

**Holographic model of superfluidity**

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We study a holographic model of a relativistic quantum system with a global  $U(1)$  symmetry, at nonzero temperature and density. When the temperature falls below a critical value, we find a second-order superfluid phase transition with mean-field critical exponents. In the symmetry-broken phase, we determine the speed of second sound as a function of temperature. As the velocity of the superfluid component relative to the normal component increases, the superfluid transition goes through a tricritical point and becomes first order.

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**I. INTRODUCTION**

Recently, gauge/gravity duality [1–3] has been used to model strongly interacting systems in terms of a gravity dual. The most important case is the strongly interacting quark gluon plasma created at the Relativistic Heavy Ion Collider. While a systematic treatment is still lacking, the  $\mathcal{N} = 4$  super-Yang-Mills plasma has been used instead, with some success. Most notably, the viscosity/entropy density ratio, universal among all theories with a gravity dual [4–6], seems to be not too far away from the value extracted from experimental data [7,8].

The long-distance behavior of a black hole horizon is captured by the same hydrodynamic equation governing the evolution of the plasma. As a result, from the gravity equations one can reconstruct the hydrodynamic equations, including transport coefficients. One can go beyond leading order in the gravity equations and capture second-order corrections, in the process discovering extra terms typically ignored in almost all implementations of the Israel-Stewart formalism [9,10]. The approach has been extended to  $R$ -charged black holes, dual to fluids with chemical potentials [11,12].

In this work, we use black holes to study relativistic superfluids. Our work is a continuation of Ref. [13], where a holographic model of a superfluid was constructed. Here, we investigate the behavior of the superfluid when a superfluid current flows through the system. It is known that in nonrelativistic superfluids there is a critical superfluid velocity, above which the superfluid phase does not exist. We discover exactly the same phenomenon in the relativistic superfluids at low temperatures: there is a first-order phase transition between the superfluid and the normal phase as one changes the superfluid current. The phase transition becomes second order at higher temperatures.

We will also be discussing the superfluid hydrodynamics of such systems, in particular, the speed of second sound. A short discussion of hydrodynamics can be found in [14]. Hydrodynamics of a system with spontaneously broken symmetries is different from hydrodynamics of normal liquids because the system contains long-range modes (Goldstone bosons) which must be included in the hydrodynamic equations.

**II. THERMODYNAMICS**

Let us first review the thermodynamics of quantum systems with a spontaneous  $U(1)$  symmetry breaking.

For simplicity consider a system with one global  $U(1)$  symmetry, such as complex  $\phi^4$  theory. We define the free energy  $F = -T \ln Z$  where

$$Z = \text{tr} e^{-\beta H}, \quad (1)$$

and  $H$  is the Hamiltonian. While we eventually take the thermodynamic limit  $V \rightarrow \infty$ , for now it is more convenient to think of the system as living inside a  $d - 1$  dimensional torus  $\mathbf{T}^{d-1}$  where the perimeters of the  $d - 1$  circles are  $L_i$ .

When the  $U(1)$  symmetry is not spontaneously broken, the free energy  $F$  depends only on the temperature, and, if we do not take the thermodynamic limit, the  $L_i$ :  $F = F(T, L_i)$ . When the symmetry is spontaneously broken, the system contains a massless (pseudo) scalar field  $\varphi$ , which is compact:  $\varphi \sim \varphi + 2\pi$ . In the example of the complex  $\phi^4$  theory,  $\varphi$  is just the phase of  $\phi$ . If one thinks about  $\varphi$  as the elementary field, periodicity in  $\varphi$  space means that we are free to impose boundary conditions such that  $\varphi|_{x_i=L_i} = \varphi|_{x_i=0} + 2\pi n_i$ . Thus there is a set of partition functions  $Z[\mathbf{n}]$ , where we integrate over all fields with

boundary conditions specified by  $\mathbf{n}$ . The usual ground state corresponds to  $n_i = 0$ .

As we shall see later,  $Z[\mathbf{n}]$  does not exist in the strict thermodynamic sense. Physically, if one sets  $n_i \neq 0$ , the system will relax over time to a state with no winding. However, the time scale for this process may be long, and the winding state is practically stable. We assume stability.

Define  $q_i \equiv \partial_i \varphi$ , an analogue of the topological charge density whose integral can distinguish various boundary conditions:

$$\oint_{C_i} \mathbf{q} \cdot d\mathbf{x} = \int_0^{L_i} q_i dx^i = \varphi|_{x_i=L_i} - \varphi|_{x_i=0} = 2\pi n_i, \quad (2)$$

where  $C_i$  is the  $i$ th spatial circle of the  $\mathbf{T}^{d-1}$ . We can therefore impose the boundary conditions on  $\varphi$  by introducing Lagrange multipliers  $\lambda_i$  which pick out the required values of  $n_i$ :

$$\begin{aligned} Z[\mathbf{n}] &= \int_{\{\mathbf{n}\}} D\phi e^{-S[\phi]} \\ &= \int D\phi d\lambda e^{-S[\phi]} \exp\left(i \sum_j \lambda_j \left( \oint_{C_j} \mathbf{q} \cdot d\mathbf{x} - 2\pi n_j \right)\right). \end{aligned} \quad (3)$$

We are taking advantage of the fact that the Euclidean path integral on a compact time circle of perimeter  $\beta = 1/T$  yields the thermal partition function at a temperature  $T$ . Having compactified the time direction, it is natural to ask what boundary conditions should be applied for  $\varphi$  in the timelike direction. The answer to this question involves introducing a chemical potential, and we will return to it shortly.

We can write the field  $\varphi$  as  $\varphi(x) = \alpha_i x^i + \delta\varphi(x)$  where  $\alpha_i$  is constant, and  $\delta\varphi(x)$  is periodic. The delta function in the path integral for  $Z[\mathbf{n}]$  will pick out  $\alpha_i = 2\pi n_i/L_i$ , so knowing  $\alpha_i$  is equivalent to knowing the boundary conditions for  $\varphi$ . In the thermodynamic limit,  $F$  depends on the  $L_i$  only through an overall multiplicative volume factor  $V$  and  $\alpha_i$  becomes a continuous variable which transforms as a vector under rotations. At zero temperature, the energy (for given  $\mathbf{n}$ ) is always minimized for  $\varphi = \alpha_i x^i$  because that is when the  $(\nabla\varphi)^2$  term is smallest. So at zero temperature, the ground state can be uniquely characterized by the  $\alpha_i$ . We assume now that the thermal state can be uniquely characterized by  $\alpha_i$  and  $T$ , and define the partition function as

$$Z[\boldsymbol{\alpha}] = \int D\phi d\lambda e^{-S[\phi]} \exp\left(i \sum_j \lambda_j \oint_{C_j} (\nabla\varphi - \boldsymbol{\alpha}) \cdot d\mathbf{x}\right). \quad (4)$$

Note that  $\boldsymbol{\alpha}$  is the equilibrium value of  $(\nabla\varphi)$ . In nonrelativistic superfluids  $\nabla\varphi$  is proportional to the superfluid velocity. We shall use the same terminology and call  $\nabla\varphi$  the superfluid velocity.  $Z[\boldsymbol{\alpha}]$  describes a stationary state with constant, nonzero superfluid velocity. The partition

function  $Z$  is a scalar, and therefore can only depend on  $\boldsymbol{\alpha}^2$  in the thermodynamic limit.

