

Simple formula for high-energy gluon bremsstrahlung in a finite, expanding medium

Peter Arnold

Department of Physics, University of Virginia, Box 400714, Charlottesville, Virginia 22904, USA

(Received 20 August 2008; published 23 March 2009)

Previous authors have considered the problem of the medium effects on single gluon bremsstrahlung associated with producing a high-energy particle in a finite, time-dependent QCD plasma. Working to leading logarithmic order, I show that the result for the bremsstrahlung gluon spectrum can be cast into a remarkably simple form in the general case. I similarly analyze the process of pair production. Also, I comment on the radius of convergence of the opacity expansion in cases where the leading-log approximation holds, showing that the opacity expansion does not converge when the thickness of the plasma is greater than roughly the bremsstrahlung formation time.

DOI: 10.1103/PhysRevD.79.065025

PACS numbers: 12.38.Mh, 11.10.Wx

I. INTRODUCTION AND MAIN RESULT

Roughly a decade ago, Baier, Dokshitzer, Mueller, and Schiff (BDMS) [1] derived a simple result for the effect of the medium on the probability of single gluon bremsstrahlung from a high-energy parton produced by some hard process in the background of a uniform, time-independent chunk of hot QCD matter (known as a “brick”). Their simple result (based on application of a more general formalism) was derived for cases where the number N_{coh} of coherent soft scatterings during gluon bremsstrahlung is large, and they looked for a result valid to leading order in $(\ln N_{\text{coh}})^{-1}$. They found

$$\omega \frac{d}{d\omega} (I - I_{\text{vac}}) = \frac{\alpha}{\pi} x P_{s \rightarrow g}(x) \ln |\cos(\omega_0 L)|, \quad (1.1)$$

where I is the probability of gluon bremsstrahlung from the high-energy particle of energy E and species s (quark or gluon), I_{vac} is the corresponding probability had the hard particle been produced in vacuum, $\omega = xE$ is the energy of the bremsstrahlung gluon, $P_{s \rightarrow g}(x)$ is the usual vacuum splitting function, L is the distance the high-energy particle travels through the (uniform) medium before abruptly exiting into vacuum, and ω_0 is a complex number with phase $\exp(-i\pi/4)$ given by

$$\omega_0^2 = -i \frac{[(1-x)C_A + x^2 C_s] \hat{q}}{2x(1-x)E}. \quad (1.2)$$

Here, $C_R \hat{q}$ is the average squared transverse momentum transfer per unit length that a high-energy particle with color representation R picks up through soft, elastic collisions with the medium, evaluated at leading-log order,

$$\hat{q} \equiv \int d^2 q_{\perp} \frac{d\bar{\Gamma}_{\text{el}}}{d^2 q_{\perp}} q_{\perp}^2, \quad (1.3)$$

where $C_R \bar{\Gamma}_{\text{el}}$ is the collision rate (which is the same at leading order for high-energy quarks and gluons, except for an overall factor of the quadratic color Casimir C_R). The leading-log approximation arises from the need to cut off

the large q_{\perp} behavior of this integral, which I will briefly review later.

In another paper [2],¹ BDMS showed that they could also find leading-log results for nonuniform, time-dependent media, such as an expanding quark-gluon plasma. The result was not as simple, however, involving a double integral of a complicated function found for the particular case they studied. In this paper, I show that there is a magically simple generalization of (1.1) to the general case of nonuniform, time-dependent media. The result is

$$\omega \frac{d}{d\omega} (I - I_{\text{vac}}) = \frac{\alpha}{\pi} x P_{s \rightarrow g}(x) \ln |c(0)|, \quad (1.4)$$

where $c(t)$ satisfies the differential equation

$$\frac{d^2 c}{dt^2} = -\omega_0^2(t) c(t) \quad (1.5)$$

with the boundary condition that $c(t)$ approach the constant 1 as $t \rightarrow \infty$, and the convention that $t = 0$ is the time of the hard collision that produced the initial high-energy particle. Here, $\omega_0^2(t)$ is (1.2) evaluated at the position of the high-energy particle at time t , and now $\hat{q} = \hat{q}(t, \mathbf{x}(t))$ is time dependent. The fact that the particle eventually ends up in vacuum means that $\omega_0^2(t) \rightarrow 0$ as $t \rightarrow \infty$.

I will later give the generalization of the result to the case $g \rightarrow q\bar{q}$ of pair production.

I should note that BDMS’s result and my generalization are not complete descriptions of the average bremsstrahlung spectrum at leading-log order [3]. For sufficiently small L , the average medium effect on bremsstrahlung is instead dominated by atypical events where there is a

¹Readers should beware that Ref. [2] investigates a slightly different problem than the one proposed here, and gets a correspondingly different answer, for example, for the brick case (1.1). Here, as in Ref. [1], I consider radiation from a high-energy parton after it leaves a hard collision that occurs inside the medium. Reference [2], in contrast, purports to study the case where the particle approaches the medium from the outside. See the discussion immediately following Eq. (42b) of Ref. [1].

single, larger-than-normal scattering from the medium. I will review this later, along with the condition on L [4].

The simple form (1.4) is peculiar to three spatial dimensions (i.e. two transverse dimensions). I do not know of a generalization that would give a comparably simple result in other dimensions.

In the next section, I review the starting point for the calculation, based on the formalism of Zakharov [5] and Baier, Dokshitzer, Mueller, Peigne, and Schiff (BDMPS) [1,6–8]. I organize the notation in a way that is a little friendlier for perturbative calculations in a QCD medium with nonstatic scatterers than the original BDMPS version. (See the discussion in the Appendix.) Then I review the leading-log approximation and the range of validity of the BDMS result (1.1). In Sec. III, I derive the basic result (1.4) of this paper. Section IV then gives various examples for some cases where the equation (1.5) for $c(t)$ has analytic solutions. Section V analyzes the general problem in the limiting cases of a QCD medium that is narrow or wide compared to the formation length for gluon bremsstrahlung. Throughout this paper, I focus on the case of bremsstrahlung in order to simplify notation, but the formalism applies equally well to pair production. In Sec. VI, I give the corresponding results for the case of pair production. Finally, in Sec. VII, I comment on implications of BDMS's original result (1.1) for the convergence of what is known as the opacity expansion—the expansion of the bremsstrahlung probability in powers of the number of elastic scatterings.

The notational conventions that I use are not exactly the same as those of BDMS or Zakharov. The relationship between my notation and various other authors is discussed in the Appendix.

II. STARTING POINT AND ASSUMPTIONS

A. Notational preliminaries

Throughout, I will use C_R to denote the quadratic Casimir $T_R^a T_R^a$ for the color representation R associated with some particle, with color generators T_R^a . For a particle of type s , I will abbreviate this as C_s . For QCD,

$$C_q \equiv C_F = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3}, \quad C_g \equiv C_A = N_c = 3, \quad (2.1)$$

where $N_c = 3$ is the number of colors. d_R will be the dimension of the color representation, so that

$$d_q \equiv d_F = N_c = 3, \quad d_g \equiv d_A = N_c^2 - 1 = 8. \quad (2.2)$$

$t_R = C_R d_R / d_A$ will be the trace normalization defined by $\text{tr}(T_R^a T_R^b) = t_R \delta^{ab}$, with

$$t_q \equiv t_F = \frac{1}{2}, \quad t_g \equiv t_A = N_c = 3. \quad (2.3)$$

The Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) splitting functions in (1.1) and (1.4) are

$$P_{q \rightarrow g}(x) = C_F \frac{[1 + (1-x)^2]}{x}, \quad (2.4)$$

$$P_{g \rightarrow g}(x) = C_A \frac{[1 + x^4 + (1-x)^4]}{x(1-x)}. \quad (2.5)$$

Throughout this paper, I will generally place a bar over quantities when I have removed an overall factor of C_R from its definition. So I work with \hat{q} , for example, instead of the more standard \hat{q} .

B. General formalism

Calculations of bremsstrahlung from sufficiently high-energy jets must take into account the Landau-Pomeranchuk-Migdal (LPM) effect, which arises when the quantum mechanical duration (formation time) of the bremsstrahlung process becomes comparable to, or exceeds, the mean-free time for small-angle elastic collisions. The basic procedure for making such calculations was laid out for QED by Migdal in 1956 [9]. The generalization to QCD requires accounting for the fact that a bremsstrahlung gluon, unlike a photon, carries (color) charge and so can also undergo collisions during the formation time. I will find it convenient to start with the particular version of this result derived by Zakharov [5,10]. This is equivalent to the BDMPS formalism of Baier *et al.* [1,11], and I will use some of that correspondence in how I choose to write Zakharov's result. The general formula is

$$\begin{aligned} \omega \frac{d}{d\omega} (I - I_{\text{vac}}) &= \frac{\alpha x P_{s \rightarrow g}(x)}{[x(1-x)E]^2} \text{Re} \int_0^\infty dt_1 \\ &\times \int_{t_1}^\infty dt_2 [\nabla_{\mathbf{B}_1} \cdot \nabla_{\mathbf{B}_2} \{G(\mathbf{B}_2, t_2; \mathbf{B}_1, t_1) \\ &- G_{\text{vac}}(\mathbf{B}_2, t_2; \mathbf{B}_1, t_1)\}]_{B_1=B_2=0}, \quad (2.6) \end{aligned}$$

where $G(\mathbf{B}_2, t_2; \mathbf{B}_1, t_1)$ is the Green's function for a two-dimensional quantum mechanics problem with the time-dependent Hamiltonian

$$H(t) = \delta E(\mathbf{p}_B, t) - i\Gamma_3(\mathbf{B}, t). \quad (2.7)$$

The two terms in H above will be described in a moment. The Green's function $G(\mathbf{B}, t; \mathbf{B}_1, t_1)$ is a solution to the Schrödinger equation

$$i\partial_t \psi(\mathbf{B}, t) = H(t) \psi(\mathbf{B}, t) \quad (2.8)$$

with initial condition

$$G(\mathbf{B}, t_1; \mathbf{B}_1, t_1) = \delta^{(2)}(\mathbf{B} - \mathbf{B}_1). \quad (2.9)$$

