq. (25) has to be compared with the term involving T_2 in Eq. (2). Therefore, contrary to the claim in Ref. [21], they only consider the *pion-pole* contribution to hadronic light-by-light scattering and not the *pion-exchange* contribution which involves fully off-shell form factors at the internal and the external vertex. Actually, also a second argument in Ref. [21] [after Eq. (20) there] in favor of a constant form factor at the external vertex is clearly based on the use of on-shell form factors. The use of a nonconstant on-shell form factor $\mathcal{F}_{\pi^0\gamma^*\gamma}(q_3^2, 0)$ at the external vertex would lead, together with the pion propagator, to an overall $1/q_3^4$ behavior, since $\mathcal{F}_{\pi^0\gamma^*\gamma}(q_3^2, 0) \sim 1/q_3^2$, for large q_3^2, according to Brodsky-Lepage; see Eq. (11). This would contradict the $1/q_3^2$ behavior observed in Eq. (25) (apart from $f_3^{\sigma\rho}$).

IV. NEW EVALUATION OF THE PSEUDOSCALAR-EXCHANGE CONTRIBUTION IN LARGE-N_C QCD

In the spirit of the minimal hadronic Ansatz for Green's functions in large-N_C QCD, on-shell $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2)$ and off-shell form factors $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2)$ have been constructed in Ref. [14]. They contain either the lowest-lying multiplet of vector resonances (LMD) or two multiplets, the ρ and the ρ' (LMD + V). Both Ansätze fulfill *all* the OPE constraints from Eqs. (12), (16), and (18) ; however, the LMD Ansatz does *not* reproduce the Brodsky-Lepage behavior from Eq. (11). Instead it behaves like $\mathcal{F}^{\mathrm{LMD}}_{\pi^0\gamma^*\gamma}(m_\pi^2, -Q^2, 0) \sim$ const. The $1/Q^2$ falloff can be achieved with the LMD + V Ansatz with a certain choice of the free parameters; see below. Note that it might not always be possible to satisfy all short-distance constraints, in particular, from the high-energy behavior of form factors, if only a finite number of resonances is included; see Ref. [54]. The on-shell form factors were later used in Ref. [9] to evaluate the pion-pole contribution; see also Ref. [27].

In the following, we reevaluate the pion-exchange contribution using *off-shell* LMD + V form factors [14] at both vertices

$$\mathcal{F}^{\mathrm{LMD+V}}_{\pi^{0*}\gamma^*\gamma^*}(q_3^2, q_1^2, q_2^2) = \frac{F_\pi}{3}\frac{q_1^2 q_2^2(q_1^2+q_2^2+q_3^2) + P_H^V(q_1^2, q_2^2, q_3^2)}{(q_1^2-M_{V_1}^2)(q_1^2-M_{V_2}^2)(q_2^2-M_{V_1}^2)(q_2^2-M_{V_2}^2)}, \tag{26}$$

$$P_H^V(q_1^2, q_2^2, q_3^2) = h_1(q_1^2+q_2^2)^2 + h_2 q_1^2 q_2^2 + h_3(q_1^2+q_2^2)q_3^2 + h_4 q_3^4 + h_5(q_1^2+q_2^2) + h_6 q_3^2 + h_7, \tag{27}$$

with $q_3^2 = (q_1+q_2)^2$. In the spirit of resonance chiral theory [28] a Lagrangian with two multiplets of vector resonances was proposed recently in Ref. [41] and references therein, which reproduces the above LMD + V Ansatz and which fulfills all the QCD short-distance constraints for the $\langle VVP\rangle$ Green's function.

The constants h_i in the Ansatz for $\mathcal{F}^{\mathrm{LMD+V}}_{\pi^{0*}\gamma^*\gamma^*}$ in Eq. (26) are determined as follows. The normalization with the pion decay amplitude $\pi^0 \to \gamma\gamma$ in Eq. (10) yields $h_7 = -N_C M_{V_1}^4 M_{V_2}^4/(4\pi^2 F_\pi^2) - h_6 m_\pi^2 - h_4 m_\pi^4 = -14.83\ \mathrm{GeV}^6 - h_6 m_\pi^2 - h_4 m_\pi^4$, where we used $M_{V_1} = M_\rho = 775.49$ MeV and $M_{V_2} = M_{\rho'} = 1.465$ GeV [32]. Note that in Refs. [9,14] the small corrections proportional to the pion mass were dropped, assuming that the $|h_i|$ are of order 1–10 in appropriate units of GeV. The Brodsky-Lepage behavior from Eq. (11) can be reproduced by choosing $h_1 = 0\ \mathrm{GeV}^2$. Furthermore, in Ref. [14] a fit to the CLEO data for the on-shell form factor $\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma}(m_\pi^2, -Q^2, 0)$ was performed, with the result $h_5 = (6.93 \pm 0.26)\ \mathrm{GeV}^4 - h_3 m_\pi^2$. Again, the correction proportional to the pion mass was omitted in Refs. [9,14]. As pointed out in Ref. [21], the constant h_2 can be obtained from the higher-twist corrections in the OPE. Comparing with Eq. (17) yields the result $h_2 = -4(M_{V_1}^2 + M_{V_2}^2) + (16/9)\delta^2 \simeq -10.63\ \mathrm{GeV}^2$.

