

# Repulsive Casimir effect from extra dimensions and Robin boundary conditions: From branes to pistons

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We evaluate the Casimir energy and force for a massive scalar field with general curvature coupling parameter, subject to Robin boundary conditions on two codimension-one parallel plates, located on a  $(D + 1)$ -dimensional background spacetime with an arbitrary internal space. The most general case of different Robin coefficients on the two separate plates is considered. With independence of the geometry of the internal space, the Casimir forces are seen to be attractive for special cases of Dirichlet or Neumann boundary conditions on both plates and repulsive for Dirichlet boundary conditions on one plate and Neumann boundary conditions on the other. For Robin boundary conditions, the Casimir forces can be either attractive or repulsive, depending on the Robin coefficients and the separation between the plates, what is actually remarkable and useful. Indeed, we demonstrate the existence of an equilibrium point for the interplate distance, which is stabilized due to the Casimir force, and show that stability is enhanced by the presence of the extra dimensions. Applications of these properties in braneworld models are discussed. Finally, the corresponding results are generalized to the geometry of a piston of arbitrary cross section.

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## I. INTRODUCTION

Many of the high-energy theories of fundamental physics are formulated in higher-dimensional spacetimes. In particular, the idea of extra dimensions has been extensively used in supergravity and superstring theories. It is commonly assumed that the extra dimensions are compactified. From the inflationary point of view, universes with compact spatial dimensions, under certain conditions, should be considered a rule rather than an exception [1]. Models involving a compact universe with nontrivial topology play a very important role by providing proper initial conditions for inflation. And compactification of spatial dimensions leads to a number of interesting quantum field theoretical effects, which include instabilities in interacting field theories, topological mass generation, and symmetry breaking.

In the case of nontrivial topology, the boundary conditions imposed on fields give rise to a modification of the spectrum for vacuum fluctuations and, as a result, to Casimir-type contributions in the vacuum expectation values of physical observables (for the topological Casimir effect and its role in cosmology see [2] and references therein). In models of the Kaluza-Klein type, the Casimir effect has been used as a stabilization mechanism for moduli fields and as a source for dynamical compactification of the extra dimensions, in particular, for quantum Kaluza-Klein gravity (see Ref. [3]). The Casimir energy

can also serve as a model for dark energy needed for the explanation of the present accelerated expansion of the universe (see [4] and references therein). In addition, recent measurements of the Casimir forces between macroscopic bodies provide a sensitive test for constraining the parameters of long-range interactions, as predicted by modern unification theories of fundamental interactions [5]. The influence of extra compactified dimensions on the Casimir effect in the classical configuration of two parallel plates has been recently discussed in [6–9], for the case of a massless scalar field with Dirichlet boundary conditions, and in [10–13], for the electromagnetic field for perfectly conducting boundary conditions.

More recently, interest has concentrated on the topic of the Casimir effect in braneworld models with large extra dimensions. In this type of models (for a review see [14]) the concept of brane is used as a submanifold embedded in a higher-dimensional spacetime, on which the standard-model particles are confined. Braneworlds naturally appear in the string/M theory context and provide a novel setup for discussing phenomenological and cosmological issues related with extra dimensions. In braneworld models the investigation of quantum effects is of considerable phenomenological interest, both in particle physics and in cosmology. The braneworld corresponds to a manifold with boundaries. All fields which propagate in the bulk will give Casimir-type contributions to the vacuum energy and, as a result, to the vacuum forces acting on the branes. Casimir forces provide a natural mechanism for stabilizing the radion field in the Randall-Sundrum model, as required

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for a complete solution of the hierarchy problem. In addition, the Casimir energy gives a contribution to both the brane and the bulk cosmological constants. Hence, it has to be taken into account in any self-consistent formulation of the braneworld dynamics. The Casimir energy and corresponding Casimir forces within the framework of the Randall-Sundrum braneworld [15] have been evaluated in Refs. [16–22] by using both dimensional and zeta function regularization methods. Local Casimir densities were considered in Refs. [23,24]. The Casimir effect in higher-dimensional generalizations of the Randall-Sundrum model with compact internal spaces has been investigated in [25–30].

