Radiative neutrino mass in type III seesaw model

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The simplest type III seesaw model as originally proposed introduces one lepton triplet. It thus contains four active neutrinos, two massive and two massless at tree level. We determine the radiative masses that the latter receive first at two loops. The masses are generally so tiny that they are definitely excluded by the oscillation data, if the heavy leptons are not very heavy, say, within the reach of the CERN LHC. To accommodate the data on masses, the seesaw scale must be as large as the scale of grand unification. This indicates that the most economical type III model would entail no new physics at low energies beyond the tiny neutrino masses.

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I. INTRODUCTION

The standard model (SM) of electroweak interactions when viewed as an effective field theory at low energies has a unique dimension five operator that can generate Majorana neutrino masses [1]. And the operator has only three possible realizations at tree level [2]. These correspond to the three celebrated types of seesaw models [3–5]. While the type I model introduces sterile neutrinos as the minimal option to operate the seesaw, the other two prescribe particles that participate in electroweak interactions. If the seesaw scale is not too high, richer phenomena are expected in the last two types of models. There have been extensive investigations on the types I and II seesaw models, but the interest in type III has been catalyzed recently by the advent of the LHC at CERN [6], where the assumed triplet leptons could be directly produced through gauge interactions if they are not too heavy [7–10]. Various other phenomenological aspects of the model have also been explored, including possible modifications to leptogenesis [10–14], low energy effects in lepton flavor changing processes [15] and anomalous magnetic moments of charged leptons [16,17], renormalization group running of neutrino parameters [18], and the potential role as dark matter [19], to mention a few.

For a seesaw model like type III to be relevant at relatively low energies, it must be capable of incorporating the data from oscillation experiments and other constraints with a not too high seesaw scale. We are thus motivated to start with the simplest type III seesaw as was originally proposed [5]. It extends the SM by one triplet of leptons, resulting in two massive and two massless neutrinos at tree level, plus a pair of heavy charged leptons. It also serves as an approximation to more general structures that contain additional sequentially heavier triplets of leptons. The massless neutrinos not being protected by any symmetry should receive radiative masses, which will be determined in this work. It would be interesting to ask whether it is possible in this minimal model to get a radiative mass at a desired level with a seesaw scale accessible at LHC.

The idea of generating a one-loop radiative mass for neutrinos was originally suggested in Ref. [20], and extended to two loops in [21,22]. It offers a nice way to induce hierarchical and tiny neutrino masses. There is vast literature that extends the idea in various aspects (see, as examples, [23–27]) and calculates radiative masses in different models [28–30]. We will not attempt to review the topic but reemphasize the point that for a mechanism of radiative mass generation to be testable at colliders [31–33], the relevant heavy mass scale cannot be too high.

The paper is organized as follows. In the next section we describe in some detail the minimal model to set up our notations. The exact constraints on the lepton masses and diagonalization matrices are highlighted. They will be extensively utilized in our analytic evaluation of radiative mass. Also listed are the Yukawa couplings of leptons that may be useful in other applications. The radiative mass is then calculated in Sec. III in a manner that facilitates later numerical analysis, and the final answer is given in terms of some loop integrals. These integrals are defined in the Appendix, and their leading terms in the heavy mass limit are given. For numerical analysis in Sec. IV, we first demonstrate the order of magnitude of radiative mass for a heavy mass scale that would be accessible at colliders. Then we consider the heavy mass limit trying to accommodate neutrino masses derived from oscillation experiments. We conclude in the last section where the main points of the work are recapitulated.

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II. TYPE III SEESAW MODEL

We describe systematically in this section the type III seesaw model proposed in Ref. [5]. While the exposed relations among the lepton mixing matrix and the lepton masses will be employed in the next section to evaluate the radiative neutrino masses, the displayed interactions may also be useful in other applications.

A. Yukawa couplings and lepton mass matrices

The model introduces a lepton multiplet, Σ, that is a triplet of $SU(2)_L$ but carries no hypercharge, on top of the fields present in the SM. We shall restrict ourselves to the leptonic sector of the model. The lepton fields are

$$F_L = \begin{pmatrix} n_L \\ f_L \end{pmatrix}, \qquad f_R, \qquad \Sigma_R = \begin{pmatrix} \frac{1}{\sqrt{2}}\Sigma_R^0 & \Sigma_R^+ \\ \Sigma_R^- & -\frac{1}{\sqrt{2}}\Sigma_R^0 \end{pmatrix}. \tag{1}$$

We have assumed without loss of generality that Σ is right handed. The Yukawa couplings plus the bare mass for Σ are

$$\begin{aligned} -\mathcal{L}_{\text{Yuk}} = {} & \tfrac{1}{2}\text{tr}(M_\Sigma \bar{\Sigma}_R \Sigma_R^C + M_\Sigma^* \bar{\Sigma}_R^C \Sigma_R) \\ & + (\bar{F}_L y_\Phi f_R \Phi + \Phi^\dagger \bar{f}_R y_\Phi^\dagger F_L) \\ & + (\bar{F}_L y_\Sigma \Sigma_R \tilde{\Phi} + \tilde{\Phi}^\dagger \bar{\Sigma}_R y_\Sigma^\dagger F_L), \end{aligned} \tag{2}$$

where Φ is the scalar doublet with $\tilde{\Phi} = i\sigma^2\Phi^*$. y_Φ and y_Σ are, respectively, 3×3 and 3×1 complex Yukawa coupling matrices. The superscript C denotes the charge conjugation, $\psi^C = \mathcal{C}\gamma^0\psi^*$ with $\mathcal{C} = i\gamma^0\gamma^2$. Our notation is such that $\psi_L^C = (\psi_L)^C$. It is not necessary to include a $F_L^C - \Sigma_R^C$ coupling since $\bar{\psi}^C\chi^C = \bar{\chi}\psi$. Note that we can choose M_Σ, which is the seesaw scale in the model, to be real positive as any phase of it may be absorbed into y_Σ.

After Φ develops a vacuum expectation value, v, the lepton mass terms become

$$\begin{aligned} -\mathcal{L}_m = {} & \frac{1}{2}M_\Sigma(\bar{\Sigma}_R^0\Sigma_R^{0C} + \bar{\Sigma}_R^-\Sigma_R^{+C} + \bar{\Sigma}_R^+\Sigma_R^{-C} + \text{H.c.}) \\ & + \frac{v}{\sqrt{2}}\left(\bar{f}_L y_\Phi f_R + \frac{1}{\sqrt{2}}\bar{n}_L y_\Sigma \Sigma_R^0 + \bar{f}_L y_\Sigma \Sigma_R^- + \text{H.c.}\right). \end{aligned} \tag{3}$$

Since $\Sigma_R^\pm$ carry electric charge, they cannot be Majorana particles. Instead, their equal bare mass suggests the combination to a Dirac field,

$$\Psi = \Sigma_R^- + \Sigma_R^{+C}, \tag{4}$$

with $\Sigma_R^{+C} = \mathcal{C}\gamma^0(\Sigma_R^+)^*$. It is then impossible to assign a lepton number to Ψ without explicitly breaking gauge symmetry. The lepton mass terms are summarized as

$$-\mathcal{L}_m = \tfrac{1}{2}\bar{N}_L m_N N_L^C + \bar{E}_L m_E E_R + \text{H.c.}, \tag{5}$$

where the neutral and charged lepton fields and their mass matrices are

$$N_L = \begin{pmatrix} n_L \\ \Sigma_R^{0C} \end{pmatrix}, \qquad N_R = N_L^C = \begin{pmatrix} n_L^C \\ \Sigma_R^0 \end{pmatrix}, \qquad E = \begin{pmatrix} f \\ \Psi \end{pmatrix}, \tag{6}$$

$$m_N = \begin{pmatrix} 0_3 & \frac{1}{2}v y_\Sigma \\ \frac{1}{2}v y_\Sigma^T & M_\Sigma \end{pmatrix}, \qquad m_E = \begin{pmatrix} \frac{1}{\sqrt{2}}v y_\Phi & \frac{1}{\sqrt{2}}v y_\Sigma \\ 0 & M_\Sigma \end{pmatrix}. \tag{7}$$

B. Gauge couplings of leptons

The kinetic term for the triplet field is

$$\mathcal{L}_{\text{kin}}^\Sigma = \text{tr}\bar{\Sigma}_R i\not{D}\Sigma_R, \tag{8}$$

where the covariant derivative is

$$D_\mu\Sigma_R = \partial_\mu\Sigma_R - ig_2\tfrac{1}{2}[A_\mu^a\sigma^a, \Sigma_R], \tag{9}$$

with A_μ^a and g_2 being the $SU(2)_L$ gauge fields and coupling. The kinetic term can be expressed in terms of the fields defined in Eq. (6). In so doing, the following relations are useful, $\bar{\psi}^C\gamma^\mu\chi^C = -\bar{\chi}\gamma^\mu\psi$, $\bar{\psi}^C\not{\partial}\chi^C = \bar{\chi}\not{\partial}\psi - \partial_\mu(\bar{\chi}\gamma^\mu\psi)$, where the total derivative may be dropped from the Lagrangian.