Within the LMD + V framework, the vector-tensor two-point function reads [14]

$$\Pi^{\mathrm{LMD+V}}_{\mathrm{VT}}(p^2) = -\langle\bar\psi\psi\rangle_0 \frac{p^2 + c_{\mathrm{VT}}}{(p^2-M_{V_1}^2)(p^2-M_{V_2}^2)}, \tag{28}$$

$$c_{\mathrm{VT}} = \frac{M_{V_1}^2 M_{V_2}^2 \chi}{2}, \tag{29}$$

where we fixed the constant c_{VT} using Eq. (21). As shown in Ref. [14] the OPE from Eq. (18) for $\mathcal{F}^{\mathrm{LMD+V}}_{\pi^{0*}\gamma^*\gamma^*}$ leads to the relation

$$h_1 + h_3 + h_4 = 2c_{\mathrm{VT}}. \tag{30}$$

As noted above, the value of the magnetic susceptibility $\chi(\mu)$ and the relevant scale μ are not precisely known. However, the LMD estimate $\chi^{\mathrm{LMD}} = -2/M_V^2 = -3.3\ \mathrm{GeV}^{-2}$ is close to $\chi(\mu = 1\ \mathrm{GeV}) = -(3.15 \pm 0.30)\ \mathrm{GeV}^{-2}$ obtained in Ref. [50] using QCD sum rules with several vector resonances ρ, ρ', and ρ''. Assuming that the LMD/LMD + V framework is self-consistent, we will therefore take $\chi = (-3.3 \pm 1.1)\ \mathrm{GeV}^{-2}$ in our numerical evaluation, with a typical large-N_C uncertainty of about 30%. This translates into the constraint $h_3 + h_4 = (-4.3 \pm 1.4)\ \mathrm{GeV}^2$, corresponding to $c_{\mathrm{VT}} = (-2.13 \pm 0.71)\ \mathrm{GeV}^2$. We will vary h_3 in the range $\pm 10\ \mathrm{GeV}^2$ and determine h_4 from Eq. (30) and vice versa.

Note that using the off-shell LMD + V form factor at the external vertex leads to a short-distance behavior in the full light-by-light scattering contribution which at least qualitatively agrees with the OPE constraint derived in Ref. [21]. As stressed earlier, Ref. [21] only considers the pion-pole contribution with on-shell form factors; therefore a direct quantitative comparison with our approach is not possible. Nevertheless, taking first $q_1^2 \sim q_2^2 \gg q_3^2$ and then q_3^2 large, one obtains, together with the pion propagator in Eq. (2) [in the term with T_2], an overall $1/q_3^2$ behavior for the pion-exchange contribution, since at the external vertex we have [55]

$$\frac{3}{F_\pi}\mathcal{F}^{\mathrm{LMD+V}}_{\pi^{0*}\gamma^*\gamma}(q_3^2, q_3^2, 0) \stackrel{q_3^2\to\infty}{\longrightarrow} \frac{h_1+h_3+h_4}{M_{V_1}^2 M_{V_2}^2} = \frac{2c_{\mathrm{VT}}}{M_{V_1}^2 M_{V_2}^2} = \chi. \tag{31}$$

In the derivation we used Eqs. (29) and (30); see also Eq. (22). This $1/q_3^2$ behavior is as expected from Eq. (25), reproduced earlier from Ref. [21]. On the other hand, if we would use a constant form factor proportional to the WZW term at the external vertex as proposed in Ref. [21], we would get [55]

$$\frac{3}{F_\pi}\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma\gamma}(0,0,0) = \frac{h_7}{M_{V_1}^4 M_{V_2}^4} = -\frac{N_C}{4\pi^2 F_\pi^2} \simeq -8.9\ \mathrm{GeV}^{-2}, \quad (32)$$

where for simplicity we considered the chiral limit. That means that with the value of $\chi = -N_C/(4\pi^2 F_\pi^2)$ from Ref. [43] we would in Eq. (31) precisely satisfy the short-distance constraint from Ref. [21].

