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Apparently noninvariant terms of nonlinear sigma models in lattice perturbation theory

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Apparently noninvariant terms (ANTs) that appear in loop diagrams for nonlinear sigma models are revisited in lattice perturbation theory. The calculations have been done mostly with dimensional regularization so far. In order to establish that the existence of ANTs is independent of the regularization scheme, and of the potential ambiguities in the definition of the Jacobian of the change of integration variables from group elements to ''pion'' fields, we employ lattice regularization, in which everything (including the Jacobian) is well defined. We show explicitly that lattice perturbation theory produces ANTs in the four-point functions of the pion fields at one-loop and the Jacobian does not play an important role in generating ANTs.

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I. INTRODUCTION

There has been a long time since apparently chiral noninvariant, divergent contributions were noticed in the loop calculations of nonlinear sigma (NLS) models. It is important to realize that there are two kinds of such contributions. The first kind, which produces the mass term of the pion field, leads to the violation of the soft-pion theorem. The second kind is more subtle. It does not violate the soft-pion theorem and is claimed to vanish on shell. It is well understood that the first kind of contributions are canceled by those from the Jacobian [1–3]. (They had been overlooked at that time.) In the dimensional regularization, this kind of noninvariant contributions are absent; it is consistent with the absence of the nontrivial Jacobian in this regularization scheme. As for the second kind contributions, although they have been discussed in the literature [4–8], there still seems to be unclear points, of which we are going to discuss in this paper.

The prescriptions of how to avoid the second kind have been proposed. Tătaru [5] showed, using dimensional regularization, that the second kind contributions are proportional to the (classical) equations of motion and do not contribute to the S matrix, following the argument by 't Hooft [9]. Honerkamp [4] and Kazakov, Pervushin, and Pushkin [6] proposed to use the background field method. This is essentially to modify the theory. Appelquist and Bernard [8] pointed out that a field redefinition removes such contributions. The most popular and practical method is to consider not the pion field but the currents [10,11]. In recent papers Ferrari *et al.* [12–15] reconsidered the renormalization problem emphasizing the symmetry point of view, heavily relying on the Ward-Takahashi identities, and gave the subtraction procedure consistent with them. They claim that the use of the dimensional regularization, in which the tadpole contributions are absent, is essential.

In this paper, we instead use lattice regularization for the following reasons: (i) Since everything is well defined in the lattice regularization, it is obvious that there is no source of the violation of chiral symmetry (up to a ''spurion'' mass term), if we start with a symmetric partition function. This fact is important for establishing that chiral symmetry is not lost despite the appearance of ANTs. Hence the name; they *do not* violate chiral symmetry, though they *appear* to be noninvariant. (ii) In the case of the first kind contributions, the Jacobian plays an essential role. It is interesting to see if the Jacobian plays any role in the second kind. The logarithm of the Jacobian is proportional to $\delta^4(0)$, thus in the dimensional regularization it is trivially set to zero, while in other continuous regularization schemes it is ill-defined. In the lattice regularization, on the other hand, it is regularized and well defined, so that one can carefully examine the effects of the Jacobian. One might suspect that the (naive) Jacobian is actually the latent source of the violation of chiral symmetry, and that a properly defined Jacobian should contain momentum-dependent terms in order for the theory to be chiral invariant, which eventually cancel the ANTs produced by loop diagrams. It is therefore important to see what happens with the well-defined, momentum-independent Jacobian in the manifestly chiral invariant theory. (iii) Lattice regularization is completely different from dimensional regularization. It is therefore useful to see if the existence of ANTs is independent of the regularization scheme. To our best knowledge, ANTs in four dimensions have never been calculated by using lattice regularization in the literature. (In $2+\epsilon$ dimensions, Symanzik [16] obtained ANTs in the lattice regularization.)

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The purpose of this paper is to establish the existence of ANTs in the lattice perturbation theory at one-loop preserving chiral symmetry manifestly. This implies that ANTs are compatible with chiral symmetry. We also see that the Jacobian does not play an important role in generating ANTs and that the appearance of ANTs is independent of regularization schemes.

Our calculation is a straightforward generalization of Shushpanov and Smilga [17], who calculated only the self-energy contributions. We consider the four-point (amputated) Green functions at low momenta ($p \ll 1/a$) to order $\mathcal{O}(p^4)$ at one-loop level. A mass term is introduced in order to regularize the IR singularities. Unlike the self-energy calculation, the IR regularization with the mass term plays an important role for the calculations of the four-point functions. We find that the divergent part of it contains ANTs, which cannot be removed by a symmetric counterterms. We also find that the Jacobian does not play an essential role. The ANTs vanish on the mass shell.

In the next section, we establish the existence of the ANTs by an explicit one-loop calculation. In Sec. III, we summarize the results and give a discussion. Appendix A contains some integration formulae.