The first term in (2.7) describes the energy difference

$$(E_{s,p} + E_{g,k}) - E_{s,p+k} \simeq \frac{p_{\perp}^2 + m_s^2}{2p} + \frac{k_{\perp}^2 + m_g^2}{2k} - \frac{|\mathbf{p}_{\perp} + \mathbf{k}_{\perp}|^2 + m_s^2}{2(p+k)} \quad (2.10)$$

between (i) a high-energy parton of momentum $\mathbf{P} = \mathbf{p} + \mathbf{k}$ and energy $E = P$ and (ii) the same parton with momentum \mathbf{p} plus a bremsstrahlung gluon with momentum \mathbf{k} . If (following Ref. [1]) one defines

$$\mathbf{p}_B \equiv \frac{p\mathbf{k}_{\perp} - k\mathbf{p}_{\perp}}{P}, \quad (2.11)$$

then this energy difference can be rewritten as

$$\begin{aligned} \delta E(\mathbf{p}_B, t) &\equiv \frac{p_B^2}{2x(1-x)P} + \frac{m_s^2(t)}{2(1-x)P} + \frac{m_g^2(t)}{2xP} - \frac{m_s^2(t)}{2P} \\ &= \frac{p_B^2 + x^2 m_s^2(t) + (1-x)m_g^2(t)}{2x(1-x)E}. \end{aligned} \quad (2.12)$$

The notation $m(t)$ accounts for the fact that the effective masses will change as the particle transverses an inhomogeneous or time-dependent medium. Qualitatively, the expectation of $1/\delta E(\mathbf{p}_B)$ is of order of the formation time for the bremsstrahlung process in the medium.² The second term in (2.7) is given by

$$\begin{aligned} \Gamma_3(\mathbf{B}, t) &= \frac{1}{2}C_A \bar{\Gamma}_2(\mathbf{B}, t) + (C_s - \frac{1}{2}C_A) \bar{\Gamma}_2(x\mathbf{B}, t) \\ &\quad + \frac{1}{2}C_A \bar{\Gamma}_2((1-x)\mathbf{B}, t), \end{aligned} \quad (2.13)$$

where $\bar{\Gamma}_2$ is related to the Fourier transform of $d\bar{\Gamma}_{\text{el}}/d^2q_{\perp}$ and defined by

$$\begin{aligned} \bar{\Gamma}_2(\mathbf{b}, t) &\equiv \int d^2q_{\perp} \frac{d\bar{\Gamma}_{\text{el}}(t)}{d^2q_{\perp}} (1 - e^{i\mathbf{b}\cdot\mathbf{q}_{\perp}}) \\ &= \frac{1}{\pi} \int d^2q_{\perp} \frac{d\bar{\Gamma}_{\text{el}}(t)}{d(q_{\perp}^2)} (1 - e^{i\mathbf{b}\cdot\mathbf{q}_{\perp}}). \end{aligned} \quad (2.14)$$

I have not used exactly the same notation as either Zakharov or BDMS, and I summarize the differences of notation in the Appendix. On a slightly more substantive matter, both implicitly assumed that the rate Γ_{el} for soft scattering of the high-energy particle could be written as a number density n of static particles in the medium times a cross section σ_{el} for scattering from such particles. However, their results do not actually depend on this assumption. If one simply writes their formulas in terms of the rate Γ_{el} rather than $n\sigma$, then they apply equally well to the case of scattering from nonstatic particles, which, for example, was analyzed for leading-order calculations in an infinite, time-independent thermal medium by Arnold, Moore, and Yaffe (AMY) [12–14] and Jeon and Moore

²There is a difference between my use of the phrase “formation time” and Zakharov’s [5]. See the Appendix.

[15]. Specifically, the differential rate is

$$\begin{aligned} \frac{d\bar{\Gamma}_{\text{el},s}}{d^2q_{\perp}} &= \int dq_z \sum_{s_2} \nu_{s_2} \int \frac{d^3p_2}{(2\pi)^3} \frac{d\bar{\sigma}_{\text{el}}}{d^3q} \\ &\quad \times f_{s_2}(\mathbf{p}_2) [1 \pm f_{s_2}(\mathbf{p}_2 - \mathbf{q})], \end{aligned} \quad (2.15)$$

Here, $C_R \bar{\sigma}_{\text{el}}$ is the soft, elastic scattering rate for a high-energy particle to scatter from a single plasma particle of momentum \mathbf{p}_2 and species s_2 . \mathbf{q}_{\perp} is the transverse momentum transfer to the high-energy particle from this single scattering. $f(\mathbf{p}_2)$ is the phase space density of plasma particles per degree of freedom, which in thermal equilibrium is the Bose or Fermi distribution for the plasma particle. ν_{s_2} is the number of spin, color, and flavor degrees of freedom for species s_2 , which would be $2d_A = 16$ for gluons and $4d_F N_f = 12N_f$ for the sum of quarks and antiquarks, where N_f is the number of quark flavors. The factor of f gives the density of plasma particles, while the factor of $1 \pm f$ is a final-state Bose or Fermi factor. Final-state factors for the high-energy particle (as opposed to the plasma particle it is scattering from) may be included at the end of the LPM calculation, if desired (see, for example, the $1 \rightarrow 2$ splitting terms in the effective kinetic theories of Refs. [12,13,16]), but in the present context I assume that the high-energy particle is an isolated particle of energy much higher than the plasma particles, so that its final-state factor can be ignored.

In terms of specifics, perturbative calculations for a QCD plasma in local equilibrium give the simple formulas³

$$\frac{d\bar{\Gamma}_{\text{el}}}{d^2q_{\perp}} \simeq \frac{1}{(2\pi)^2} \times \begin{cases} \frac{g^2 T m_D^2}{q_{\perp}^2 (q_{\perp}^2 + m_D^2)}, & q_{\perp} \ll T, \\ \frac{g^4 \mathcal{N}}{q_{\perp}^4}, & q_{\perp} \gg T, \end{cases} \quad (2.16)$$

in the limits of q_{\perp} small or large compared to the temperature T . Here m_D is the Debye mass,

$$m_D^2 = (t_A + N_f t_F) \frac{1}{3} g^2 T^2 = (1 + \frac{1}{6} N_f) g^2 T^2, \quad (2.17)$$

and \mathcal{N} is the weighted number density

$$\begin{aligned} \mathcal{N} &\equiv \sum_{s_2} \nu_{s_2} t_{s_2} \int \frac{d^3p_2}{(2\pi)^3} f_{s_2}(\mathbf{p}_2) \\ &= \frac{\zeta(3)}{\zeta(2)} \left(t_A + \frac{3}{2} N_f t_F \right) \frac{1}{3} T^3 = \frac{\zeta(3)}{\zeta(2)} \left(1 + \frac{1}{4} N_f \right) T^3, \end{aligned} \quad (2.18)$$

where $\zeta(z)$ is the Riemann zeta function.

The formalism reviewed above assumes that the characteristics of the medium do not change significantly over a Debye screening length. It is not restricted to equilibrium

³The simple form of the $q_{\perp} \ll T$ formula comes from Ref. [17]. This is the formula used by AMY [14] in studying the LPM effect in hydrodynamic transport coefficients, where the relevant particle energies are $E \sim T$.

situations, but I will assume that the differential elastic cross section is isotropic in the transverse plane. The formalism also assumes that the final bremsstrahlung gluon and accompanying particle are energetic enough that transverse momentum transfers from the medium will be small compared to their momenta.

More generally, all calculations based on variations of Migdal's procedure require that the mean-free path for soft, elastic collisions be large compared to the screening length.⁴ This assumption holds for a thermal plasma in the weak coupling limit, where the mean-free path is order $1/g^2T$ and the screening length is order $1/gT$. (See, for example, the discussion in Ref. [12].)

If running of the coupling constant α is included in the analysis, then $d\bar{\Gamma}_{\text{el}}/d^2q_{\perp}$ should plausibly be evaluated with $g^2(q_{\perp})$.⁵ In Ref. [19], it is argued that the overall factor of α associated with the coupling of the bremsstrahlung gluon [here the explicit α in (1.4) or (2.6)], should plausibly be evaluated as $\alpha(Q_{\perp})$, where Q_{\perp} is the typical transverse momentum transfer over the formation time and is discussed below. This last prescription is in the spirit of earlier suggestions by BDMPS [8].⁶

C. Leading-log (harmonic oscillator) approximation

Consider a medium that is thick enough that the total number of soft scatterings with individual momentum transfers $q_{\perp} \gtrsim m_D$, as the particle traverses the medium, is large.⁷ In the high-energy limit, the number N_{coh} of such scatterings in a formation time also becomes large. As noted long ago by Migdal [9], the calculation of the LPM effect simplifies significantly if one works to leading order

⁴More precisely, it is the mean-free path for the subset of soft, elastic collisions which contribute to the result at the desired accuracy. In the thermal case, for example, ultrasoft magnetic interactions with $q_{\perp} \sim g^2T$ do not affect results at leading order in coupling.

⁵See, for example, Refs. [18,19]. In order to avoid an unphysical infrared divergence of the calculation when the definition of g^2 blows up at Λ_{QCD} , one should appropriately cut off the running in the infrared. One possibility would be to use $g^2(\sqrt{q_{\perp}^2 + m_D^2})$.

⁶Specifically, after Eq. (3.12) of Ref. [8], they suggest taking $\alpha_s(k)$ with $k \propto L^{1/2}$ for the calculation of average bremsstrahlung energy loss in a thin QCD medium. For that problem, the energy loss is dominated by gluons whose formation time is of order of the length L of the medium. In that case, $Q_{\perp} \propto L^{1/2}$ as in (2.20) below.

⁷This statement contains the restriction $q_{\perp} \gtrsim m_D$ because, in the weak coupling limit, the most common scatterings, by a parametric factor of $\ln(\alpha^{-1})$, have momentum $g^2T \lesssim q_{\perp} \ll m_D$ and are mediated by the exchange of low-frequency magnetic gluons. These low-frequency magnetic gluons are not Debye screened, and their contribution is cut off only by nonperturbative effects. However, these ultralow momentum scatterings do not contribute at leading order to (1.3) and (2.14) [because of the factor of q_{\perp}^2 in (1.3)] and so do not have an effect on bremsstrahlung at leading order in coupling.

in $(\ln N_{\text{coh}})^{-1}$. In the context of QCD, such leading-log calculations were explored by Baier *et al.* using their BDMPS formalism and what is known as the harmonic oscillator approximation. Following BDMS [2], I will focus on leading-log calculations in this paper as well.