The coefficient h_6 in the LMD + V Ansatz is undetermined as well. We can obtain some indirect information on its size and sign in the following way. Low-energy constants in chiral Lagrangians can be estimated by starting with some resonance Lagrangian and then integrating out the heavy resonance states, usually at tree level. In particular for low-energy constants which are given by the exchanges of vector and axial-vector mesons, this procedure works in general quite well [28,56]. Although for instance for vector mesons one can write down many different Lagrangians, it was shown in Ref. [28] that imposing QCD short-distance constraints on the resonance Lagrangian itself leads to unique estimates for the low-energy constants at order p^4 in the chiral Lagrangian. At order p^6 this is not true anymore [14,16]; nevertheless, it seems reasonable to reduce the model dependence by imposing again short-distance constraints on such resonance Lagrangians.

Usually, only the exchange of the lightest resonance state in each channel is considered in this approach. One expects, however, some corrections to these estimates, with a typical large-N_C error of about 30%, if also the exchanges of heavier resonance states are taken into account. In Ref. [14] this was shown to be true for one of two linear combinations of low-energy constants from the chiral Lagrangian of odd intrinsic parity at order p^6 which enter in the low-energy expansion of the Green's function $\langle VVP \rangle$. With the LMD and LMD + V Ansätze for this Green's function, the relevant combination of low-energy constants is given by

$$A^{\mathrm{LMD}}_{V,p^2} = \frac{F_\pi^2}{8M_V^4} - \frac{N_C}{32\pi^2 M_V^2} = -1.11\frac{10^{-4}}{F_\pi^2}, \quad (33)$$

$$A^{\mathrm{LMD+V}}_{V,p^2} = \frac{F_\pi^2}{8M_{V_1}^4}\frac{h_5}{M_{V_2}^4} - \frac{N_C}{32\pi^2 M_{V_1}^2}\left(1 + \frac{M_{V_1}^2}{M_{V_2}^2}\right) = -1.36\frac{10^{-4}}{F_\pi^2}. \quad (34)$$

The constant h_5 which enters $A^{\mathrm{LMD+V}}_{V,p^2}$ is directly related to the Brodsky-Lepage behavior of the form factor. Even though this falloff behavior cannot be reproduced with the LMD Ansatz, the change in the low-energy constant when going from LMD to LMD + V is only about 20%, well within the expected large-N_C uncertainty.

On the other hand, the coefficient h_6 determines a second linear combination of low-energy constants at order p^6:

$$A^{\mathrm{LMD}}_{V,(p+q)^2} = -\frac{F_\pi^2}{8M_V^4} = -0.26\frac{10^{-4}}{F_\pi^2}, \qquad A^{\mathrm{LMD+V}}_{V,(p+q)^2} = -\frac{F_\pi^2}{8M_{V_1}^4 M_{V_2}^4}h_6. \quad (35)$$

Note that using the resonance Lagrangian of Ref. [57], one would obtain $A^{\mathrm{res}}_{V,(p+q)^2} = 0$ instead. However, this resonance Lagrangian in general fails to reproduce the short-distance constraints from QCD, in contrast to the LMD and LMD + V Ansätze; see Ref. [14]. In particular, the prediction for $A_{V,(p+q)^2}$ in the LMD model follows directly from the implementation of these short-distance constraints. Note that there is no problem with the short-distance behavior for the LMD form factor in the relevant channel where at low energies $A_{V,(p+q)^2}$ enters. If we would assume a 30% error on the LMD estimate in Eq. (35), we would obtain the quite narrow range $h_6 = M_{V_2}^4(1 \pm 0.3) = (4.6 \pm 1.4)\ \mathrm{GeV}^4$. However, this procedure might underestimate the potential variation of h_6, since the low-energy constant $A^{\mathrm{LMD}}_{V,(p+q)^2}$ happens to be small compared to A^{LMD}_{V,p^2}; see Eq. (33). The magnitude of the shift of A_{V,p^2} when going from LMD to LMD + V is $-0.25(10^{-4}/F_\pi^2)$. That is, the shift is of the same size as $A^{\mathrm{LMD}}_{V,(p+q)^2}$ itself. Assuming again that the LMD/LMD + V framework is self-consistent, but, to be conservative, allowing for a 100% uncertainty of $A^{\mathrm{LMD}}_{V,(p+q)^2}$, we get the range $h_6 = (5 \pm 5)\ \mathrm{GeV}^4$.

Of course, the uncertainties of the values of the undetermined parameters h_3, h_4 and h_6 and of the magnetic susceptibility $\chi(\mu)$ is a drawback when using the off-shell LMD + V form factor and will limit the precision of the final estimate.