II. LATTICE PERTURBATION THEORY

A. Setup

In this section, we give an explicit one-loop calculation for the four-point amputated Green function in the $SU(2)\times SU(2)$ NLS model in four dimensions. In the NLS as an effective theory there are infinitely many terms with increasing number of derivatives. We are however interested only in whether there arises an ANT of $\mathcal{O}(p^2)$ or of $\mathcal{O}(p^4)$ at the one-loop level. [Note that, unlike the dimensional regularization, there are contributions of $\mathcal{O}(p^2)$ from the one-loop diagrams in the lattice regularization.] To see this, we will consider the one-loop contributions only with vertices of $\mathcal{O}(p^2)$ and examine whether the contributions of $\mathcal{O}(p^2)$ and of $\mathcal{O}(p^4)$ can be absorbed in the symmetric terms. There may be other ANTs involving higher derivative vertices, but they are not related to the lower order contributions by the symmetry, and cannot cancel the ANTs that may arise to this lowest order.

In the continuum, the action of $\mathcal{O}(p^2)$ is given by

$$\mathcal{L}_2 = \frac{F^2}{4}\,\mathrm{Tr}(\partial_\mu U^\dagger \partial_\mu U) - \frac{F^2 m^2}{4}\,\mathrm{Tr}(U + U^\dagger), \quad (2.1)$$

where U is an $SU(2)$-valued field and F is the coupling constant. (In the dimensional regularization, it is the pion decay constant in the chiral limit.) We also introduce the mass term to regularize the IR singularities.

On the hypercubic lattice with a being the lattice constant, the action may be written as

$$S_2^{\mathrm{lat}}[U] = \frac{F^2 a^2}{4}\sum_n\Big[\sum_\mu \mathrm{Tr}(2 - U_n^\dagger U_{n+\mu} - U_{n+\mu}^\dagger U_n) - m^2 a^2\,\mathrm{Tr}(U_n^\dagger + U_n)\Big], \quad (2.2)$$

which is obtained by the simple replacement

$$\partial_\mu U(x) \to (U_{n+\mu} - U_n)/a. \quad (2.3)$$

There are many other discretization methods, but the choice does not make a crucial difference in the following discussions, so we stick to this simplest choice.

The partition function is given by

$$Z = \int \prod_n DU_n e^{-S_2^{\mathrm{lat}}[U]}, \quad (2.4)$$

where DU_n stands for the invariant measure under the global $SU(2)_L \times SU(2)_R$ transformations

$$U_n \to g_L U_n g_R^\dagger, \quad (2.5)$$

where g_L and g_R are $SU(2)_{L,R}$ elements. Note that if the mass term is treated as a ''spurion'' field [10], and transformed properly, the theory is manifestly invariant under $SU(2)_L \times SU(2)_R$.

We introduce pion fields to do perturbation theory. We employ the following parameterization:

$$U_n = \sigma_n + i\pi_n^a \tau^a/F, \qquad \sigma_n = \sqrt{1-(\pi_n^a)^2/F^2}. \quad (2.6)$$

There are of course other parameterizations. But the main results are independent of the choice.

In terms of the pion fields, the measure is written as

$$\prod_n DU_n = e^{-S_{\mathrm{Jacob}}^{\mathrm{latt}}}\prod_{n,a} D\pi_n^a, \quad (2.7)$$

with [18]

$$S_{\mathrm{Jacob}}^{\mathrm{lat}} = -\frac{1}{2}a^4\sum_n \frac{1}{a^4}\,\mathrm{Tr}\ln\Big[\delta_{ab} + \frac{\pi_n^a\pi_n^b}{F^2 - (\pi_n^c)^2}\Big]. \quad (2.8)$$

Note that the $\delta^4(0)$ is regularized as $1/a^4$ on the lattice. It is important to note that the vertices from the Jacobian is momentum independent.

Expanding S_2^{latt} and $S_{\mathrm{Jacob}}^{\mathrm{latt}}$ in terms of the pion fields π, we obtain

$$\begin{aligned} S_2^{\mathrm{lat}} = {}& \frac{a^2}{2}\sum_n\Big[\sum_\mu(\pi_{n+\mu}^a - \pi_n^a)^2 + m^2a^2(\pi_n^a)^2\Big] \\ & - \frac{a^2}{4F^2}\sum_{n,\mu}(\pi_n^a)^2(\pi_{n+\mu}^b)^2 + \frac{a^2}{8F^2}(m^2a^2+8)\sum_n[(\pi_n^a)^2]^2 \\ & - \frac{a^2}{16F^4}\sum_{n,\mu}(\pi_n^a)^2(\pi_{n+\mu}^b)^2[(\pi_n^c)^2 + (\pi_{n+\mu}^c)^2] \\ & + \frac{a^2}{16F^4}(m^2a^2+8)\sum_n[(\pi_n^a)^2]^3 + \cdots, \end{aligned} \quad (2.9)$$

$$S_{\text{Jacob}}^{\text{lat}} = \sum_n \left[-\frac{1}{2} \frac{(\pi_n^a)^2}{F^2} - \frac{1}{4} \frac{[(\pi_n^a)^2]^2}{F^4} + \cdots \right], \quad (2.10)$$

where we retain only the terms that contribute to the two- and four-point Green functions up to including $\mathcal{O}(p^4/F^4)$. Note that, because of the discretization, it is difficult to count the power of momenta buried, say, in $1 - \cos(ap)$. Instead, we count the power of $1/F$. There are no terms with positive power of F.