The large N_{coh} limit corresponds to the case where the total transverse momentum transfer Q_{\perp} to a high-energy particle during the formation time is large compared to the screening mass m_D . One consequence of large transverse momentum is that we can ignore the effective particle masses m_s and m_g in (2.12). Another consequence is that large Q_{\perp} corresponds in Fourier space to small B . Naively, Eq. (2.14) for $\bar{\Gamma}_2$ can then be replaced by its small b limit, which is formally

$$\bar{\Gamma}_2(\mathbf{b}, t) \simeq \frac{1}{4} \int d^2q_{\perp} \frac{d\bar{\Gamma}_{\text{el}}(t)}{d^2q_{\perp}} q_{\perp}^2 b^2 = \frac{1}{4} \hat{q}(t) b^2. \quad (2.19)$$

This is known as the harmonic oscillator approximation because of the form of (2.19). The problem is that the above integral is logarithmically divergent because of the large q_{\perp} behavior of (2.16). For a leading-log analysis of typical events, it should be cut off at order of the typical total momentum transfer Q_{\perp} in a formation time. Parametrically, recalling the definition of \hat{q} ,

$$Q_{\perp} \sim \begin{cases} (C_R \hat{q} L)^{1/2}, & L \lesssim L_{\text{cr}}, \\ (C_R \hat{q} L_{\text{cr}})^{1/2}, & L \gtrsim L_{\text{cr}}, \end{cases} \quad (2.20)$$

where L is the characteristic thickness of the medium and L_{cr} is the infinite-medium formation time⁸

$$L_{\text{cr}} \sim \left(\frac{E_i}{C_R \hat{q}} \right)^{1/2}. \quad (2.21)$$

Above, E_i is the energy E , xE , or $(1-x)E$ of a particular parton in the splitting process, and one should use whichever parton gives the smallest Q_{\perp} . For small x , that will be the bremsstrahlung gluon, giving $L_{\text{cr}} \sim (\omega/C_A \hat{q})^{1/2}$.

Using (1.3) and (2.16), the leading-log value of \hat{q} for a weakly coupled thermal QCD plasma is then⁹

$$\hat{q} \simeq \alpha T m_D^2 \ln \left(\frac{Q_{\perp}^2}{m_D^2} \right) \quad (2.22a)$$

if $Q_{\perp} \lesssim T$ and

$$\hat{q} \simeq \alpha T m_D^2 \ln \left(\frac{T^2}{m_D^2} \right) + 4\pi\alpha^2 \mathcal{N} \ln \left(\frac{Q_{\perp}^2}{T^2} \right) \quad (2.22b)$$

otherwise. For 3-flavor QCD, $\alpha T m_D^2$ and $4\pi\alpha^2 \mathcal{N}$ differ by only about 15%, and so one could combine the logarithms of (2.22) into either $\alpha T m_D^2 \ln(Q_{\perp}^2/m_D^2)$ or $4\pi\alpha^2 \mathcal{N} \ln(Q_{\perp}^2/m_D^2)$ without much error.

⁸See, for example, the discussion in Sec. 3 of Ref. [7].

⁹For (2.22a), see also Eq. (13) of Ref. [20] and the relation to Ref. [21] discussed after Eq. (61) of Ref. [20].

If Q_\perp is so large that $\alpha(Q_\perp)$ is significantly different from $\alpha(m_D)$, then one should include one-loop running of the coupling when integrating (2.16). The result can be put into the form¹⁰

$$\hat{q} \simeq \alpha(T) T m_D^2(m_D) \ln\left(\frac{T^2}{m_D^2}\right) + 4\pi\alpha(Q_\perp)\alpha(T)\mathcal{N} \ln\left(\frac{Q_\perp^2}{T^2}\right), \quad (2.23)$$

where $m_D(m_D)$ indicates the Debye mass (2.17) evaluated with running coupling $g^2(m_D)$.

Note that the leading-log formula (2.22) for \hat{q} depends logarithmically on Q_\perp , which in turn depends on \hat{q} . One could determine \hat{q} self-consistently, but it should be kept in mind that a precise value of \hat{q} inside the logarithm is not called for because we are only pursuing a leading-log result. For an example of how things work out at next-to-leading logarithmic order, see the infinite-medium calculation of Ref. [19].

In any case, in the leading-log approximation (2.19), the two-dimensional Hamiltonian of (2.6) becomes a two-dimensional harmonic oscillator problem with time-dependent frequency:

$$H(t) \simeq \frac{p_B^2}{2M} + \frac{1}{2} M \omega_0^2(t) B^2, \quad (2.24)$$

with

$$M = x(1-x)E, \quad (2.25)$$

$$\omega_0^2(t) = -i \frac{[(1-x)C_A + x^2 C_s] \hat{q}(t)}{2x(1-x)E}. \quad (2.26)$$

Note that ω_0^2 is imaginary. Its inverse magnitude $1/|\omega_0|$ is of order the infinite-medium formation time L_{cr} of (2.22).

The harmonic oscillator approximation breaks down for sufficiently small L , even when logarithms are large. Using the typical total momentum transfer (2.20) as an upper cutoff to determine the integral in (2.19) ignores the possibility of bremsstrahlung from rare, atypical scatterings with larger q_\perp , which turn out to be important for sufficiently small L . There has been some confusion about the resulting range of validity of the harmonic oscillator approximation used by BDMS for a leading-log analysis of the spectrum. Zakharov [3] suggested that the harmonic oscillator approximation outlined in this section is only valid when $L \gg L_{\text{cr}}$, which is equivalent to $|\omega_0 L| \gg 1$. In Ref. [4], however, I argue that the validity extends to

¹⁰See, for example, the discussion in Sec. VI of Ref. [19]. Though the form of (2.23) is convenient, it can be misleading. In the limit that Q_\perp is so large that $\alpha(Q_\perp) \ll \alpha(T)$, the answer does not actually depend on q_\perp of order Q_\perp —it is instead dominated by those q_\perp for which $\alpha(q_\perp)$ is of order $\alpha(T)$ [19]. Also, the simple formula (2.23) is only valid if there are no vacuum mass thresholds between m_D and Q_\perp .

$$L \gg \frac{L_{\text{cr}}}{[\ln(Q_\perp^2/m_D^2)]^{1/2}}. \quad (2.27)$$

This includes the interesting region $L \sim L_{\text{cr}}$ (equivalently $|\omega_0 L| \sim 1$) in a leading-log analysis, which treats the logarithm as large.

III. DERIVATION

A. A double integral

If G is the Green's function, then the two components of the vector function $\nabla_{\mathbf{B}_1} G(\mathbf{B}, t; \mathbf{B}_1, t_1)$ will also satisfy the Schrödinger equation (2.8) but with initial condition

$$\nabla_{\mathbf{B}_1} G(\mathbf{B}, t_1; \mathbf{B}_1, t_1) = \nabla_{\mathbf{B}_1} \delta^{(2)}(\mathbf{B} - \mathbf{B}_1). \quad (3.1)$$

In (2.6), we are interested in the particular case $\mathbf{B}_1 = 0$, which then corresponds to the initial condition

$$\nabla_{\mathbf{B}_1} G(\mathbf{B}, t_1; \mathbf{B}_1, t_1)|_{\mathbf{B}_1=0} = -\nabla_{\mathbf{B}} \delta^{(2)}(\mathbf{B}). \quad (3.2)$$

The desired solution in the harmonic oscillator approximation (2.24) is

$$\begin{aligned} & \nabla_{\mathbf{B}_1} G(\mathbf{B}, t; \mathbf{B}_1, t_1)|_{\mathbf{B}_1=0} \\ &= -\frac{M^2}{2\pi S^2(t; t_1)} \mathbf{B} \exp\left(\frac{iM \partial_t S(t; t_1)}{2S(t; t_1)} B^2\right), \end{aligned} \quad (3.3)$$

where $S(t; t_1)$ satisfies the differential equation

$$\partial_t^2 S = -\omega_0^2(t) S \quad (3.4)$$

with boundary conditions

$$S(t_1; t_1) = 0, \quad \partial_t S(t; t_1)|_{t=t_1} = 1. \quad (3.5)$$

One may check this by (i) plugging it into the Schrödinger equation and noting that it is a solution, and (ii) checking the initial condition by solving for t infinitesimally close to t_1 , where $S(t; t_1) \rightarrow t - t_1$ and (3.3) becomes

$$\begin{aligned} & \nabla_{\mathbf{B}_1} G(\mathbf{B}, t; \mathbf{B}_1, t_1)|_{\mathbf{B}_1=0} \\ & \rightarrow -\frac{M^2}{2\pi(t-t_1)^2} \mathbf{B} \exp\left(\frac{iM}{2(t-t_1)} B^2\right) \\ &= -\nabla_{\mathbf{B}} \frac{M}{2\pi i(t-t_1)} \exp\left(-\frac{M}{2i(t-t_1)} B^2\right), \end{aligned} \quad (3.6)$$

which is a representation of $-\nabla_{\mathbf{B}} \delta^{(2)}(\mathbf{B})$ for infinitesimal $t - t_1$.

Substituting (3.3) into (2.6) gives

$$\omega \frac{d}{d\omega} (I - I_{\text{vac}}) = \frac{\alpha}{\pi} x P_{s \rightarrow g}(x) \text{Re} J, \quad (3.7)$$

where

$$I \equiv - \int_0^\infty dt_1 \int_{t_1}^\infty dt_2 \left[\frac{1}{S^2(t_2; t_1)} - \frac{1}{(t_2 - t_1)^2} \right]. \quad (3.8)$$

This gives an answer in terms of a double integral involving the function $S(t_2; t_1)$. But both integrals can be done explicitly, even for the case of arbitrary $\omega_0^2(t)$.