Including the standard kinetic terms for the SM fields F_L and f_R, the complete kinetic terms for leptons are

$$\begin{aligned} \mathcal{L}_{\text{kin}} = {} & \frac{1}{2}\bar{N}i\not{\partial}N + \bar{E}i\not{\partial}E + \frac{g_2}{\sqrt{2}}(J_W^{+\mu}W_\mu^+ + J_W^{-\mu}W_\mu^-) \\ & + \frac{g_2}{c_W}J_Z^\mu Z_\mu + eJ_{\text{em}}^\mu A_\mu, \end{aligned} \tag{10}$$

where $W_\mu^\pm$, Z_μ, and A_μ are the weak and electromagnetic fields coupled to the currents

$$\begin{aligned} J_W^{+\mu} &= \bar{N}\gamma^\mu(w_L P_L + w_R P_R)E, \\ J_Z^\mu &= \bar{N}\gamma^\mu z_L^N P_L N + \bar{E}\gamma^\mu(z_L^E P_L + z_R^E P_R)E, \\ J_{\text{em}}^\mu &= -\bar{E}\gamma^\mu E, \end{aligned} \tag{11}$$

and $J_W^{-\mu} = (J_W^{+\mu})^\dagger$, with the coupling matrices being

$$w_L = \begin{pmatrix} 1_3 & \\ & \sqrt{2} \end{pmatrix}, \qquad w_R = \begin{pmatrix} 0_3 & \\ & \sqrt{2} \end{pmatrix},$$

$$z_L^N = \begin{pmatrix} \frac{1}{2}1_3 & \\ & 0 \end{pmatrix}, \qquad z_L^E = \begin{pmatrix} (-\frac{1}{2} + s_W^2)1_3 & \\ & -c_W^2 \end{pmatrix},$$

$$z_R^E = \begin{pmatrix} s_W^2 1_3 & \\ & -c_W^2 \end{pmatrix}. \tag{12}$$

We have used the conventional notations $c_W = \cos\theta_W$, $s_W = \sin\theta_W$, with θ_W being the weak angle.

C. Diagonalization of lepton mass matrices

Noting that the upper left 3×3 block of m_N is zero, we can make m_N standardized as follows. A unitary transformation in family space, $F_L \to \mathcal{U}^\dagger F_L$, only modifies the Yukawa couplings, $y_\Sigma \to \mathcal{U} y_\Sigma$ and $y_\Phi \to \mathcal{U} y_\Phi$. One can choose $\mathcal{U}$ to rotate the column vector y_Σ to its third component, so that

$$m_N = \begin{pmatrix} 0_2 & & \\ & 0 & \frac{1}{2} v r_\Sigma \\ & \frac{1}{2} v r_\Sigma & M_\Sigma \end{pmatrix}, \tag{13}$$

where r_Σ is real positive. There are thus two massless neutrinos (named 1 and 2) at tree level. They will generally get a radiative mass as their masslessness is not protected by any symmetry. The other two neutrinos (3 and 4) get the masses

$$m_{3,4} = \tfrac{1}{2}\left[\sqrt{M_\Sigma^2 + (v r_\Sigma)^2} \mp M_\Sigma\right]. \tag{14}$$

The mass eigenstate fields of neutrinos are therefore

$$\nu_L = U_N^T N_L, \qquad \nu_R = \nu_L^C = U_N^\dagger N_R, \tag{15}$$

where

$$U_N = \begin{pmatrix} 1_2 & & \\ & i c_\theta & s_\theta \\ & -i s_\theta & c_\theta \end{pmatrix}, \tag{16}$$

with $c_\theta = \cos\theta$, $s_\theta = \sin\theta$, and $\tan\theta = \sqrt{m_3/m_4}$.

The mass matrix of the charged leptons is diagonalized by bi-unitary transformations,

$$E_{L,R} = U_{L,R} \ell_{L,R}, \qquad U_L^\dagger m_E U_R = \mathrm{diag}(m_e, m_\mu, m_\tau, m_\chi). \tag{17}$$

Here ν_4 and χ are the new neutral and charged leptons beyond the SM. They must be very heavy to evade the experimental detection so far. The tiny (small) mass of the observed neutrinos (charged leptons) then implies that, to very good precision, we have approximately

$$m_4 \approx m_\chi, \qquad \theta^2 \approx \frac{m_3}{m_4}, \tag{18}$$

which will be employed in later numerical analysis.

D. Summary of lepton interactions

We can now express the interactions of leptons in terms of their mass eigenstate fields, $\nu_i (i = 1, 2, 3, 4)$ and $\ell_\alpha (\alpha = e, \mu, \tau, \chi)$. The currents in Eq. (11) become

$$\begin{aligned} J_W^{+\mu} &= \bar\nu \gamma^\mu (\mathcal{W}_L P_L + \mathcal{W}_R P_R)\ell, \\ J_Z^\mu &= \bar\nu \gamma^\mu Z_L^\nu P_L \nu + \bar\ell \gamma^\mu (Z_L^\ell P_L + Z_R^\ell P_R)\ell, \\ J_{\rm em}^\mu &= -\bar\ell \gamma^\mu \ell, \end{aligned} \tag{19}$$

where

$$\begin{aligned} \mathcal{W}_L &= U_N^T w_L U_L, \qquad \mathcal{W}_R = U_N^\dagger w_R U_R, \\ Z_L^\nu &= U_N^T z_L^N U_N^*, \qquad Z_L^\ell = U_L^\dagger z_L^E U_L, \\ Z_R^\ell &= U_R^\dagger z_R^E U_R. \end{aligned} \tag{20}$$

Note that there is a degree of freedom in presenting the neutral current of Majorana neutrinos. Using $\nu = \nu_L + \nu_L^C = \nu^C$ and $\bar\psi^C \gamma^\mu P_{L,R} \chi^C = -\bar\chi \gamma^\mu P_{R,L} \psi$, we can write

$$\bar\nu \gamma^\mu Z_L^\nu P_L \nu = \tfrac{1}{2} \bar\nu \gamma^\mu (Z_L^\nu P_L - Z_L^{\nu T} P_R)\nu. \tag{21}$$

Since U_N and w, z are known, the following explicit results are useful:

$$\begin{aligned} \mathcal{W}_L &= \begin{pmatrix} 1_2 & & \\ & i c_\theta & -i\sqrt{2} s_\theta \\ & s_\theta & \sqrt{2} c_\theta \end{pmatrix} U_L, \\ \mathcal{W}_R &= \sqrt{2} \begin{pmatrix} 0_2 & & \\ & 0 & i s_\theta \\ & 0 & c_\theta \end{pmatrix} U_R, \\ Z_L^\nu &= \frac{1}{2} \begin{pmatrix} 1_2 & & \\ & c_\theta^2 & i c_\theta s_\theta \\ & -i c_\theta s_\theta & s_\theta^2 \end{pmatrix}. \end{aligned} \tag{22}$$

One observes from the above that the right-handed charged current involves only the massive neutrinos $\nu_{3,4}$ while the flavor changing neutral currents occur for both charged leptons and (massive) neutrinos.