The integral to be performed in Eq. (2) is eight-dimensional, thereof 3 integrations can be done trivially. In general, one then has to deal with a five-dimensional integration over 3 angles and 2 moduli. We have performed these integrations numerically after a rotation to Euclidean momenta using the program VEGAS [58]. As a check we have reproduced the values of $a_\mu^{\mathrm{LbyL};\pi^0}$ for various form factors which have been used earlier in the literature. For instance, using a simple VMD form factor, we obtain $a^{\mathrm{LbyL};\pi^0}_{\mu;\mathrm{VMD}} = 57 \times 10^{-11}$ for $m_\mu = 105.658\,369$ MeV and $m_{\pi^0} = 134.9766$ MeV and with the value $M_\rho = 775.49$ MeV.

The results for $a_\mu^{\mathrm{LbyL};\pi^0}$ for some selected values of h_3, h_4 and h_6, varied in the ranges discussed above, for $\chi = -3.3\ \mathrm{GeV}^{-2}$, $h_1 = 0\ \mathrm{GeV}^2$, $h_2 = -10.63\ \mathrm{GeV}^2$ and $h_5 = 6.93\ \mathrm{GeV}^4 - h_3 m_\pi^2$ are collected in Table I.

Varying χ by $\pm 1.1\ \mathrm{GeV}^{-2}$ changes the result for $a_\mu^{\mathrm{LbyL};\pi^0}$ by $\pm 2.1 \times 10^{-11}$ at most. One observes from the table that the uncertainty in h_6 affects the result by up to $\pm 6.4 \times 10^{-11}$. If we would use instead $h_6 = (0 \pm 10)\ \mathrm{GeV}^4$, the result would vary by about $\pm 12 \times 10^{-11}$ around the central value. The variation of $a_\mu^{\mathrm{LbyL};\pi^0}$ with h_3 [with h_4 determined from the constraint in Eq. (30) or vice versa] is much smaller, at most $\pm 2.5 \times 10^{-11}$. The variation of h_5 by $\pm 0.26\ \mathrm{GeV}^4$ only leads to changes of $\pm 0.6 \times 10^{-11}$ in the final result. Within the scanned region, we obtain a minimal value of $a_\mu^{\mathrm{LbyL};\pi^0} = 63.2 \times 10^{-11}$ for $\chi = -2.2\ \mathrm{GeV}^{-2}$, $h_3 = 10\ \mathrm{GeV}^2$, and $h_6 = 0\ \mathrm{GeV}^4$ and a maximum of $a_\mu^{\mathrm{LbyL};\pi^0} = 83.3 \times 10^{-11}$ for $\chi = -4.4\ \mathrm{GeV}^{-2}$, $h_4 = 10\ \mathrm{GeV}^2$, and $h_6 = 10\ \mathrm{GeV}^4$. In the absence of more information on the precise values of the constants h_3, h_4 and h_6, we take the average of the results obtained with $h_6 = 5\ \mathrm{GeV}^4$ for $h_3 = 0\ \mathrm{GeV}^2$, i.e. 71.9×10^{-11}, and for $h_4 = 0\ \mathrm{GeV}^2$, i.e. 72.8×10^{-11}, as our central value, 72.3×10^{-11}. To estimate the error, we add all the uncertainties from the variations of χ, h_3 (or h_4), h_5 and h_6 linearly to cover the full range of values obtained with our scan of parameters. Note that the uncertainties of χ and the coefficients h_3, h_4 and h_6 do not follow a Gaussian distribution. In this way we obtain our final estimate

$$a_\mu^{\mathrm{LbyL};\pi^0} = (72 \pm 12) \times 10^{-11}. \tag{36}$$

We think the 16% error should fairly well describe the inherent model uncertainty using the *off-shell* LMD + V form factor. In order to facilitate future updates of our result in case some of the parameters h_i in the LMD + V Ansatz in Eq. (26) or the value (and the relevant scale) of the magnetic susceptibility $\chi(\mu)$ will be known more precisely, we present in the appendix a parametrization of $a_\mu^{\mathrm{LbyL};\pi^0}$ for arbitrary coefficients h_i.

TABLE I. Results for $a_\mu^{\mathrm{LbyL};\pi^0} \times 10^{11}$ obtained with the off-shell LMD + V form factor for $\chi = -3.3\ \mathrm{GeV}^{-2}$ and the given values for h_3, h_4 and h_6. When varying h_3 (upper half of the table), the parameter h_4 is fixed by the constraint in Eq. (30). In the lower half the procedure is reversed. The values of the other parameters are given in the text.