B. The t_2 integration

Now consider the other solution to the 2nd-order differential Eq. (3.4), which I will call $C(t; t_1)$ and take to have boundary conditions

$$C(t_1; t_1) = 1, \quad \partial_t C(t; t_1)|_{t=t_1} = 0. \quad (3.9)$$

If $\omega_0^2(t)$ were a constant, then the two solutions would be $S = \omega_0^{-1} \sin(\omega_0(t - t_1))$ and $C = \cos(\omega_0(t - t_1))$, which is the motivation for the labels S and C .

The form of the differential equation implies that the Wronskian

$$W = C \partial_t S - S \partial_t C \quad (3.10)$$

is independent of time and so always equal to its value at $t = t_1$:

$$C \partial_t S - S \partial_t C = 1. \quad (3.11)$$

Dividing both sides by S^2 then gives

$$- \partial_t \left(\frac{C}{S} \right) = \frac{1}{S^2}. \quad (3.12)$$

We can use this to do the t_2 integral in (3.8). Rewrite the time integrals in (3.8) to have upper limit t , taking the limit $t \rightarrow \infty$ at the end. Then rewrite the t_2 integral as

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \int_{t_1 + \epsilon}^t dt_2 \left(\frac{1}{S^2(t_2; t_1)} - \frac{1}{(t_2 - t_1)^2} \right) \\ &= - \frac{C(t; t_1)}{S(t; t_1)} + \frac{1}{t - t_1} + \lim_{\epsilon \rightarrow 0} \left[\frac{C(t_1 + \epsilon; t_1)}{S(t_1 + \epsilon; t_1)} - \frac{1}{\epsilon} \right] \\ &= - \frac{C(t; t_1)}{S(t; t_1)} + \frac{1}{t - t_1}. \end{aligned} \quad (3.13)$$

So

$$I = \lim_{t \rightarrow \infty} \int_0^t dt_1 \left[\frac{C(t; t_1)}{S(t; t_1)} - \frac{1}{t - t_1} \right]. \quad (3.14)$$

C. The t_1 integration

Now note that

$$C(t; t_1) = - \partial_{t_1} S(t; t_1). \quad (3.15)$$

This follows because (i) $-\partial_{t_1} S$ will satisfy the same equation (3.4) that S does, and (ii) the boundary conditions work out correctly. The boundary conditions (3.9) can be con-

firmed from the small $t - t_1$ expansion of $S(t; t_1)$, which is

$$S(t; t_1) = (t - t_1) - \frac{1}{3!} \omega_0^2(t_1) (t - t_1)^3 + O[(t - t_1)^5], \quad (3.16)$$

so that

$$C(t; t_1) = 1 - \frac{1}{2!} \omega_0^2(t_1) (t - t_1)^2 + O[(t - t_1)^4]. \quad (3.17)$$

I shall not need it, but the corresponding derivative of C is

$$\omega_0^2(t_1) S(t; t_1) = \partial_{t_1} C(t; t_1). \quad (3.18)$$

Note that the relations (3.15) and (3.18) involve t_1 derivatives—I will discuss the case of t derivatives later.

Now substitute (3.15) into (3.14):

$$\begin{aligned} I &= - \lim_{t \rightarrow \infty} \ln \left[\frac{S(t; t_1)}{t - t_1} \right] \Bigg|_{t_1=0}^{t_1=t} = \lim_{t \rightarrow \infty} \ln \left[\frac{S(t; 0)}{t} \right] \\ &= \lim_{t \rightarrow \infty} \ln [\partial_t S(t; 0)]. \end{aligned} \quad (3.19)$$

Combining with (3.7),

$$\omega \frac{d}{d\omega} (I - I_{\text{vac}}) = \frac{\alpha}{\pi} x P_{s \rightarrow g}(x) \lim_{t \rightarrow \infty} \ln |\partial_t S(t; 0)|. \quad (3.20)$$

D. Final simplification

The result (3.20) is perfectly adequate, but it is amusing to put it in a final form that is even more closely analogous to the result (1.1) for the brick problem.

Note that any solution to a linear differential equation can be written as a superposition of others. So $S(t; t_1)$ and $C(t; t_1)$ can be expressed as superpositions of $S(t; t_0)$ and $C(t; t_0)$ for any t_0 . Specifically,

$$S(t; t_1) = C(t_1; t_0) S(t; t_0) - S(t_1; t_0) C(t; t_0), \quad (3.21)$$

$$C(t; t_1) = - \partial_{t_1} C(t_1; t_0) S(t; t_0) + \partial_{t_1} S(t_1; t_0) C(t; t_0). \quad (3.22)$$

To verify these formulas, one just needs to check the boundary conditions. The conditions $S(t_1; t_1) = 0$ and $\partial_t C(t; t_1)|_{t=t_1} = 0$ are easy. The other two, $\partial_t S(t; t_1)|_{t=t_1} = 1$ and $C(t_1; t_1) = 1$, follow from the time independence of the Wronskian,

$$\begin{aligned} & [C(t; t_0) \partial_t S(t; t_0) - S(t; t_0) \partial_t C(t; t_0)]_{t=t_1} \\ &= [C(t; t_0) \partial_t S(t; t_0) - S(t; t_0) \partial_t C(t; t_0)]_{t=t_0} = 1. \end{aligned} \quad (3.23)$$

From (3.21), we see that $S(t; t_1)$ is antisymmetric in its arguments:

$$S(t_2; t_1) = -S(t_1; t_2). \quad (3.24)$$

We can then combine this with (3.15) for the t_1 derivative of S to get a formula for the t derivative:

$$\partial_t S(t; t_1) = -\partial_t S(t_1; t) = C(t_1; t). \quad (3.25)$$

[Eq. (3.22) does not allow us to deduce any comparable symmetry property of C .] We can now use (3.25) to rewrite (3.20) in the form

$$\omega \frac{d}{d\omega} (I - I_{\text{vac}}) = \frac{\alpha}{\pi} x P_{s \rightarrow g}(x) \ln |C(0; \infty)|, \quad (3.26)$$

which is Eq. (1.4) of the introduction.

IV. EXAMPLES

One can of course solve the differential equation (1.5) numerically for any desired time dependence of $\hat{q}(t)$ along the path of the particle. In this section, I give a few examples that have analytic solutions.

A. The brick problem

Consider the case where the particle travels distance L through a uniform medium and then emerges into vacuum. So

$$\omega_0^2(t) = \begin{cases} \omega_0^2, & t < L; \\ 0, & t > L. \end{cases} \quad (4.1)$$

The solution $c(t)$ to (1.5) is then

$$c(t) = \begin{cases} \cos(\omega_0(L - t)), & t < L; \\ 1, & t > L. \end{cases} \quad (4.2)$$

Equation (1.4) then reproduces the result (1.1) of BDMS [1].

Using the fact that ω_0 is proportional to $(-i)^{1/2}$, one can alternatively write the result solely in terms of real quantities using the identity

$$\ln |\cos(e^{-i\pi/4} x)| = \frac{1}{2} \ln \left[\frac{1}{2} \cosh(\sqrt{2}x) + \frac{1}{2} \cos(\sqrt{2}x) \right] \quad (x \text{ real}). \quad (4.3)$$

The large L behavior is

$$\ln |\cos(\omega_0 L)| \simeq \frac{|\omega_0|L}{\sqrt{2}} - \ln 2, \quad (4.4)$$

up to exponentially small corrections. (But you should not take seriously the $\ln 2$ term because remember that I have only treated ω_0 itself up to leading-log order.) In this limit, one can write

$$\omega \frac{d}{d\omega} (I - I_{\text{vac}}) \simeq \omega \frac{d\Gamma_{\text{bulk}}}{d\omega} L, \quad (4.5)$$

with

$$\omega \frac{d\Gamma_{\text{bulk}}}{d\omega} \equiv \frac{\alpha}{\pi\sqrt{2}} x P_{s \rightarrow g}(x) |\omega_0|. \quad (4.6)$$

For fixed x , the small L behavior is¹¹

$$\ln |\cos(\omega_0 L)| \simeq \frac{1}{12} (|\omega_0|L)^4. \quad (4.7)$$

Small L in this context means $|\omega_0|L \ll 1$, equivalent to $L \ll L_{\text{cr}}$. But keep in mind that the harmonic oscillator approximation breaks down for calculations of the spectrum when $L \lesssim L_{\text{cr}} / [\ln(Q_{\perp}^2/m_{\text{D}}^2)]^{1/2}$ [4].

B. Exponential profile

Consider an exponential profile

$$\omega_0^2(t) = \omega_0^2(0) e^{-t/L}. \quad (4.8)$$

The solution is

$$c(t) = J_0(2\omega_0(0)L) e^{-t/2L}, \quad (4.9)$$

giving

$$\omega \frac{d}{d\omega} (I - I_{\text{vac}}) = \frac{\alpha}{\pi} x P_{s \rightarrow g}(x) \ln |J_0(2\omega_0(0)L)|. \quad (4.10)$$

C. Power law relaxation

Motivated by modeling Bjorken expansion, BDMS [2] considered the case where \hat{q} falls like a power of time and then suddenly vanishes (the particle emerges into vacuum) at time L . So

$$\omega_0^2(t) = \begin{cases} \omega_0^2(t_0) \left(\frac{t_0}{t}\right)^a, & t_0 < t < t_0 + L; \\ 0, & t_0 + L < t; \end{cases} \quad (4.11)$$

where a is some power and I have now labeled the time of the initial hard process as t_0 rather than zero. The solution to (1.5) is then $c(t) = 1$ for $t > t_0 + L$ (the vacuum solution) and

$$c(t) = \left(\frac{z}{z_L}\right)^\nu \frac{J_\nu(z) Y_{\nu-1}(z_L) - Y_\nu(z) J_{\nu-1}(z_L)}{J_\nu(z_L) Y_{\nu-1}(z_L) - Y_\nu(z_L) J_{\nu-1}(z_L)} \\ = \frac{\pi z_L}{2} \left(\frac{z}{z_L}\right)^\nu [J_\nu(z) Y_{\nu-1}(z_L) - Y_\nu(z) J_{\nu-1}(z_L)] \\ (t < t_0 + L) \quad (4.12)$$

