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Analytic, nonperturbative, almost exact QED: The two-point functions

H. M. Fried*

Department of Physics, Brown University, Providence, Rhode Island 02912 USA

Y. Gabellini

Institut Non Linéaire de Nice, UMR 6618 CNRS, 1361 Route des Lucioles, 06560 Valbonne France

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Based on the choice of a special gauge, in which a useful form of scaling invariance holds, a new method is suggested for the analytic, nonperturbative calculation of the n-point functions of QED. A modified functional analysis is employed in configuration space, where the dressed electron and photon propagators (in quenched approximation) are each found to be simple products of the relevant free propagator with an appropriate function of configuration space variables containing all powers of the square of the coupling constant.

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I. INTRODUCTION AND THE ELECTRON PROPAGATOR

This paper is a somewhat amplified version of a recent workshop presentation [1], describing a new method of attack on gauge field theories, which should have extension to QCD. The essence of the method is here illustrated by an estimation of the 2-point functions of QED, and consists of the choice of a special, relativistic gauge, which provides an invariance under a rescaling of relevant functional integrals; and this, in turn, leads to a surprising, computational simplification, in configuration space, for the sum of all perturbative contributions. The electron propagator is defined in quenched approximation as the sum over all possible emitted and reabsorbed virtual photons, while the photon propagator is restricted to all possible virtual photons exchanged inside a closed-electron loop, including self-energy corrections on the electron lines which comprise the closed loops. Gauge invariance, or, more properly, strict current conservation, is an essential, second part of the method when applied to the calculation of gauge invariant objects. It should be noted that our treatment of the photon propagator is approximate in the sense that we are only interested in obtaining, for the sum of all such perturbative corrections, the general form of that sum as a function of the square of the virtual photon 4-momentum, $f(k^2)$; the present, simplified analysis can serve to specify the exact value of $f(k^2)$ at a particular value of k^2, while displaying the generic form of the answer for other k^2 values.

This analysis is in four dimensions, using the Minkowski metric, and begins with a modified Schwinger representation [2] for the dressed electron propagator in quenched approximation:

$$S'_c(x-y) \equiv e^{\mathcal{D}_A} G_c(x,y|A) \frac{e^{L[A]}}{\langle S \rangle} \to e^{\mathcal{D}_A} G_c(x,y|A)|_{A\to 0} \tag{1.1}$$

where

$$\mathcal{D}_A = -\frac{i}{2}\int \frac{\delta}{\delta A_\mu} D_c^{\mu\nu} \frac{\delta}{\delta A_\nu}, \qquad [m_0 + \gamma\cdot(\partial_x - igA(x))]G_c(x,y|A) = \delta^{(4)}(x-y),$$
$$\tilde{D}_c^{\mu\nu}(k) = \frac{1}{k^2 - i\varepsilon}\left[\delta_{\mu\nu} - \rho\frac{k_\mu k_\nu}{k^2 - i\varepsilon}\right], \qquad L[A] = \mathrm{Tr}\ln[1 - ig\gamma\cdot A S_c],$$
$$S_c = G_c[A]|_{gA\to 0}, \qquad \langle S\rangle \equiv e^{\mathcal{D}_A} e^{L[A]}]|_{A\to 0}.$$

One then introduces the exact Fradkin representation [3] for the Green's function $G(x, y|A)$ corresponding to the propagator of a relativistic electron in a 4-vector potential field $A(x)$:

$$G_c(x,y|A) = i\int_0^\infty ds e^{-ism_0^2} e^{i\int_0^s ds'(\delta^2/(\delta v_\mu^2(s')))}\left(m_0 - \gamma\cdot\frac{\delta}{\delta v(s)}\right)\delta^{(4)}\left(x - y + \int_0^s ds' v(s')\right)$$
$$\times\, e^{-ig\int_0^s ds' v_\mu(s') A_\mu(y - \int_0^{s'} v)} (e^{g\int_0^s ds' \sigma_{\mu\nu} F_{\mu\nu}(y - \int_0^{s'} v)})_+|_{v_\mu\to 0}. \tag{1.2}$$

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The existence of the ordered exponential (OE) in (1.2) is the main reason why it has never been possible to formulate in configuration space an analytic, nonperturbative mechanism for solving QED, for analytic approximations to general OEs have only been given in adiabatic and stochastic limits [4], which are not applicable to the requirements of (1.2). It will be useful to apply a functional translation operator to the OE of (1.2) in order to extract its A dependence, and so be able to perform the linkage operation of (1.1). With the convenient variable change to $u(s') = \int_0^{s'} ds'' v(s'')$, and with $z = x - y$, $u(s) + z = 0$, $u(0) = 0$, and with $S'_c(z) = $ "$(m - \gamma \cdot \partial)$"$\mathcal{S}'_c(z)$, one obtains:

$$\mathcal{S}'_c(z) = i \int_0^{\infty} ds e^{-ism_0^2} e^{-(1/2)\,\mathrm{Tr}\ln(2ah)} N' \int d[u] e^{(i/2)\iint u(2ah)^{-1}u} \delta^{(4)}(z + u(s)) e^{i((g^2)/2)\iint_0^s ds_1 ds_2 u'_\mu(s_1) D_c^{\mu\nu}(\zeta) u'_\nu(s_2)} \times e^{-2ig^2 \iint_0^s ds_1 ds_2 (\delta/(\delta\chi_{\mu\nu}(s_1)))\vec{\partial}_\mu D_c^{\nu\nu'}(\zeta)\overleftarrow{\partial}_{\mu'}(\delta/(\delta\chi_{\mu'\nu'}(s_2)))} e^{-g^2 \iint_0^s ds_1 ds_2 u'_\mu(s_1) D_c^{\mu\lambda}(\zeta)\overleftarrow{\partial}_\nu (\delta/(\delta\chi_{\nu\lambda}(s_2)))} (e^{\int_0^s ds' \sigma_{\mu\nu}\chi_{\mu\nu}(s')})_+ |_{\chi_{ab}\to 0}, \tag{1.3}$$

where $\zeta_\mu = u_\mu(s_1) - u_\mu(s_2) \equiv \Delta u_\mu(s_1, s_2)$, $h(s_1, s_2) = \frac{1}{2}(s_1 + s_2 - |s_1 - s_2|)$, N' is a normalization constant depending on the Δs partitions of the functional integral, and a is a real, positive number to be set equal to 1 at the end of the calculation. The notation "$(m - \gamma \cdot \partial)$" means that a mass-renormalization δm has been defined (Appendix A), but suppressed for this presentation, since the object of true interest is the remaining $\mathcal{S}'_c(z)$.