The Feynman rules are obtained in the usual way, treating all the contributions from $S_{\text{Jacob}}^{\text{lat}}$ as interactions. (They are of higher order in $1/F$.) The propagator is the usual one,

$$\langle \pi_n^a \pi_m^b \rangle_0 = \delta^{ab} \int_\square \frac{d^4k}{(2\pi)^4} \frac{e^{ik(n-m)a}}{m^2 + [k]_a^2}, \quad (2.11)$$

where $\int_\square d^4k$ stands for the integration over the hypercube

$$\{k | k_\mu \in [-\pi/a, \pi/a], \mu = 1, \cdots, 4\}, \quad (2.12)$$

and we have introduced a useful notation

$$[k]_a^2 \equiv \frac{2}{a^2} \sum_\mu (1 - \cos(k_\mu a)), \quad (2.13)$$

which goes to k^2 in the continuum limit $a \to 0$. S_2^{latt} leads to the following four-point and six-point vertices

$$-\frac{1}{F^2} \sum_\mu \{ \delta^{ab}\delta^{cd}([k_a + k_b]_a^2 + m^2) + \delta^{ac}\delta^{bd}([k_a + k_c]_a^2 + m^2) + \delta^{ad}\delta^{bc}([k_a + k_d]_a^2 + m^2)\}, \quad (2.14)$$

and

$$-\frac{1}{F^4} \{ \delta^{ab}([k_a + k_b]_a^2 + m^2)[\delta^{cd}\delta^{ef} + \delta^{ce}\delta^{df} + \delta^{cf}\delta^{de}] + 14 \text{ similar terms}\}, \quad (2.15)$$

respectively. See Figs. 1 and 2.

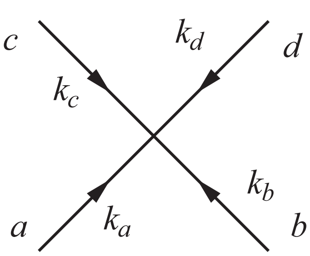

FIG. 1. Four-point vertex from S_2^{lat}. The indices $a, \cdots, d$ stand for the isospin of the pion field, and $k_a, \cdots, k_d$ are corresponding incoming momenta.

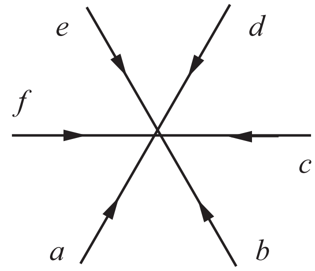

FIG. 2. Six-point vertex from S_2^{lat}. The indices $a, \cdots, f$ stand for the isospin of the pion field. The momentum labels are omitted.

B. Self-energy

Shushpanov and Smilga [17] calculated the self-energy contribution from the four-point vertex with a massless propagator. We do the same calculation with a finite mass (See Fig. 3),

$$\begin{aligned} -\Sigma^{ab}(p) &= -\delta^{ab}\Sigma(p) \\ &= -\frac{\delta^{ab}}{2F^2} \int_\square \frac{d^4k}{(2\pi)^4} \\ &\quad \times \frac{[k+p]_a^2 + [k-p]_a^2 + 5m^2}{m^2 + [k]_a^2}. \end{aligned} \quad (2.16)$$

Note that this leading order contribution is of order $1/F^2$. Following their calculations, we find

$$\begin{aligned} \Sigma(p) &= \left[\frac{1}{2F^2a^2} \left(1 + \frac{m^2a^2}{8}\right) [p]_a^2 + \frac{3m^2}{4F^2a^2} \right] \mathcal{I}_0 \\ &\quad - \frac{1}{8F^2a^2}[p]_a^2 + \frac{1}{F^2a^4}, \end{aligned} \quad (2.17)$$

where we have introduced $\mathcal{I}_0$,

$$\mathcal{I}_n \equiv \int_0^\infty ds s^n e^{-(s/2)(m^2a^2+8)} [I_0(s)]^4, \quad (2.18)$$

and $I_0(s)$ is the modified Bessel function,

$$I_0(s) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{s \cos k}. \quad (2.19)$$

The last term of Eq. (2.17) is quartically divergent, and it is cancelled by the $\mathcal{O}(1/F^2)$ contribution from $S_{\text{Jacob}}^{\text{lat}}$, giving no ANTs. This cancellation mechanism is well known [1–3].

Note that modified Bessel function behaves for $s \gg 1$ as

$$I_0(s) = \frac{e^s}{\sqrt{2\pi s}} \left(1 + \mathcal{O}\left(\frac{1}{s}\right)\right), \quad (2.20)$$

and for $0 < s \ll 1$ as

$$I_0(s) = 1 + \mathcal{O}(s^2), \quad (2.21)$$

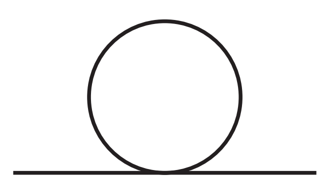

FIG. 3. Self-energy contribution from the four-point vertex from S_2^{lat}.

so that the integral I_n is finite as far as m is kept finite. Although it is finite, but is not analytic at $m^2a^2 = 0$. One cannot expand the result in terms of m^2a^2. This kind of singularity at $m^2a^2 = 0$ persists in the calculations of the four-point functions, which we discuss in the next subsection. We therefore keep the mass terms in the exponents (which come from the propagators) intact.