¹¹Readers familiar with the fact that the medium-induced contribution to energy loss is proportional to $\hat{q}L^2$ for small L [7] may wonder how the L^4 behavior in the spectrum (4.7) is consistent. In (4.7), the limit is that L is small compared to the formation time, which is of order $(xE/\hat{q})^{1/2} = (\omega/\hat{q})^{1/2}$ for x not close to 1. In contrast, the small L formula for energy loss assumes $L \ll (E/\hat{q})^{1/2}$. In the latter limit, when the energy loss is determined by integrating $\omega dI/d\omega$ over ω , the integral is dominated by ω 's for which the formation time is of order L ($\omega \sim \hat{q}L^2$), where the small L assumption of (4.7) has just started to fail. Using (4.7) merely as a parametric estimate then yields $\Delta E \sim \alpha\omega|\omega_0|^4 L^4 \sim \alpha\omega(\hat{q}/\omega)^2 L^4 \sim \alpha\hat{q}L^2$.

where¹²

$$\nu \equiv \frac{1}{2-a}, \quad (4.13)$$

$$z = z(t) \equiv 2\nu\omega_0(t_0)t_0\left(\frac{t}{t_0}\right)^{1/2\nu}, \quad (4.14)$$

$$z_0 \equiv z(t_0), \quad (4.15)$$

$$z_L \equiv z(t_0 + L). \quad (4.16)$$

The final result is then

$$\begin{aligned} \omega \frac{d}{d\omega}(I - I_{\text{vac}}) &= \frac{\alpha}{\pi} x P_{s \rightarrow g}(x) \ln \left| \left(\frac{t_0}{t_0 + L} \right)^{1/2} \right. \\ &\times \left. \frac{J_\nu(z_0)Y_{\nu-1}(z_L) - Y_\nu(z_0)J_{\nu-1}(z_L)}{J_\nu(z_L)Y_{\nu-1}(z_0) - Y_\nu(z_L)J_{\nu-1}(z_0)} \right|. \end{aligned} \quad (4.17)$$

D. sech² Profile

As a final analytic example, consider a hard particle starting at $t = t_0$ with profile

$$\omega_0^2(t) = \Omega^2 \text{sech}^2\left(\frac{t}{L}\right). \quad (4.18)$$

The solution is

$$\begin{aligned} \omega \frac{d}{d\omega}(I - I_{\text{vac}}) &= \frac{\alpha}{\pi} x P_{s \rightarrow g}(x) \\ &\times \ln \left| F\left(a_+, a_-; 1; \frac{1}{e^{2t_0/L} + 1}\right) \right|, \end{aligned} \quad (4.19)$$

where F is the hypergeometric function and

$$a_\pm \equiv \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + (2\Omega L)^2}. \quad (4.20)$$

V. GENERAL SOLUTION: LIMITING CASES

I now turn to the behavior of the general solution (1.4) for the limits of small or large width of the medium for fixed x .

A. Small width

In this case, we can solve the differential equation

$$\ddot{c}(t) = -\omega_0^2(t)c(t) \quad (5.1)$$

¹²My z differs by a factor of i from that of Ref. [2], which is why they have modified Bessel functions K and I instead of J and Y . Also, an equivalent way of writing (4.12) is to replace $Y_{\nu-1}$ and Y_ν by $J_{1-\nu}$ and $J_{-\nu}$. If comparing to Ref. [2], keep in mind that they solve a slightly different problem, as explained in footnote 1.

by perturbing around the vacuum solution $c_{\text{vac}}(t) = 1$. The solution is

$$c(t) = 1 + c_1(t) + c_2(t) + O(\omega_0^3), \quad (5.2)$$

where

$$c_1(t) = -\int_t^\infty dt'(t' - t)\omega_0^2(t'), \quad (5.3)$$

$$c_2(t) = -\int_t^\infty dt'(t' - t)\omega_0^2(t')c_1(t'). \quad (5.4)$$

Now recall that ω_0^2 is proportional to $-i$ so that c_1 is imaginary and c_2 is real. Then

$$\begin{aligned} \ln|c(0)| &\simeq \frac{1}{2} \ln[(1 + c_2(0))^2 + |c_1(0)|^2] \\ &\simeq \frac{1}{2} |c_1(0)|^2 + c_2(0) \\ &\simeq \frac{1}{2} \left(\int_0^\infty dt't' |\omega_0^2(t')| \right)^2 \\ &\quad - \int_0^\infty dt't' |\omega_0^2(t)| \int_{t'}^\infty dt''(t'' - t') |\omega_0^2(t'')|. \end{aligned} \quad (5.5)$$

This is the general form of the small-width answer, of which (4.7) is a specific case.

B. Large width

Now consider the case where $\omega_0^2(t)$ is a very slowly varying function of t . Then we can make an adiabatic approximation, and the most important feature of the solution for $s(t)$ will be a ‘‘phase factor’’ that is approximately¹³

$$\begin{aligned} c(t) &\sim \exp\left(i \int_t^\infty dt' \omega_0(t')\right) \\ &= \exp\left(\frac{1}{\sqrt{2}} \int_t^\infty dt' |\omega_0(t')|\right) \exp\left(\frac{i}{\sqrt{2}} \int_t^\infty dt' |\omega_0(t')|\right). \end{aligned} \quad (5.6)$$

Neglecting prefactors (whose effect is parametrically smaller than the exponent),

$$\ln|c(0)| \simeq \frac{1}{\sqrt{2}} \int_0^\infty dt |\omega_0(t)|. \quad (5.7)$$

Comparing to (4.6), this gives

$$\omega \frac{d}{d\omega}(I - I_{\text{vac}}) = \int_0^\infty dt \omega \frac{d\Gamma_{\text{bulk}}}{d\omega}(t), \quad (5.8)$$

as you would expect: In the limit of very thick, slowly varying media, you just treat the problem as locally uni-

¹³Because ω_0 is proportional to $\exp(-i\pi/4)$, the other solution $\exp(-i \int_t^\infty dt' \omega_0(t'))$ is, in the large width limit, exponentially small at $t = 0$, and so its contribution to $c(0)$ can be neglected.

form, use the result for the bremsstrahlung rate in an infinite, uniform medium, and integrate.

There is a technical subtlety if one goes to next order in the adiabatic expansion and looks at the prefactor. The assumption of the adiabatic expansion is that $|\dot{\omega}_0| \ll |\omega_0^2|$. To first order in the prefactor, the solution for $C(t_1; t_2)$ is

$$C(t; t_2) \simeq \left[\frac{\omega_0(t_2)}{\omega_0(t)} \right]^{1/2} \exp\left(i \int_t^{t_2} dt' \omega_0(t') \right). \quad (5.9)$$

Then

$$\ln|c(0)| \simeq \frac{1}{\sqrt{2}} \int_0^\infty dt |\omega_0(t)| + \frac{1}{2} \ln \left| \frac{\omega_0(\infty)}{\omega_0(0)} \right|. \quad (5.10)$$

Since $\omega_0(\infty) = 0$, this answer suffers from a logarithmic divergence.

The problem is that the adiabatic assumption $|\dot{\omega}_0| \ll |\omega_0^2|$ must break down at sufficiently late times. As an example, consider the exponential distribution of Sec. IV B. The adiabatic assumption first breaks down when $|\omega_0|$ drops to $|\omega_0| \sim 1/L$. If we use this value of ω_0 to cut off the logarithm in (5.10), then we find a correction to the bulk result of size $-\frac{1}{2} \ln(|\omega_0(0)|L)$. And in fact, exactly such a correction appears in the large L expansion of the exact result in (4.10), which gives

$$\ln|J_0(2\omega_0(0)L)| = \sqrt{2}|\omega_0(0)L| - \frac{1}{2} \ln(|\omega_0(0)|L) + O(1). \quad (5.11)$$

VI. PAIR PRODUCTION $g \rightarrow q\bar{q}$

Previous results are easily modified for the case of pair production $g \rightarrow q\bar{q}$. First, one uses the appropriate DGLAP vacuum splitting function, so the overall result (1.4) becomes

$$x \frac{d}{dx} (I - I_{\text{vac}}) = \frac{\alpha}{\pi} x P_{g \rightarrow q}(x) \ln|c(0)|, \quad (6.1)$$

with

$$P_{g \rightarrow q}(x) = N_f t_F [x^2 + (1-x)^2] \quad (6.2)$$

if one sums over all quark flavors. Here x is the momentum fraction of the quark. One must also change the factors in the definition (1.2) of ω_0^2 , as I shall discuss. The only other change necessary is to appropriately change the group factors in Eq. (2.13) for Γ_3 to reflect the different arrangement of color representations in the splitting process from $F \rightarrow AF$ to $A \rightarrow F\bar{F}$. The generalization of (2.13) is¹⁴

¹⁴For this form, see the discussion surrounding Eq. (6.11) and footnote 24 of Ref. [12].

$$\begin{aligned} \Gamma_3(\mathbf{B}, t) &= \frac{1}{2}(C_{R_2} + C_{R_3} - C_{R_1})\bar{\Gamma}_2(x_1\mathbf{B}, t) \\ &+ \frac{1}{2}(C_{R_3} + C_{R_1} - C_{R_2})\bar{\Gamma}_2(x_2\mathbf{B}, t) \\ &+ \frac{1}{2}(C_{R_1} + C_{R_2} - C_{R_3})\bar{\Gamma}_2(x_3\mathbf{B}, t) \end{aligned} \quad (6.3)$$

for a $R_1 \rightarrow R_2 R_3$ splitting process with corresponding momentum fractions

$$x_1 = 1, \quad x_2 = x, \quad x_3 = 1 - x. \quad (6.4)$$

For $s \rightarrow gs$ processes, this gives (2.13). For $g \rightarrow q\bar{q}$, the color factors of (2.13) (or equivalently the momentum fractions) are permuted to

$$\begin{aligned} \Gamma_3(\mathbf{B}, t) &= (C_F - \frac{1}{2}C_A)\bar{\Gamma}_2(\mathbf{B}, t) + \frac{1}{2}C_A\bar{\Gamma}_2(x\mathbf{B}, t) \\ &+ \frac{1}{2}C_A\bar{\Gamma}_2((1-x)\mathbf{B}, t). \end{aligned} \quad (6.5)$$

The resulting value of ω_0^2 replacing (2.26) is then

$$\omega_0^2 = -i \frac{[C_F - x(1-x)C_A]\hat{q}}{2x(1-x)E}. \quad (6.6)$$

VII. CONVERGENCE OF THE OPACITY EXPANSION

The opacity expansion investigated by Wiedemann [22] and Gyulassy, Levai, and Vitev (GLV) [23] involves analyzing bremsstrahlung in the QCD medium by expanding order by order in the number of elastic scatterings. It is interesting to ask what happens if such an expansion is made in a case where the harmonic oscillator calculation of BDMS is valid. An expansion in powers of elastic collisions is equivalent to an expansion in powers of Γ_3 (2.13), which in harmonic oscillator approximation is equivalent to an expansion in powers of ω_0^2 (2.26).