Note that the state described by  $Z[\boldsymbol{\alpha}]$  is in fact metastable, rather than stable. This is because  $\varphi$ , as the phase of a complex field  $\phi$ , is ill defined when  $\phi = 0$ ; hence the winding numbers  $n_i$  are not topological and can change with time. The process of ‘‘unwinding’’ can be visualized as follows. Imagine a state with only one of the  $n_i$  equal to 1, and other winding numbers equal 0. The state can be thought of as containing a  $(d-2)$  dimensional domain wall, across which  $\varphi$  changes by  $2\pi$ . However, in  $d > 2$  dimensions, there is no topological conservation law which would ensure the stability of the domain wall, and the domain wall will decay through hole nucleation. In terms of superfluid hydrodynamics in  $3+1$  dimensions, nucleating a hole in the two-dimensional domain wall corresponds to producing a vortex loop. Production of vortex loops violates Landau’s criterion for superfluidity [15], and renders the state with nonzero  $\boldsymbol{\alpha}$  metastable.

In a theory with large  $N$ , similar to the one we will be considering, the rate of vortex loop production is exponentially suppressed, as both the energy of a vortex loop and the action of the configuration describing vortex nucleations are proportional to  $N$ . Therefore, provided that the volume and time scales are not too large, one can treat states with a nonzero superfluid velocity as thermal equilibrium states.

We now would like to introduce a chemical potential  $\mu$  and work in the grand canonical ensemble. We define the potential function  $\Omega \equiv F - \mu N = -PV = -T \ln Z$  where

$$Z = \text{tr} e^{-\beta(H - \mu Q)} \quad (5)$$

and  $Q$  is the conserved charge corresponding to the  $U(1)$  symmetry. To introduce  $\mu$  at the level of the path integral, it is useful to gauge the  $U(1)$  symmetry by coupling the system to a nondynamical gauge field  $A_\mu$ :

$$Z[A] = e^{W[A]} = \int D\phi e^{-S[\phi, A]}. \quad (6)$$

We are temporarily ignoring the dependence of  $Z$  on the  $\boldsymbol{\alpha}$ . The one point function of the  $U(1)$  current is then generated by  $W[A]$ :

$$\frac{\delta W[A]}{\delta A_\mu(x)} = \langle J^\mu(x) \rangle. \quad (7)$$

In a system where the external field strength is zero  $F^{\mu\nu} = 0$ , we may pick a gauge in which  $A_\mu$  is constant and it makes sense, given (7), to interpret  $A_0 = \mu$  as the chemical potential.

Gauge transformations which send  $A_\mu \rightarrow A_\mu + \partial_\mu \lambda$  have a nontrivial effect on the compact scalar:  $\varphi \rightarrow \varphi + \lambda$ . In the presence of the background gauge field, the pressure  $P$  can depend only on gauge invariant quantities such as  $D_\mu \varphi = \partial_\mu \varphi - A_\mu$ . The grand canonical partition

function thus becomes

$$Z[A, \boldsymbol{\alpha}] = \int D\phi d\lambda e^{-S[\phi, A]} \times \exp\left(i \sum_j \lambda_j \oint_{C_j} (\nabla\phi - \mathbf{A} - \boldsymbol{\alpha}) \cdot d\mathbf{x}\right). \quad (8)$$

Similar to  $Z[\boldsymbol{\alpha}]$ , by rotational symmetry  $Z[\boldsymbol{\alpha}]$  can only depend on  $\boldsymbol{\alpha}^2$  in the thermodynamic limit. We are assuming that the pressure  $P = -TV^{-1} \ln Z$  depends on three parameters,  $P = P(T, \mu, \frac{1}{2}(D_i\phi)^2)$ .

We can now choose a gauge in which  $\phi = 0$ , so that  $(D_i\phi)^2 = A_i^2$ . The chemical potential  $\mu$  can be interpreted as the zero component of the gauge field, and we have  $P = P(T, A_0, A_i^2)$ . At zero temperature, the system is Lorentz invariant, provided one treats  $A_0$  and  $A_i$  as spurion fields that transform as a four-vector under the Lorentz group. As a result, the pressure can only depend on  $A_\mu^2 = A_i^2 - A_0^2$ . At finite temperature, the pressure depends on  $A_0$  and  $A_i^2$  separately, which we can write as  $P = P(T, A_0, A_\mu^2)$ . In other words, gauging the symmetry allows us to trade the dependence of pressure on the condensate phase for the dependence of pressure on the background gauge field. In a general gauge,  $P = P(T, -D_0\phi, (D_\mu\phi)^2)$ . Denoting  $\chi \equiv \frac{1}{2}(D_\mu\phi)^2$ , we have the pressure as a function of three thermodynamic variables:  $P = P(T, \mu, \chi)$ . From it one can define the conjugate variables,

$$dP = s dT + n d\mu - f^2 d\chi. \quad (9)$$

Notice that  $s$ ,  $n$ , and  $f^2$  are functions of  $T$ ,  $\mu$ , and  $\chi$ . If the third term was absent from Eq. (9), then  $s$  would be the entropy density, and  $n$  the charge density. We will give the interpretation of  $s$ ,  $n$ , and  $f^2$  in the next section.

### III. HYDRODYNAMICS

Let us set the external gauge field  $A_\mu$  to 0. The equations of ideal relativistic superfluid hydrodynamics are known. We will use the form written in [14]. In this formulation, the degrees of freedom are  $T$ ,  $\mu$ ,  $\varphi$ , and a unit four-vector  $u^\mu$  satisfying  $\eta_{\mu\nu} u^\mu u^\nu = -1$  (recall that we use the mostly plus convention for the flat metric tensor  $\eta_{\mu\nu}$ ). We will later identify  $u^\mu$  with the velocity of the normal component. The equations consist of the conservations of

the energy-momentum tensor  $T^{\mu\nu}$  and of the  $U(1)$  symmetry current  $j^\mu$ ,

$$\partial_\mu T^{\mu\nu} = 0, \quad (10)$$

$$\partial_\mu j^\mu = 0, \quad (11)$$

and a ‘‘Josephson equation’’ describing time evolution of  $\varphi$ ,

$$u^\mu \partial_\mu \varphi + \mu = 0. \quad (12)$$

The stress-energy tensor and the current are expressed in terms of the hydrodynamic variables through

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P\eta^{\mu\nu} + f^2 \partial^\mu \varphi \partial^\nu \varphi, \quad (13)$$

$$j^\mu = nu^\mu + f^2 \partial^\mu \varphi, \quad (14)$$

where the energy density  $\epsilon$  is defined by  $\epsilon + P = Ts + n\mu$ . Thus we have  $(d+2)$  hydrodynamic equations (10)–(12) for  $(d+2)$  variables  $T$ ,  $\mu$ ,  $\varphi$ , and  $u^\mu$ . One can derive from Eqs. (10)–(14)

$$\partial_\mu (su^\mu) = 0. \quad (15)$$

One can interpret this equation as the equation of entropy conservation (entropy is conserved since we are working at the level of ideal, nonviscous hydrodynamics). Thus  $s$  is the entropy density and  $u^\mu$  is the velocity of entropy flow. In the two-fluid model only the normal component carries entropy; therefore  $u^\mu$  is interpreted as the normal velocity. The two contributions to the current in Eq. (14) can be interpreted as the normal and superfluid currents. Therefore  $n$  is the normal density, and  $f^2$  is the analogue of the pion decay constant. (The equivalent of the superfluid density would be  $f^2\mu$ .)