The purpose of the present paper is to study the Casimir energy and force for a massive scalar field with an arbitrary curvature coupling parameter, obeying Robin boundary conditions on two codimension one parallel plates which are embedded in the background spacetime  $R^{(D_1-1,1)} \times \Sigma$ , being  $\Sigma$  an arbitrary compact internal space. The most general case is considered, where the constants in the boundary conditions are different for the two separate plates. It will be shown that Robin boundary conditions with different coefficients are necessary to obtain repulsive Casimir forces. Robin type conditions are an extension of Dirichlet and Neumann boundary conditions and genuinely appear in a variety of situations, including vacuum effects for a confined charged scalar field in external fields [31], spinor and gauge field theories, quantum gravity and supergravity [32]. Robin conditions can be made conformally invariant, while purely-Neumann conditions cannot. Therefore, Robin type conditions are needed when one deals with conformally invariant theories in the presence of boundaries and wishes to preserve this invariance. It is interesting to note that a quantum scalar field satisfying Robin conditions on the boundary of a cavity violates the Bekenstein's entropy-to-energy bound near certain points in the space of the parameter defining the boundary conditions [33]. Robin boundary conditions are an extension of those imposed on perfectly conducting boundaries and may, in some geometries, be useful for modelling the finite penetration of the field through the boundary, the skin-depth parameter being related to the Robin coefficient [34,35]. In other words, those are the boundary conditions which are more suitable to describe physically realistic situations. This type of boundary conditions naturally arise for scalar and fermion bulk fields in the Randall-Sundrum model [18,24,36] and the corresponding Robin coefficients are related to the curvature scale and to the boundary mass terms of the field. Robin boundary conditions also appear in the study of Casimir forces between the boundary planes of films (for a recent discussion with references see, for instance, [37]). The Casimir effect in the geometry of two parallel plates with Robin boundary condition was investigated in Refs. [35,38–41]. Note moreover that boundary problems with nonlocal boundary conditions can also be

reduced to corresponding ones with Robin conditions, with the coefficients depending on the wave vector components along the plates [42].

The outline of the paper is as follows. In the next section we will consider the geometry of the problem and the corresponding eigenfunctions. The Casimir energy for two parallel plates in the general case for the internal subspace is evaluated in Sec. III. The boundary-free and single plate parts will be extracted in a cutoff independent way. Applications to braneworlds are then discussed. In Sec. IV we consider the Casimir forces and show that depending on the coefficients in the boundary conditions these forces can be either attractive or repulsive. The asymptotic behavior of the forces, for small and large interplate distances, is given. As an application of the general results, in Sec. V, a simple example with an internal space  $S^1$  is discussed, for general periodicity condition along the compactified dimension. For this special example, we also present the boundary-free part and extract from the single plate parts the topological contributions. The corresponding generalizations for the internal spaces  $(S^1)^N$  and  $S^N$  are also given in detail. In Sec. VI we extend the results for the Casimir energy and force to the case of the geometry of a piston with arbitrary cross section. Section VII contains a summary of the work.

## II. GEOMETRY OF THE PROBLEM AND EIGENFUNCTIONS

We consider a scalar field  $\varphi(x)$ , with arbitrary curvature coupling parameter  $\zeta$ , satisfying the equation of motion

$$(g^{MN}\nabla_M\nabla_N + m^2 + \zeta R)\varphi(x) = 0, \quad (1)$$

$M, N = 0, 1, \dots, D$ , with  $R$  being the scalar curvature for a  $(D + 1)$ -dimensional background spacetime (for the metric signature and the curvature tensor we adopt the conventions of Ref. [43]). For the special cases of minimally and of conformally coupled scalars one has, respectively,  $\zeta = 0$  and  $\zeta = \zeta_D \equiv (D - 1)/4D$ . We will assume that the background spacetime has a topology  $R^{(D_1,1)} \times \Sigma$ , where  $R^{(D_1,1)}$  is  $(D_1 + 1)$ -dimensional Minkowski spacetime and  $\Sigma$  a  $D_2$ -dimensional internal manifold,  $D = D_1 + D_2$ . The corresponding line element has the form

$$ds^2 = g_{MN}dx^M dx^N = \eta_{\mu\nu}dx^\mu dx^\nu - \gamma_{il}dX^i dX^l, \quad (2)$$

with  $\eta_{\mu\nu} = \text{diag}(1, -1, \dots, -1)$  being the metric for the  $(D_1 + 1)$ -dimensional Minkowski spacetime and the coordinates  $X^i$  cover the manifold  $\Sigma$ . Here and below  $\mu, \nu = 0, 1, \dots, D_1$  and  $i, l = 1, \dots, D_2$ . For the scalar curvature of the metric tensor, from (2) one has  $R = -R_{(\gamma)}$ , where  $R_{(\gamma)}$  is the scalar curvature for the metric tensor  $\gamma_{il}$ .