	$h_6 = 0\ \mathrm{GeV}^4$	$h_6 = 5\ \mathrm{GeV}^4$	$h_6 = 10\ \mathrm{GeV}^4$
$h_3 = -10\ \mathrm{GeV}^2$	68.4	74.1	80.2
$h_3 = 0\ \mathrm{GeV}^2$	66.4	71.9	77.8
$h_3 = 10\ \mathrm{GeV}^2$	64.4	69.7	75.4
$h_4 = -10\ \mathrm{GeV}^2$	65.3	70.7	76.4
$h_4 = 0\ \mathrm{GeV}^2$	67.3	72.8	78.8
$h_4 = 10\ \mathrm{GeV}^2$	69.2	75.0	81.2

As far as the contribution to a_μ from the exchanges of the other light pseudoscalars η and η' is concerned, it is not so straightforward to apply the above analysis within the LMD + V framework to these resonances. In particular, the short-distance analysis in Ref. [14] was performed in the chiral limit and assumed octet symmetry. For the η the effect of nonzero quark masses has definitely to be taken into account. Furthermore, the η' has a large admixture from the singlet state and the gluonic contribution to the axial anomaly will play an important role. We therefore resort to a simplified approach which was also adopted in other works [6–9,21] and take a simple VMD form factor

$$\mathcal{F}^{\mathrm{VMD}}_{\mathrm{PS}^*\gamma^*\gamma^*}(q_3^2, q_1^2, q_2^2) = -\frac{N_C}{12\pi^2 F_{\mathrm{PS}}} \frac{M_V^2}{(q_1^2 - M_V^2)} \times \frac{M_V^2}{(q_2^2 - M_V^2)}, \qquad \mathrm{PS} = \eta, \eta', \tag{37}$$

normalized to the experimental decay width $\Gamma(\mathrm{PS} \to \gamma\gamma)$. We can fix the normalization by adjusting the (effective) pseudoscalar decay constant F_{PS} in Eq. (37). Using the latest values $\Gamma(\eta \to \gamma\gamma) = (0.510 \pm 0.026)$ keV and $\Gamma(\eta' \to \gamma\gamma) = (4.30 \pm 0.15)$ keV from Ref. [32], one obtains $F_{\eta,\mathrm{eff}} = 93.0$ MeV with $m_\eta = 547.853$ MeV and $F_{\eta',\mathrm{eff}} = 74.0$ MeV with $m_{\eta'} = 957.66$ MeV. However, we do not follow the approach of Ref. [21] and will also take a VMD form factor at the external vertex.

Note that the on- and off-shell VMD form factors are identical, since the form factor does not depend on the momentum q_3^2 which flows through the pion leg. The problem with the VMD form factor is that the damping is too strong as it behaves like $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, -Q^2, -Q^2) \sim 1/Q^4$, instead of $\sim 1/Q^2$ deduced from the OPE; see Eq. (16). This effect might lead to an underestimating of the contribution. However, the relevant integrals for $a_\mu^{\mathrm{LbyL;PS}}$ do not seem to be very sensitive to the correct asymptotic behavior for large momenta. This can be seen from the weight functions which multiply the form factors in the integral and which are displayed in Ref. [9]. It seems more important to have a good description at small and intermediate energies below 1 GeV, e.g. by reproducing the slope of the form factor $\mathcal{F}_{\mathrm{PS}\gamma^*\gamma}(-Q^2, 0)$ at the origin. The CLEO Collaboration [34] has made a fit of the on-shell form factors $\mathcal{F}_{\eta\gamma^*\gamma}(-Q^2, 0)$ and $\mathcal{F}_{\eta'\gamma^*\gamma}(-Q^2, 0)$, normalized to the corresponding experimental width $\Gamma(\mathrm{PS} \to \gamma\gamma)$, using a VMD Ansatz with an adjustable parameter Λ_{PS} in place of the vector-meson mass M_V in Eq. (37). Taking their values $\Lambda_\eta = (774 \pm 29)$ MeV and $\Lambda_{\eta'} = (859 \pm 28)$ MeV, we then obtain the results $a_\mu^{\mathrm{LbyL};\eta} = 14.5 \times 10^{-11}$ and $a_\mu^{\mathrm{LbyL};\eta'} = 12.5 \times 10^{-11}$, which update the values given in Ref. [9].[6] Only a more detailed analysis, along the line of the LMD + V framework, will show whether these values are realistic. Thus, adding up the contributions from all the light pseudoscalar exchanges (π^0, η, η'), we obtain the estimate

$$a_\mu^{\mathrm{LbyL;PS}} = (99 \pm 16) \times 10^{-11}, \tag{38}$$

where we have assumed a 16% error, as inferred above for pion-exchange contribution using the off-shell LMD + V form factor.[7]

V. DISCUSSION AND CONCLUSIONS

Following the observation in Refs. [25,26] we have reevaluated the pion-exchange contribution to hadronic light-by-light scattering in the muon $g-2$ using fully off-shell form factors $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ at both vertices. We used a model based on the large-N_C QCD framework with two multiplets of vector mesons (LMD + V) [14] which fulfills all QCD short-distance constraints on the form factor and reproduces the experimentally confirmed Brodsky-Lepage behavior. We also derived a new short-distance constraint on the form factor at the external vertex, relating it to the quark condensate magnetic susceptibility χ. The obtained value $a_\mu^{\mathrm{LbyL};\pi^0} = (72 \pm 12) \times 10^{-11}$ replaces the result obtained in Ref. [9] with on-shell LMD + V form factors at both vertices. Adding the contribution from the exchanges of the η and η' evaluated with simple VMD form factors, we obtain $a_\mu^{\mathrm{LbyL;PS}} = (99 \pm 16) \times 10^{-11}$ for the sum of all light pseudoscalars. These values for the pion and all pseudoscalars are about 20% larger than the estimates obtained in Refs. [6–8] which used other hadronic models for the form factor.