Now consider BDMS's result (1.1) for the brick problem, and rewrite it in the form

$$\begin{aligned} \omega \frac{d}{d\omega} (I - I_{\text{vac}}) &= \frac{\alpha}{2\pi} x P_{s \rightarrow g}(x) \ln[\cos(e^{i\pi/4} z^{1/2}) \\ &\times \cos(e^{-i\pi/4} z^{1/2})], \end{aligned} \quad (7.1)$$

where

$$z \equiv |\omega_0^2|L^2. \quad (7.2)$$

The opacity expansion of this result is its Taylor series in z , proportional to

$$\begin{aligned} &\ln[\cos(e^{i\pi/4} z^{1/2}) \cos(e^{-i\pi/4} z^{1/2})] \\ &= \frac{1}{6} z^2 - \frac{17}{1260} z^4 + \frac{691}{467775} z^6 - \dots \end{aligned} \quad (7.3)$$

Mathematically, the expression (7.1) is an analytic function of z , and therefore its radius of convergence is given by the distance to the nearest singularity in the complex z plane. The nearest singularities are the branch points of the logarithm where either of the cosines vanish, at $z = \pm i(\pi/2)^2$. In this example, the opacity expansion therefore

only convergences for $|z| < (\pi/2)^2$, which corresponds to

$$L < \frac{\pi/2}{|\omega_0|} \quad (\text{brick}). \quad (7.4)$$

Recall that, qualitatively, $1/|\omega_0|$ is of order the formation time. The conclusion is that the opacity expansion does not converge when the medium is thicker than roughly the formation time.

In Fig. 1, I show the function (7.3) vs its expansion to n th order in the opacity expansion for several n . One can see the failure of convergence beyond $z = (\pi/2)^2$.

One of the uses of the opacity expansion has been as a hook to derive general results by summing up the expansion to all orders, arriving at formalism related to BDMPS and Zakharov (for example, as in Ref. [22]). In this case, the lack of convergence of the Taylor series for large L does not matter.

Readers may wonder at the juxtaposition of the opacity expansion and the harmonic oscillator approximation. In the large N_{coh} limit necessary for the harmonic oscillator approximation, $z = 1$ in Fig. 1 represents a very large number of elastic scatterings. But the answer is nonetheless reproduced well by the $n = 4$ curve, which only includes up to four scatterings. How can this be? The reason is that the LPM effect causes even a large number of scatterings to behave like a single scattering if they occur within a distance small compared to the formation time. For this reason, it is possible for just four scatterings, spread out across L , to reproduce the same total bremsstrahlung rate as a large number of scatterings, in leading-log approximation.

I should clarify that the expansion discussed here depends on first making the harmonic oscillator approximation, treating \hat{q} as a constant, and only then making the opacity expansion. So, for instance, I have ignored the fact that the upper limit Q_{\perp} of the logarithm in (2.22) depends on the number n of collisions. In particular, readers familiar with the opacity expansion may wonder at the absence

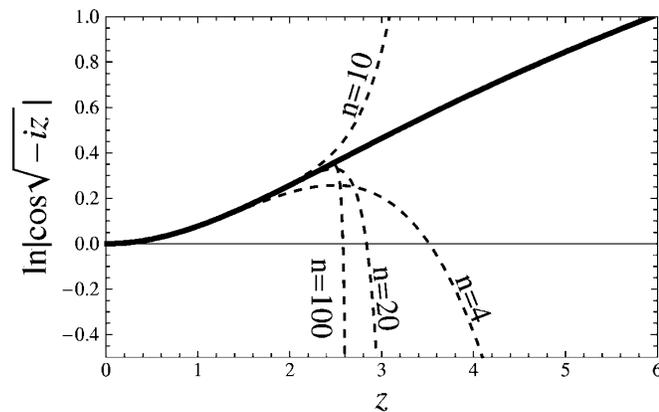


FIG. 1. The function of (7.3) (solid line) vs $z = |\omega_0|^2 L^2$ compared to its Taylor series expansion to n th order for selected values of n (dashed lines).

of a leading $n = 1$ term in the expansion (7.3), proportional to z . This is a special consequence of the leading-log approximation [3,4].¹⁵

One might wonder whether the lack of convergence is an artifact of the brick problem, where $\omega_0^2(t)$ is not an analytic function of time. However, one can draw the same conclusion from the exponential profile (4.10). In this case, the singularity occurs at the first zero of the Bessel function, when its argument is $2.40482\dots$. The corresponding condition for convergence of the opacity expansion in this case is

$$L < \frac{2.40482}{2|\omega_0(0)|} \quad (\text{exponential}). \quad (7.5)$$

These results have been derived in the leading-log approximation, assuming (2.27). In situations where corrections to the leading-log approximation are small, one expects similar conclusions since small perturbations will not remove the presence of singularities. The nonconvergence of the opacity expansion might possibly be related to the observed poor convergence in numerical results at small x by Wicks, shown in Appendix B of Ref. [24], since $L/L_{\text{cr}} \propto x^{-1/2}$ in the small x limit.

ACKNOWLEDGMENTS

I am indebted to Guy Moore and Rudolph Baier for useful discussions. I am also grateful to the referee for bringing my attention to the failure of the harmonic oscillator approximation at sufficiently small L . This work was supported, in part, by the U.S. Department of Energy under Grant No. DE-FG02-97ER41027.

APPENDIX: RELATION OF NOTATION TO OTHER AUTHORS

1. Zakharov

The equations I give in Sec. II B are organized slightly differently than by Zakharov [5]. I will compare my conventions specifically to Ref. [5]. There, Zakharov ignores the fact that effective particle masses in a nonuniform medium will depend on position and time. If one treats them as constant, then their contribution to the Hamiltonian defined by (2.7), (2.12), and (2.13) is an additive constant, and their sole effect is to contribute a simple phase $\exp[-i(\text{constant})\Delta t]$ in the Green function, which Zakharov explicitly factors out. Specifically, the relationship between my Hamiltonian and Green function and those of Zakharov (Z) [5] is

¹⁵Specifically, consider Eq. (6.7) of Ref. [22], using definitions (3.39), (5.6–9) and (5.11) of that reference. The leading-log approximation is $\bar{\sigma}(\rho) \propto \rho^2$, which corresponds to $\bar{\Sigma}(\mathbf{q}_{\perp}) \propto \nabla^2 \delta^{(2)}(\mathbf{q}_{\perp})$. If one uses this form of $\bar{\Sigma}$ and integrates Eq. (6.7) of Ref. [22] over all bremsstrahlung gluon transverse momenta \mathbf{k}_{\perp} (making the $k_{\perp} \ll k$ approximation by integrating all the way up to $k_{\perp} = \infty$), one finds a zero result.

$$H^{(\text{my})} = H^{(Z)} + \frac{1}{L_f^{(Z)}}, \quad (\text{A1})$$

$$G^{(\text{my})}(\mathbf{B}_2, t_2; \mathbf{B}_1, t_1) = \exp\left[-\frac{i(t_2 - t_1)}{L_f^{(Z)}}\right] K^{(Z)}(\mathbf{B}_2, t_2 | \mathbf{B}_1, t_1), \quad (\text{A2})$$

where

$$L_f^{(Z)} \equiv \frac{2x(1-x)E}{x^2 m_s^2 + (1-x)m_g^2} \quad (\text{A3})$$

is what Zakharov calls the formation time. Zakharov chooses to incorporate $1/L_f$ into his Hamiltonian in the later work of Ref. [11].

There is a difference between his and my use of the phrase ‘‘formation time.’’ Zakharov uses it to mean the formation time in vacuum in the case of exactly collinear bremsstrahlung, which is given by the inverse of (2.12) with p_B set to zero. I use it to mean the formation time of typical bremsstrahlung in the medium, consistently accounting for the LPM effect, which is the inverse of (2.12) including the expectation of p_B^2 .

The gradients in (2.6) correspond (up to factors of $+i$ and $-i$) to the operators \mathbf{p} in Zakharov’s definition of $g(\xi_1, \xi_2, x)$.

Finally, the way I have written Zakharov’s three-parton and dipole cross sections σ_3 and σ_2 can be taken from BDMS’s discussion of the equivalence of BDMPS and Zakharov formalisms in Ref. [1], as I shall discuss below.

2. BDMS

Throughout, where BDMS [1] or the earlier works of BDMPS [6–8] expresses rates in terms of density ρ times a

cross section σ , I instead write a rate Γ . This allows one to more easily apply the formulas to calculations that account for the dynamical nature of screening in the plasma.