Our main interests in this paper will be to study the Casimir energy density and the mutual forces occurring for the geometry of two parallel infinite plates of codimension one, located at  $x^{D_1} = a_1$  and  $x^{D_1} = a_2$ ,  $a_1 < a_2$ . As most

general set up, we assume that on these boundaries the scalar field obeys Robin boundary conditions

$$(1 + \beta_j n^M \nabla_M) \varphi(x) = [1 + \beta_j (-1)^{j-1} \partial_{D_1}] \varphi(x) = 0, \\ x^{D_1} = a_j, \quad j = 1, 2, \quad (3)$$

with constant coefficients  $\beta_j$ . For  $\beta_j = 0$  these boundary conditions are reduced to Dirichlet one and for  $\beta_j = \infty$  to Neumann boundary conditions. The choice of different boundary conditions on the plates may correspond physically to use of different materials for plates. The imposition of boundary conditions on the quantum field changes the spectrum for the zero-point fluctuations and leads to the modification of the vacuum expectation values for physical quantities, as compared with the same situation without boundaries.

In the region between the plates,  $a_1 < x^{D_1} < a_2$ , the corresponding eigenfunctions, satisfying the boundary condition on the plate at  $x^{D_1} = a_j$ , can be expressed in the decomposed form:

$$\varphi_\alpha(x^M) = C_\alpha \exp\left(-i \sum_{\mu, \nu=0}^{D_1-1} \eta_{\mu\nu} k^\mu x^\nu\right) \\ \times \cos[k^{D_1} |x^{D_1} - a_j| + \alpha_j] \psi_\beta(X), \quad (4)$$

where  $\alpha$  denotes a set of quantum numbers specifying the solution and

$$k^0 = \omega = \sqrt{k^2 + (k^{D_1})^2 + m_\beta^2}, \quad m_\beta^2 = \lambda_\beta^2 + m^2, \\ k = |\mathbf{k}|, \quad \mathbf{k} = (k^1, \dots, k^{D_1-1}). \quad (5)$$

In Eq. (4), the  $\alpha_j$ ,  $j = 1, 2$ , are defined by the relations

$$\sin \alpha_j = \frac{1}{\sqrt{(k^{D_1})^2 \beta_j^2 + 1}}, \quad \cos \alpha_j = \frac{k^{D_1} \beta_j}{\sqrt{(k^{D_1})^2 \beta_j^2 + 1}}. \quad (6)$$

The modes  $\psi_\beta(X)$  are the eigenfunctions of the operator  $\Delta_{(\gamma)} + \zeta R_{(\gamma)}$ :

$$[\Delta_{(\gamma)} + \zeta R_{(\gamma)}] \psi_\beta(X) = -\lambda_\beta^2 \psi_\beta(X), \quad (7)$$

with eigenvalues  $\lambda_\beta^2$ , and fulfill the normalization condition

$$\int d^{D_2} X \sqrt{\gamma} \psi_\beta(X) \psi_{\beta'}^*(X) = \delta_{\beta\beta'}. \quad (8)$$

In Eq. (7),  $\Delta_{(\gamma)}$  is the Laplace-Beltrami operator for the metric  $\gamma_{il}$ . In the consideration below we will assume that  $\lambda_\beta \geq 0$ .