It is instructive to compare the cutoff integral (for $m \ll \Lambda$)

$$2\pi^2 \int_0^\Lambda \frac{k^3 dk}{(2\pi)^4} \frac{1}{k^2 + m^2} \sim c_2\Lambda^2 + c_0 m^2 \ln\left(\frac{m^2}{\Lambda^2}\right), \quad (2.22)$$

where c_2 and c_0 are numerical constants, with the corresponding lattice version

$$\int_\Box \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2]_a + m^2} = \frac{1}{2a^2} I_0. \quad (2.23)$$

By identifying $\Lambda \sim 1/a$, we see

$$I_0 \sim \tilde{c}_2 + \tilde{c}_0 (m^2a^2) \ln(m^2a^2) \quad (2.24)$$

for $ma \ll 1$, where $\tilde{c}_2$ and $\tilde{c}_0$ are other numerical constants. The second term causes the nonanalyticity of I_0. Similarly for I_1, we have

$$I_1 \sim \tilde{d}_2 + \tilde{d}_0 \ln(m^2a^2), \quad (2.25)$$

with some numerical constants $\tilde{d}_2$ and $\tilde{d}_0$.

C. Four-point function

There are two kinds of contributions to the four-point function besides the ones from $S^{\text{lat}}_{\text{Jacob}}$: the ones involving a six-point vertex (Fig. 4) and the ones involving two four-point vertices (Fig. 5).

In general, the four-point function in the continuum has the following structure:

$$\delta_{ab}\delta_{cd}A(p_a, p_b, p_c, p_d) + \delta_{ac}\delta_{bd}A(p_a, p_c, p_b, p_d) + \delta_{ad}\delta_{bc}A(p_a, p_b, p_d, p_c). \quad (2.26)$$

It has the same structure on the lattice. Since the amplitude is symmetric under the crossing, it is sufficient to calculate only the contributions $A_L(p_a, p_b, p_c, p_d)$ on the lattice that correspond to the first term $A(p_a, p_b, p_c, p_d)$ of Eq. (2.26).

From Fig. 4, we have the contribution

$$A_L(p_a, p_b, p_c, p_d)^{\text{Fig. 4}} = -\frac{1}{2F^4} \int_\Box \frac{d^4k}{(2\pi)^4} \frac{1}{m^2 + [k]_a^2} \left\{ 10[p_a + p_b]_a^2 + \sum_{i=a,b,c,d} ([k + p_i]_a^2 + [k - p_i]_a^2) + 21m^2 \right\}, \quad (2.27)$$

and from Fig. 5,

$$\begin{aligned} A_L(p_a, p_b, p_c, p_d)^{\text{Fig. 5}} = {} & \frac{1}{2F^4} \int_\Box \frac{d^4k}{(2\pi)^4} \frac{1}{m^2 + [k]_a^2} \frac{1}{m^2 + [p_a + p_b - k]_a^2} (3([p_a + p_b]_a^2 + m^2)^2 + 2([p_a + p_b]_a^2 + m^2) \\ & \times ([k + p_d]^2 + [k - p_a]^2 + 2m^2)) + \frac{1}{2F^4} \int_\Box \frac{d^4k}{(2\pi)^4} \frac{1}{m^2 + [k]_a^2} \frac{1}{m^2 + [p_a + p_c - k]_a^2} \\ & \times 2([p_a - k]_a^2 + m^2)([k + p_b]_a^2 + m^2) + \frac{1}{2F^4} \int_\Box \frac{d^4k}{(2\pi)^4} \frac{1}{m^2 + [k]_a^2} \frac{1}{m^2 + [p_a + p_d - k]_a^2} \\ & \times 2([p_a - k]_a^2 + m^2)([k + p_b]_a^2 + m^2). \end{aligned} \quad (2.28)$$

Note that these are of $\mathcal{O}(1/F^4)$.

If we set all the external momenta and the mass m to be zero, we have

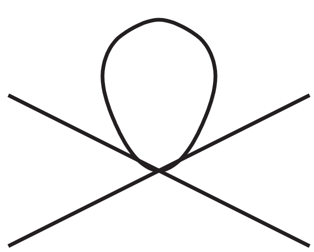

FIG. 4. Contribution from the six-point vertex from S_2^{lat} to the four-point function.

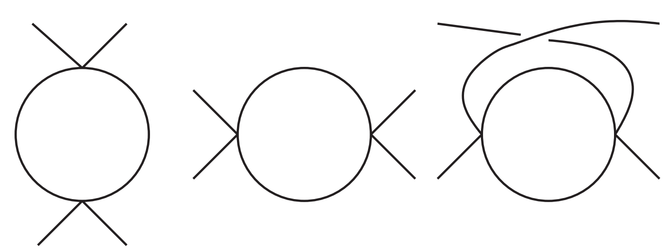

FIG. 5. Three s-, t-, u-channel contributions from the four-point vertex from S_2^{latt} to the four-point function.

$$A_L(p_a, p_b, p_c, p_d)^{\text{Fig. 4}}|_{p_i=m=0} = -\frac{4}{F^4 a^4}, \tag{2.29}$$

$$A_L(p_a, p_b, p_c, p_d)^{\text{Fig. 5}}|_{p_i=m=0} = \frac{2}{F^4 a^4}. \tag{2.30}$$

The sum of them exactly cancels the $\mathcal{O}(1/F^4)$ contribution from the Jacobian, $2/F^4a^4$. Thus, the amplitude satisfies the soft-pion theorem. There is no momentum (or mass) independent ANT. Note that all the Jacobian contributions are used up to cancel the momentum (and mass) independent contributions to this order. The vertices from the Jacobian are now shown not to produce ANTs.