For completeness, we present some additional results that may be useful in other applications of the model. First of all, one can construct the coupling matrices in the neutral currents in terms of those in the charged currents:

$$\begin{aligned} Z_L^\nu &= 1_4 - \tfrac{1}{2} \mathcal{W}_L \mathcal{W}_L^\dagger, \\ Z_L^\ell &= s_W^2 1_4 - \tfrac{1}{2} \mathcal{W}_L^\dagger \mathcal{W}_L, \\ Z_R^\ell &= s_W^2 1_4 - \tfrac{1}{2} \mathcal{W}_R^\dagger \mathcal{W}_R. \end{aligned} \tag{23}$$

The Yukawa couplings of the would-be Goldstone bosons $G^{\pm,0}$ are

$$\begin{aligned} \mathcal{L}_{\rm Yuk}^{G^{0,\pm}} = &+ \frac{g_2}{\sqrt{2} m_W} G^+ \bar\nu [m_\nu (\mathcal{W}_L P_L + \mathcal{W}_R P_R) \\ &- (\mathcal{W}_L P_R + \mathcal{W}_R P_L) m_\ell]\ell + \text{H.c.} \\ &- \frac{i g_2}{c_W m_Z} G^0 \bar\nu [m_\nu Z_L^\nu P_L - Z_L^\nu m_\nu P_R]\nu \\ &- \frac{i g_2}{c_W m_Z} G^0 \bar\ell [m_\ell (Z_L^\ell P_L + Z_R^\ell P_R) \\ &- (Z_L^\ell P_R + Z_R^\ell P_L) m_\ell]\ell, \end{aligned} \tag{24}$$

where m_ν and m_ℓ are the diagonal mass matrices of the neutrinos and charged leptons. The above simple structure is dictated by the nature of $G^{\pm,0}$ although the intermediate results in a direct derivation from $\mathcal{L}_{\text{Yuk}}$ may look cumbersome. In constrast, the Yukawa couplings to the physical Higgs field h are quite different since the leptons obtain masses from both the bare mass term and the Yukawa couplings:

$$\begin{aligned}\mathcal{L}^{h}_{\text{Yuk}} = &-\frac{h}{v}m_3c^2_\theta(\bar{\nu}_3\nu_3+\bar{\nu}_4\nu_4)\\ &-i\frac{h}{v}(m_4-m_3)c_\theta s_\theta(\bar{\nu}_{3L}\nu_{4R}-\bar{\nu}_{4R}\nu_{3L})\\ &-\frac{h}{v}\{\bar{\ell}_{\alpha L}[m_\alpha\delta_{\alpha\beta}-(m_4-m_3)U^*_{L\chi\alpha}U_{R\chi\beta}]\ell_{\beta R}\\ &+\text{H.c.}\}. \end{aligned} \tag{25}$$

E. Constraints on mixing matrices and lepton masses

For convenience in the next section, we collect here the constraints on $U_{L,R}$, m_α, and m_i:

$$C1\colon\ m_3^2c_\theta = m_4^2 s_\theta, \tag{26}$$

$$C2\colon\ \sum_\alpha U^*_{Li\alpha}U_{L3\alpha} = \sum_\alpha U^*_{Li\alpha}U_{L4\alpha} = 0, \tag{27}$$

$$C3\colon\ \sum_\alpha m_\alpha U^*_{Li\alpha}U_{R4\alpha} = 0, \tag{28}$$

$$C4\colon\ \sum_\alpha m_\alpha^2 U^*_{Li\alpha}U_{L4\alpha} = 0, \tag{29}$$

where $i = 1, 2$ in $C2$, $C3$, and $C4$. They will be extensively used to improve the apparent convergence of the loop integrals and extract the leading terms in the large mass limit of heavy leptons. These constraints are exact and can be readily derived. The constraint $C1$ is from the diagonalization of m_N while $C2$ represents unitarity of U_L. After rotating the column vector y_Σ to its third component, the first two columns in the last row of $m_E^\dagger$ vanish. This yields $(U_R m_\ell^\dagger U_L^\dagger)_{4i} = (m_E^\dagger)_{4i} = 0$ for $i = 1, 2$ which is $C3$. In addition, we find that $(m_E m_E^\dagger)_{4i}$ also vanishes for $i = 1, 2$ which gives the last constraint $C4$. For the sake of notational simplicity, we sometimes also use the Latin letters i, j and numbers, which enter through the charged current matrices $\mathcal{W}_{L,R}$, as the indices for the corresponding charged leptons.

III. TWO-LOOP INDUCED NEUTRINO MASSES

Now we calculate the radiative mass of the neutrinos $\nu_{1,2}$ that are massless at tree level. This is given by their minus self-energy evaluated at the zero momentum. We thus need to calculate the amplitude for the transition, $\nu_{iL}\to\nu^C_{jL}$, with $i, j = 1, 2$. There is no contribution at one loop. This arises because, while the neutral current does not couple $\nu_{1,2}$ to the massive ones $\nu_{3,4}$, the charged current involving $\nu_{1,2}$ is purely left handed and thus cannot induce a mass for a massless particle.

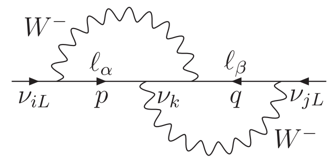

FIG. 1. Diagram contributing to $-i\Sigma_{ji}$.

At two loops, we note first that a diagram with at least one of the two external lines connected to a virtual Z boson cannot contribute. This is because, if it did, removing this virtual Z line would also contribute since Z couples diagonally to $\nu_{1,2}$ and conserves chirality. But this would contradict our claim at one loop. The external lines must therefore all connect to virtual $W^\pm$ bosons. Finally, the two external lines cannot connect to the same virtual $W^\pm$ due to charge conservation. This leaves us with the single diagram shown in Fig. 1.

We shall evaluate the radiative mass in unitarity gauge. We first simplify and classify the contributions from the diagram. Then we apply the constraints $C1$–$C4$ to reach manifest convergence in loop integrals and to get prepared for isolating leading terms in the seesaw limit. Finally, the contributions are expressed in terms of some standard parameter integrals.

To start with, we note that the external $\nu_{i,j}$ $(i, j = 1, 2)$ have no right-handed couplings to the corresponding virtual charged leptons $\ell_{\alpha,\beta}$. The diagram then decomposes into four terms according to the chiralities of the two vertices involving the virtual neutrino ν_k. After some algebraic work, we can remove all γ matrices in favor of the products of loop momenta and obtain

$$\begin{aligned} u_j^T\Sigma_{ji}u_i &= \mathcal{M}_{ji}u_j^T\mathcal{C}P_Lu_i,\\ \mathcal{M}_{ji} &= \frac{g_2^4}{4(4\pi)^4}[T^{LL}+T^{RR}+T^{RL}+T^{RL}|_{i\leftrightarrow j}], \end{aligned}\tag{30}$$

where $u_{i,j}$ are the spinors for external neutrinos, and $\mathcal{M}$ gives the radiative neutrino mass. The T functions are

$$\begin{aligned} T^{LL} &= m_k\mathcal{W}^*_{Li\alpha}\mathcal{W}^*_{Lj\beta}\mathcal{W}_{Lk\alpha}\mathcal{W}_{Lk\beta}F^{LL}(\alpha,\beta;k),\\ T^{RR} &= m_km_\alpha m_\beta m_W^{-2}\mathcal{W}^*_{Li\alpha}\mathcal{W}^*_{Lj\beta}\mathcal{W}_{Rk\alpha}\\ &\quad\times\mathcal{W}_{Rk\beta}F^{RR}(\alpha,\beta;k),\\ T^{RL} &= m_\alpha\mathcal{W}^*_{Li\alpha}\mathcal{W}^*_{Lj\beta}\mathcal{W}_{Rk\alpha}\mathcal{W}_{Lk\beta}F^{RL}(\alpha,\beta;k), \end{aligned}\tag{31}$$