One may now make a first, and somewhat remarkable observation: because of the nature of the Dirac $\sigma_{\mu\nu} = \frac{1}{4} \times [\gamma_\mu, \gamma_\nu]$, and the asymmetry of the $\chi_{\mu\nu}$ one can prove (Appendix B) that, to all g^2 order:

$$e^{-2ig^2 \iint_0^s ds_1 ds_2 (\delta/(\delta\chi))(\vec{\partial} D_c \overleftarrow{\partial})(\delta/(\delta\chi))} (e^{\int_0^s \sigma\cdot\chi})_+ |_{\chi\to 0} = 1. \tag{1.4}$$

Equation (1.4) was a surprise, first seen in simple demonstrations that its terms of order g^2, g^4, g^6 all vanished by algebraic cancellation. However, the $u' \ldots$(OE) linkages do not appear to vanish in a similar way, but generate log divergent terms in every g^2 order; and the question of how to handle these in a nonperturbative way remains.

A second and most useful observation may now be made: as written following (1.1), a general relativistic gauge for the bare photon propagator may be defined by the choice of a parameter, ρ, with $\rho = 0$, 1 and -2 defining the Feynman, Landau and Yennie gauges, respectively:

$$\tilde{D}_c^{\mu\nu}(k) = \frac{1}{k^2 - i\varepsilon}\left(\delta_{\mu\nu} - \rho \frac{k_\mu k_\nu}{k^2 - i\varepsilon}\right).$$

In configuration space, this becomes

$$D_c^{\mu\nu}(z) = \frac{i}{4\pi^2}\left(\frac{\delta_{\mu\nu}(1 - \rho/2)}{z^2 + i\varepsilon} + \rho \frac{z_\mu z_\nu}{(z^2 + i\varepsilon)^2}\right).$$

A most useful, special gauge is here defined by the choice: $\rho = 2$. Why is this gauge useful? Because the $u' \ldots u'$ term of (1.3) may then be rewritten in the form:

$$\exp\left[-\frac{g^2}{4\pi^2}\iint_0^s ds_1 ds_2 \frac{u'_\mu(s_1)\Delta u_\mu}{(\Delta u)^2 + i\varepsilon}\frac{u'_\nu(s_2)\Delta u_\nu}{(\Delta u)^2 + i\varepsilon}\right] = \exp\left[\gamma \iint_0^s ds_1 ds_2 \frac{\partial}{\partial s_1}\ln Z \cdot \frac{\partial}{\partial s_2}\ln Z\right] \tag{1.5}$$

with $Z = M^2[(\Delta u)^2 + i\varepsilon]$, $\gamma = g^2/4\pi^2$, and M an arbitrary (for the moment) mass parameter introduced for dimensional reasons. This is almost a perfect (double) differential, but not quite; we can, however, rewrite it as

$$\exp\left[\gamma \iint_0^s ds_1 ds_2 \left(\frac{1}{2}\frac{\partial}{\partial s_1}\frac{\partial}{\partial s_2}\ln^2 Z - \ln Z \frac{\partial}{\partial s_1}\frac{\partial}{\partial s_2}\ln Z\right)\right] \tag{1.6}$$

in which the first term of (1.6) is a perfect differential, and as such, its u-fluctuations of the functional integral all cancel away, and its value is given by the endpoint $u(s) = -z$, $u(0) = 0$ quantities as

$$\exp\left[\frac{\gamma}{2}\iint_0^s ds_1 ds_2 \frac{\partial^2}{\partial s_1 \partial s_2}\ln^2 Z\right] = \exp\left[-\gamma \ln^2\left(\frac{z^2 + i\varepsilon}{i\varepsilon}\right) - 2\gamma \ln\left(\frac{z^2 + i\varepsilon}{i\varepsilon}\right)\ln(i\varepsilon M^2)\right]. \tag{1.7}$$

Evaluation of the second term of (1.6) was first attempted by approximation: since the ε of $\ln Z$ acts as a cutoff parameter in configuration space (in momentum space, $\varepsilon \simeq \Lambda^{-2}$), one expects that $\ln Z$ should be "slowly varying," and the second term of (1.6) could reasonably be approximated by

$$\exp\left[-\gamma \langle \ln Z \rangle \iint_0^s ds_1 ds_2 \frac{\partial^2}{\partial s_1 \partial s_2}\ln Z\right], \tag{1.8}$$

where the $\langle$average$\rangle$ is taken over $s_{1,2}$ and over the u-fluctuations. The integrals of (1.8) are now perfect differentials, which can be evaluated immediately:

$$\iint_0^s ds_1 ds_2 \frac{\partial^2}{\partial s_1 \partial s_2}\ln Z = -2\ln\left(\frac{z^2 + i\varepsilon}{i\varepsilon}\right) \tag{1.9}$$

and the remaining functional integral is trivial, yielding the free particle result: $I_0 = \frac{1}{(4\pi as)^2} e^{iz^2/4as}$. This procedure was originally thought to be the beginning of a strong-coupling approximation; but one can do much better, as follows.

Add and substract to the second term of (1.6) the quantity

$$- \gamma \ln Q \iint_0^s ds_1 ds_2 \frac{\partial^2}{\partial s_1 \partial s_2} \ln Z, \tag{1.10}$$

where Q is a real, positive number >1; in the remaining functional integral

$$R(z^2, a) = e^{-(1/2)\,\mathrm{Tr}\ln(2ah)} N' \int d[u] e^{(i/2)\iint u(2ah)^{-1}u} \delta^{(4)}(z + u(s)) \exp\Big[-\gamma \iint_0^s ds_1 ds_2 \ln Z \frac{\partial^2}{\partial s_1 \partial s_2} \ln Z\Big]$$
$$\times e^{-g^2 \iint_0^s ds_1 ds_2 u'_\mu(s_1) D_c^{\mu\lambda}(\Delta u)) \overleftarrow{\partial}_\nu (\delta/(\delta\chi_{\nu\lambda}(s_2)))} (e^{\int_0^s ds' \sigma_{\mu\nu}\chi_{\mu\nu}(s')})_+ |_{\chi_{ab}\to 0} \tag{1.11}$$

This replaces the term $\exp[-\gamma \iint ds_1 ds_2 \ln Z \frac{\partial^2}{\partial s_1 \partial s_2} \ln Z]$ by

$$\exp\Big[-\gamma \ln Q \iint ds_1 ds_2 \frac{\partial^2}{\partial s_1 \partial s_2} \ln Z - \gamma \iint ds_1 ds_2 \ln(Z/Q) \frac{\partial^2}{\partial s_1 \partial s_2} \ln(Z/Q)\Big]. \tag{1.12}$$