A straightforward but tedious calculation leads to the following result for the one-loop contributions:

$$\begin{aligned} A_L(p_a, p_b, p_c, p_d) = &-\frac{3\mathcal{I}_0}{4F^4a^2}(2s + 3m^2) + \frac{s}{8F^4a^2}\left(1 - \frac{1}{2}(8 + m^2a^2)\mathcal{I}_0\right) \\ &+ \frac{\mathcal{I}_1}{24F^4}[9(s + m^2)^2 - 3(s + m^2)(2s - \Delta) + 2Z(p_a, p_b, p_c, p_d)] \\ &+ \frac{\mathcal{I}_0}{288F^4}\bigg[9(s + m^2)(2s - \Delta) - 8Z(p_a, p_b, p_c, p_d) \\ &+ 48\sum_\mu (p_a)_\mu (p_b)_\mu (p_c)_\mu (p_d)_\mu, \bigg], \end{aligned} \tag{2.31}$$

where $s = (p_a + p_b)^2$, $t = (p_a + p_c)^2$, and $u = (p_a + p_d)^2$ expanded in powers of the external momenta up to including $\mathcal{O}(p^4/F^4)$. Here, we have introduced the notation

$$\begin{aligned} \Delta &\equiv s + t + u, \\ Z(p_a, p_b, p_c, p_d) &\equiv \tfrac{1}{2}[s(t + u) + 2(t^2 + u^2) \\ &\quad - 2(t + u)\Delta + 2(\Delta_{ac}\Delta_{bd} + \Delta_{ad}\Delta_{bc}) \\ &\quad - \Delta_{ab}\Delta_{cd}], \\ \Delta_{ij} &\equiv p_i^2 + p_j^2. \end{aligned} \tag{2.32}$$

Some useful formulae to calculate Eq. (2.31) are given in Appendix A.

The terms proportional to $1/a^2$ correspond to quadratically divergent ones. The chiral logarithms are contained in $\mathcal{I}_n$. The last term in Eq. (2.31) is not rotational invariant. It is not a surprise, because the lattice regularization breaks rotational invariance.

In order to see if the result is manifestly chiral invariant, we need to relate the expression to local operators. The terms in the first line of Eq. (2.31) are proportional to $1/a^2$ (i.e., quadratically divergent) and quadratic in external momenta. It is important to notice that they depend only on s except for the mass m. Note that there is only one chiral invariant operator of $\mathcal{O}(p^2)$; Eq. (2.1) in the continuum. It produces terms of exactly the same form as those in the first line, and thus may cancel the divergence. That is, the terms in the first line do not contain ANTs.

A vigilant reader may notice that we have already considered the same counterterm to cancel the divergence in the self-energy contribution, thus its coefficient has been fixed. Here comes an important feature of the perturbation theory; in terms of U; there is only one parameter, i.e., the coupling constant F. On the other hand, when we introduce the pion field, we have another parameter, the wave function renormalization constant. Introducing the renormalized coupling constant F_R and the renormalized field π^a_{Rn}, we have

$$\frac{\pi_n^a}{F} = \left(\frac{1 + \delta_\pi}{1 + \delta_F}\right)\frac{\pi^a_{Rn}}{F_R}. \tag{2.33}$$

By tuning only the parameter δ_π, one can cancel the divergence in the self-energy contribution. The parameter δ_F is now determined to cancel the divergence in the first line of Eq. (2.31).

Note that we consider the continuum action in order to see if ANTs emerge. In momentum space, the difference between the continuum and the lattice regularized ones is of higher order in momenta, and is not rotational invariant. In order to cancel the divergence coming from the difference, we need more counterterms, which are of higher order in momenta. Since they are not rotational invariant, the existence of such counterterms do not interfere with the following argument for the existence of ANTs, which, as we will see shortly, are rotational invariant.

The terms in the second and third lines of Eq. (2.31) are quartic in momenta (and the mass). The terms in the second line contain logarithmic divergence due to $\mathcal{I}_1$, while those in the third line are finite. There are only three chiral invariant operators of $\mathcal{O}(p^4)$ available in the continuum;

$$\mathcal{O}_1 = \mathrm{Tr}(\partial_\mu U^\dagger \partial_\mu U)\mathrm{Tr}(\partial_\nu U^\dagger \partial_\nu U), \tag{2.34}$$

$$\mathcal{O}_2 = \mathrm{Tr}(\partial_\mu U^\dagger \partial_\nu U)\mathrm{Tr}(\partial_\mu U^\dagger \partial_\nu U), \tag{2.35}$$

$$\mathcal{O}_3 = \mathrm{Tr}(\partial_\mu^2 U^\dagger \partial_\nu^2 U). \tag{2.36}$$