As mentioned earlier, the identification of individual contributions, like pion exchange, in hadronic light-by-light scattering is model-dependent as soon as one uses off-shell form factors. We view our evaluation as being a part of a full calculation based on a resonance Lagrangian, which fulfills all the relevant QCD short-distance constraints, along the lines of the resonance chiral theory approach developed in Ref. [28].

We would like to stress that although our result for the pion-exchange contribution is not too far from the value $a_\mu^{\mathrm{LbyL};\pi^0\text{-pole}} = (76.5 \pm 6.7) \times 10^{-11}$ given in Ref. [21],[8] this is *pure coincidence*. We have used off-shell LMD + V form factors at both vertices, whereas the authors of Ref. [21] evaluated the *pion-pole* contribution using the on-shell LMD + V form factor at the internal vertex and a constant (WZW) form factor at the external vertex. On the other hand, as has been observed in Refs. [6–9,22], it is the region of momenta below about 2 GeV which gives the bulk of the contribution to the final result in the pion-exchange or pion-pole correction to hadronic light-by-light scattering. This is also clearly visible from the weight functions that multiply the form factors in the integrals and which have been presented in Ref. [9]; see also Ref. [22]. They have a peak around 0.5 GeV. Therefore, as long as the absolute values of the model parameters h_3, h_4 and h_6, which control the off-shellness of the pion in the LMD + V form factor in Eq. (26),[9] are not too large, i.e. below about 10 in appropriate units of GeV, one obtains a result which will not be too far from the one obtained with on-shell LMD + V form factors. We have given some arguments for our choice of the parameters h_i and the ranges in which we vary them and they fulfill this constraint on their size. Recall that the constant term in the numerator of the form factor in Eq. (26) has the value $h_7 \simeq -14.8\ \mathrm{GeV}^6$. As pointed out before, our Ansatz for the neutral pion contribution to hadronic light-by-light scattering with two off-shell LMD + V form factors agrees qualitatively with the short-distance behavior derived in Ref. [21]. However, since only the pion-pole contribution was considered throughout that paper, a direct quantitative comparison is not possible.

Recently, an evaluation of the pion-exchange contribution appeared [29] which uses a nonlocal chiral quark model for the off-shell form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$. In that model, off-shell effects of the pion always lead to a rather strong damping in the form factor and the result $a_\mu^{\mathrm{LbyL};\pi^0} = (65 \pm 2) \times 10^{-11}$ is therefore smaller than the pion-pole contribution obtained in Ref. [21]. Although we also get a value which is (slightly) smaller than the pion-pole contribution, our result depends on the chosen model parameters, i.e. the constants h_i in the LMD + V Ansatz and on the value for

[6]If we use a constant (WZW) form factor at the external vertex, as proposed in Ref. [21], we would obtain $a_\mu^{\mathrm{LbyL};\eta\text{-pole}} = 21.5 \times 10^{-11}$ and $a_\mu^{\mathrm{LbyL};\eta'\text{-pole}} = 20.1 \times 10^{-11}$. Note that these values are somewhat larger than $a_\mu^{\mathrm{LbyL};\eta\text{-pole}} = 18 \times 10^{-11}$ and $a_\mu^{\mathrm{LbyL};\eta'\text{-pole}} = 18 \times 10^{-11}$ given in Ref. [21].

[7]Applying the same procedure to the electron, we obtain $a_e^{\mathrm{LbyL};\pi^0} = (2.98 \pm 0.34) \times 10^{-14}$ with off-shell LMD + V form factors at both vertices. This number supersedes the value given in Ref. [9]. Note that the naive rescaling $a_e^{\mathrm{LbyL};\pi^0}(\text{rescaled}) = (m_e/m_\mu)^2 a_\mu^{\mathrm{LbyL};\pi^0} = 1.7 \times 10^{-14}$ yields a value which is almost a factor of 2 too small. Our estimates for the other pseudoscalars contributions using VMD form factors at both vertices are $a_e^{\mathrm{LbyL};\eta} = 0.49 \times 10^{-14}$ and $a_e^{\mathrm{LbyL};\eta'} = 0.39 \times 10^{-14}$. Therefore we get $a_e^{\mathrm{LbyL;PS}} = (3.9 \pm 0.5) \times 10^{-14}$, where the relative error of about 12% is again taken over from the pion-exchange contribution.