From the boundary condition on the second plate one obtains that the eigenvalues for  $k^{D_1}$  are solutions of the equation

$$F(z) = (1 - b_1 b_2 z^2) \sin z - (b_1 + b_2) z \cos z = 0, \\ z = a k^{D_1}, \quad a = a_2 - a_1, \quad b_j = \beta_j / a. \quad (9)$$

We denote by  $z = z_n$ ,  $n = 1, 2, \dots$ , the zeros of the function  $F(z)$  in the right half-plane of the complex variable  $z$ , arranged in ascending order,  $z_n < z_{n+1}$ . In the discussion below we will assume that all these zeros are real. This is the case for the conditions (see [38])  $\{b_1 + b_2 \geq 1, b_1 b_2 \leq 0\} \cup \{b_{1,2} \leq 0\}$ . The coefficient  $C_\alpha$  in (4) is determined from the orthonormality condition for the eigenfunctions, and is equal to

$$C_\alpha^2 = \frac{(2\pi)^{1-D_1}}{\omega(z_n) a} \left[ 1 + \frac{1}{z_n} \sin(z_n) \cos(z_n + 2\alpha_j) \right]^{-1}, \quad (10)$$

being  $\omega(z_n) = \sqrt{k^2 + z_n^2/a^2 + m_\beta^2}$  the eigenfrequencies.

### III. THE CASIMIR ENERGY

The vacuum energy in the region between the plates (per unit volume along the directions  $x^1, \dots, x^{D_1-1}$ ) is given by the formal expression

$$E_{[a_1, a_2]} = \frac{1}{2} \int \frac{d\mathbf{k}}{(2\pi)^{D_1-1}} \sum_{\beta} \sum_{n=1}^{\infty} \sqrt{k^2 + z_n^2/a^2 + m_\beta^2}. \quad (11)$$

In the discussion below we will assume that some cutoff function is present, without writing it explicitly. Alternatively, one can use zeta function regularization, that yields the same result. For the sum over  $n$  we use the summation formula [38,44]

$$\sum_{n=1}^{\infty} \frac{\pi f(z_n)}{1 + \sin(z_n) \cos(z_n + 2\alpha_j)/z_n} \\ = -\frac{\pi}{2} \frac{f(0)}{1 - b_2 - b_1} + \int_0^\infty dz f(z) \\ + i \int_0^\infty dz \frac{f(iz) - f(-iz)}{\frac{(b_1 z - 1)(b_2 z - 1)}{(b_1 z + 1)(b_2 z + 1)} e^{2z} - 1}. \quad (12)$$

By taking into account the relation

$$1 + \frac{\sin(z_n)}{z_n} \cos(z_n + 2\alpha_j) = 1 - \sum_{j=1}^2 \frac{b_j}{1 + b_j^2 z_n^2}, \quad (13)$$

we see that the sum on the left-hand side of (12) coincides with the corresponding sum in the vacuum energy, if we take

$$f(z) = \sqrt{k^2 + z^2/a^2 + m_\beta^2} \left( 1 - \sum_{j=1}^2 \frac{b_j}{1 + b_j^2 z^2} \right). \quad (14)$$

The application of the summation formula (12) with (14) allows us to write the vacuum energy from (11) in the decomposed form

$$E_{[a_1, a_2]} = aE_{R^{(D_1,1)} \times \Sigma} + \sum_{j=1,2} E_j + \Delta E_{[a_1, a_2]}, \quad (15)$$

where we have introduced the notations

$$E_j = -\frac{1}{8} \int \frac{d\mathbf{k}}{(2\pi)^{D_1-1}} \sum_{\beta} \sqrt{k^2 + m_{\beta}^2} - \frac{\beta_j}{2\pi} \\ \times \int \frac{d\mathbf{k}}{(2\pi)^{D_1-1}} \sum_{\beta} \int_0^{\infty} dx \frac{\sqrt{k^2 + x^2 + m_{\beta}^2}}{1 + \beta_j^2 x^2}, \quad (16)$$

and

$$\Delta E_{[a_1, a_2]} = -\frac{1}{\pi} \int \frac{d\mathbf{k}}{(2\pi)^{D_1-1}} \sum_{\beta} \int_{\sqrt{k^2 + m_{\beta}^2}}^{\infty} dz \\ \times \frac{\sqrt{z^2 - k^2 - m_{\beta}^2}}{\frac{(\beta_1 z - 1)(\beta_2 z - 1)}{(\beta_1 z + 1)(\beta_2 z + 1)} e^{2az} - 1} \left( a + \sum_{j=1}^2 \frac{\beta_j}{\beta_j^2 z^2 - 1} \right). \quad (17)$$