BDMS and BDMPS characterize the differential elastic cross section in terms of a normalized quantity

$$V(Q^2) \equiv \frac{1}{\sigma_{\text{el}}} \frac{d\sigma_{\text{el}}}{d^2Q}, \quad (\text{A4})$$

where they define $Q \equiv q_{\perp}/m_D$. Translating to the language of rates, one may equivalently write

$$V(Q^2) \equiv \frac{1}{\Gamma_{\text{el}}} \frac{d\Gamma_{\text{el}}}{d^2Q}, \quad (\text{A5})$$

where Γ_{el} is written $\lambda^{-1} = \rho\sigma$ in the BDMPS formalism and λ is the mean-free path for elastic collisions. This expression is problematical for full, leading-order perturbative calculations, however, because the total elastic scattering rate Γ_{el} for a high-energy parton traveling through a QCD plasma has a logarithmic infrared divergence in perturbation theory, as can be seen by integrating (2.16) over d^2q_{\perp} . [The divergence does not appear in the discussions of BDMS and BDMPS because, when they specialize to the case of Coulomb scattering, they model $V(Q^2)$ as proportional to $1/(q_{\perp}^2 + m_D^2)^2$ rather than the actual low-momentum perturbative behavior of (2.16).] The divergence arises from the exchange of low-frequency magnetic gluons, which are not screened, and is cut off only by the nonperturbative physics of magnetic confinement in hot QCD at a momentum scale $q_{\perp} \sim g^2 T$. Formally, it is not clear whether there is any rigorous, convention independent, nonperturbative definition of the total rate $\lambda^{-1} = \Gamma_{\text{el}}$, and so it is best to avoid the quantity altogether. Fortunately, this is just an issue of normalization convention. The various quantities in the BDMPS formulas for the

TABLE I. Translation between notation of this paper and various authors. The entries in the \hat{q} line are logarithmically divergent in the ultraviolet and should be understood as appropriately cut off for a leading-log approximation, as discussed in the text, or equivalently evaluated at some small effective value of B (BDMS) or ρ (Zakharov) of order $1/Q_{\perp}$.

This paper	BDMS [1]	Zakharov [5]	AMY [12]
m_D	μ		m_D
q_{\perp}	$\mu \underline{Q}$		q_{\perp}
$\bar{\Gamma}_{\text{el}}$	$\frac{1}{\lambda C_F} = \frac{\rho\sigma}{C_F}$		
$\frac{d\bar{\Gamma}_{\text{el}}}{d^2q_{\perp}}$	$\frac{V(Q^2)}{\mu^2 \lambda C_F}$		$\frac{g^2}{(2\pi)^2} \int \frac{dq^z}{2\pi} \langle\langle A^-(Q)[A^-(Q)]^* \rangle\rangle_{q^0=q^z}$
p_B	$\mu(\underline{U} - x\underline{V})$	\mathbf{p}	\mathbf{h}/p'
\mathbf{B}	\underline{B}/μ	$\boldsymbol{\rho}$	
\hat{q}	$\frac{\mu^2 \hat{v}(0)}{C_F \lambda}$	$\frac{3}{2} n C_2(0)$	
ω_0	$\frac{\omega_0^* N_c}{2C_F \lambda}$		
t	$\frac{2C_F \lambda \tau}{N_c}$	z or ξ	
M	$\frac{\bar{\kappa} N_c}{2\lambda C_F} = -\frac{2\lambda C_F \mu^2}{N_c} m$	$\mu(x)$	
$G(\mathbf{B}_2, t_2; \mathbf{B}_1, t_1)$	$\mu^2 G^*(\underline{B}_2, z_2; \underline{B}_1, z_1)$	$e^{-i(\xi_2 - \xi_1)/L_f} K(\boldsymbol{\rho}_2, \xi_2 \boldsymbol{\rho}_1, \xi_1)$	
$\bar{\Gamma}_2(\mathbf{B})$	$\frac{1 - \hat{V}(\mathbf{B})}{\lambda C_F}$	$\frac{3}{8} n \sigma_2(\boldsymbol{\rho})$	

bremstrahlung rate appear in combinations where the factors of λ cancel, and I have chosen to avoid them in the formulas of Sec. II B.

A notational translation table is provided in Table I. The reasons for the complex conjugation that appears in some entries of the BDMS column is that BDMS pick conventions where their analog of the Schrödinger equation (2.8) corresponds to a problem with negative mass M . One can convert to a Schrödinger equation with a positive mass (Zakharov's convention, which I adopt) by taking the complex conjugate of the equation, which takes $\psi \rightarrow \psi^*$, $M \rightarrow -M$, and $\omega_0 \rightarrow \omega_0^*$.

In BDMS [1], the quark and gluon masses are ignored. This is parametrically valid when $Q_\perp \gg m_q$ and m_g , which for a *thick* medium ($L \gtrsim L_{\text{cr}}$) corresponds to the high-energy limit $E \gg m_R^4/\hat{q}$. In perturbation theory, where $m_q \sim m_g \sim gT$, this condition is parametrically $E \gg T$. However, in applications of the LPM effect where $E \sim T$ is of interest (such as leading-order calculations of viscosity and other transport coefficients [14]), one should include the mass terms.

Finally, there is an overall minus sign difference between my (2.6) and the comparable Eq. (59) of Ref. [1]. One quick way to resolve minus sign issues is to check that the final answer for the effect of the medium is positive in the limit of a very thick medium, as in (5.8).¹⁶

3. AMY

Next, I wish to make contact with the notation used in my previous work with Moore and Yaffe [12–14]. That analysis was for the case of an infinite, uniform, time-independent medium. Following Migdal [9], one can treat this case by starting with the nonvacuum part of (2.6), changing integration variables from t_2 to the time difference $\Delta t \equiv t_2 - t_1$, and then using time invariance to note that the Green function depends only on Δt . The t_1 integral then just gives a factor of the total time, and the resulting equation for the bremsstrahlung rate is

$$\omega \frac{d\Gamma_{\text{brem}}}{d\omega} = \frac{\alpha x P_{s \rightarrow g}(x)}{[x(1-x)E]^2} \text{Re} \int_0^\infty d(\Delta t) \times [\nabla_{\mathbf{B}_1} \cdot \nabla_{\mathbf{B}_2} G(\mathbf{B}_2, \Delta t; \mathbf{B}_1, 0)]_{B_1=B_2=0}. \quad (\text{A6})$$

Now define

$$\mathbf{f}(\mathbf{B}, t) = 2i[\nabla_{\mathbf{B}_1} G(\mathbf{B}, t; \mathbf{B}_1, 0)]_{B_1=0}, \quad (\text{A7})$$

where the overall normalization of $2i$ is chosen to make contact with AMY conventions. Each component of \mathbf{f} satisfies the same Schrödinger equation (2.8) that G does, so that

¹⁶There appears to be a lost minus sign in the transition from Eqs. (31) and (33) to (51) of Ref. [1], which then propagates to their (59).

$$i\partial_t \mathbf{f}(\mathbf{B}, t) = H\mathbf{f}(\mathbf{B}, t) \quad (\text{A8})$$

with initial condition

$$\mathbf{f}(\mathbf{B}, 0) = -2i\nabla_{\mathbf{B}} \delta^{(2)}(\mathbf{B}). \quad (\text{A9})$$

Now define the time-integrated amplitude

$$\mathbf{f}(\mathbf{B}) \equiv \int_0^\infty dt \mathbf{f}(\mathbf{B}, t). \quad (\text{A10})$$

Integrating both sides of (A8) over time [and noting that $\mathbf{f}(\mathbf{B}, t)$ decays with time because of the $-i\Gamma_3$ piece of H],

$$-2\nabla_{\mathbf{B}} \delta^{(2)}(\mathbf{B}) = H\mathbf{f}(\mathbf{B}). \quad (\text{A11})$$

The rate (A6) can be written in terms of $\mathbf{f}(\mathbf{B})$ as

$$\omega \frac{d\Gamma_{\text{brem}}}{d\omega} = \frac{\alpha x P_{s \rightarrow g}(x)}{[x(1-x)E]^2} \text{Re}[(2i)^{-1} \nabla_{\mathbf{B}} \cdot \mathbf{f}(\mathbf{B})]_{B=0}. \quad (\text{A12})$$

Now Fourier transform from \mathbf{B} to \mathbf{p}_B . Using the form (2.7) of H , the equation for \mathbf{f} becomes

$$\begin{aligned} -2i\mathbf{p}_B &= \delta E(p_B) \mathbf{f}(\mathbf{p}_B) - i \int d^2 q_\perp \frac{d\bar{\Gamma}_{\text{el}}}{d^2 q_\perp} \\ &\times \left[\frac{1}{2} C_A [\mathbf{f}(\mathbf{p}_B) - \mathbf{f}(\mathbf{p}_B + \mathbf{q}_\perp)] + \left(C_s - \frac{1}{2} C_A \right) \right. \\ &\times [\mathbf{f}(\mathbf{p}_B) - \mathbf{f}(\mathbf{p}_B + x\mathbf{q}_\perp)] \\ &\left. + \frac{1}{2} C_A [\mathbf{f}(\mathbf{p}_B) - \mathbf{f}(\mathbf{p}_B + (1-x)\mathbf{q}_\perp)] \right]. \quad (\text{A13}) \end{aligned}$$

Instead of \mathbf{p}_B , AMY uses the variable $\mathbf{h} \equiv \mathbf{p}_B P$. In the case of bremsstrahlung, they define the momenta of the splitting particles as $p' = P$, $k = xP$, and $p = (1-x)P$. If one defines

$$\mathbf{F}(\mathbf{h}) = P\mathbf{f}(\mathbf{h}/P), \quad (\text{A14})$$

then (A13) becomes

$$\begin{aligned} -2i\mathbf{h} &= \delta E\mathbf{F}(\mathbf{h}) - i \int d^2 q_\perp \frac{d\bar{\Gamma}_{\text{el}}}{d^2 q_\perp} \\ &\times \left[\frac{1}{2} C_A [\mathbf{F}(\mathbf{h}) - \mathbf{F}(\mathbf{h} + p'\mathbf{q}_\perp)] + \left(C_s - \frac{1}{2} C_A \right) \right. \\ &\times [\mathbf{F}(\mathbf{h}) - \mathbf{F}(\mathbf{h} + k\mathbf{q}_\perp)] \\ &\left. + \frac{1}{2} C_A [\mathbf{F}(\mathbf{h}) - \mathbf{F}(\mathbf{h} + p\mathbf{q}_\perp)] \right]. \quad (\text{A15}) \end{aligned}$$