[8]Actually, using the on-shell LMD + V form factor at the internal vertex with $h_2 = -10\ \mathrm{GeV}^2$ and $h_5 = 6.93\ \mathrm{GeV}^4$ and a constant (WZW) form factor at the external vertex, we obtain 79.8×10^{-11}, close to the value 79.6×10^{-11} given in Ref. [22] and 79.7×10^{-11} in Ref. [29].

[9]Note, however, that even if h_3, h_4 and h_6 are put to zero, which actually violates the short-distance constraint from Eq. (30), one does not recover the on-shell LMD + V form factor because of a term proportional to $q_1^2 q_2^2 (q_1 + q_2)^2$ in the numerator in Eq. (26).

the magnetic susceptibility $\chi(\mu)$. We have given arguments for our preferred choice of these parameters. In some other corner of the parameter space one can, however, obtain a result which is larger than the pion-pole contribution; i.e. we get a maximal value of $a_\mu^{\mathrm{LbyL};\pi^0} = 83.3 \times 10^{-11}$ in the scanned range. Of course, any additional information to pin down these model parameters and the "correct" value of $\chi(\mu)$ and the relevant scale μ would be highly welcome to obtain a more precise prediction for the pion-exchange contribution. At this point it is not clear whether the nonlocal chiral quark model used in Ref. [29] or our LMD + V model for the form factor better represents the strongly interacting region of QCD below about 2 GeV. At least the LMD + V form factor fulfills all the relevant QCD short-distance constraints. In any case, we think that the error of $\pm 2 \times 10^{-11}$ given in Ref. [29] probably underestimates the inherent uncertainly of any hadronic model.

In Ref. [21] an improved evaluation of the axial-vector contribution to hadronic light-by-light scattering was given compared to Refs. [6–8], with the result $(22 \pm 5) \times 10^{-11}$. Note, however, that this seems to be again only the pole contribution. Furthermore, Ref. [6] obtained the following results for the remaining contributions: $(-7 \pm 2) \times 10^{-11}$ for scalar exchanges, $(-19 \pm 13) \times 10^{-11}$ for the dressed pion and kaon loops and $(21 \pm 3) \times 10^{-11}$ for the dressed quark loops. These estimates have more conservative errors than those given in Refs. [7,8]. Furthermore, the scalar-exchange contribution is not evaluated in the latter references. If we combine our value for the pseudoscalars with these results, we obtain the new estimate

$$a_\mu^{\mathrm{LbyL;had}} = (116 \pm 40) \times 10^{-11} \tag{39}$$

for the total hadronic light-by-light scattering contribution to the anomalous magnetic moment of the muon. To be conservative, we have added all the errors linearly, as has become customary in recent years, and rounded up the obtained value $\pm 39 \times 10^{-11}$. In the very recent review [24] the central values of some of the individual contributions to hadronic light-by-light scattering are adjusted and some errors are enlarged to cover the results obtained by various groups which used different models. The errors are finally added in quadrature to yield the estimate $a_\mu^{\mathrm{LbyL;had}} = (105 \pm 26) \times 10^{-11}$. Note that the dressed light quark loops are not included as a separate contribution in Ref. [24]. They are assumed to be already covered by using the short-distance constraint from Ref. [21] on the pseudoscalar-pole contribution.

Some progress has been achieved in recent years to better understand the hadronic light-by-light scattering contribution to the muon $g-2$. We hope that our new short-distance constraint on the off-shell form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma}(q^2, q^2, 0)$ at the external vertex will further help to control the numerically dominant pion-exchange contribution. We should not forget, however, that the contribution of the exchanges of η and η' are theoretically not that well understood. We have simply used VMD form factors as has been done in most other evaluations. A new analysis, along the lines of the approach for the pion, is definitely needed. Finally, as stressed in Refs. [22,24] a better control of the numerically subdominant but non-negligible other contributions is also needed, if we fully want to profit from a potential future $g-2$ experiment.

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APPENDIX

We provide in this appendix a parametrization of $a_\mu^{\mathrm{LbyL};\pi^0}$ for arbitrary coefficients h_i in the LMD + V Ansatz for the off-shell form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$ in Eq. (26). This will facilitate future updates of our result for the pion-exchange contribution in Eq. (36) in case some of the parameters h_i or the value (and the relevant scale) of the magnetic susceptibility $\chi(\mu)$ will be known more precisely.