Let us look at small fluctuations about an equilibrium state at fixed temperature, chemical potential and zero normal and superfluid velocities, i.e., we write  $T = T_0 + T'$ ,  $\mu = \mu_0 + \mu'$ ,  $u^\mu = (1, v^i)$ ,  $\xi_i \equiv \partial_i \varphi = \xi'_i$ , where  $T'$ ,  $\mu'$ ,  $v^i$ , and  $\xi'_i$  are small. In terms of  $\xi$  the variation of pressure is  $dP = s dT + (n + \mu f^2) d\mu - f^2 \xi d\xi$ , where  $\xi = |\vec{\xi}|$ . Further, let us assume that pressure is a smooth function of  $\xi^2$  at small  $\xi$ , so that  $\partial P / \partial \xi = 0$  in equilibrium. The linearized hydrodynamic equations become (omitting subscript ‘‘0’’ on equilibrium quantities)

$$\frac{\partial^2 P}{\partial T \partial \mu} \partial_i T' + \frac{\partial^2 P}{\partial \mu^2} \partial_i \mu' + f^2 \partial_i \xi'_i + n \partial_i v'_i = 0, \quad (16a)$$

$$\left(\mu \frac{\partial^2 P}{\partial T \partial \mu} + T \frac{\partial^2 P}{\partial T^2}\right) \partial_i T' + \left(\mu \frac{\partial^2 P}{\partial \mu^2} + T \frac{\partial^2 P}{\partial T \partial \mu}\right) \partial_i \mu' + w \partial_i v'_i + f^2 \mu \partial_i \xi'_i = 0, \quad (16b)$$

$$w \partial_i v'_i + f^2 \mu \partial_i \xi'_i + s \partial_i T' + (n + \mu f^2) \partial_i \mu' = 0, \quad (16c)$$

$$\partial_i \xi'_i + \partial_i \mu' = 0. \quad (16d)$$

In this system of equations, pressure is taken as  $P = P(T, \mu, \xi)$ , and  $w = \epsilon + P$  is the density of enthalpy. The first equation is the linearized current conservation equation  $\partial_\mu j^\mu = 0$ . The second equation is the linearized energy

conservation  $\partial_\mu T^{\mu 0} = 0$ . The third equation is the linearized momentum conservation,  $\partial_\mu T^{\mu i} = 0$ . Finally, the fourth equation says that  $\mu$  and  $\xi_i$  are not independent because  $\mu = -\partial_i \varphi$  while  $\xi_i = \partial_i \varphi$ .

An interesting feature of the linearized hydrodynamic equations (16) is that they admit propagating mode solutions even if one ignores the fluctuations of the energy-momentum tensor, i.e., if one ignores the (normal) sound fluctuations. If one were to ignore the condition  $\partial_\mu T^{\mu\nu} = 0$ , together with fluctuations of temperature  $T'$  and velocity of the normal component  $v'_i$ , then the system (16) becomes

$$\frac{\partial^2 P}{\partial \mu^2} \partial_i \mu' + f^2 \partial_i \xi'_i = 0, \quad (17a)$$

$$\partial_i \xi'_i + \partial_i \mu' = 0. \quad (17b)$$

Fourier transforming all variables as  $e^{-i\omega t + ik \cdot x}$ , we find a propagating mode with frequency

$$\omega^2 = v_2^2 k^2, \quad v_2^2 = \frac{f^2}{\left(\frac{\partial^2 P}{\partial \mu^2}\right)} = -\frac{\left(\frac{\partial^2 P}{\partial \xi^2}\right)_{T, \mu}}{\left(\frac{\partial^2 P}{\partial \mu^2}\right)_{T, \xi}} > 0. \quad (18)$$

When expressing  $f^2$  in terms of  $(\partial^2 P / \partial \xi^2)$ , we have assumed that  $P = O(\xi^2)$  at small  $\xi$ . The thermodynamic derivatives in Eq. (18) are to be evaluated at  $\xi = 0$ . The propagating mode (18) is the leftover of the second sound in superfluids which survives even if one ignores the  $\partial_\mu T^{\mu\nu} = 0$  part of the hydrodynamic equations. One expects that this mode is captured by the dual gravitational description which ignores the backreaction of the gauge fields on the metric.

It is not difficult to find the eigenmodes of the full system (16). Taking all variables proportional to  $e^{-i\omega t + ik \cdot x}$ , one finds a fourth order equation for frequency,

$$a\omega^4 - bk^2\omega^2 + ck^4 = 0 \quad (19)$$

where the coefficients  $a$ ,  $b$ , and  $c$  are independent of  $k$ . When  $b^2 > 4ac$ , the equation for  $\omega^2$  has two real positive roots  $\omega^2 = v_s^2 k^2$  and  $\omega^2 = v_2^2 k^2$ . The first solution is the normal sound, and the second solution is the second sound. In terms of thermodynamic derivatives, the coefficients  $a$ ,  $b$ , and  $c$  are given by

$$a = Tw \left[ \left( \frac{\partial^2 P}{\partial T^2} \right) \left( \frac{\partial^2 P}{\partial \mu^2} \right) - \left( \frac{\partial^2 P}{\partial T \partial \mu} \right)^2 \right], \quad (20)$$

$$b = \left( \frac{\partial^2 P}{\partial T^2} \right) T(n^2 + wf^2) + \left( \frac{\partial^2 P}{\partial \mu^2} \right) Ts^2 - 2 \left( \frac{\partial^2 P}{\partial T \partial \mu} \right) Tsn, \quad (21)$$

$$c = f^2 Ts^2. \quad (22)$$

In particular,  $a$ ,  $b$ , and  $c$  are positive. Let us now look at simple examples. In the nonsuperfluid phase  $f^2 = 0$ , and therefore  $v_s^2 = b/a$ , while the second sound is absent,  $v_2^2 = 0$ . Even in the nonsuperfluid phase, the speed of the

normal sound looks like a complicated expression in terms of the derivatives of pressure  $P(T, \mu)$ . The expression for  $v_s^2$  simplifies if instead of  $P(T, \mu)$  we work with  $P(s, n)$ . Indeed, the coefficient  $a$  is proportional to the Jacobian of the transformation from the  $(s, n)$  variables to the  $(T, \mu)$  variables. In terms of  $P(s, n)$ , the speed of sound in the normal phase becomes

$$v_s^2 = \frac{n}{w} \left( \frac{\partial P}{\partial n} \right)_s + \frac{s}{w} \left( \frac{\partial P}{\partial s} \right)_n. \quad (23)$$

Now that we have  $v_s^2$  expressed in terms of  $P(s, n)$  without reference to the chemical potential, we can evaluate the thermodynamic derivatives in the canonical ensemble instead of the grand canonical. In the canonical ensemble, the total charge (number of particles)  $N = nV$  is fixed, and therefore  $dn/n = -dV/V$ . It will be convenient to go from the  $(s, n)$  to the  $(S, \epsilon)$  variables. For the total entropy  $S$  we have  $dS/V = d(sV)/V = ds - sdn/n$ , while on the other hand the relation  $TdS = dE + PdV$  gives  $TdS/V = d\epsilon - wdn/n$ . This implies that the speed of sound in the normal phase (23) can be written as

$$v_s^2 = \left( \frac{\partial P}{\partial \epsilon} \right)_{S, N}. \quad (24)$$

As another example, consider a scale-invariant theory, in which case pressure has the form  $P(T, \mu) = T^d g(T/\mu)$ , where  $g(T/\mu)$  is a dimensionless function. The speed of normal sound evaluated from Eq. (19) is  $v_s^2 = 1/(d-1)$ , in either normal or superfluid phase. The speed of the second sound evaluated from Eq. (19) can be written as

$$v_2^2 = f^2 \left[ \left( 1 + \frac{\mu n}{Ts} \right) \left( \frac{\partial^2 P}{\partial \mu^2} - \frac{n + \mu f^2}{s} \frac{\partial^2 P}{\partial T \partial \mu} \right) \right]^{-1}. \quad (25)$$

We can see that in scale-invariant theories, the cartoon expression (18) can be recovered in the formal ‘‘large-entropy’’ limit  $Ts \gg \mu n$ , together with

$$s \left( \frac{\partial^2 P}{\partial \mu^2} \right) \gg (n + \mu f^2) \left( \frac{\partial^2 P}{\partial T \partial \mu} \right).$$

In theories which do not have scale invariance, the speed of the second sound does not have the simple form (25), but can be easily determined from Eq. (19) given the equation of state  $P(T, \mu)$ .