where the loop functions F are dimensionless functions of the mass ratios. Upon Wick rotation to Euclidean space, they become

$$F^{LL}(\alpha,\beta;k) = -\iint \frac{p\cdot q}{D(\alpha,\beta;k)}[4 + p^2q^2 + 4(p^2+q^2)],$$
$$F^{RR}(\alpha,\beta;k) = \iint \frac{1}{D(\alpha,\beta;k)}[-8 + 2(p\cdot q)^2 - q^2p^2 - 2(p^2+q^2)],$$
$$F^{RL}(\alpha,\beta;k) = \iint \frac{1}{D(\alpha,\beta;k)}[-4(p\cdot q + q^2) - p^2q^2(p\cdot q + q^2) + 2(p^2 p\cdot q + 2(p\cdot q)^2 - q^2p^2) - 4q^2(p\cdot q + q^2)], \tag{32}$$

where the notations are

$$D(\alpha,\beta;k) = [(p+q)^2 + r_k][p^2 + r_\alpha][p^2+1][q^2 + r_\beta]\times[q^2+1],$$
$$r_k = \frac{m_k^2}{m_W^2}, \qquad r_\alpha = \frac{m_\alpha^2}{m_W^2}, \qquad \iint = \frac{1}{\pi^4}\int d^4p \int d^4q. \tag{33}$$

Here the summation over the virtual lepton flavors α, β, k is implied in the T functions, and $\mathcal{M}_{ji}$ is manifestly symmetric as expected for Majorana particles.

Since only massive neutrinos enter the right-handed charged current, the virtual ν_k in the T functions is actually restricted to $\nu_{3,4}$. Using the explicit forms of $\mathcal{W}_{L,R}$ shown in Eq. (22), the T functions decompose into

$$\begin{aligned} T^{LL} &= U^*_{Li\alpha}U^*_{Lj\beta}\{U_{L3\alpha}U_{L3\beta}[m_4 s_\theta^2 F_4^{LL} - m_3 c_\theta^2 F_3^{LL}] \\ &\quad + 2U_{L4\alpha}U_{L4\beta}[m_4 c_\theta^2 F_4^{LL} - m_3 s_\theta^2 F_3^{LL}] \\ &\quad + \sqrt{2}c_\theta s_\theta(U_{L4\alpha}U_{L3\beta} + U_{L3\alpha}U_{L4\beta}) \\ &\quad \times [m_4F_4^{LL} + m_3F_3^{LL}]\}, \\ T^{RR} &= 2m_\alpha m_\beta m_W^{-2}U^*_{Li\alpha}U^*_{Lj\beta}U_{R4\alpha}U_{R4\beta} \\ &\quad \times[m_4c_\theta^2F_4^{RR} - m_3s_\theta^2F_3^{RR}], \\ T^{RL} &= m_\alpha U^*_{Li\alpha}U^*_{Lj\beta}U_{R4\alpha}\{\sqrt{2}c_\theta s_\theta U_{L3\beta}[F_4^{RL} - F_3^{RL}] \\ &\quad + 2U_{L4\beta}[s_\theta^2F_3^{RL} + c_\theta^2F_4^{RL}]\}, \end{aligned} \tag{34}$$

where for brevity the first two arguments α, β of the F functions are suppressed while the third one k appears as a subscript 3 or 4. In addition to improving apparent convergence, the main merit of applying the constraints $C1$–$C4$ is to subtract heavy leptons ν_4, χ from the loops. This manifestly avoids in the contributing terms some large numbers that are actually balanced by the small matrix elements mixing the light and heavy leptons. Furthermore, this facilitates the extraction of the leading terms that can survive upon being multiplied by the mixing matrix elements and summing over light flavors α, β, for which the hierarchical limit $1 \gg r_\alpha \gg r_3$ works very well. We stress that we are not discarding the contributions from heavy leptons but are combining them in a judicious manner with those from light leptons before numerical analysis is done. In the following sections we shall reduce the T functions using the constraints.

A. Reduction of T^{LL}

We note first of all that the numerator of F^{LL} is separately linear in p^2 and q^2. Take p^2 as an example. By decomposing $p^2 = (p^2 + r_\alpha) - r_\alpha$, the first term cancels the corresponding factor in $D(\alpha,\beta;k)$ so that its contribution to F^{LL} is independent of α. The constraint $C2$ then implies that it does not survive in T^{LL} upon summing over α. We can thus effectively set in the numerator of F^{LL}, $p^2 \to -r_\alpha$ and similarly $q^2 \to -r_\beta$:

$$F^{LL}(\alpha,\beta;k) \to -[4 + r_\alpha r_\beta - 4(r_\alpha + r_\beta)] \times \iint \frac{p\cdot q}{D(\alpha,\beta;k)}, \tag{35}$$

where the arrow means equality when multiplied by U factors and summing over α, β. To go further, we have to cope separately with the four terms in T^{LL} according to the U_L factors involved:

$$T^{LL} = U^*_{Li\alpha}U^*_{Lj\beta}\{U_{L3\alpha}U_{L3\beta}T_{33}^{LL} + 2U_{L4\alpha}U_{L4\beta}T_{44}^{LL} + \sqrt{2}[U_{L4\alpha}U_{L3\beta}T_{43}^{LL} + U_{L3\alpha}U_{L4\beta}T_{34}^{LL}]\} \tag{36}$$

with obvious definitions on T_{33}^{LL} etc. by comparing with Eq. (34).

Although the first term, T_{33}^{LL}, is already convergent upon applying $C1$ due to the subtraction between F_4^{LL} and F_3^{LL}, we can do better by subtracting explicitly the contribution from the heavy charged lepton χ. The trick is that, for a term in F^{LL} that is not proportional to r_α we make the substitution

$$\frac{1}{p^2 + r_\alpha} \to \frac{1}{p^2 + r_\alpha} - \frac{1}{p^2 + r_4} \equiv d_\alpha(p), \tag{37}$$

while for a term that is proportional to r_α, we do as follows:

$$\frac{r_\alpha}{p^2 + r_\alpha} \to \frac{r_\alpha}{p^2 + r_\alpha} - \frac{r_4}{p^2 + r_4} \equiv e_\alpha(p). \tag{38}$$

The legitimacy of the substitutions is guaranteed by the constraint $C2$. Thus,

$$T_{33}^{LL} \to m_3c_\theta^2 \iint \frac{p\cdot q}{[p^2+1][q^2+1]}d_3(p+q)[4d_\alpha(p)d_\beta(q) + e_\alpha(p)e_\beta(q) - 4e_\alpha(p)d_\beta(q) - 4d_\alpha(p)e_\beta(q)]. \tag{39}$$

The second term, T_{44}^{LL}, is multiplied by $U_{L4\alpha}U_{L4\beta}$ so that we have a choice of whether to use the constraint $C2$ [i.e., Eq. (37)] or $C4$ [Eq. (38)] for the terms proportional to r_α or r_β. It turns out that the latter is better as it can reduce the amount of work by bringing down more factors of $r_{\alpha,\beta}$ for light leptons α, β, which makes the corresponding term subdominant in the hierarchical limit. The last two terms may be similarly manipulated. The results are summarized as follows:

$$\begin{aligned} T_{44}^{LL} \to & [4 + r_\alpha r_\beta - 4(r_\alpha + r_\beta)] \\ & \times \iint \frac{p \cdot q}{[p^2+1][q^2+1]} d_\alpha(p) d_\beta(q) \\ & \times \left[\frac{m_3 s_\theta^2}{(p+q)^2 + r_3} - \frac{m_4 c_\theta^2}{(p+q)^2 + r_4}\right], \\ T_{43}^{LL} \to & \iint \frac{p \cdot q}{[p^2+1][q^2+1]} [4(r_\alpha - 1) d_\beta(q) \\ & + (4 - r_\alpha) e_\beta(q)] d_\alpha(p) \\ & \times \left[\frac{m_4}{(p+q)^2 + r_4} + \frac{m_3}{(p+q)^2 + r_3}\right] c_\theta s_\theta, \end{aligned} \tag{40}$$

while T_{34}^{LL} is obtained from T_{43}^{LL} by $\alpha \leftrightarrow \beta$ and $i \leftrightarrow j$. Since α, β are summed over, this amounts to symmetrizing T_{43}^{LL} in i, j.