[Note that for $SU(2)$ there are some nontrivial relations which reduce the number of independent operators. For example, $\mathrm{Tr}[(\partial_\mu U^\dagger \partial_\mu U)^2]$ is proportional to $\mathcal{O}_1$.] If the terms in the second and third lines of Eq. (2.31) are of the same form as those produced by some linear combinations of these operators, then these divergences may be cancelled by manifestly chiral invariant operators. Let $C_i(p_a, p_b, p_c, p_d)/F^4$ $(i = 1, 2, 3)$ denote the contributions of these operators to the amplitude, $A_L(p_a, p_b, p_c, p_d)$, to $\mathcal{O}(p^4/F^4)$. They are given by

$$C_1(p_a, p_b, p_c, p_d) = (s - \Delta_{ab})(s - \Delta_{cd}), \tag{2.37}$$

$$C_2(p_a, p_b, p_c, p_d) = (t - \Delta_{ac})(t - \Delta_{bd}) + (u - \Delta_{bc})(u - \Delta_{ad}), \tag{2.38}$$

$$C_3(p_a, p_b, p_c, p_d) = s^2, \tag{2.39}$$

respectively. In the massless limit, the terms in the square bracket in the second line of Eq. (2.31) may be written as

$$- C_1(p_a, p_b, p_c, p_d) + 2C_2(p_a, p_b, p_c, p_d) + 3C_3(p_a, p_b, p_c, p_d) + 3s\Delta, \tag{2.40}$$

and those in the third line as

$$4C_1(p_a, p_b, p_c, p_d) - 8C_2(p_a, p_b, p_c, p_d) + 18C_3(p_a, p_b, p_c, p_d) - 9s\Delta. \tag{2.41}$$

It is important to note that the last terms of Eqs. (2.40) and (2.41) cannot be expressed as a contribution of chiral invariant operators. We have thus established the existence of ANTs.

We remark that the terms that correspond to the logarithmic divergence, Eq. (2.40), are different from those in the continuum. Compare Eq. (2.40) with Eq. (3.3) in Ref. [8].

It is interesting to note that the ANTs are rotational invariant. We also note that these are proportional to Δ, i.e., the ANTs vanish if the (massless) on-shell conditions are imposed for all the external momenta.

The term in the fourth line of Eq. (2.31) is finite. It is manifestly chiral invariant, though it is not rotational invariant. Actually, it can be obtained from the chiral invariant operator of the form

$$\sum_\mu \mathrm{Tr}(\partial_\mu U^\dagger \partial_\mu U \partial_\mu U^\dagger \partial_\mu U). \tag{2.42}$$

Even though it is uneasy to have such a rotational noninvariant term, it has nothing to do with ANTs. We add the operator (2.42) with the coefficient $\mathcal{I}_0/96$ as the counterterm to cancel the term in the fourth line of Eq. (2.31) so that the amplitude is rotational invariant.

III. CONCLUSION

In this paper, we have established the existence of ANTs in lattice chiral perturbation theory. Since the definition of the partition function regularized on a lattice is manifestly chiral invariant (up to the mass which regularizes the infrared singularities), and the calculations are consistent with chiral symmetry, the symmetry is not broken at all. Nevertheless the one-loop diagrams generate ANTs. ANTs are compatible with chiral symmetry. The existence has been known in the literature. Our contribution is the first demonstration of it in the explicit lattice calculation.

On a lattice the Jacobian is well regularized, and we have shown that it is not responsible for the appearance of ANTs. The role played by the Jacobian is just to cancel the momentum-independent, chirally noninvariant contributions of the first kind mentioned in the introduction.

The result of the present paper has also given support for that the appearance of ANTs is independent of regularization scheme.

We find that the ANTs vanish when all the external momenta are on-shell, consistent with the results obtained with dimensional regularization. It means that the ANTs do not contribute to the S matrix for the two-pion scattering at least at the one-loop level.

Finally, we discuss a few points concerning ANTs, which are still unclear to us.

Our original motivation for this study is related to setting up the Wilsonian renormalization group calculation for the nonlinear sigma model. The appearance of ANTs would cause a problem to the standard program of the approach, even though they are compatible with chiral symmetry. It would be desired to have a better statement of symmetry than just the manifest invariance of the Wilsonian effective action. In other words, we should seek for the combination of the Wilsonian program and the Ward-Takahashi identities.

It is not clear to us if the ANTs in general [i.e., in higher order, and/or in $n(>4)$-point functions] do not contribute to the S matrix. Ferrari *et al.* [13] discussed general forms of ANTs in the effective action, which is the generating function of the one-particle irreducible Green functions. In order to see how these terms contribute to the S matrix, one needs to examine the effects of one-particle reducible diagrams.

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APPENDIX: SOME INTEGRATION FORMULAE

In this appendix, we give some useful integration formulae for the evaluation of $A_L(p_a, p_b, p_c, p_d)$ up to and including $\mathcal{O}(p^4/F^4)$ discussed in Sec. II.