Note that the simple expression (18) ceases to be a good approximation to the speed of the second sound at  $T \ll \mu$ . This is because at small temperatures we expect that the system can be described as a gas of massless Goldstone bosons, with entropy density  $s \sim T^{d-1}$ . On the other hand, we expect in the low-temperature region that  $n \sim \mu^{d-1}$  which implies that the condition of large entropy cannot be satisfied at  $T \ll \mu$ . At low temperatures (high densities) the speed of the second sound has to be determined from the full equation (19).



#### IV. DUAL GRAVITY DESCRIPTION

On the gravity side, we study the Einstein-Maxwell theory with a complex scalar field. Fluctuations of the bulk metric correspond to fluctuations of  $T^{\mu\nu}$  on the boundary, while fluctuations of the  $U(1)$  gauge field correspond to fluctuations of  $J^\mu$  on the boundary. The scalar field is charged under the bulk  $U(1)$ , and its background value corresponds to the condensate on the boundary. The mass of the scalar is a free parameter. We also need to specify the boundary conditions. The metric will be asymptotically anti-de Sitter (AdS) because we want to study the superfluid system in flat space. The boundary conditions for the scalar correspond to the “normalizable” mode because we want to describe the system in which a charged operator has an expectation value. However, the boundary conditions for the  $U(1)$  gauge field should correspond to the “non-normalizable” mode because we want to describe the system in the background gauge field, as discussed above. In other words, we fix the value of  $A_\mu$  at the asymptotic AdS infinity. Fixing the value of  $A_0$  at infinity amounts to fixing the chemical potential in the boundary theory, and leads to a charged black hole in AdS. Fixing the value of  $A_i$  at infinity does not introduce additional “hair” for the black hole because we expect the new black hole solution to be only metastable, in accord with field theory expectations. So we want to find a (meta-stable) stationary solution with the above boundary conditions, and then study small fluctuations around this solution, corresponding to hydrodynamic fluctuations in the boundary theory.

In the following, we will ignore the backreaction of the gauge and scalar fields on the metric. It would be nice to include the backreaction, at least perturbatively to leading order. The action is

$$S = - \int d^{d+1}x \sqrt{-g} \left[ \frac{1}{4e^2} F_{MN} F^{MN} + (D_M \phi)(D^M \phi)^* + m^2 \phi \phi^* \right], \quad (26)$$

where  $D_M \phi = \partial_M \phi - iA_M \phi$ ,  $F_{MN} = \partial_M A_N - \partial_N A_M$ , and capital Latin indices run from 0 to  $d$ . The equations of motion are

$$\frac{1}{\sqrt{-g}} D_A (\sqrt{-g} g^{AB} D_B \phi) = m^2 \phi, \quad (27)$$

$$\frac{1}{\sqrt{-g}} \partial_M (\sqrt{-g} g^{MA} g^{NB} F_{AB}) = e^2 g^{NA} J_A, \quad (28)$$

where the current is  $J_A = i[\phi^*(D_A \phi) - \phi(D_A \phi)^*]$ . We write the bulk scalar as  $\phi = \frac{1}{\sqrt{2}} \rho e^{i\varphi}$ , and make a gauge transformation  $A_M \rightarrow A_M + \partial_M \varphi$ . In the new gauge, the phase  $\varphi$  disappears from the equations of motion, and the current becomes

$$J_M = \rho^2 A_M. \quad (29)$$

From the Maxwell equations (28) one can see that the background value for  $\rho$  would induce a (position-dependent) mass for the gauge field. This is the Higgs mechanism in the bulk. We take the  $(d+1)$  dimensional background metric of the following form:

$$ds^2 = \frac{1}{z^2} \left( -f(z) dt^2 + d\mathbf{x}^2 + \frac{dz^2}{f(z)} \right). \quad (30)$$

The metric with  $f(z) = 1$  corresponds to pure AdS and  $f(z) = 1 - (z/z_h)^d$  corresponds to our black hole solution. To find the background solution for  $\rho$  and  $A_M$ , we take all fields independent of  $t$  and  $\mathbf{x}$ . The  $z$ -component of the Maxwell equations (28) now gives  $\rho^2 A_z = 0$  which means we can choose  $A_z = 0$ . The equations of motion become

$$z^{d-1} \partial_z \left[ \frac{f}{z^{d-1}} \rho' \right] = \left( A_i^2 - \frac{A_i^2}{f} + \frac{m^2}{z^2} \right) \rho, \quad (31)$$

$$z^{d-3} \partial_z \left[ \frac{1}{z^{d-3}} A_i' \right] = \frac{e^2}{z^2 f} \rho^2 A_i, \quad (32)$$

$$z^{d-3} \partial_z \left[ \frac{f}{z^{d-3}} A_i' \right] = \frac{e^2}{z^2} \rho^2 A_i. \quad (33)$$

In the limit when  $A_i = 0$  they reduce to the coupled  $(A_i, \rho)$  system of equations studied recently in [13].

#### A. The free energy

Up to boundary counterterms, the free energy of the field theory is determined by the value of the action (26) evaluated on shell,  $\Omega = -TS_{\text{os}} + \dots$ , where the ellipsis denotes boundary terms that we will introduce. Employing the equations of motion, we may rewrite (26) as

$$S_{\text{os}} = \int d^d x \left[ \frac{\sqrt{-g} g^{zz}}{2} \left( \frac{1}{e^2} g^{\mu\nu} A_\mu A_\nu' + \rho \rho' \right) \right]_{z=\epsilon} + \frac{1}{2} \int_\epsilon^{z_h} dz \sqrt{-g} A_\mu A^\mu \rho^2. \quad (34)$$

We have included a cutoff  $\epsilon$  because this on-shell action is naively divergent and needs to be regularized through the addition of boundary terms. It is difficult to treat the general case both succinctly and clearly and several full treatments already exist in the literature (see, for example, [16]). We proceed to regularize this action in the simple case  $d = 3$  and  $m^2 = -2$ . Using the explicit form of the metric (30), the on-shell action reduces to

$$S_{\text{os}} = \int d^d x \left[ \frac{f}{2} \left( \frac{1}{e^2} \eta^{\mu\nu} A_\mu A_\nu' + \frac{1}{z^2} \rho \rho' \right) \right]_{z=\epsilon} + \frac{1}{2} \int_\epsilon^{z_h} dz \sqrt{-g} A_\mu A^\mu \rho^2. \quad (35)$$

The near boundary behavior of the fields takes the form

$$A_\mu = (a_\mu + \mathcal{O}(z^2)) + z(b_\mu + \mathcal{O}(z^2)), \quad (36)$$

$$\rho = z(a + \mathcal{O}(z^2)) + z^2(b + \mathcal{O}(z^2)). \quad (37)$$

These boundary values have various reinterpretations in the field theory. For the gauge field,  $a_0 = \mu$  is the chemical potential while  $a_i = \xi_i$  is a superfluid velocity. Then  $b_0 \sim \rho$  is proportional to the charge density while  $b_i \sim J_i$  are charge currents. For the scalar field, there exists an ambiguity [17]. For a scalar operator  $O_1$  of conformal dimension one,  $a = \langle O_1 \rangle$  while  $b$  is a source. For a scalar  $O_2$  of conformal dimension two  $b = \langle O_2 \rangle$  while  $a$  is a source.