This is equation (6.7) of Ref. [12] if one changes the integration variable from \mathbf{q}_\perp to $-\mathbf{q}_\perp$ in some of the terms and recognizes that

$$\begin{aligned} \frac{d\bar{\Gamma}_{\text{el}}}{d^2 q_\perp} &= \frac{g^2}{(2\pi)^2} \mathcal{A}(q_\perp) \\ &\equiv \frac{g^2}{(2\pi)^2} \int_{-\infty}^{+\infty} \frac{dq^z}{2\pi} \langle\langle A^-(Q)[A^-(Q)]^* \rangle\rangle_{q^0=q^z}. \quad (\text{A16}) \end{aligned}$$

With the same notation, the rate (A12) becomes

$$\begin{aligned}
\omega \frac{d\Gamma_{\text{brem}}}{d\omega} &= x \frac{d\Gamma_{\text{brem}}}{dx} \\
&= \frac{\alpha x P_{s \rightarrow g}(x)}{4[x(1-x)E]^2} \int \frac{d^2 p_B}{(2\pi)^2} \text{Re}[2\mathbf{p}_B \cdot \mathbf{f}(\mathbf{p}_B)] \\
&= \frac{\alpha x P_{s \rightarrow g}(x)}{4x^2(1-x)^2 E^6} \int \frac{d^2 h}{(2\pi)^2} \text{Re}[2\mathbf{h} \cdot \mathbf{F}(\mathbf{h})].
\end{aligned} \tag{A17}$$

This formula can be extracted from the rates per unit volume presented for kinetic theory in AMY Ref. [13], for example, with

$$\begin{aligned}
\gamma_{s \rightarrow gs}(E; xE, (1-x)E) &= \frac{d_s \alpha P_{s \rightarrow g}(x)}{(2\pi)^3 2x^2(1-x)^2 E^5} \\
&\quad \times \int \frac{d^2 h}{(2\pi)^2} \text{Re}[2\mathbf{h} \cdot \mathbf{F}(\mathbf{h})].
\end{aligned} \tag{A18}$$

More simply, if final-state factors of $[1 \pm f(xE)] \times [1 \pm f((1-x)E)]$ are included, it corresponds to Eq. (5) of Jeon and Moore [15] or Eqs. (1.1) and (4.1–2) of Ref. [19]. Jeon and Moore use the symbol $d\Gamma/dt$ to denote rate rather than Γ .

Readers comparing to AMY should beware that AMY uses the symbol Γ to indicate the rate per unit volume, integrating what I call Γ_{brem} over the initial particle's momentum with a factor of its distribution function f and including final-state factors.¹⁷

4. Wiedemann

Finally, I will translate to the notation of Wiedemann and collaborators [22,25–27] as presented in Salgado and Wiedemann [26]. They specialize to the $x \ll 1$ limit of soft bremsstrahlung gluons, but they study more properties of the process, such as the angle between the emitted gluons and the high-energy parton, and what happens when the gluon momentum is so small that the approximation $k_\perp \ll k$ is no longer valid. The basic result, Eq. (2.1) of Ref. [26], is

¹⁷A pernicious factor of 2 that arises when comparing to AMY expressions is that they sum formulas for splitting of particle types $a \rightarrow bc$ over the types b and c . For bremsstrahlung from a quark, this gives rise to a factor of 2 because both $q \rightarrow qg$ and the identical $q \rightarrow gq$ are summed over. For $g \rightarrow gg$ there is no such factor of 2, accounting for the relative factor of 1/2 one needs to include when integrating over the final momentum fractions of two identical particles.

TABLE II. Translation between notation of this paper and Salgado and Wiedemann [26], which studies the $x \ll 1$ limit.

This paper	Salgado and Wiedemann [26]
t_1	y_l
t_2	\bar{y}_l
\mathbf{B}_1	\mathbf{y}
\mathbf{B}_2	\mathbf{u}
\mathbf{B}	\mathbf{r}
\hat{q}	\hat{q}/C_A
M	ω
ω_0	$[(1+i)\sqrt{\frac{\hat{q}}{4\omega}}]^*$
$\omega_0 L$	$[(1+i)\sqrt{\frac{\omega_c}{2\omega}}]^*$
$\Gamma_3(\mathbf{B})$	$\frac{n\sigma(\mathbf{r})}{2}$
$\bar{\Gamma}_2(\mathbf{B})$	$\frac{n\sigma(\mathbf{r})}{2C_A}$

$$\begin{aligned}
\omega \frac{dI}{d\omega} &= \frac{\alpha_s C_R}{(2\pi)^2 \omega^2} 2 \text{Re} \int_{\xi_0}^{\infty} dy_l \int_{y_l}^{\infty} d\bar{y}_l \int du \\
&\quad \times \int_0^{\chi\omega} dk_\perp e^{-ik_\perp \cdot \mathbf{u}} e^{-(1/2) \int_{\bar{y}_l}^{\infty} d\xi n(\xi)\sigma(\mathbf{u})} \frac{\partial}{\partial \mathbf{y}} \cdot \frac{\partial}{\partial \mathbf{u}} \\
&\quad \times \int_{y=0=r(y_l)}^{u=r(\bar{y}_l)} \mathcal{D}\mathbf{r} \\
&\quad \times \exp\left[i \int_{y_l}^{\bar{y}_l} d\xi \frac{\omega}{2} \left(\dot{r}^2 - \frac{n(\xi)\sigma(\mathbf{r})}{i\omega} \right) \right].
\end{aligned} \tag{A19}$$

The limit $k_\perp \leq \chi\omega$ is used to restrict attention to gluon bremsstrahlung in a finite opening angle Θ with $\chi = \sin\Theta$. In this paper, I have put no such restriction, and I have assumed k sufficiently large that $k_\perp \ll k$ dominates. This corresponds to replacing the upper limit $\chi\omega$ on the k_\perp integration by infinity. That integral then generates a factor of $\delta^{(2)}(\mathbf{u})$, which makes the \mathbf{u} integration trivial. Using the fact that their definition of $\sigma(\mathbf{u})$ has $\sigma(0) = 0$, one then obtains

$$\begin{aligned}
\omega \frac{dI}{d\omega} &= \frac{\alpha_s C_R}{\omega^2} 2 \text{Re} \int_{\xi_0}^{\infty} dy_l \int_{y_l}^{\infty} d\bar{y}_l \frac{\partial}{\partial \mathbf{y}} \cdot \frac{\partial}{\partial \mathbf{u}} \int_{y=r(y_l)}^{u=r(\bar{y}_l)} \mathcal{D}\mathbf{r} \\
&\quad \times \exp\left[i \int_{y_l}^{\bar{y}_l} d\xi \frac{\omega}{2} \left(\dot{r}^2 - \frac{n(\xi)\sigma(\mathbf{r})}{i\omega} \right) \right] \Big|_{u=y=0}.
\end{aligned} \tag{A20}$$

This is the small x approximation to (2.6) of this paper, with the notational translations shown in Table II, my convention $\xi_0 = 0$, the bremsstrahlung gluon mass ignored, and the Green function expressed as a path integral.

- [1] R. Baier, Y.L. Dokshitzer, A.H. Mueller, and D. Schiff, Nucl. Phys. **B531**, 403 (1998).
- [2] R. Baier, Y.L. Dokshitzer, A.H. Mueller, and D. Schiff, Phys. Rev. C **58**, 1706 (1998).
- [3] B.G. Zakharov, Pis'ma Zh. Eksp. Teor. Fiz. **73**, 55 (2001) [JETP Lett. **73**, 49 (2001)].
- [4] P. Arnold, arXiv:0903.1081.
- [5] B.G. Zakharov, Pis'ma Zh. Eksp. Teor. Fiz. **63**, 906 (1996) [JETP Lett. **63**, 952 (1996)].
- [6] R. Baier, Y.L. Dokshitzer, A.H. Mueller, S. Peigne, and D. Schiff, Nucl. Phys. **B478**, 577 (1996).
- [7] R. Baier, Y.L. Dokshitzer, A.H. Mueller, S. Peigne, and D. Schiff, Nucl. Phys. **B483**, 291 (1997).
- [8] R. Baier, Y.L. Dokshitzer, A.H. Mueller, S. Peigne, and D. Schiff, Nucl. Phys. **B484**, 265 (1997).
- [9] A.B. Migdal, Phys. Rev. **103**, 1811 (1956).
- [10] B.G. Zakharov, Pis'ma Zh. Eksp. Teor. Fiz. **65**, 585 (1997) [JETP Lett. **65**, 615 (1997)].
- [11] R. Baier, D. Schiff, and B.G. Zakharov, Annu. Rev. Nucl. Part. Sci. **50**, 37 (2000).
- [12] P. Arnold, G.D. Moore, and L.G. Yaffe, J. High Energy Phys. 06 (2002) 030.
- [13] P. Arnold, G.D. Moore, and L.G. Yaffe, J. High Energy Phys. 01 (2003) 030.
- [14] P. Arnold, G.D. Moore, and L.G. Yaffe, J. High Energy Phys. 05 (2003) 051.
- [15] S. Jeon and G.D. Moore, Phys. Rev. C **71**, 034901 (2005).
- [16] R. Baier, A.H. Mueller, D. Schiff, and D.T. Son, Phys. Lett. B **502**, 51 (2001).
- [17] P. Aurenche, F. Gelis, and H. Zaraket, J. High Energy Phys. 05 (2002) 043.
- [18] A. Peshier, J. Phys. G **35**, 044028 (2008).
- [19] P. Arnold and C. Dogan, Phys. Rev. D **78**, 065008 (2008).
- [20] R. Baier and Y. Mehtar-Tani, Phys. Rev. C **78**, 064906 (2008).
- [21] G.D. Moore and D. Teaney, Phys. Rev. C **71**, 064904 (2005).
- [22] U.A. Wiedemann, Nucl. Phys. **B588**, 303 (2000).
- [23] M. Gyulassy, P. Levai, and I. Vitev, Nucl. Phys. **B594**, 371 (2001); Phys. Rev. D **66**, 014005 (2002).
- [24] S. Wicks, arXiv:0804.4704.
- [25] U.A. Wiedemann, Nucl. Phys. **A690**, 731 (2001).
- [26] C.A. Salgado and U.A. Wiedemann, Phys. Rev. D **68**, 014008 (2003).
- [27] A. Kovner and U.A. Wiedemann, arXiv:hep-ph/0304151.