In Eq. (15),

$$E_{R^{(D_1,1)} \times \Sigma} = \frac{1}{2} \int \frac{d\mathbf{k}_{D_1}}{(2\pi)^{D_1}} \sum_{\beta} \sqrt{k_{D_1}^2 + m_{\beta}^2} \quad (18)$$

is the vacuum energy (per unit volume along the directions  $x^1, \dots, x^{D_1}$ ) in the spacetime of topology  $R^{(D_1,1)} \times \Sigma$  for the case when the plates are absent. In the limit  $a \rightarrow \infty$  the term  $\Delta E_{[a_1, a_2]}$  vanishes and the contribution  $E_j$  can be interpreted as the vacuum energy (per unit volume along the directions  $x^1, \dots, x^{D_1-1}$ ) induced by the presence of the plate located at  $x^{D_1} = a_j$  in the half-space  $x^{D_1} \geq a_j$ . These single plate components do not depend on the location of the plate and do not contribute to the vacuum force acting on the plates. As it will be shown below, the latter is determined by the term  $\Delta E_{[a_1, a_2]}$ . Note that this contribution is finite and that the cutoff function is strictly necessary for the terms  $E_{R^{(D_1,1)} \times \Sigma}$  and  $E_j$  only. In the discussion below we will refer to  $\Delta E_{[a_1, a_2]}$  as the interaction term.

For further simplification of the corresponding expression, we use the relation

$$\int \frac{d\mathbf{k}}{(2\pi)^{D_1-1}} \int_{\sqrt{k^2 + m_{\beta}^2}}^{\infty} dz \sqrt{z^2 - k^2 - m_{\beta}^2} g(z) \\ = \frac{(4\pi)^{1-D_1/2}}{2D_1 \Gamma(D_1/2)} \int_{m_{\beta}}^{\infty} dx (x^2 - m_{\beta}^2)^{D_1/2} g(x). \quad (19)$$

In order to derive this formula, we must first integrate the left-hand side over the angular part of the vector  $\mathbf{k}$  and then change to a new integration variable,  $y = \sqrt{z^2 - k^2 - m_{\beta}^2}$ . After introducing polar coordinates in the  $(k, y)$ -plane and integrating over the polar angle, we get Eq. (19). By using this relation, for the interaction part of the vacuum energy we find

$$\Delta E_{[a_1, a_2]} = -\frac{(4\pi)^{-D_1/2}}{\Gamma(D_1/2 + 1)} \sum_{\beta} \int_{m_{\beta}}^{\infty} dx \\ \times \frac{(x^2 - m_{\beta}^2)^{D_1/2}}{\frac{(\beta_1 x - 1)(\beta_2 x - 1)}{(\beta_1 x + 1)(\beta_2 x + 1)} e^{2ax} - 1} \left( a + \sum_{j=1}^2 \frac{\beta_j}{\beta_j^2 x^2 - 1} \right). \quad (20)$$

Using

$$\left( a + \sum_{j=1}^2 \frac{\beta_j}{\beta_j^2 x^2 - 1} \right) \frac{2}{\frac{(\beta_1 x - 1)(\beta_2 x - 1)}{(\beta_1 x + 1)(\beta_2 x + 1)} e^{2ax} - 1} \\ = \frac{d}{dx} \ln \left[ 1 - \frac{(\beta_1 x - 1)(\beta_2 x - 1)}{(\beta_1 x + 1)(\beta_2 x + 1)} e^{-2ax} \right], \quad (21)$$

and integrating by parts, the interaction term in the vacuum energy can be written as

$$\Delta E_{[a_1, a_2]} = \frac{(4\pi)^{-D_1/2}}{\Gamma(D_1/2)} \sum_{\beta} \int_{m_{\beta}}^{\infty} dx x (x^2 - m_{\beta}^2)^{D_1/2-1} \\ \times \ln \left[ 1 - \frac{(\beta_1 x + 1)(\beta_2 x + 1)}{(\beta_1 x - 1)(\beta_2 x - 1)} e^{-2ax} \right]. \quad (22)$$

In the case when the internal space is absent and for a massless scalar field this result reduces to the one derived in [38]. Note that the bulk divergences in the vacuum energy between the plates are contained in the first term on the right-hand side of (15) and the boundary divergences are contained in the single plate contributions  $E_j$ . The interaction part is unambiguously defined. In particular, it does not depend on the regularization scheme used (see, for example, Ref. [38] for the case without the internal space, where exactly the same result is obtained with zeta function techniques).