But $\ln(Z/Q)$ can be written as $\ln(M^2 \frac{(\Delta u)^2 + i\varepsilon}{Q}) = \ln(M^2[\frac{(\Delta u)^2}{Q} + i\varepsilon])$, and rescaling the dummy variables $u_\mu(s_i) = \sqrt{Q}\bar{u}_\mu(s_i)$ consistently in (1.11) produces the scaling statement:

$$R(z^2, a) = \frac{1}{Q^2} e^{2\gamma \ln Q \cdot \ln((z^2 + i\varepsilon)/i\varepsilon)} I\Big(\frac{z^2}{Q}, \frac{a}{Q}; \frac{1}{Q}\Big), \tag{1.13}$$

where R is, by construction, explicitly independent of Q, and

$$I\Big(\frac{z^2}{Q}, \frac{a}{Q}; \frac{1}{Q}\Big) = e^{-(1/2)\,\mathrm{Tr}\ln(2(a/Q)h)} N' \int d[\bar{u}] e^{(i/2)\iint \bar{u}(2(a/Q)h)^{-1}\bar{u}} \delta^{(4)}\Big(\bar{u}(s) + \frac{z}{\sqrt{Q}}\Big)$$
$$\times e^{-\gamma \iint \ln Z(\Delta\bar{u})((\partial^2)/(\partial s_1 \partial s_2)) \ln Z(\Delta\bar{u})} e^{-((g^2)/Q) \iint \bar{u}'_\mu D_c^{\mu\lambda}(\Delta\bar{u})) \overleftarrow{\partial}_\nu (\delta/(\delta\chi_{\nu\lambda}))} (e^{\int \sigma\cdot\chi})_+ |_{\chi\to 0}. \tag{1.14}$$

Because of the $1/Q$ dependence of the $u' \ldots$OE term, this is not a useful scaling relation. But if Q is taken as arbitrarily large—on the order of a dimensionless cutoff associated with the logarithmically divergent $u' \ldots$OE perturbative terms—then all of those terms are removed from the integrand, and their sum appears in the exponential factor $\exp[2\gamma \ln Q \cdot \ln(\frac{z^2 + i\varepsilon}{i\varepsilon})]$.

Perturbatively, every g^2 order of the original $u' \ldots$OE cross-terms, when calculated in momentum space, has a log divergence; and the Fourier transform of our final result, (1.20), displays exactly this property. Therefore, although we cannot (yet) prove rigorously our claim that in the large Q limit the effect of the $u' \ldots$OE term in the integrand of $I(z^2/Q, a/Q; 1/Q)$ is given by the multiplicative form $Q^{-2}\exp(2\gamma \ln Q)$ of (1.13), we can point to the effective realization of this limit. Functionally, what has occurred is that the "free-field" portion of the complete functional integral yields the function $I_0((z^2/Q)/(a/Q)) = I_0(z^2/a)$, independent of Q, while the "interaction" part of the functional integrand mixes the z^2, a, Q, ε, and M^2 dependence to produce (1.20). Each term of the perturbation expansion must be sequentially renormalized, whereas the complete renormalization of our result is performed directly in the steps leading to (1.20) and to (1.21) and (C3).

This represents a magnificent calculational tool, for now the OE dependence, which has always blocked nonperturbative estimations (except in Bloch-Nordsieck approximations, where such terms are neglected), has been effectively summed. And now, with $R(z^2, a)$ independent of Q, one has a useful scaling relation

$$R(z^2, a) = \frac{1}{Q^2} e^{2\gamma \ln Q \cdot \ln((z^2 + i\varepsilon)/(i\varepsilon))} I\Big(\frac{z^2}{Q}, \frac{a}{Q}\Big), \tag{1.15}$$

where $I(\frac{z^2}{Q}, \frac{a}{Q})$ denotes the functional integral (1.14) without the $u' \ldots$OE term. It should be noted that the OE terms are themselves gauge invariant, since they stem from an initial $F_{\mu\nu}$ dependence; but this special gauge provides the framework for their summation. This procedure of rescaling and taking the limit of arbitrarily large Q is seemingly reasonable.

One can question whether this procedure is really valid. For example, since the scaling replacement $u_\mu \to \sqrt{Q}\bar{u}_\mu$ leaves $\bar{u}$ with the dimensions of length, and with the condition that $\bar{u}(s) + z/\sqrt{Q} = 0$, and since z_μ is the only 4-vector present, could it not be possible that the $\bar{u}_\mu(s_i)$, $0 < s_i < s$ could eventually be expressed as proportional to $z/\sqrt{Q}$, so that the final $u' \ldots$OE term ends up independent of Q? The answer would seem to be: probably not, because

(i) the "proper-time" variables s_i essential to the computation, carry the dimension of $(\text{length})^2$; and (ii) although the $u(s')$ variables in the integrand of the functional integral may be considered as continuous, the FI process of integration destroys that continuity, so that there is no obvious reason why the proportionality of $u(s)$ to z may be transferred to the other $u(s_i)$, $s_i < s$. Such questions must be answered definitively, before this procedure can be considered as valid; and it is here presented as a conjecture, but one which, if true, can lead to spectacular results. In the following, we shall assume that this conjecture is true.

Renormalization group methods can now be employed to provide a differential equation for $I(\frac{z^2}{Q}, \frac{a}{Q})$, by considering small variations of the very large Q, and calculating $0 = Q\frac{\partial}{\partial Q}R(z^2, a)$:

$$0 = \left[2\gamma \ln\left(\frac{z^2 + i\varepsilon}{i\varepsilon}\right) - 2\right] I\left(\frac{z^2}{Q}, \frac{a}{Q}\right) - \left(z^2 \frac{\partial}{\partial z^2} + a\frac{\partial}{\partial a}\right) I\left(\frac{z^2}{Q}, \frac{a}{Q}\right). \tag{1.16}$$

It is simplest to rescale $z^2 \to z^2 Q$, $a \to aQ$, to obtain the differential equation:

$$\left[2\gamma \ln Q + 2\gamma \ln\left(\frac{z^2 + i\varepsilon}{i\varepsilon}\right) - 2\right] I(z^2, a) = \left(z^2 \frac{\partial}{\partial z^2} + a\frac{\partial}{\partial a}\right) I(z^2, a). \tag{1.17}$$

We look for a solution of form $I(z^2, a) = I_0(z^2, a)J(z^2, a)$, where $I_0(z^2, a) = \mathcal{S}'_c|_{g\to 0}$, because out of the "landscape" of possible solutions to the functional Schwinger equations, this is the one desired. Further, one can demand that $J(z^2, a)|_{a\to 1, z^2\to 0} = 1$ so that the equal time, anti commutation relations originally assumed for free and dressed fermion field operators remain the same. With $J = \exp\Omega$, (1.17) becomes

$$\left(z^2 \frac{\partial}{\partial z^2} + a\frac{\partial}{\partial a}\right)\Omega(z^2, a) = 2\gamma\left[\ln Q + \ln\left(\frac{z^2 + i\varepsilon}{i\varepsilon}\right)\right], \tag{1.18}$$

and this is the relation one must now solve.

Inspection shows that to any solution of (1.18) satisfying the above "boundary conditions" can be added an almost arbitrary function of z^2/a, which sum will produce another solution to (1.18). This lack of mathematical uniqueness, however, is not detrimental to our physical description, for there is yet one more condition which must be applied and interpreted; and the final result will have a satisfactory "physical uniqueness". At this point one can realize that, whatever the solution chosen, the essence of this special gauge + rescaling method is that the nonperturbative solutions to this dressed propagator are essentially multiplicative in coordinate space. In contrast, Feynman graphs in higher orders of perturbation theory involve horrendous and overlapping integrals in momentum space. This special gauge is the bridge that leads to analytically obtainable, exact solutions in configuration space. Of course, Fourier transforms must finally be taken; but that is separate matter.