The basic technique that we make use of is Schwinger parameterization of the propagator

$$\frac{1}{m^2 + [k]_a^2} = \int_0^\infty ds e^{-s(m^2 + [k]_a^2)}. \tag{A1}$$

To illustrate the method, let us consider the simple example

$$\int_\square \frac{d^4k}{(2\pi)^4} \frac{1}{m^2 + [k]_a^2} \frac{1}{m^2 + [k+p]_a^2}. \tag{A2}$$

By using Eq. (A1), it can be written as

$$\int_0^\infty du \int_0^\infty dv \int_\Box \frac{d^4k}{(2\pi)^4} e^{-u(m^2+[k]_a^2)} e^{-v(m^2+[k+p]_a^2)}. \quad (A3)$$

Here, we insert the identity

$$1 = \int_0^\infty ds \delta(s-u-v), \quad (A4)$$

and making a change of variables, $v = s\alpha$, $k \to k/a$, and $s \to sa^2/2$, we have

$$\frac{1}{4}\int_0^1 d\alpha \int_0^\infty sds \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \times e^{-s(M-\sum_\mu \cos k_\mu)} e^{s\alpha(\sum_\mu \cos(k_\mu + p_\mu a) - \sum_\mu \cos k_\mu)}, \quad (A5)$$

where $M \equiv (\frac{8+m^2a^2}{2})$ is introduced.

In this way, all the necessary integrals may be written as the form

$$\int_0^\infty dse^{-sM}\langle\langle X(p,k)\rangle\rangle, \quad (A6)$$

where we have introduced a useful notation $\langle\langle X(p,k)\rangle\rangle$,

$$\langle\langle X(p,k)\rangle\rangle \equiv \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} e^{s\sum_\mu \cos k_\mu} X(p,k), \quad (A7)$$

with $X(p,k)$ being a function of the external momentum p and the dimensionless (i.e., rescaled) loop momentum k.

The diagrams we are interested in contain either a single propagator or two propagators. For those involving a single propagator, the following two integrals are relevant:

$$\int_\Box \frac{d^4k}{(2\pi)^4} \frac{1}{m^2+[k]_a^2} = \frac{1}{2a^2}\int_0^\infty dse^{-sM}\langle\langle 1\rangle\rangle, \quad (A8)$$

$$\int_\Box \frac{d^4k}{(2\pi)^4} \frac{1}{m^2+[k]_a^2}(m^2+[k+p]_a^2) = \frac{1}{a^4}\int_0^\infty dse^{-sM}\left\langle\left\langle M - \sum_\mu \cos(k_\mu + p_\mu a)\right\rangle\right\rangle. \quad (A9)$$

There are three types of integral that are relevant for one-loop diagrams involving two propagators:

$$\int_\Box \frac{d^4k}{(2\pi)^4} \frac{1}{m^2+[k]_a^2}\frac{1}{m^2+[k+p]_a^2} = \frac{1}{4}\int_0^1 d\alpha \int_0^\infty sdse^{-sM}\langle\langle e^{-s\alpha N(p,k)}\rangle\rangle, \quad (A10)$$

$$\int_\Box \frac{d^4k}{(2\pi)^4} \frac{1}{m^2+[k]_a^2}\frac{1}{m^2+[k+p]_a^2}(m^2+[k+q]_a^2) = \frac{1}{2a^2}\int_0^1 d\alpha \int_0^\infty sdse^{-sM} \times \left\langle\left\langle e^{-s\alpha N(p,k)}\left(M - \sum_\mu \cos(k_\mu + q_\mu a)\right)\right\rangle\right\rangle, \quad (A11)$$

$$\int_\Box \frac{d^4k}{(2\pi)^4} \frac{1}{m^2+[k]_a^2}\frac{1}{m^2+[k+p]_a^2}(m^2+[k+q]_a^2) \times (m^2+[k+l]_a^2) = \frac{1}{a^4}\int_0^1 d\alpha \int_0^\infty sdse^{-sM}\left\langle\left\langle e^{-s\alpha N(p,k)}\left(M - \sum_\mu \cos(k_\mu + q_\mu a)\right)\left(M - \sum_\nu \cos(k_\nu + l_\nu a)\right)\right\rangle\right\rangle, \quad (A12)$$

where $N(p,k)$ is defined as

$$N(p,k) \equiv \sum_\mu [(1-\cos(p_\mu a))\cos k_\mu + \sin(p_\mu a)\sin k_\mu]. \quad (A13)$$

We can calculate $\langle\langle\cdots\rangle\rangle$'s, by expanding $e^{-s\alpha N(p,k)}$ in powers of external momenta and using the following formulae:

$$\langle\langle 1\rangle\rangle = I_0^4, \quad (A14)$$

$$\langle\langle \cos k_\mu\rangle\rangle = I_0' I_0^3, \quad (A15)$$

$$\langle\langle \cos k_\mu \cos k_\nu\rangle\rangle = \delta_{\mu\nu}\left(I_0^4 - \frac{1}{4s}(I_0^4)' - I_0^2(I_0')^2\right) + I_0^2(I_0')^2, \quad (A16)$$

$$\langle\langle \sin k_\mu \sin k_\nu\rangle\rangle = \frac{1}{4s}(I_0^4)'\delta_{\mu\nu}, \quad (A17)$$