The advantage of the above results can be understood by recalling that we now only need to sum over light flavors α, β in T^{LL}. Since $1 \gg r_{\alpha,\beta} \gg r_3$, it is numerically very good to set $r_\alpha = r_\beta = r_3 = 0$. For instance, the largest $r_\tau \sim 5 \times 10^{-4}$ while $r_3 \sim 6 \times 10^{-24}$ for $m_3 \sim 0.2$ eV. This will not introduce mass singularities in the loop integrals. In addition, when a term proportional to m_3 is accompanied by one proportional to m_4, we ignore the former since it cannot make a significant contribution to the radiative mass. (Note that T^{LL} is exceptional since $m_3 c_\theta^2 = m_4 s_\theta^2$.) Although the above argument is self-evident, we have inspected and compared carefully all of the terms to verify it. This considerably simplifies the integrals to compute

$$\begin{aligned} T_{33}^{LL} \to & m_3 c_\theta^2 r_4^2 \{4[\mathcal{X}_2(0) - \mathcal{X}_2(r_4)] \\ & + 8[\mathcal{X}_1(0) - \mathcal{X}_1(r_4)] + \mathcal{X}_0\}, \\ T_{44}^{LL} \to & -m_4 c_\theta^2 4 r_4^2 \mathcal{X}_2(r_4), \\ T_{43}^{LL} \to & -m_4 c_\theta s_\theta 4 r_4^2 [\mathcal{X}_2(r_4) + \mathcal{X}_1(r_4)], \end{aligned} \tag{41}$$

and $T_{34}^{LL} = T_{43}^{LL}$, where the loop integrals $\mathcal{X}$ are defined in the Appendix. These functions are independent of α, β and depend only on r_4.

B. Reduction of T^{RR}

The second term in T^{RR} is doubly suppressed by $m_3 s_\theta^2$ compared to the first one and will be ignored from the start. Since the numerator in the integrand of F_4^{RR} is again linear in p^2 and q^2, they may be replaced by $-r_\alpha$ and $-r_\beta$, respectively, employing the constraint $C3$. For the $2(p \cdot q)^2$ term in the numerator, we decompose as follows:

$$\begin{aligned} 2(p \cdot q)^2 = & p \cdot q([(p+q)^2 + r_4] - [p^2 + r_\alpha] \\ & - [q^2 + r_\beta] + [r_\alpha + r_\beta - r_4]). \end{aligned}$$

The first term is canceled by the same factor in the denominator D making the integrand odd in p, and thus vanishes upon integration. The second term again cancels a same factor from D and is killed upon summing over α by the constraint $C3$, and the same happens with the third term as well. The numerator now becomes effectively,

$$-(8 + r_4 p \cdot q) + (p \cdot q + 2)(r_\alpha + r_\beta) - r_\alpha r_\beta.$$

Now we make the substitutions in Eqs. (37) and (38) as we did in the previous section, though employing this time the constraint $C3$, to obtain

$$\begin{aligned} F_4^{RR} \to & \iint \frac{1}{[p^2+1][q^2+1][(p+q)^2 + r_4]} \\ & \times \{-(8 + r_4 p \cdot q) d_\alpha(p) d_\beta(q) + (p \cdot q + 2) \\ & \times [e_\alpha(p) d_\beta(q) + e_\beta(p) d_\alpha(q)] - e_\alpha(p) e_\beta(q)\}. \end{aligned} \tag{42}$$

Since it is now legitimate to sum only over light flavors α, β, the above simplifies to

$$\begin{aligned} F_4^{RR} \to & -r_4^2 \{8\mathcal{Y}_2(r_4) + 4\mathcal{Y}_1(r_4) + \mathcal{Y}_0(r_4) + r_4 \mathcal{X}_2(r_4) \\ & + 2\mathcal{X}_1(r_4)\}, \end{aligned} \tag{43}$$

where the new integrals $\mathcal{Y}$ are also defined in the Appendix.

C. Reduction of T^{RL}

This chirality-mixed part from the two vertices involving the virtual neutrino ν_k contains the most number of terms in F_k^{RL}:

$$\begin{aligned} T^{RL} = & m_\alpha U_{Li\alpha}^* U_{Lj\beta}^* U_{R4\alpha} \{\sqrt{2} c_\theta s_\theta U_{L3\beta} [F_4^{RL} - F_3^{RL}] \\ & + 2 c_\theta^2 U_{L4\beta} F_4^{RL}\}, \end{aligned} \tag{44}$$

where we have dropped the $s_\theta^2 F_3^{RL}$ term as one cannot rely on it to induce a reasonable mass due to a tiny $s_\theta^2 \sim 10^{-12}$ at $m_3 \sim 0.2$ eV and $m_4 \sim 200$ GeV, for instance.

The numerator of the integrand in F^{RL} is linear in p^2, which can thus be replaced by $-r_\alpha$ using the contraint $C3$. On the other hand, since the numerator is quadratic in q^2, we must distinguish between the two terms in T^{RL} which are proportional to $U_{L3\beta}$ and $U_{L4\beta}$, respectively. For the first one, we can only set one factor of q^2 to $-r_\beta$ using $C2$. After this, we apply $C2$ and $C3$ via the substitutions in Eqs. (37) and (38) and obtain

$$\begin{aligned} F_4^{RL} - F_3^{RL} \to & \iint \frac{d_3(p+q)}{[p^2+1][q^2+1]} [(p \cdot q + q^2 + 2) \\ & \times e_\alpha(p) e_\beta(q) + 2 p \cdot q e_\alpha(p) d_\beta(q) \\ & - 4(p \cdot q + q^2 + 1) d_\alpha(p) e_\beta(q) \\ & + 4(p \cdot q - (p \cdot q)^2) d_\alpha(p) d_\beta(q)]. \end{aligned} \tag{45}$$

The summation over light flavors α, β then yields the result in terms of the standard integrals:

$$\begin{aligned}F_4^{RL} - F_3^{RL} \to r_4^2\{ & r_4[\mathcal{U}_0 + 4\mathcal{U}_1] + \mathcal{X}_0 \\ & + 6[\mathcal{X}_1(0) - \mathcal{X}_1(r_4)] \\ & + 4[\mathcal{X}_2(0) - \mathcal{X}_2(r_4)] - 2r_4\mathcal{X}_2(r_4) \\ & + [\mathcal{Y}_0(0) - \mathcal{Y}_0(r_4)]\}. \end{aligned} \quad (46)$$

For the second term proportional to $U_{L4\beta}$ in T^{RL}, we can set two factors of q^2 to $-r_\beta$ because of $C2$ and $C4$. The subsequent manipulation based on the constraints and Eqs. (37) and (38) is similar, and gives

$$\begin{aligned}F_4^{RL} \to r_4^2\{ & r_\beta(\mathcal{Y}_0(r_4) + [\mathcal{X}_1(r_4) + 2\mathcal{Y}_1(r_4)] + 4\mathcal{Y}_1(r_4) \\ & + 4[\mathcal{Y}_2(r_4) + \mathcal{X}_2(r_4)]) - 4\mathcal{X}_2(r_4) \\ & - 2[\mathcal{X}_1(r_4) + r_4\mathcal{X}_2(r_4)]\}, \end{aligned} \quad (47)$$

where the terms suppressed by r_β will be ignored from now on.