Measuring the parameters h_i in appropriate units of GeV, i.e. defining $\tilde{h}_i = h_i/\mathrm{GeV}^2$ for $i = 1, 2, 3, 4$, $\tilde{h}_i = h_i/\mathrm{GeV}^4$ for $i = 5, 6$ and $\tilde{h}_7 = h_7/\mathrm{GeV}^6$, we can write

$$a_\mu^{\mathrm{LbyL};\pi^0} = \left(\frac{\alpha}{\pi}\right)^3 \left[\sum_{i=1}^{7} c_i \tilde{h}_i + \sum_{i=1}^{7}\sum_{j=i}^{7} c_{ij}\tilde{h}_i\tilde{h}_j\right], \tag{A1}$$

where the coefficients c_i and c_{ij} are given in Table II.

This representation follows immediately from the general expression for the pion-exchange contribution in Eq. (2) and from the LMD + V Ansatz for the form factor in Eq. (26). Because of our new short-distance constraint at the external vertex, the parameters h_1, h_3 and h_4 are not independent, but must obey the relation $h_1 + h_3 + h_4 = M_{V_1}^2 M_{V_2}^2 \chi$; see Eq. (30). Note the absence of a term without the constants h_i in Eq. (A1). This follows from the fact that at the external vertex with the soft photon the form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma}(q_3^2, q_3^2, 0)$ enters, e.g. in the term with T_2 in Eq. (2). This also leads to $c_2 = 0$ and $c_{22} = 0$ in Table II. For

TABLE II. Values of the coefficients c_i and c_{ij} which appear in Eq. (A1).

	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$	$i = 7$
$c_i \times 10^4$	−1.4530	0	−1.4530	−1.4530	0.4547	0.4547	−1.2048

$c_{ij} \times 10^4$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$	$j = 7$
$i = 1$	0 4447	0.0729	0.7428	0.7120	−0.3620	−0.3332	0.9916
$i = 2$	⋯	0	0.0730	0.0729	−0.0557	−0.0557	0.2221
$i = 3$	⋯	⋯	0.2980	0.5653	−0.1967	−0.1679	0.1796
$i = 4$	⋯	⋯	⋯	0.2672	−0.1679	−0.1391	0.1162
$i = 5$	⋯	⋯	⋯	⋯	0.1215	0.1796	−0.8072
$i = 6$	⋯	⋯	⋯	⋯	⋯	0.0581	−0.3052
$i = 7$	⋯	⋯	⋯	⋯	⋯	⋯	1.6122

evaluating the integrals, we used $m_\mu = 105.658\,369$ MeV, $m_{\pi^0} = 134.9766$ MeV, $F_\pi = 92.4$ MeV, $M_{V_1} = M_\rho = 775.49$ MeV and $M_{V_2} = M_{\rho'} = 1.465$ GeV. The decay constant F_π only enters as an overall factor in $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$.

Note, however, that some of the model parameters h_i are quite well determined from experimental data and theoretical constraints. The normalization with the pion decay amplitude $\pi^0 \to \gamma\gamma$ yields $h_7 = -N_C M_{V_1}^4 M_{V_2}^4/(4\pi^2 F_\pi^2) - h_6 m_\pi^2 - h_4 m_\pi^4$. The Brodsky-Lepage behavior can be reproduced by choosing $h_1 = 0$ GeV^2. Furthermore, a fit to the CLEO data for the on-shell form factor $\mathcal{F}_{\pi^0\gamma^*\gamma}^{\rm LMD+V}(m_\pi^2, -Q^2, 0)$ leads to $h_5 = (6.93 \pm 0.26)\,\mathrm{GeV}^4 - h_3 m_\pi^2$. Finally, the constant h_2 can be obtained from higher-twist corrections in the OPE, with the result $h_2 = -4(M_{V_1}^2 + M_{V_2}^2) + (16/9)\delta^2 \simeq -10.63$ GeV^2.

If we use the above informations to fix h_1, h_2, h_5 and h_7, we obtain the simplified expression

$$a_\mu^{\mathrm{LbyL};\pi^0} = \left(\frac{\alpha}{\pi}\right)^3 [503.3764 - 6.5223\tilde{h}_3 - 5.0962\tilde{h}_4 + 7.8557\tilde{h}_6 + 0.3017\tilde{h}_3^2 + 0.5683\tilde{h}_3\tilde{h}_4 - 0.1747\tilde{h}_3\tilde{h}_6 + 0.2672\tilde{h}_4^2 - 0.1411\tilde{h}_4\tilde{h}_6 + 0.0642\tilde{h}_6^2] \times 10^{-4}, \tag{A2}$$

where only h_3, h_4 and h_6 enter as free parameters. Note again, however, that h_3 and h_4 are not independent, but now obey the relation $h_3 + h_4 = M_{V_1}^2 M_{V_2}^2 \chi$.

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