$$\langle\langle \cos k_\mu \sin k_\nu \sin k_\lambda\rangle\rangle = \frac{1}{s}\delta_{\nu\lambda}\left[\delta_{\mu\nu}\left(I_0^4 - \frac{1}{2s}(I_0^4)' - I_0^2(I_0')^2\right) + I_0^2(I_0')^2\right], \quad (A18)$$

$$\langle\langle \sin k_\mu \sin k_\nu \sin k_\lambda \sin k_\gamma\rangle\rangle = \frac{1}{s^2}\delta_{\mu\nu}\delta_{\lambda\gamma}\left[\delta_{\mu\lambda}\left(I_0^4 - \frac{1}{2s}(I_0^4)' - I_0^2(I_0')^2\right) + I_0^2(I_0')^2\right] + \frac{1}{s^2}\delta_{\mu\lambda}\delta_{\nu\gamma}\left[\delta_{\mu\nu}\left(I_0^4 - \frac{1}{2s}(I_0^4)' - I_0^2(I_0')^2\right) + I_0^2(I_0')^2\right] + \frac{1}{s^2}\delta_{\mu\gamma}\delta_{\nu\lambda}\left[\delta_{\mu\lambda}\left(I_0^4 - \frac{1}{2s}(I_0^4)' - I_0^2(I_0')^2\right) + I_0^2(I_0')^2\right], \quad (A19)$$

where $I_0(s)$ is the modified Bessel function given in Eq. (2.19). The prime stands for a derivative with respect to s. Note that a bracket $\langle\langle\cdots\rangle\rangle$ containing an odd number of $(\sin k_\mu)$'s vanishes because of parity.

It is important to notice that all integrands in Eqs. (A10)–(A12) contain the exponential suppression factor e^{-sM} with $M > 4$. It justifies the expansion of $e^{-s\alpha N(p,k)}$ in powers of (ap_μ) within the integrals even though $N(p,k)$ is multiplied by s, since it effectively cuts off the domain of integration where s is large.

Now Eqs. (A10)–(A12) can be expressed in terms of $\mathcal{I}_n$ defined in Eq. (2.18). In doing so, we extensively use the identity

$$I_0''(s) = I_0(s) - \frac{1}{s} I_0'(s), \tag{A20}$$

which is nothing but the modified Bessel differential equation satisfied by $I_0(s)$.

Finally we obtain the integrals involving a single propagator,

$$\int_\Box \frac{d^4k}{(2\pi)^4} \frac{1}{m^2 + [k]_a^2} = \frac{1}{2a^2} \mathcal{I}_0, \tag{A21}$$

$$\int_\Box \frac{d^4k}{(2\pi)^4} \frac{1}{m^2 + [k]_a^2} (m^2 + [k+p]_a^2) = \frac{1}{a^4}\left[M\mathcal{I}_0 + \frac{1}{4}(1 - M\mathcal{I}_0) \sum_\mu \cos(p_\mu a) \right], \tag{A22}$$

and those involving two propagators,

$$\int_\Box \frac{d^4k}{(2\pi)^4} \frac{1}{m^2 + [k]_a^2} \frac{1}{m^2 + [k+p]_a^2} = \frac{1}{4}[\mathcal{I}_1 + O((ap)^2)], \tag{A23}$$

$$\int_\Box \frac{d^4k}{(2\pi)^4} \frac{1}{m^2 + [k]_a^2} \frac{1}{m^2 + [k+p]_a^2} (m^2 + [k+q]_a^2) = \frac{1}{2a^2}\left[\mathcal{I}_0 + \frac{1}{8}\sum_\mu a^2 (p_\mu q_\mu - q_\mu^2)(\mathcal{I}_0 - M\mathcal{I}_1) + O((ap)^4) \right],$$

$$\begin{aligned}\int_\Box \frac{d^4k}{(2\pi)^4} \frac{1}{m^2 + [k]_a^2} \frac{1}{m^2 + [k+p]_a^2} (m^2 + [k+q]_a^2)(m^2 + [k+l]_a^2) = \frac{1}{a^4}\Bigg[& 1 - \frac{a^2}{8}[p-(q+l)]_a^2 (1 - M\mathcal{I}_0) + \frac{1}{4}\mathcal{I}_1 \sum_\mu a^4 \\ & \times \{p_\mu^2 q_\mu l_\mu - p_\mu(q_\mu l_\mu^2 + l_\mu q_\mu^2) + q_\mu^2 l_\mu^2\} \\ & + \frac{1}{144}(M\mathcal{I}_0 + (4 - M^2)\mathcal{I}_1) \sum_{\mu,\nu} a^4 \{3p_\mu^2 q_\mu l_\mu + p_\mu^2 \\ & \times (q_\nu l_\nu) - 4(p_\mu q_\mu)(p_\nu l_\nu) - 3p_\mu(q_\mu l_\mu^2 + l_\mu q_\mu^2) \\ & + 3(p_\mu q_\mu) l_\nu^2 + 3(p_\mu l_\mu) q_\nu^2 + 3(q_\mu^2 l_\mu^2 - q_\mu^2 l_\nu^2)\} \\ & + O((ap)^6) \Bigg],\end{aligned} \tag{A24}$$

where only the necessary terms to calculate $A_L(p_a, p_b, p_c, p_d)$ to order $\mathcal{O}(p^4/F^4)$ are retained.

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