To finish this section, we summarize the terms in the radiative neutrino mass as follows:

$$\begin{aligned}\mathcal{M}_{ji} = \frac{m_W^4 G_F^2}{2^5\pi^4} U_{Li\alpha}^* U_{Lj\beta}^* \{ & U_{L3\alpha}U_{L3\beta}T_{33}^{LL} + 2U_{L4\alpha}U_{L4\beta}T_{44}^{LL} + \sqrt{2}(U_{L4\alpha}U_{L3\beta} + U_{L3\alpha}U_{L4\beta})T_{43}^{LL} \\ & + 2\sqrt{r_\alpha r_\beta}U_{R4\alpha}U_{R4\beta}m_4 c_\theta^2 F_4^{RR} + \sqrt{2}c_\theta s_\theta(m_\alpha U_{R4\alpha}U_{L3\beta} + m_\beta U_{R4\beta}U_{L3\alpha})(F_4^{RL} - F_3^{RL}) \\ & + 2c_\theta^2(m_\alpha U_{R4\alpha}U_{L4\beta} + m_\beta U_{R4\beta}U_{L4\alpha})F_4^{RL}\}, \end{aligned} \quad (48)$$

where the relevant functions are given in Eqs. (41), (43), (46), and (47), in terms of the standard integrals calculated in the Appendix. The summation over the light charged leptons ℓ_α and ℓ_β is understood in the above.

IV. NUMERICAL ANALYSIS

Now we investigate whether we can accommodate the neutrino masses measured in oscillation experiments. Our starting formula was given in (48) which involves the light-heavy mixing parameters in addition to the upper left 3×3 submatrix of U_L. From Eq. (22) we see that the latter is just the leptonic mixing matrix measured in oscillation experiments to very good precision. However it is no more exactly unitary, and the deviation from unitarity is determined by the light-heavy mixing. A realistic numerical estimate should take all this into account to avoid a misleading conclusion. Although a global fitting to the lepton mixing parameters is possible with radiative corrections included, our main result on the seesaw scale required to reproduce the neutrino masses is independent of this fitting.

Both matrices $m_E m_E^\dagger$ and $m_E^\dagger m_E$ for the charged leptons have the hierarchical structure

$$M = \begin{pmatrix} B & d \\ d^\dagger & A \end{pmatrix}, \quad (49)$$

where B and d are, respectively, a 3×3 and a 3×1 matrix, whose entries are much smaller in magnitude than the positive number A. Then, the submatrix of the diagonalization matrix that mixes the small and large entries can be estimated as $\kappa \approx dA^{-1}$. Application of this to $m_E m_E^\dagger$ and $m_E^\dagger m_E$ yields for $\alpha = e, \mu, \tau$:

$$\begin{aligned} U_{L4\alpha} &\sim (m_3/m_4)^{1/2} = (r_3/r_4)^{1/4} = \theta, \\ U_{R4\alpha} &\sim (m_3/m_4)^{1/2}(m_\alpha/m_4) = \theta(r_\alpha/r_4)^{1/2}, \end{aligned} \quad (50)$$

where (18) is used. And the unitarity violation in the submatrix of light leptons is, for $i = 1, 2$

$$\sum_{\alpha=e,\mu,\tau} U_{Li\alpha}^* U_{L3\alpha} = -U_{Li\chi}^* U_{L3\chi} \sim \theta^2. \quad (51)$$

Consider first the case in which m_4 is not very large. This is the range of parameters that is particularly relevant to LHC physics. A heavy active lepton, especially the charged one χ, is supposed to be accessible if it is not much heavier than several hundred GeV. Our estimate of heavy-light mixing parameters is still good enough since m_4 is much larger than the light lepton masses. Using the estimates in Eqs. (18) and (50) [but not yet the one in (51)], we find that the three classes of contributions to $\mathcal{M}_{ji}$ in Eq. (48) consist of the following terms in units of $2^{-5}\pi^{-4}m_W^4 G_F^2 m_3$:

$$\begin{aligned} LL&: U_{Li\alpha}^* U_{Lj\beta}^* U_{L3\alpha}U_{L3\beta},\ U_{Li\alpha}^* U_{Lj\beta}^*, \\ & (U_{Li\alpha}^* U_{Lj\beta}^* + U_{Lj\alpha}^* U_{Li\beta}^*)U_{L3\beta}, \\ RR&: r_\alpha r_\beta U_{Li\alpha}^* U_{Lj\beta}^*, \\ RL&: r_\alpha(U_{Li\alpha}^* U_{Lj\beta}^* + U_{Lj\alpha}^* U_{Li\beta}^*)U_{L3\beta}, \\ & r_\alpha(U_{Li\alpha}^* U_{Lj\beta}^* + U_{Lj\alpha}^* U_{Li\beta}^*), \end{aligned} \quad (52)$$

where each term is to be multiplied by a coefficient that is a sum of integrals as can be obtained from Eqs. (41), (43), (46), and (47). The point is that these coefficients are order 1 numbers for r_4 not very large. Then, independent of the mixing matrix of light leptons, it is safe to say that

$$|\mathcal{M}_{ji}| < 1.8 \times 10^{-6} m_3. \quad (53)$$

Since no light neutrinos can be heavier than an eV from cosmological considerations, there is no hope to induce a large enough radiative mass m_1 or m_2 from m_3. Therefore, the minimal type III seesaw model cannot accommodate oscillation data if the heavy leptons have an intermediate mass. To put it another way, the oscillation data already exclude the possibility that the active heavy leptons in the model would be accessible at the LHC.

It is interesting to ask whether there is a chance at all in the model to induce a large enough neutrino mass. For this purpose, we study the seesaw limit in which m_4 blows up. Then $\mathcal{M}_{ji}$ is a sum of the following terms (again in units of $2^{-5}\pi^{-4}m_W^4G_F^2m_3$):

$$\begin{aligned} LL:\ & r_3r_4[\mathcal{X}_0] - U^*_{Li\alpha}U^*_{Lj\beta}8[r_4^2\mathcal{X}_2(r_4)] \\ & - (U^*_{Li\alpha} + U^*_{Lj\alpha})4\sqrt{2r_3r_4}[r_4\mathcal{X}_1(r_4)], \\ RR:\ & - r_\alpha U^*_{Li\alpha} r_\beta U^*_{Lj\beta}2[r_4\mathcal{Y}_0(r_4) + r_4^2\mathcal{X}_2(r_4) \\ & + 2r_4\mathcal{X}_1(r_4)], \\ RL:\ & (r_\alpha U^*_{Li\alpha}\sqrt{2r_3r_4}[r_4\mathcal{U}_0 + \mathcal{X}_0] \\ & - r_\alpha U^*_{Li\alpha}U^*_{Lj\beta}4[r_4\mathcal{X}_1(r_4) + r_4^2\mathcal{X}_2(r_4)]) \\ & + (i \leftrightarrow j). \end{aligned} \tag{54}$$

All combinations of loop integrals in the square brackets are $O(1)$ constants up to logarithmic corrections in the large r_4 limit. We have also taken into account the unitarity violation estimated in Eq. (51). Because of the estimates employed, the relative sign and factors of 2 between terms in the above cannot be taken seriously. But this does not preclude us from making a definite conclusion as shown below.

To induce a mass of $O(m_3)$, some terms in Eq. (54) must be above 10^5. This obviously requires a large r_4. But even this is insufficient. On the one hand, the terms not multiplied by r_4 factors outside the square brackets can be safely ignored; on the other hand, all remaining terms are controlled by r_3r_4. We must therefore require $r_3r_4 \gg 1$. This corresponds to the combined limit in terms of the original parameters in Lagrangian, $M_\Sigma \gg v r_\Sigma \gg m_W$. In the limit, only the first term in the LL class is relevant:

$$\mathcal{M}_{ji} \sim 2^{-5}\pi^{-4}m_W^4G_F^2m_3r_3r_4[\mathcal{X}_0]. \tag{55}$$

Inspection of our derivation shows that this is the term that is doubly suppressed by the unitarity violation between the third row and the first two rows of the light lepton mixing matrix. But unfortunately it is impractical to measure the violation down to the level that we are interested in, i.e., $\sim\theta^2 = r_3/r_4 = m_3^2/m_4^2$. The information on the indices (i, j) is lost also because of the estimates employed. This means in passing that our analysis on the neutrino masses in the above limit is independent of a detailed fitting to the leptonic mixing parameters. We find it is natural for the model to favor the normal hierarchy scenario; namely, a larger m_3 seeds a smaller $m_{1,2}$. For the purpose of illustration, we assume $m_1 = 0$. The solar and atmospherical oscillation data then give $m_2 \approx 8.7 \times 10^{-3}$ eV and $m_3 \approx 4.9 \times 10^{-2}$ eV, respectively, which can be fulfilled by requiring

$$m_4 \sim 4 \times 10^{16}\ \text{GeV}. \tag{56}$$

This is roughly the scale of grand unification.