In regulating  $S_{\text{os}}$ , we must carefully formulate the boundary conditions. For example, do we wish to keep  $A_\mu$  or  $A'_\mu$  fixed on the boundary? Keeping  $A_0$  fixed corresponds to keeping the chemical potential fixed and thus working in the grand canonical ensemble in the field theory. In the spatially homogenous case where we can set  $A_z = 0$ , varying  $A_M$  in the bulk leads to a boundary term proportional to  $A'_\mu \delta A_\mu$ . Without an additional boundary term, we are working in an ensemble where  $a_\mu$  is held fixed. If we would like to work in the canonical ensemble, at fixed charge, it is  $b_0$  that must be held fixed at the boundary ( $\delta A'_0 = 0$ ). To accommodate this change, we would need to make what amounts to a Legendre transform and add the boundary term

$$\frac{1}{e^2} \int d^d x A_0 A'_0 \Big|_{z=\epsilon} = -\mu Q/T, \quad (38)$$

to the action (26). We will work in an ensemble where  $a_\mu$  is held fixed and thus will need no such further boundary terms.

A similar decision needs to be made about the scalar operator. It is most natural to work in an ensemble where the value of  $\langle O_i \rangle$  is fixed instead of the source for the operator. However, we still must decide whether we want our scalar operator to have conformal dimension one or two. As the ensemble where the source for  $O_1$  is fixed is equivalent to the ensemble where  $\langle O_2 \rangle$  is fixed, we know that the ensembles where  $\langle O_1 \rangle$  and  $\langle O_2 \rangle$  are fixed must be related by a Legendre transform [17].

Consider first the case where  $\langle O_1 \rangle$  is fixed. The on-shell action is naively divergent, and we must add a counterterm. The counterterm and the  $\rho\rho'$  term in the on-shell action combine to give

$$\left( \frac{1}{2z^2} \rho\rho' - \frac{1}{2z^3} \rho^2 \right) \Big|_{z=\epsilon} = \frac{1}{2} ab + \mathcal{O}(\epsilon). \quad (39)$$

In the Legendre transformed case, where  $\langle O_2 \rangle$  is fixed, we add two counterterms, one to ensure that we hold  $\delta\rho'$  fixed at the boundary instead of  $\delta\rho$  and one to control the divergence:

$$\left( \frac{1}{2z^2} \rho\rho' - \frac{1}{z^2} \rho\rho' + \frac{1}{2z^3} \rho^2 \right) \Big|_{z=\epsilon} = -\frac{1}{2} ab + \mathcal{O}(\epsilon). \quad (40)$$

Recalling that this regularized on-shell action is the negative of the free energy and assuming a spatially homogenous system so that we may divide out by a factor of the volume  $V$ , we find for the free energies that

$$\Omega_i(\mu, \xi, O_i)/V = \frac{1}{2} \left[ \mu\rho - \xi \cdot J + (-1)^i O_1 O_2 - \int_0^{z_h} A_\mu A_\nu g^{\mu\nu} \rho^2 \sqrt{-g} dz \right]. \quad (41)$$

Recall  $\Omega = -PV$ . Having fixed  $O_i$ ,  $\mu$ , and  $\xi$ , the conjugate boundary values  $\epsilon_{ij} O_j$ ,  $Q$ , and  $J_s$  are then determined through the dynamics of the gravitational theory.

Since  $\Omega_1(O_1)$  and  $\Omega_2(O_2) = \Omega_1(O_1) + O_1 O_2 V$  are Legendre transforms of each other, we have

$$\frac{1}{V} \frac{\partial \Omega_1}{\partial O_1} = -O_2 \quad \text{and} \quad \frac{1}{V} \frac{\partial \Omega_2}{\partial O_2} = O_1. \quad (42)$$

Thus critical points of the free energies correspond to gravitational solutions where at least one of the two  $O_i$  vanish.

## B. Numerical results

The nonlinear differential equations (31)–(33) appear to be intractable analytically. However, it is relatively straightforward to integrate the equations numerically. The results of this section are most succinctly summarized by the two phase diagrams for the scalars  $O_1$  and  $O_2$  shown in Fig. 1.<sup>1</sup>

To see where these phase diagrams come from, we begin by reviewing the case  $\xi = 0$  studied in the canonical ensemble in [13] while here we choose to work at fixed  $\mu$ . Given  $\xi = 0$ , the third differential equation (33) drops out. Plots of the expectation value  $\langle O_i \rangle$  versus temperature are shown in Fig. 2, as the black curves on the far right. At high temperature,  $\langle O_i \rangle = 0$ , but at the critical temperature  $T_c$ , there is a second-order phase transition where the expectation values become nonzero. For  $O_1$ ,  $T_c = 0.213|\mu|$  while for  $O_2$ ,  $T_c = 0.0587|\mu|$ . Near but slightly below  $T_c$  the scalars exhibit the standard mean-field scaling with the reduced temperature

$$\langle O_i \rangle \sim (T_c - T)^{1/2}. \quad (43)$$

The most straightforward way to see that the phase transition is second order is to examine a plot of the free energy versus temperature:  $\Omega_i$  is smooth at  $T_c$ . Using Eq. (41), we have produced Fig. 3(a) for the scalar  $O_1$ , which indeed shows this smooth behavior. We do not show a very similar plot for the second scalar  $O_2$ .

<sup>1</sup>We set  $e = 1$  in this section.