At the end of this calculation, the parameter $a \to 1$, and any a-dependence in the solution becomes a constant (a general constant will be chosen below). Since there is no a-dependence on the RHS of (1.18), one may take as the simplest solution that obtained by assuming $\Omega = \Omega(y)$, $y = \ln[M^2(z^2 + i\varepsilon)]$, and (1.18) then becomes $\frac{d\Omega}{dy} = 2\gamma[y + \ln(\frac{Q}{i\varepsilon M^2})]$, with solution $\Omega(y) = \gamma y^2 + 2\gamma y \ln(Q/i\varepsilon M^2) + \kappa$, with κ constant, so that

$$J(z^2, a) = \exp[\gamma \ln^2(M^2(z^2 + i\varepsilon)) + 2\gamma \ln(M^2(z^2 + i\varepsilon)) \times \ln(Q/i\varepsilon M^2) + \kappa] \tag{1.19}$$

Remembering to rescale: $z^2 \to z^2/Q$ (in order to undo the passage from (1.16) to (1.17), and including the previous, perfect differential term of (1.7),

$$\frac{\gamma}{2}\iint_0^s ds_1 ds_2 \frac{\partial^2}{\partial s_1 \partial s_2} \ln^2 Z = -\gamma \ln^2\left(\frac{z^2 + i\varepsilon}{i\varepsilon}\right) - 2\gamma \ln\left(\frac{z^2 + i\varepsilon}{i\varepsilon}\right)\ln(i\varepsilon M^2)$$

the entire answer becomes

$$\mathcal{S}'_c(z) = I_0(z^2, 1)\exp\left[2\gamma \ln\left(\frac{z^2 + i\varepsilon}{i\varepsilon}\right)\ln\left(\frac{Q}{i\varepsilon M^2}\right) + 2\gamma \ln Q \ln(i\varepsilon M^2) - \gamma \ln^2(i\varepsilon M^2) + \kappa\right].$$

If the exponential factor multiplying I_0 is to become unity when $z^2 \to 0$, then the constant of this solution must be chosen as $\kappa = \gamma \ln^2(i\varepsilon M^2) - 2\gamma \ln Q \ln(i\varepsilon M^2) - is\delta m^2$, where the term $-is\delta m^2$ has been included in this constant (obtained from our previous, suppressed knowledge of δm) so as to renormalize the bare mass sitting in the exponential factor of (1.2). The entire result, with m_0 everywhere replaced by m, is then

$$\mathcal{S}'_c(z) = I_0(z^2, 1)\exp\left[2\gamma \ln\left(\frac{z^2 + i\varepsilon}{i\varepsilon}\right)\ln\left(\frac{Q}{i\varepsilon M^2}\right)\right], \tag{1.20}$$

where the result of all interactions is, in configuration space, simply a multiplicative, log divergent, exponential factor multiplying the free-field propagator.

Without actually performing the Fourier transform into momentum space (Appendix C), let us try to guess the electron's wave function renormalization (WFR) constant. Going to the mass shell in momentum space corresponds to taking the limit $z^2 \to \infty$ in coordinate space. This can be represented as an infrared limit in momentum space, if z^2 is represented by $1/\mu^2$; also, as noted above, $\varepsilon \sim \Lambda^{-2}$ can be thought of as an UV cutoff in momentum space. The argument of the multiplicative exponential factor of (1.20) then becomes

$$2\gamma\left[-\left(\frac{\pi}{2}\right)^2 + \ln\left(\frac{\Lambda^2}{\mu^2}\right)\ln\left(\frac{Q\Lambda^2}{M^2}\right)\right] - i\pi\gamma\ln\left(\frac{\Lambda^4 Q}{\mu^2 M^2}\right),$$

and in order for the Z_2 of this solution to be real, one must choose $(\frac{\Lambda^4 Q}{\mu^2 M^2}) = 1$, or $M^2 = Q\frac{\Lambda^4}{\mu^2}$, so that

$$Z_2 = \exp\left[-2\gamma\left(\left(\frac{\pi}{2}\right)^2 + \ln^2\left(\frac{\Lambda^2}{\mu^2}\right)\right)\right] \tag{1.21}$$

is not only real, but is bounded between 0 and 1, as expected from the formal theory.

Finally, one can argue that the lack of mathematical uniqueness of the solution (1.19) is really not physically relevant, because the only task of the dressed electron propagator is to produce a WFR constant which can be identified as needed in every n-point function, so as to cancel with an equal Z_1, and help provide a gauge independent renormalization of the electron's charge; Z_2 is not a measurable quantity. Any other solution chosen in place of (1.19), which the above $g^2 \to 0$ and $z^2 \to 0$ requirements, will just correspond to a change in the unmeasurable Z_2. The really relevant points of this analysis are that (i) it provides a straightforward method of choosing a non perturbative Z_2 which can be used in conjunction with a similar analysis of higher n-point functions to produce gauge invariant, nonperturbative results; and (ii) this analysis of the two-point function suggests that one will, in other n-point functions, find a simple description of nonperturbative Physics in configuration space, with the use of the special gauge + rescaling.

II. THE PHOTON PROPAGATOR

This second expectation is almost realized by the related computation of the dressed photon propagator, where the sum over all virtual photons exchanged across a closed electron loop is provided by the gauge invariant, functional expression:

$$K_{\mu\nu}(z) = -ig^2 e^{\mathcal{D}_A}\,\mathrm{tr}[\gamma_\mu G_c(x, y|A)\gamma_\nu G_c(y, x|A)]|_{A\to 0}, \tag{2.1}$$

where again, $z = x - y$. In lowest order g^2 approximation, an additional limiting process must be employed [5] to maintain the essential requirement of current conservation, $\partial_\mu K_{\mu\nu} = 0$; and in a somewhat less dramatic way, this is also true in the calculation of the fourth order term [6]; subsequent orders of the conventional perturbative calculations should automatically satisfy this requirement. The present treatment will suggest an alternate method, to all g^2 orders, based on the functional representation of the previous Section, in which strict current conservation is a necessary part of the procedure. The extension of these arguments to include the sum of all such connected bubbles, and more generally to the entire sum of all functional, cluster coefficients [6], is a nontrivial exercise best left for future consideration.

The present procedure requires the following steps.

(1) Insert a functional representation for each propagator, in its modified Fradkin form, which we indicate by

$$i\int_0^\infty ds e^{-ism_0^2}\int d[u]\delta^{(4)}(z + u(s))\dots i\int_0^\infty dt e^{-itm_0^2}\int d[w]\delta^{(4)}(z + w(t)).$$

(2) Perform the linkage operations on each $G_c[A]$ and the connections between them, using the special gauge. Note that all $g^2\int\frac{\delta}{\delta\chi}(\vec{\partial}D_c\overleftarrow{\partial})\frac{\delta}{\delta\chi}$ contributions exactly vanish, as in the previous section.

(3) Perform mass renormalization within each functional integral, so that $m_0 \to m$, everywhere. Rewrite the functional integrands as in Sec. I, by extracting those parts which are total, double derivatives, dependent upon z and not upon the fluctuating u, w variables.