V. CONCLUSION

The minimal type III seesaw model introduces a lepton triplet on top of the particles in the SM. Two neutrinos out of four are massless at the tree level, but they are not protected by any symmetry from getting a radiative mass at the quantum level. We have shown that the latter takes place first at two loops, and determined it in terms of some parameter functions. By employing realistic estimates of the mixing parameters between the light and heavy leptons, we studied the pattern of the neutrino masses. We found that it is not possible to accommodate the spectrum determined in oscillation experiments if the heavy leptons have a mass that would be within the reach of the LHC. However, if the seesaw scale is as large as that of grand unification, it is still possible to accommodate the spectrum in a nice manner: one light neutrino gets mass directly from seesaw while the other two get a radiative mass. The model would then contain nothing new but the tiny neutrino masses. The main message extracted from this work is therefore, if the LHC sees something like a triplet lepton, it definitely comes from a structure that goes beyond the economical one as originally suggested.

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APPENDIX: LOOP INTEGRALS

The loop integrals in the final result of T^{LL} [see Eq. (41)] are defined as

$$\mathcal{X}_2(r) = \iint \frac{p\cdot q}{D_1(r)p^2q^2}, \qquad \mathcal{X}_1(r) = \iint \frac{p\cdot q}{D_1(r)q^2}, \qquad \mathcal{X}_0 = \iint \frac{p\cdot q r_4}{D_1(r_4)(p+q)^2}, \tag{A1}$$

where

$$\begin{aligned} D_1(r) = {}& [p^2 + r_4][p^2 + 1][q^2 + r_4][q^2 + 1] \\ & \times [(p+q)^2 + r]. \end{aligned} \tag{A2}$$

The new integrals appearing in T^{RR} and T^{RL} are, respectively,

$$\mathcal{Y}_2(r)=\iint\frac{1}{D_1(r)p^2q^2},\qquad \mathcal{Y}_1(r)=\iint\frac{1}{D_1(r)q^2},\qquad \mathcal{Y}_0(r)=\iint\frac{1}{D_1(r)},\tag{A3}$$

and

$$\mathcal{U}_0=\iint\frac{1}{D_2},\qquad \mathcal{U}_1=\iint\frac{1}{D_2q^2},\tag{A4}$$

where

$$D_2=(p+q)^2[(p+q)^2+r_4][q^2+1][q^2+r_4][p^2+r_4].\tag{A5}$$

There is another integral in calculating T^{RL} that can be related to those already defined:

$$\iint\frac{(p\cdot q)^2}{D_1(r)p^2q^2}=-\mathcal{X}_1(r)-\frac{r}{2}\mathcal{X}_2(r).$$

The basic technique to compute the above integrals is to use fractions and the one-loop integrals in $n=4-2\epsilon$ dimensions:

$$\begin{aligned}&(4\pi)^2\int\frac{d^np}{(2\pi)^n}\frac{1}{[(p+q)^2+r][p^2+a]}\\&=(4\pi)^\epsilon\bigg[\Gamma(\epsilon)-\int_0^1dx\ln g(a,r)\bigg],\\&(4\pi)^2\int\frac{d^np}{(2\pi)^n}\frac{p\cdot q}{[(p+q)^2+r][p^2+a]}\\&=(4\pi)^\epsilon q^2\bigg[-\frac{1}{2}\Gamma(\epsilon)+\int_0^1dxx\ln g(a,r)\bigg],\end{aligned}\tag{A6}$$

where

$$g(a,r)=q^2x(1-x)+rx+a(1-x).\tag{A7}$$

Introducing the abbreviations,

$$\begin{aligned}&\bar{g}(a,r)=x(1-x)(1-y)+[rx+a(1-x)]y,\\&\tilde{g}_0=\frac{\bar{g}(0,r_4)}{\bar{g}(r_4,r_4)},\qquad \tilde{g}_1(r)=\frac{\bar{g}(1,r)}{\bar{g}(r_4,r)},\\&\mathcal{G}(r)=\frac{\ln\bar{g}(r_4,r)}{(r_4-1)r_4}+\frac{\ln\bar{g}(1,r)}{1-r_4}+\frac{\ln\bar{g}(0,r)}{r_4},\end{aligned}\tag{A8}$$

and denoting the parameter integrals in the form,

$$X=\int_0^1dx\int_0^1dyI[X],\tag{A9}$$

where X enumerates all of the defined integrals, the integrands are

$$\begin{aligned}&I[\mathcal{X}_2(r)]=\frac{x(1-y)\mathcal{G}(r)}{y(1-y+r_4y)},\\&I[\mathcal{X}_1(r)]=\frac{1}{r_4-1}\frac{x(1-y)}{y(1-y+r_4y)}\ln\tilde{g}_1(r),\\&I[\mathcal{X}_0]=\frac{1}{r_4-1}\frac{x(1-y)^2}{y^2(1-y+r_4y)}\ln\frac{\tilde{g}_1(0)}{\tilde{g}_1(r_4)},\end{aligned}\tag{A10}$$

for the $\mathcal{X}$ sequence, and

$$\begin{aligned}&I[\mathcal{Y}_2(r)]=-\frac{\mathcal{G}(r)}{1-y+r_4y},\\&I[\mathcal{Y}_1(r)]=\frac{1}{1-r_4}\frac{1}{1-y+r_4y}\ln\tilde{g}_1(r),\\&I[\mathcal{Y}_0(r)]=\frac{1}{1-r_4}\frac{1-y}{y(1-y+r_4y)}\ln\tilde{g}_1(r),\\&I[\mathcal{U}_0]=-\frac{1}{r_4}\frac{1-y}{y(1-y+r_4y)}\ln\tilde{g}_0,\\&I[\mathcal{U}_1]=-\frac{1}{r_4}\frac{1}{1-y+r_4y}\ln\tilde{g}_0,\end{aligned}\tag{A11}$$

for the $\mathcal{Y}$ and $\mathcal{U}$ sequences.

The above integrals have a magnitude of order 1 or smaller for r_4 not very large, and can be readily integrated numerically. This is a sufficient message for the first part of our numerical analysis in Sec. IV. For the analysis in the heavy mass limit, we need the leading terms of the integrals. We obtain them in two ways. One is to use the techniques and formulas developed already in the literature [30,34], and extend them slightly to cover all cases occurring in our integrals. [There is a typographic error in expansion (ii) on page 230 in Ref. [34]: $\frac{1}{2}\ln^2 a$ should have a plus sign instead of a minus.] The leading terms can also be extracted directly. For illustration, we calculate below the integrals $\mathcal{X}_1(r_4)$ and $\mathcal{X}_2(r_4)$ that appear most frequently in Eq. (54). We finish first the integration over y in terms of logarithm and dilogarithm functions using

$$\begin{aligned}&I(b)=\int_0^1\frac{dy}{y}\ln[1+(b-1)y]=-\mathrm{Li}_2(1-b),\\&J(b,r)=(r-1)\int_0^1dy\frac{\ln[1+(b-1)y]}{1+(r-1)y}\\&=\mathrm{Li}_2\bigg(\frac{b-r}{(b-1)r}\bigg)-\mathrm{Li}_2\bigg(\frac{b-r}{b-1}\bigg)\\&-\ln\frac{r-1}{b-1}\ln r+\frac{1}{2}\ln^2r,\end{aligned}\tag{A12}$$

where $b>1$, $r>1$. Denoting

$$b_1=\frac{r_4x+1-x}{x(1-x)},\qquad b_2=\frac{r_4}{x(1-x)},\qquad b_3=\frac{r_4}{1-x},\qquad a_i=\frac{b_i-r_4}{b_i-1},\tag{A13}$$

with $b_2\geq b_1\geq b_3\geq r_4>1>a_i>0$ for $x\in(0,1)$, and using the abbreviations

$$I_i = I(b_i), \qquad J_i = J(b_i, r_4), \tag{A14}$$

we express the integrals as follows:

$$\begin{aligned}(r_4-1)\mathcal{X}_1(r_4) &= \int xdx\Big\{([I_1-I_2]-[J_1-J_2]) \\ &\quad -\frac{J_1-J_2}{r_4-1}\Big\}, \\ r_4(r_4-1)\mathcal{X}_2(r_4) &= \int xdx\Big\{r_4([I_2-I_1]-[J_2-J_1]) \\ &\quad -(r_4-1)([I_2-I_3]-[J_2-J_3]) \\ &\quad -r_4\Big[\frac{J_2-J_1}{r_4-1}-\frac{J_2-J_3}{r_4}\Big]\Big\}.\end{aligned} \tag{A15}$$