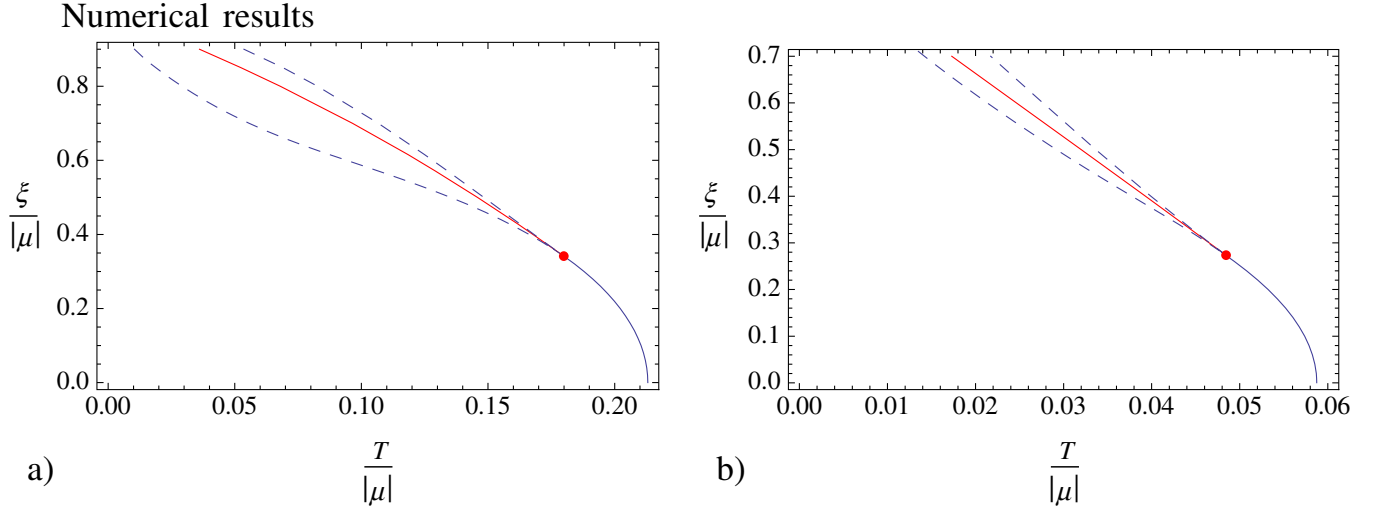


FIG. 1 (color online). The phase diagrams for the theory with a scalar with (a) conformal dimension one and (b) conformal dimension two. The solid blue lines indicate a second-order phase transition while the solid red lines (in between the dashed lines) indicate a first-order phase transition. The dashed blue lines are spinodal curves, while the red dot indicates the tricritical point.

A more elaborate demonstration that the phase transition is second order comes from an investigation of  $\Omega_i$  as a function of the order parameter  $O_i$  near  $T_c$ , recovering completely standard results in the Landau-Ginzburg mean-field theory of phase transitions. We have found numerically that

$$\frac{\Delta\Omega_i}{V|\mu|^3} = \alpha_i(T)\left(\frac{\langle O_i \rangle}{|\mu|^i}\right)^2 + \beta_i(T)\left(\frac{\langle O_i \rangle}{|\mu|^i}\right)^4 \quad (44)$$

fits the free energy curves extremely well. Moreover,  $\alpha_i$  and  $\beta_i$  are nearly linear in  $T$ :

$$\begin{aligned} \alpha_1 &= 3.07(T - T_c)/T_c, \\ \beta_1 &= 0.743 - 0.899(T - T_c)/T_c, \end{aligned} \quad (45)$$

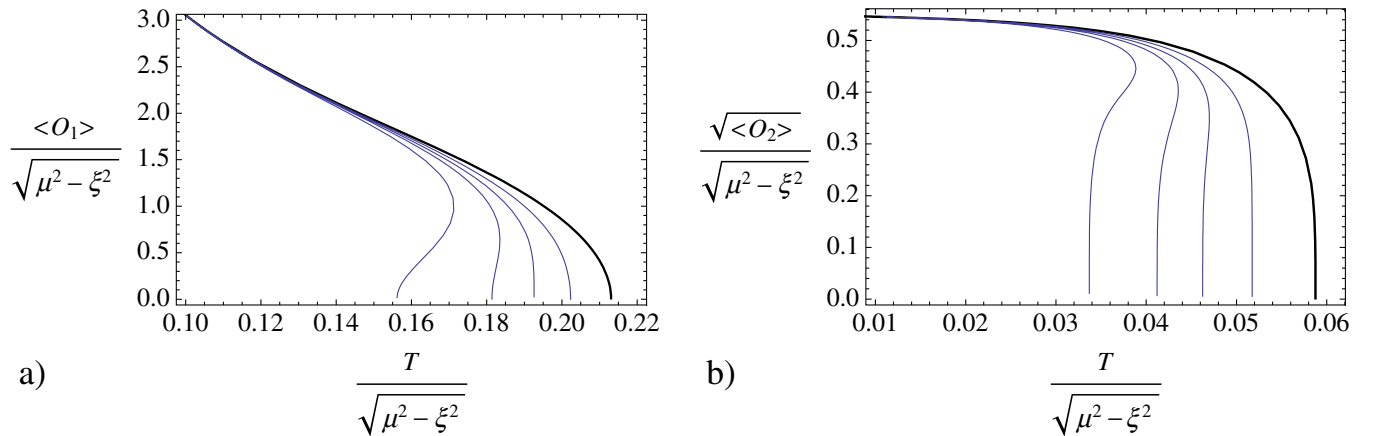


FIG. 2 (color online). The condensate as a function of temperature for the two operators: (a)  $O_1$  and (b)  $O_2$ . The curves in the plots, from right to left, are for  $|\xi/\mu| = 0, 1/4, 1/3, 2/5$ , and  $1/2$ .

$$\begin{aligned} \alpha_2 &= 5.13(T - T_c)/T_c, \\ \beta_2 &= 1.34 - 1.45(T - T_c)/T_c. \end{aligned} \quad (46)$$

By definition,  $\alpha_i$  passes through zero at the phase transition.

As we increase the superfluid velocity  $\xi$ , nothing dramatic happens immediately. The phase transition remains second order although  $T_c$  decreases as can be seen from Fig. 2. Because the decrease is due to the additional kinetic energy of the system, we expect the decrease to be quadratic in  $\xi$ , which is born out by the shape of the second-order lines in the phase diagram, Fig. 1. Numerically, we find that  $T_c(\xi) \approx T_c(0) - \lambda\xi^2/|\mu|$  where  $\lambda \approx 0.27$  for  $O_1$  and  $\lambda \approx 0.14$  for  $O_2$ .

However, there exists a critical  $\xi$  above which the phase transition becomes first order. For  $O_1$ , this critical velocity

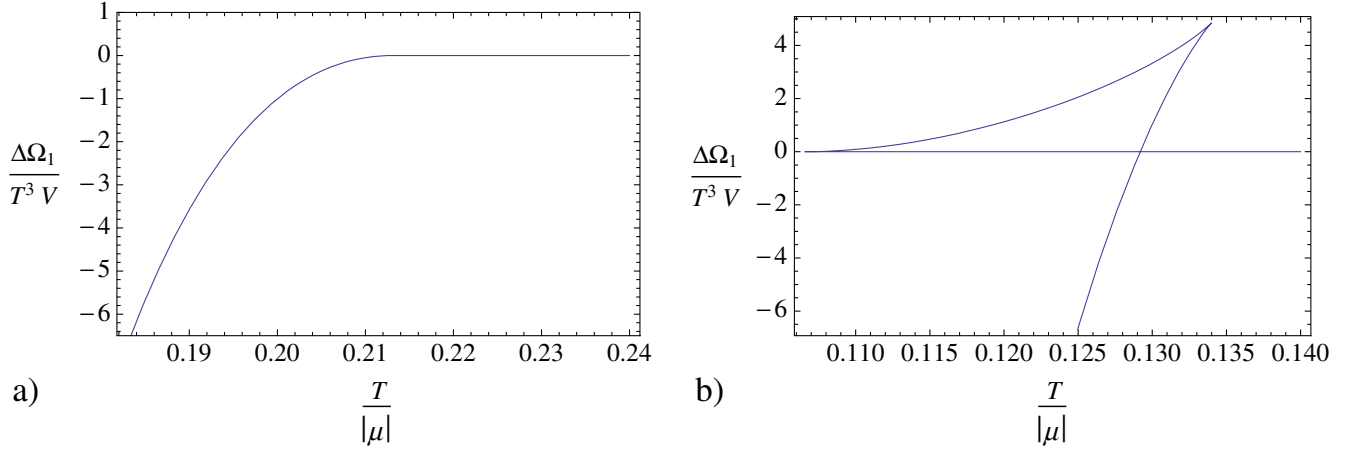


FIG. 3 (color online). The difference in free energy  $\Delta\Omega_1$  between the phase with a scalar condensate and without one as a function of  $T/|\mu|$ : (a)  $\xi = 0$  and (b)  $\xi/|\mu| = 4/7$ .

is  $\xi = 0.342|\mu|$  while for  $O_2$  it is  $\xi = 0.274|\mu|$ . From Fig. 2, it is clear that something interesting must happen because the curves  $\langle O_i \rangle$  become multivalued for sufficiently large  $\xi$ . It is possible to see that the phase transition is first order in different ways. The simplest is to look at the free energy as a function of temperature. Figure 3(b) presents this classic swallow tail shape for  $O_1$  and  $\xi/|\mu| = 4/7$ . At the phase transition, the free energy is continuous but not differentiable. The two nonanalytic points in the free energy curve are “spinodal points” or points beyond which one of the phases ceases to exist even as a metastable minimum of the free energy.