(4) Add and substract exponential factors of those exponential terms which would become total, double derivatives, were the relevant Z factors replaced by averaged constants

$$\exp\left[-\gamma\ln Q\left(\iint ds_1 ds_2\frac{\partial^2}{\partial s_1\partial s_2}\ln Z(\Delta(u)) + \iint dt_1 dt_2\frac{\partial^2}{\partial t_1\partial t_2}\ln Z(\Delta(w)) + 2\iint ds_1 dt_1\frac{\partial^2}{\partial s_1\partial t_1}\ln Z(u(s_1) - w(t_1))\right)\right]$$

so that the complete representation is independent of Q.

(5) Move the $\ln Q$ total derivative terms—which are independent of the $u(s_i)$ and $w(t_i)$—outside of the functional integrals; and absorb the remaining $\ln Q$ terms by rescaling the variables $u \to \sqrt{Q}\bar{u}$, $w \to \sqrt{Q}\bar{w}$, with $Q \gg 1$.

(6) Observe that ALL total derivative terms cancel, and the remainder, in the limit of arbitrarily large Q, can be succinctly written, suppressing the factors of $i\int_0^\infty ds e^{-ism^2} i\int_0^\infty dt e^{-itm^2}$ as:

$$i\left(\frac{g^2}{Q^4}\right)e^{-(1/2)\,\mathrm{Tr}\ln(2(a/Q)h)}N'\int d[\bar{u}]e^{(i/2)\iint \bar{u}(2(a/Q)h)^{-1}\bar{u}}e^{-(1/2)\,\mathrm{Tr}\ln(2(a/Q)h)}N'\int d[\bar{w}]e^{(i/2)\iint \bar{w}(2(a/Q)h)^{-1}\bar{w}}\delta^{(4)}\left(\frac{z}{Q}+\bar{u}(s)\right)$$
$$\times\,\delta^{(4)}\left(\frac{z}{Q}+\bar{w}(t)\right)[4m^2\delta_{\mu\nu}+(u'(s),w'(t))_{\mu\nu}+g^2[z]_{\mu\nu}]$$
$$\times\exp\left[-\frac{\alpha}{2\pi}\iint ds_1ds_2\ln\bar{Z}\frac{\partial}{\partial s_1}\frac{\partial}{\partial s_2}\ln\bar{Z}-\frac{\alpha}{2\pi}\iint dt_1dt_2\ln\bar{Z}\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}\ln\bar{Z}-\frac{\alpha}{\pi}\iint ds_1dt_1\ln\bar{Z}\frac{\partial}{\partial s_1}\frac{\partial}{\partial t_1}\ln\bar{Z}\right], \quad (2.2)$$

where $[z]_{\mu\nu}=\frac{2}{\pi^2}[\frac{\delta_{\mu\nu}}{z^2+i\varepsilon}-2\frac{z_\mu z_\nu}{(z^2+i\varepsilon)^2}]$, and the $\bar{Z}$ denotes the appropriate function of the rescaled $\bar{u}$, $\bar{w}$. The bracketed terms of (2.2) arise from the action of

$$\mathrm{tr}\left[\gamma_\mu\left(m-\gamma\cdot\frac{\delta}{\delta u'(s)}\right)\gamma_\nu\left(m-\gamma\cdot\frac{\delta}{\delta w'(t)}\right)\right] \quad (2.3)$$

upon the u' and w' dependence of the original functional integrals, before rescaling. Upon rescaling and permitting Q to become arbitrarily large, a set of terms produced by the trace operator of (2.3) is removed, and those which remain appear in (2.2) (in an unscaled fashion).

The combination $(u'(s),w'(t))_{\mu\nu}$ represents a symmetric (in μ, ν) pair of basically divergent terms that violate gauge symmetry (and current conservation), and have previously been defined [4] in such fashion that $K^{(2)}_{\mu\nu}$ and $K^{(4)}_{\mu\nu}$ have a vanishing 4-divergence. Because these objects are not directly calculable, and must be redefined, rescaling has not been applied to them. An alternate way of writing (2.2) is

$$K_{\mu\nu}(z|s,t)=ig^2[4m^2\delta_{\mu\nu}+\langle(u'(s),w'(t))_{\mu\nu}\rangle+g^2[z]_{\mu\nu}]\frac{1}{Q^4}C\left(\frac{z^2}{Q},\frac{a}{Q}\right), \quad (2.4)$$

where $C(z^2/Q,a/Q)/Q^4$ denotes the complete, rescaled functional integrals of (2.2), and the notation $\langle(u'(s),w'(t))_{\mu\nu}\rangle$ defines an "averaged" redefinition of those basically divergent quantities. One should note an implicit, simplifying assumption made at this point, that such an averaged redefinition is essentially independent of g^2, for all orders higher than g^2. This assumption allows us to require gauge invariance in a simple way but only at the price of generating a qualitatively correct form for the sum of all such radiative corrections.

Because $K_{\mu\nu}(z|s,t)$ is independent of Q, one can apply the RG analysis with the statement that $0=(\partial/\partial Q)K_{\mu\nu}$ produces the requirement

$$\left[4+z^2\frac{\partial}{\partial z^2}+a\frac{\partial}{\partial a}\right]C\left(\frac{z^2}{Q},\frac{a}{Q}\right)=0. \quad (2.5)$$

To understand how (2.5) can be satisfied, consider the $g=0$ limit of C, called C_0, which, after performing the elementary functional integrals, yields

$$C_0\left(\frac{z^2}{Q},\frac{a}{Q}\right)=\frac{Q^4}{a^4}e^{iz^2/4a\bar{s}}, \qquad \frac{1}{\bar{s}}=\frac{1}{s}+\frac{1}{t}. \quad (2.6)$$

In this way, with a normalization factor from each integral of $1/a^2$, and as a function of z^2/a, C/Q^4 is independent of Q. We shall now assume that every g^{2n} order of C, called C_{2n} contains the same normalization factor of $1/a^4$, as well as the combination $\exp iz^2/4a\bar{s}$. With such an assumption, $C(z^2,a)=\sum_{n=0}^{\infty}g^{2n}C_{2n}(z^2,a)$, and

$$C_{2n}(z^2,a)=\frac{e^{-im^2(s+t)}}{(4\pi a)^4s^2t^2}C_{2n}(z^2/a)e^{iz^2/4a\bar{s}}$$
$$\equiv\phi(s,t)e^{iz^2/4a\bar{s}}\frac{1}{a^4}C_{2n}(z^2/a) \quad (2.7)$$

so that every term of this expansion, when rescaled, satisfies (2.5) identically. With this notation, the quantity called C_0 of (2.6) has had its normalizing factors removed, and is now simply: $C_0=1$.

In contrast to (1.17), there is no "inhomogeneous" term, a term dependent only upon z^2, in (2.5), and there is no further input from this RG-like relation; by construction, it is satisfied identically. The parameter Q may be chosen to be arbitrarily large, but the g^2 expansions of C are independent of Q; and if so, it is surely simplest to consider $Q=1$ in subsequent manipulations. And since the parameter a is eventually to be set equal to 1, it can be done at this stage; at the end of the calculation, one may multiply all terms by $1/a^4$, and divide every z^2 factor by a, and the result will be independent of Q.