Since none of I_i and J_i diverge as a power as r_4 becomes large, we have for $r_4 \gg 1$,

$$\begin{aligned}(r_4-1)\mathcal{X}_1(r_4) &= \int xdx\{[I_1-I_2]-[J_1-J_2]\} \\ &\quad + O(r_4^{-1}), \\ r_4(r_4-1)\mathcal{X}_2(r_4) &= \int xdx\{r_4([I_3-I_1]-[J_3-J_1]) \\ &\quad +[I_2-I_3]+[J_1-J_2]\}+O(r_4^{-1}).\end{aligned} \tag{A16}$$

To extract the leading terms, we have to expand the first combination in $\mathcal{X}_2(r_4)$ to $O(r_4^{-1})$ and all others to $O(1)$. Consider the latter first. Since all $b_i \gg 1$ for $r_4 \gg 1$, we use the Landen identity of dilogarithm for the last two combinations in $\mathcal{X}_2$:

$$\begin{aligned}I_2-I_3 &= \frac{1}{2}\ln(b_2b_3)\ln\frac{b_2}{b_3}+\mathrm{Li}_2(1-b_2^{-1})-\mathrm{Li}_2(1-b_3^{-1}) \\ &= -\frac{1}{2}[2\ln r_4-\ln x-2\ln(1-x)]\ln x+O(r_4^{-1}), \\ J_1-J_2 &= [\mathrm{Li}_2(a_1/r_4)-\mathrm{Li}_2(a_1)+\ln(b_1-1)\ln r_4] \\ &\quad -(a_1\to a_2; b_1\to b_2) \\ &= \mathrm{Li}_2(1-x(1-x))-\mathrm{Li}_2(x)+\ln x\ln r_4+O(r_4^{-1}).\end{aligned} \tag{A17}$$

Then

$$\begin{aligned}B_2 &\equiv \int xdx\{[I_2-I_3]+[J_1-J_2]\} \\ &= -\frac{1}{2}-\frac{11\pi^2}{36}+\frac{1}{12}\psi_1(1/6)+\frac{1}{12}\psi_1(1/3)+O(r_4^{-1}) \\ &\approx 0.435+O(r_4^{-1}),\end{aligned} \tag{A18}$$

where $\psi_1(z)=\frac{d^2}{dz^2}\ln\Gamma(z)$ is the trigamma function. Since

$$[I_1-I_2]-[J_1-J_2] = -[I_2-I_3]-[J_1-J_2]+O(r_4^{-1}), \tag{A19}$$

this also gives the leading term

$$r_4\mathcal{X}_1(r_4) = -B_2+O(r_4^{-1}). \tag{A20}$$

The first combination in $\mathcal{X}_2$ is more complicated. Using the Landen identity and expansions of $\mathrm{Li}_2(z)$ at $z=0$ and $z=1^-$, we have

$$\begin{aligned}I_3-I_1 &= -\frac{1}{r_4}\frac{1-x}{x}\ln\frac{r_4}{1-x}+O(r_4^{-2}), \\ J_3-J_1 &= \mathrm{Li}_2(a_1)-\mathrm{Li}_2(a_3)-\frac{1}{r_4}\frac{1-x}{x}\ln r_4+O(r_4^{-2}).\end{aligned} \tag{A21}$$

The combination is thus

$$\begin{aligned}B_1 &\equiv r_4\int xdx\{[I_3-I_1]-[J_3-J_1]\} \\ &= \int dx\{(1-x)\ln(1-x) \\ &\quad + r_4x[\mathrm{Li}_2(a_3)-\mathrm{Li}_2(a_1)]\}+O(r_4^{-1}).\end{aligned} \tag{A22}$$

The $\mathrm{Li}_2(a_i)$ terms can be worked out by integration by parts, noting that $a_{1,3}=1$ at $x=1$ while $a_1=1$ and $a_3=0$ at $x=0$:

$$\int xdx\mathrm{Li}_2(a_{1,3}) = \frac{\pi^2}{12}+\frac{1}{2}\int dxx^2\ln(1-a_{1,3})\frac{d\ln a_{1,3}}{dx}, \tag{A23}$$

where $\frac{d}{dz}\mathrm{Li}_2(z)=-\frac{\ln(1-z)}{z}$ is applied. Upon expanding the integrand in r_4^{-1}, we arrive at

$$\begin{aligned}&\int xdx[\mathrm{Li}_2(a_3)-\mathrm{Li}_2(a_1)] \\ &= \frac{1}{2r_4}\int dx\Big[-\frac{\ln(1-x)}{x}(2x-2)+(1-x)\Big]+O(r_4^{-2}) \\ &= \frac{5}{4r_4}-\frac{\pi^2}{6r_4}+O(r_4^{-2}),\end{aligned} \tag{A24}$$

so that $B_1 = 1-\pi^2/6+O(r_4^{-1})$.

We collect below the leading terms for all integrals:

$$\begin{aligned}\mathcal{X}_0 &= \frac{\pi^2}{12}-\frac{1}{2}C_0+O(r_4^{-1}), \\ r_4\mathcal{X}_1(0) &= \frac{1}{2}-\frac{\pi^2}{6}+O(r_4^{-1}), \\ r_4\mathcal{X}_1(r_4) &= \frac{1}{2}+\frac{\pi^2}{12}-\frac{1}{2}C_0+O(r_4^{-1}), \\ r_4^2\mathcal{X}_2(0) &= -1+\frac{1}{3}\pi^2-\ln r_4+O(r_4^{-1}), \\ r_4^2\mathcal{X}_2(r_4) &= \frac{1}{2}-\frac{\pi^2}{4}+\frac{1}{2}C_0+O(r_4^{-1}),\end{aligned} \tag{A25}$$

$$r_4\mathcal{Y}_0(0) = \frac{\pi^2}{3} + O(r_4^{-1}),$$
$$r_4\mathcal{Y}_0(r_4) = -\frac{\pi^2}{6} + C_0 + O(r_4^{-1}),$$
$$r_4^2\mathcal{Y}_1(0) = 1 - \frac{\pi^2}{3} + \ln r_4 + \frac{1}{2}\ln^2 r_4 + O(r_4^{-1}),$$
$$r_4^2\mathcal{Y}_1(r_4) = 3 - C_0 + \ln r_4 + O(r_4^{-1}), \tag{A26}$$
$$r_4^2\mathcal{Y}_2(0) = \frac{\pi^2}{3} + O(r_4^{-1}),$$
$$r_4^3\mathcal{Y}_2(r_4) = -7 + \frac{\pi^2}{2} + C_0 - 2\ln r_4 + \ln^2 r_4 + O(r_4^{-1}),$$

and

$$r_4\mathcal{U}_0 = -\frac{\pi^2}{6} + C_0 + O(r_4^{-1}),$$
$$r_4^2\mathcal{U}_1 = 3 - C_0 + \ln r_4 + O(r_4^{-1}), \tag{A27}$$

with $C_0 = 2\sqrt{3}\mathrm{Cl}(\pi/3) = -4\pi^2/9 + 1/6\psi_1(1/6) + 1/6\psi_1(1/3) \approx 3.515\,86$, where Cl is the Clausen function. These leading terms have been numerically verified.

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