To demonstrate more convincingly that the phase transition becomes first order, we computed the free energy as a function of the order parameter near the putative tricritical point. We found numerically that the free energy is well described by the sixth order polynomial

$$\frac{\Delta\Omega_i}{V|\mu|^3} = \alpha_i(T, \xi) \left( \frac{\langle O_i \rangle}{|\mu|^i} \right)^2 + \beta_i(T, \xi) \left( \frac{\langle O_i \rangle}{|\mu|^i} \right)^4 + \gamma_i(T, \xi) \left( \frac{\langle O_i \rangle}{|\mu|^i} \right)^6. \quad (47)$$

At the tricritical point,  $\alpha_i$  and  $\beta_i$  both vanish. Moreover, near the tricritical point,  $\alpha_i$  and  $\beta_i$  vary linearly with  $T$  and  $\xi$  while  $\gamma_i$  is positive and roughly constant.

Before moving to a discussion of second sound, we would like to point out one nice feature of Fig. 2: The curves approach each other at low temperature. This agreement is a consequence of the fact that at  $T = 0$ , the pressure  $P$  can only be a function of the Lorentz invariant quantity  $\mu^2 - \xi^2$ , as discussed in Sec. II.

The speed of second sound as a function of temperature is shown in Fig. 4. The plots were computed in the probe approximation using Eq. (18). We have set the superfluid

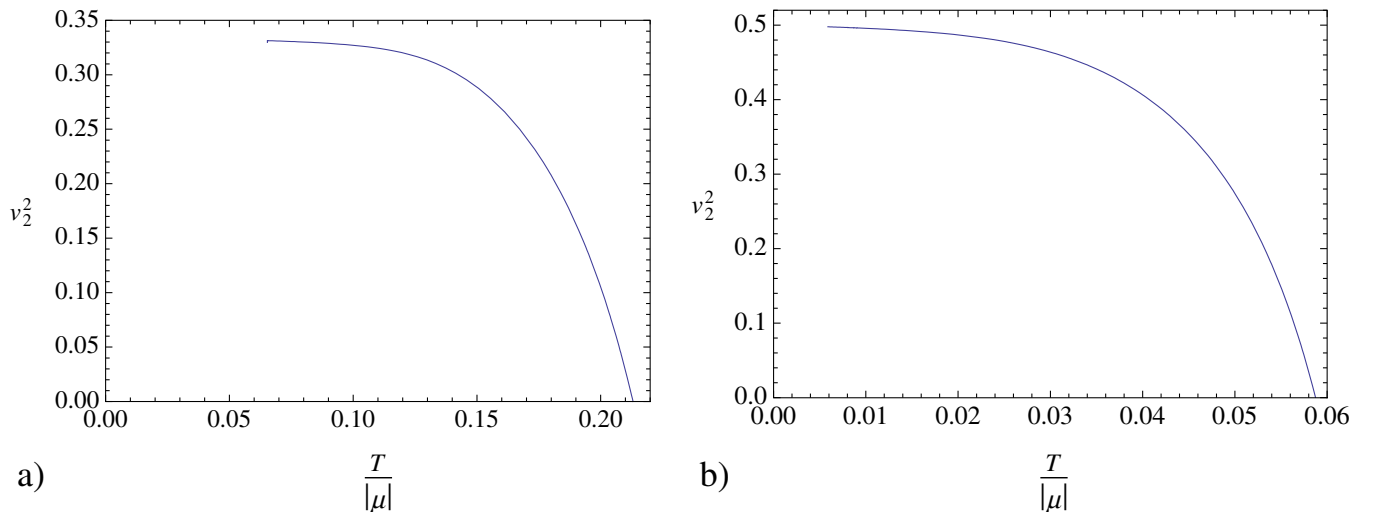


FIG. 4 (color online). The speed of second sound as a function of  $T/|\mu|$ , computed by evaluating thermodynamic derivatives in Eq. (18): (a)  $O_1$  scalar, (b)  $O_2$  scalar. The speed of second sound vanishes as  $T \rightarrow T_c$  and appears to approach a constant value as  $T \rightarrow 0$ .



velocity  $\xi = 0$ . The behavior close to  $T_c$  is qualitatively similar to that of superfluid  $^4\text{He}$  [18]. As the temperature is decreased from  $T_c$ , the speed of second sound rises rapidly from zero and eventually levels off. Experimentally, it is difficult to go to very low temperature and remain within the hydrodynamic limit. The scattering length for the phonons approaches the system size. Numerically, we have also had difficulty finding solutions at very low temperatures; the curves in Fig. 4 terminate where our numerics fail. Theoretically, however, the expectation for  $^4\text{He}$  is that second sound at  $T \approx 0$  is a sound wave supported by a gas of phonons [19]. Thus, the speed of second sound close to  $T = 0$  should approach  $v_2^2 = v_s^2/(d-1)$ .

The naive extrapolation of the curves in Fig. 4 suggests that  $v_2^2$  is  $1/3$  in the  $O_1$  case and  $1/2$  in the  $O_2$  case. The fact that the curves in Fig. 4 level off at low temperature is caused numerically by a similar growth in the susceptibility and the pion decay constant in Eq. (18). That we work in the probe approximation suggests our model may not be reliable at low temperature anyway. We have neglected the coupling between our Abelian-Higgs sector and the metric. The Abelian-Higgs sector on its own does not support ordinary sound.

## V. CONCLUDING REMARKS

We have discussed the hydrodynamics of relativistic superfluids. Moreover, we presented a holographic model that reproduces many of the familiar features of superfluids, including a critical superfluid velocity above which the system returns to its normal phase and a second-order phase transition at zero superfluid velocity between the normal and superfluid phases.

In many respects, our holographic model is similar to the Landau-Ginzburg mean-field treatment of phase transitions. Near the phase transition, we saw that the potential function for the order parameter has the familiar polynomial expansion  $V(\phi) = \alpha\phi^2 + \beta\phi^4 + \dots$ . In our case, however, the values of  $\alpha$  and  $\beta$  and indeed the entire structure of  $V(\phi)$  are not set directly by the field theorist but are instead encoded in a nontrivial fashion by the bulk gravitational solution. By assuming a simple gravitational model, we are led to a particular  $V(\phi)$  that is valid not just near  $T_c$  but at all temperatures in the superfluid phase. Moreover, from the model one can extract not only the static thermodynamic properties, but also quantities relevant for time-dependent, dynamic processes, such as the kinetic coefficients and the correlation functions.