An alternative way of understanding this simplification is to observe that the functional integrals defining $Q^{-4}C(z^2/Q,a/Q)$, for very large Q, may have their $\bar{u}(s')$, $\bar{w}(t')$ variables rescaled by a real, positive number P such that: $\bar{u}(s')\to\frac{1}{P}\bar{\bar{u}}(s')$, $\bar{w}(t')\to\frac{1}{P}\bar{\bar{w}}(t')$. All that will then change are the normalization factors, where the Q dependence of (2.2) is everywhere replaced by Q/P. But P is arbitrary; and if we choose $P=Q$, all such scaling factors disappear from (2.2). And since the parameter a is eventually to be set equal to 1, that limit may now be adopted; and the result is the same as that of the preceding paragraph.

Our problem then reduces to finding such perturbative solutions—and then the sum of all such perturbative terms—to the functional integral representation for $K_{\mu\nu}(z|s,t)$ of (2.4) where

$$C(z^2|s, t) = \sum_{n=0}^{\infty} g^{2n}\phi(s, t)e^{iz^2/4\bar{s}}C_{2n}(z^2|s, t) \quad (2.8)$$

with $\phi(s, t) = \frac{e^{-im^2(s+t)}}{(4\pi)^4 s^2 t^2}$. Of course, one could have written down such a representation before introducing the ln$Q$ factors and the rescaling procedure; but then it would not have been clear how all the gauge dependent terms originating from the electron propagators exactly cancel away. We still must require the solutions of (2.4) to be strictly gauge invariant, and can turn that requirement into a calculational tool, as follows.

Starting from (2.4), now written as

$$K_{\mu\nu}(z|s, t) = ig^2[4m^2\delta_{\mu\nu} + \langle(u'(s), w'(t))_{\mu\nu}\rangle + g^2[z]_{\mu\nu}]C(z^2|s, t) \quad (2.9)$$

and with the understanding of (2.8), let us now write the corresponding relations for every $K^{(2n)}_{\mu\nu}(z|s, t)$ of $K_{\mu\nu}(z|s, t) = \sum K^{(2n)}_{\mu\nu}(z|s, t)$, beginning with

$$K^{(2)}_{\mu\nu}(z|s, t) = ig^2[4m^2\delta_{\mu\nu} + \langle(u'(s), w'(t))_{\mu\nu}\rangle]\phi(s, t)e^{iz^2/4\bar{s}}C_0. \quad (2.10)$$

As it stands, (2.10) gives no information, because, as noted above, a definition built from the requirement of current conservation must be given for the divergent factors $u'(s)$, $w'(t)$. The proper definition, given in different contexts [4] a half century ago, can be written in the present context as

$$K^{(2)}_{\mu\nu}(z|s, t) = 8ig^2\phi\eta_2(\delta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)e^{iz^2/4\bar{s}} \quad (2.11)$$

with $\eta_2 = \bar{s}/(s+t)$; that is, the Fourier transform of (2.11) exactly reproduces the correct $\tilde{K}^{(2)}_{\mu\nu}(k)$, before renormalization.

Returning to (2.9), the g^4 contribution may be written as

$$K^{(4)}_{\mu\nu}(z|s, t) = ig^2[4m^2\delta_{\mu\nu} + \langle(u'(s), w'(t))_{\mu\nu}\rangle]\phi e^{iz^2/4\bar{s}}C_2 + ig^4[z]_{\mu\nu}\phi e^{iz^2/4\bar{s}} \quad (2.12)$$

or, by a comparison of (2.12) and (2.10), as

$$K^{(4)}_{\mu\nu}(z|s, t) = C_2(z^2|s, t)K^{(2)}_{\mu\nu}(z|s, t) + ig^4[z]_{\mu\nu}\phi e^{iz^2/4\bar{s}}. \quad (2.13)$$

[At this point, one should reemphasize the remarks made in the paragraph after (2.4)]. Since both $K^{(2)}_{\mu\nu}$ and $K^{(4)}_{\mu\nu}$ are to have a zero divergence, the application of $\sum_\mu\partial_\mu$ to both sides of (2.13) generates a first order differential equation (DE) for $C_2(z|s, t)$, and when this is solved, one has the solution for $K^{(4)}_{\mu\nu}$. (As it turns out, it is the choice of the constant of integration for C_2 which is determined by the input of $K^{(4)}_{\mu\nu}$).

Repeating the process for the g^6 expansion of (2.9), one writes

$$K^{(6)}_{\mu\nu}(z|s, t) = C_4(z^2|s, t)K^{(2)}_{\mu\nu}(z|s, t) + ig^4[z]_{\mu\nu}C_2(z^2|s, t)\phi e^{iz^2/4\bar{s}}. \quad (2.14)$$

Again, because $K^{(6)}_{\mu\nu}$ must be divergence-free, application of $\sum_\mu\partial_\mu$ to (2.14) generates a DE for C_4, which can be solved to within a constant of integration; and when C_4 is known, $K^{(6)}_{\mu\nu}$ has been determined. This procedure can be continued indefinitely, and can be applied to the sum of all such $K^{(2n)}_{\mu\nu}$: if $C(z^2|s, t) = \sum_n g^{2n}C_{2n}(z^2|s, t)$, one can write

$$K^{(2n)}_{\mu\nu} = C_{2(n-1)}K^{(2)}_{\mu\nu} + ig^4[z]_{\mu\nu}C_{2(n-2)}\phi e^{iz^2/4\bar{s}}$$

so that

$$(C-1)K^{(2)}_{\mu\nu} + ig^4[z]_{\mu\nu}C\phi e^{iz^2/4\bar{s}} = \sum_{n=2}^{\infty} K^{(2n)}_{\mu\nu}$$

or

$$CK^{(2)}_{\mu\nu} + ig^4[z]_{\mu\nu}C\phi e^{iz^2/4\bar{s}} = \sum_{n=1}^{\infty} K^{(2n)}_{\mu\nu} = K_{\mu\nu}. \quad (2.15)$$

Requiring $\sum_\mu\partial_\mu K_{\mu\nu} = 0$, the DE for C follows: $(T+U)C' + RC = 0$, where R, T and U are functions of z^2, defined by: $z_\nu T = z_\mu K^{(2)}_{\mu\nu}$, $z_\nu U = ig^4 z_\mu[z]_{\mu\nu}\phi e^{iz^2/4\bar{s}}$, $z_\nu R = ig^4\partial_\mu([z]_{\mu\nu}\phi e^{iz^2/4\bar{s}})$, and with solution

$$C = \exp\left[-\int_{z_0^2}^{z^2} du \frac{R(u)}{U(u)+T(u)}\right], \quad (2.16)$$

where z_0^2 is a constant to be determined. Since the lowest g^2 order dependence of R, T, U are $R = \mathcal{O}(g^4)$, $T = \mathcal{O}(g^2)$, $U = \mathcal{O}(g^4)$, the exponential factor of (2.16) begins as $\mathcal{O}(g^2)$; and therefore, z_0^2 is to be determined by a comparison with the $\mathcal{O}(g^2)$ term of C; that is from C_2 or from $K^{(4)}_{\mu\nu}$. Finally, when C is known, then from (2.15), the complete $K_{\mu\nu}$ is known as well; but, as stressed above, since for a particular choice of C_2, $\tilde{K}^{(4)}_{\mu\nu}(k)$ will be exact at only one value of k^2, that statement is also true for the final $\tilde{K}_{\mu\nu}(k)$ obtained from (2.17) below.

The thrust of the present argument is that the complete $K_{\mu\nu}$, to all orders in g^2, is given in configuration space, by z^2 dependence built from the z dependence of the functions which define $K^{(2)}_{\mu\nu}$, $K^{(4)}_{\mu\nu}$, and $[z]_{\mu\nu}$. Renormalization has not yet been performed; and the Fourier transform to momentum space is needed for applications to practical problems; but if this simple, procedural analysis is correct, it suggests a tremendous simplification by performing the analysis in configuration space, where the result is essentially multiplicative, as in (2.15), or the equivalent

$$K_{\mu\nu}(z|s, t) = C(1-C_2)K^{(2)}_{\mu\nu} + CK^{(4)}_{\mu\nu}. \quad (2.17)$$

III. SUMMARY

Work is now beginning on the application of these techniques to the vertex function of QED; but before any detailed study is begun, one can foresee certain relevant properties of its nonperturbative representation. It has always seemed somewhat suggestive that eikonal representations of the QED vertex function [7], as well as the sum of its leading perturbative contributions [8] should produce an exponential of either a log or a $\log^2$ dependence upon momentum transfer (depending upon the way in which large momentum transfers are limited); but this has now become clear from the forms found using the special gauge. What will surely happen in this coming calculation of the QED vertex, to all orders of its coupling constant, is that the gauge dependent exponential factors will cancel away, leaving—after Fourier transforms into momentum space are performed—the exponential of finite, logarithmic dependence upon relevant momentum transfer.

Finally, a word on the extension to QCD. Color complications may be expected, compared to the simpler, Abelian QED, but there are both approximations and additional Gaussian integrals which can be employed to bring a measure of success to the enterprise. One must use functional techniques which avoid integration over gauge copies, as well as introduce techniques to handle the ever present ordered exponentials of the theory; and one expects that such complications can be overcome. The non Abelian gauge groups will require current conservation relations between functional representations of different n-point functions, rather than involving only the same n-point function, as in the QED photon propagator. But as long as the (bare) gluons are massless, a special gauge can be found for their propagator's coordinate space representations; and the main structure of the QED analysis suggested above should be possible. This direct approach to nonperturbative solutions of 4-dimensional QCD may well be worth trying.

APPENDIX A

Mass renormalization in the present framework can be understood by returning to (1.3) with the correct factor of $(m_0 - \gamma \cdot \frac{\delta}{\delta u'(s)})$ inserted under the functional integral. Using the representation $\langle s_1 | h^{-1} | s_2 \rangle = \frac{\vec{\partial}}{\partial s_1} \delta(s_1 - s_2) \frac{\overleftarrow{\partial}}{\partial s_2}$, the exponential factor $\frac{i}{2} \iint_0^s u(2ah)^{-1} u$ can be rewritten as $\frac{i}{4a} \int_0^s ds' u'^2(s')$. By inspection, the operator $\frac{\partial}{\partial u'(s)}$ will then generate a factor of

$$\frac{i}{2} u'_\mu(s) + ig^2 \int_0^s ds' D_c^{\mu\nu}(u(s) - u(s')) \times \left[u'_\nu(s') + i\overleftarrow{\partial}_\lambda \frac{\delta}{\delta \chi_{\lambda\nu}(s)} \right] \tag{A1}$$

under the FI.

Upon rescaling, with Q arbitrarily large, the integral of (A1), will become arbitrarily small, and can be discarded, effectively removing the divergences inherent in those terms, as s' approaches s. But the $u'(s)$ term cannot be scaled away; and quite apart from any scaling consideration, this term requires a definition which can only be given precisely in momentum space, as done in conventional perturbation theory; in configuration space, it is inherently meaningless, and requires a definition external to the FI itself. For example, if one introduces a "momentum" variable q, conjugate to $u(s)$, by inserting a factor $\exp[iqu(s)]$ under the FI, and computes $\langle u'(s) \rangle$, using the normalized expression of (1.3) as probability distribution, with $g = 0$ for simplicity, one finds and indeterminate result, of form q/s, in the limit as both q and s vanish. (A similar situation is presented by the factors $u'(s)w'(t)$ of Sec. II, a doubly meaningless combination whose proper definition is given by the requirement of current conservation, or gauge invariance). The needed definition is straightforward in momentum space, but in the present context it can only be formally defined, as follows.

Since $u'_\mu(s)$ has the dimensions of a four momentum, and in momentum space the only relevant four momentum is that of the physical particle p_μ, where $p^2 + m^2 = 0$, it is reasonable to assume that, in momentum space, $u'(s)$ must be proportional to p, which proportionality constant we write as $1 + \xi$. The parameter ξ, which in momentum space is a log divergent quantity, here represents an effective mass renormalization, since this term enters in the combination $m_0 - i(1 + \xi)\gamma \cdot p$, and when touching a Dirac spinor (constructed with the physical mass m) can be rewritten as the combination $m_0 + \xi m - i\gamma \cdot p$, with $\xi m = \delta m$, and $m_0 + \delta m = m$. With this understanding, the quantity $S'_c(z) = (m - \gamma \cdot \partial)\mathcal{S}'_c(z)$, and the Fourier transform of (1.20), near its mass shell, is given by $Z_2(m - i\gamma \cdot p)(m^2 + p^2)^{-1}$ as required. While this definition is not particularly appealing, it is only meant to suggest in configuration space what can only be properly obtained in momentum space.

APPENDIX B

The twice differentiated propagator of (1.4) appears in a gauge invariant combination, and its value can be calculated most simply in the Feynman gauge, where one finds the exponential of (1.4) to be

$$\iint_0^s ds_1 ds_2 \frac{\delta}{\delta \chi_{\lambda\nu}(s_1)} \Omega_{\lambda\mu}(z) \frac{\delta}{\delta \chi_{\mu\nu}(s_2)}, \quad \Omega_{\lambda\mu}(z) = \left(\frac{g}{\pi}\right)^2 \frac{1}{(z^2 + i\varepsilon)^2} \left[\delta_{\mu\nu} - 4 \frac{z_\mu z_\lambda}{(z^2 + i\varepsilon)} \right] \tag{B1}$$

with $z_\mu = \Delta u_\mu = u_\mu(s_1) - u_\mu(s_2)$. The action of this operator upon the ordered exponential of (1.4) is to exhibit an arbitrary number of $\sigma_{\mu\nu}$ matrices in some well defined s_i-order, yielding for any such operation a quantity of the form

$$\iint_0^s ds_1 ds_2 \Omega_{\lambda\mu}(z)(\ldots \sigma_{\lambda\nu} \ldots \sigma_{\mu\nu} \ldots), \tag{B2}$$

where the triple dots of (B2) denote other $\sigma_{\alpha\beta}$ matrices which have been arranged before, between, and after the $\sigma_{\lambda\nu}$ and $\sigma_{\mu\nu}$ displayed; the precise ordering depending on the s-labels of the $\chi_{ab}(s_i)$ and the $\frac{\delta}{\delta\chi_{\alpha\beta}(s_2)}$, in such a manner that all arrangements correspond to the (proper) time ordered expansion of (B2). There will be an appropriate Ω_{ab} term linking each such pair of σ matrices comprising the triple dots of (B2); and we here focus attention on one such pair, as written in (B2), and will prove that this combination vanishes for any number, including zero, of matrices which lie between $\sigma_{\lambda\nu}$ and $\sigma_{\mu\nu}$.