In the future, it would be very interesting to extend the numerical results above. Two obvious directions present themselves. In the calculation of the speed of second sound from derivatives of the pressure  $P$ , one could go beyond the probe approximation and include the backreaction of the Abelian-Higgs sector on the metric. We hope that the resulting speed of second sound will be phenomenologi-

cally meaningful not only near  $T = T_c$  but all the way down to  $T = 0$ .

Another interesting project for the future would be to find the sound wave poles in the density-density correlation function. By studying fluctuations of the scalar and gauge field of the form  $e^{-i\omega t + ikx}$  in the probe limit, one should be able to isolate a pole of the form  $1/(\omega^2 - v_2^2 k^2 + \dots)$  in the Fourier transform of the retarded Green's function for the charge density. Here the ellipsis denotes higher order (damping) terms in  $k$ . Beyond the probe limit, there should also be a pole corresponding to the propagation of ordinary sound.

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*Note added.*—While we were completing this paper, we learned of Ref. [26] which has some overlap with this work.

## APPENDIX: COMPARISON WITH THE CARTER-KHALATNIKOV-LEBEDEV FORMULATION OF SUPERFLUID HYDRODYNAMICS

In this Appendix we show that the hydrodynamic equations written in Sec. III are equivalent to the set of equations proposed previously by Carter, Khalatnikov, and Lebedev [20–23] (see also [24,25]). The formulations of Refs. [20,21] have been shown to be equivalent in Refs. [22,23]. We will follow the notation of Ref. [23].

In Ref. [23], superfluid hydrodynamics is formulated as follows. First, one postulates that the thermodynamic properties of the superfluid are defined by a scalar function  $\Lambda$ , which is a function of two 4-vectors  $s^\mu$  and  $n^\mu$ , which are the entropy density and the particle number density. Since  $\Lambda$  is a Lorentz scalar, the number of variables that it depends on is three:  $s^\mu s_\mu$ ,  $n^\mu n_\mu$ , and  $s^\mu n_\mu$ .

One defines two Lorentz vectors  $\Theta^\mu$  and  $\mu^\mu$  from

$$d\Lambda = \Theta_\mu ds^\mu + \mu_\mu dn^\mu. \quad (\text{A1})$$

The superfluid hydrodynamic equations are

$$\partial_\mu s^\mu = 0, \quad (\text{A2})$$

$$\partial_\mu n^\mu = 0, \quad (\text{A3})$$

$$\partial_\mu \mu_\nu - \partial_\nu \mu_\mu = 0, \quad (\text{A4})$$

$$s^\mu(\partial_\mu \Theta_\nu - \partial_\nu \Theta_\mu) = 0. \quad (\text{A5})$$

At first sight, Eqs. (A2)–(A5) do not bear much resemblance to Eqs. (10)–(15). However, these systems of equations are in fact equivalent. The variables appearing in the Carter-Khalatnikov formulation can be identified with the hydrodynamic variables used in this paper as in the following relations:

$$\Lambda = \epsilon - f^2(\partial_\mu \varphi)^2, \quad (\text{A6})$$

$$s^\mu = su^\mu, \quad (\text{A7})$$

$$n^\mu = nu^\mu + f^2 \partial^\mu \varphi, \quad (\text{A8})$$

$$\mu_\mu = -\partial_\mu \varphi, \quad (\text{A9})$$

$$\Theta_\mu = \frac{1}{s} \left[ -(Ts + \mu n)u_\mu + n \partial_\mu \varphi \right]. \quad (\text{A10})$$

We now show that any solution to the hydrodynamic equations (10)–(15) satisfies Eqs. (A1)–(A5), upon the substitutions (A6)–(A10). First, Eqs. (A2) and (A3) coincide with Eqs. (11) and (15) due to Eqs. (A7) and (A8). Furthermore, Eq. (A4) is trivially satisfied by Eq. (A9).

Now let us check Eq. (A1). The left-hand side is

$$\begin{aligned} d\Lambda &= d\epsilon - d(f^2(\partial_\mu \varphi)^2) \\ &= Tds + \mu dn - \partial_\mu \varphi d(f^2(\partial^\mu \varphi)), \end{aligned} \quad (\text{A11})$$

where the thermodynamic relation  $\epsilon + P = sT + \mu n$  and Eq. (9) have been used, while the right-hand side is

$$\begin{aligned} \Theta_\mu ds^\mu + \mu_\mu dn^\mu &= \frac{1}{s} \left[ -(Ts + \mu n)u_\mu + n \partial_\mu \varphi \right] \\ &\quad \times (u^\mu ds + sdu^\mu) \\ &\quad - \partial_\mu \varphi [u^\mu dn + ndu^\mu + d(f^2 \partial^\mu \varphi)] \\ &= \left( T + \frac{\mu n}{s} \right) ds + \frac{n}{s} u^\mu \partial_\mu \varphi ds \\ &\quad - u^\mu \partial_\mu \varphi dn - \partial_\mu \varphi d(f^2 \partial^\mu \varphi) \end{aligned} \quad (\text{A12})$$

where we have used  $u^2 = -1$ ,  $u_\mu du^\mu = 0$ . Now from the Josephson equation  $u^\mu \partial_\mu \varphi = -\mu$  one sees immediately that (A12) is identical to (A11). Thus, Eq. (A1) is verified.

The last equation that has to be checked is Eq. (A5). We first write

$$s^\mu(\partial_\mu \Theta_\nu - \partial_\nu \Theta_\mu) = \partial_\mu (s^\mu \Theta_\nu) - s^\mu \partial_\nu \Theta_\mu \quad (\text{A13})$$

where  $\partial_\mu s^\mu = \partial_\mu (su^\mu) = 0$  has been used. We expand

$$\begin{aligned} \partial_\mu (s^\mu \Theta_\nu) &= \partial_\mu [-(Ts + \mu n)u^\mu u_\nu + nu^\mu \partial_\nu \varphi] \\ &= -\partial_\mu [(Ts + \mu n)u^\mu u_\nu] + nu^\mu \partial_\mu \partial_\nu \varphi \\ &\quad - \partial_\mu (f^2 \partial^\mu \varphi) \partial_\nu \varphi \end{aligned} \quad (\text{A14})$$

where Eq. (11) has been used;

$$\begin{aligned} s^\mu \partial_\nu \Theta_\mu &= su^\mu \partial_\nu \left[ -\left( T + \frac{\mu n}{s} \right) u_\mu + \frac{n}{s} \partial_\mu \varphi \right] \\ &= s \partial_\nu T + n \partial_\nu \mu + s \mu \partial_\nu \left( \frac{n}{s} \right) + su^\mu \partial_\mu \varphi \partial_\nu \left( \frac{n}{s} \right) \\ &\quad + nu^\mu \partial_\mu \partial_\nu \varphi. \end{aligned} \quad (\text{A15})$$

Combining Eqs. (A14) and (A15), using the Josephson equation, one finds

$$\begin{aligned} s^\mu(\partial_\mu \Theta_\nu - \partial_\nu \Theta_\mu) &= -\partial_\mu [(Ts + \mu n)u^\mu u_\nu] \\ &\quad - \partial_\mu (f^2 \partial^\mu \varphi) \partial_\nu \varphi - s \partial_\nu T \\ &\quad - n \partial_\nu \mu. \end{aligned} \quad (\text{A16})$$

It is easy to check that the right-hand side is equal to  $\partial_\mu T^\mu_\nu$  up to an overall sign, with  $T^\mu_\nu$  defined in Eq. (13), and hence is equal to zero.

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