Summing over λ, μ, ν of (B2), with $\not{z} = \gamma \cdot z$, leads to

$$\left(\frac{g}{2\pi}\right)^2 \frac{1}{(z^2 + i\varepsilon)^2} \sum_{\lambda,\nu} \left[-\gamma_\lambda \gamma_\nu \cdots \gamma_\nu \gamma_\lambda + 4\not{z}\gamma_\nu \cdots \gamma_\nu \not{z} \frac{1}{(z^2 + i\varepsilon)} \right]. \tag{B3}$$

If the triple dots refer to no matrices at all, (B3) obviously vanishes, since $\not{z}\not{z} = z^2$ and $\sum_\nu \gamma_\nu^2 = 4$. If the triple dots denote a single $\sigma_{\alpha\beta}$, then we have

$$\sum_\nu \gamma_\nu \sigma_{\alpha\beta} \gamma_\nu = \delta_{\alpha\beta}. \tag{B4}$$

Inserting (B4) into the remaining operations of (B3) gives zero. Apparently, if the $\sum_\nu$ operation of (B3) generates a number, rather than a matrix, insertion of that number into the remaining $\sum_\lambda$ of (B3) will produce an algebraic zero. That the $\sum_\nu$ operation will always lead to a pure number when the triple dots between $\sigma_{\lambda\nu}$ and $\sigma_{\mu\nu}$ contain any products of σ, or any even number of matrices, can be seen by writing the general expression for any 4×4 Dirac matrix in the form

$$M = M_0 + M_\alpha \gamma_\alpha + M_{\alpha\beta}\sigma_{\alpha\beta} + M_5 \gamma_5 + M_{5\alpha}\gamma_5\gamma_\alpha,$$

where sums over all repeated indices are understood, and where the γ_5 terms are irrelevant to these considerations and may be discarded. The numerical coefficients M_0, M_α, $M_{\alpha\beta}$ are obtained from the traces

$$M_0 = \tfrac{1}{4}\mathrm{Tr}M, \qquad M_\alpha = \tfrac{1}{4}\mathrm{Tr}[\gamma_\alpha M], \qquad M_{\alpha\beta} = \tfrac{1}{4}\mathrm{Tr}[\sigma_{\alpha\beta} M].$$

Because the trace of an odd number of γ matrices must vanish, $M_\alpha = 0$; and we have seen above that M_0 and $M_{\alpha\beta}\sigma_{\alpha\beta}$ inserted into the $\sum$ of (B3) do vanish; and therefore all such quantities of the form of (B2) are zero.

Were this peculiar algebraic property not true, examination of the perturbative divergences which would result from such terms suggests that singularities more divergent than logarithmic would appear in higher orders of QED, although the large Q cancellation conjecture suggests that each of these terms would vanish under rescaling, proportional to Q^{-2}. Another peculiarity of terms of the form of (B2) is that they vanish upon integration over a four dimensional euclidian volume [8].

APPENDIX C

Rewriting $\mathcal{S}'_c(z)$ in terms of the assumed Z_2 of (1.21)

$$\mathcal{S}'_c(z) = \frac{Z_2}{(4\pi)^2} \int_0^\infty \frac{ds}{s^2} e^{-ism^2 + iz^2/4s} e^{-\Gamma \ln(\mu^2 z^2)}, \tag{C1}$$

where $\Gamma = \frac{\alpha}{2\pi}[\ln(\frac{\Lambda^2}{\mu^2}) + i\frac{\pi}{2}]$, one then calculates the Fourier transform

$$\tilde{\mathcal{S}}'_c(p) = \int d^4z e^{-ip\cdot z} \mathcal{S}'_c(z^2). \tag{C2}$$

The computation is most simply performed by continuing the real z_0 integration $\int_{-\infty}^{+\infty} dz_0$ to run along the imaginary z_0 axis $\int_{+i\infty}^{-i\infty} dz_0$, so that $\int d^4z \to -i\int_E d^4z$. With this change from Minkowski to Euclidean space, one may use the convenient 4D angular integral

$$\int d\Omega_4 e^{-ip\cdot z} = \frac{4\pi^2}{pz} J_1(pz)$$

together with the well-known, "proper-time" integral

$$\int_0^\infty \frac{ds}{s^2} e^{-ism^2 + iz^2/4s} = 4i\left(\frac{m}{z}\right) K_1(mz)$$

to reduce the four-fold integral of (C2) to a single integral

$$\mathcal{S}'_c(z) = \left(\frac{m}{p}\right) Z_2 \int_0^\infty z dz J_1(pz) K_1(mz) (\mu^2 z^2)^{-\Gamma}. \tag{C3}$$

Now expand the $(\mu^2 z^2)^{-\Gamma}$ factor of (C3) in powers of α

$$(\mu^2 z^2)^{-\Gamma} = 1 - \Gamma \ln(\mu^2 z^2) + \cdots \tag{C4}$$

and consider the leading term on the RHS of (C4). Using the formula [9]

$$\int_0^\infty x dx J_1(\lambda x) K_1(\mu x) = \left(\frac{\lambda}{\mu}\right) \frac{1}{\lambda^2 + \mu^2},$$

this leading term produces exactly $Z_2/(p^2 + m^2)$, so that multiplying this term by $p^2 + m^2$ yields just Z_2. The integral of (C3) over each of the $(\Gamma \ln(\mu^2 z^2))^q$, $q > 1$ corrections to this result is finite, and will, in general, produce a result different from that of the leading term, without the factor $(p^2 + m^2)^{-2}$; so that when p^2 is continued back to its Minkowski form, and each of these corrections is multiplied by $(p^2 + m^2)$ in the mass shell limit, each of these higher g^2 terms vanishes, as in conventional perturbation theory. That is, under mass shell amputation, the Z_2 guessed in Sec. I is the correct wave function renormalization constant of that solution.

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[8] One may ask if is there a physical reason why such terms vanish? This question was put to one of us (HMF) by R. Stora, in a recent discussion in his office at LAPP, Annecy; and this author could not, and is still unable, to answer it satisfactorily.

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