For the special cases of Dirichlet and Neumann boundary conditions on both plates, from (22) one finds

$$\Delta E_{[a_1, a_2]}^{(J,J)} = -\frac{(4\pi)^{-D_1/2} a}{\Gamma(D_1/2 + 1)} \sum_{\beta} \int_{m_{\beta}}^{\infty} dx \frac{(x^2 - m_{\beta}^2)^{D_1/2}}{e^{2ax} - 1}, \quad (23)$$

with  $J = D$  and  $J = N$  for Dirichlet and Neumann boundary conditions, respectively. By expanding the factor  $1/(e^{2ax} - 1)$  in the integrand one gets

$$\int_{m_{\beta}}^{\infty} dx \frac{(x^2 - m_{\beta}^2)^{D_1/2}}{e^{2ax} - 1} \\ = \frac{\Gamma(D_1/2 + 1)}{\sqrt{\pi} a^{D_1+1}} \sum_{n=1}^{\infty} \left( \frac{am_{\beta}}{n} \right)^{(D_1+1)/2} K_{(D_1+1)/2}(2nam_{\beta}), \quad (24)$$

being  $K_{\nu}(z)$  the Mac-Donald (or modified Bessel) function. This allows us to write the corresponding vacuum energy for Dirichlet and Neumann scalars as

$$\Delta E_{[a_1, a_2]}^{(J,I)} = -\frac{2a^{-D_1}}{(8\pi)^{(D_1+1)/2}} \sum_{\beta} \sum_{n=1}^{\infty} \frac{f_{(D_1+1)/2}(2nam_{\beta})}{n^{D_1+1}}, \quad (25)$$

with the notation

$$f_{\nu}(z) = z^{\nu} K_{\nu}(z). \quad (26)$$

The energy given by Eq. (25) is always negative and the corresponding Casimir forces are attractive for all interplate distances, as will be shown below. In the case  $D_1 = 3$  and for a massless scalar field, Eq. (25) reduces to the expression given in Ref. [8], where the zeta function method was used.

For Dirichlet boundary conditions on one plate and Neumann boundary conditions on the other, similarly to (20) we get

$$\begin{aligned} \Delta E_{[a_1, a_2]}^{(D,N)} &= \frac{(4\pi)^{-D_1/2} a}{\Gamma(D_1/2 + 1)} \sum_{\beta} \int_{m_{\beta}}^{\infty} dx \frac{(x^2 - m_{\beta}^2)^{D_1/2}}{e^{2ax} + 1} \\ &= -\frac{2a^{-D_1}}{(8\pi)^{(D_1+1)/2}} \sum_{\beta} \sum_{n=1}^{\infty} \frac{f_{(D_1+1)/2}(2nam_{\beta})}{(-1)^n n^{D_1+1}}. \end{aligned} \quad (27)$$

In this case the energy  $\Delta E_{[a_1, a_2]}$  is always positive and the corresponding vacuum forces are repulsive for all distances between the plates.

By using the result (22), in a similar way as in [21], we obtain the corresponding Casimir energy for a conformally coupled massless scalar field  $\bar{\varphi}(x)$  on the background of a spacetime with metric tensor  $\bar{g}_{MN} = \Omega^2(x^{D_1})g_{MN}$ , where the metric  $g_{MN}$  is defined by the line element (2). We assume that the field obeys the boundary conditions:

$$(1 + \bar{\beta}_j n^M \nabla_M) \bar{\varphi}(x) = [1 + (-1)^{j-1} \Omega_j^{-1} \bar{\beta}_j \partial_{D_1}] \bar{\varphi}(x) = 0, \quad (28)$$

$$\Omega_j = \Omega(x_j^{D_1}),$$

on two codimension-one branes with coordinates  $x^{D_1} = a_j$ ,  $j = 1, 2$ . The corresponding results for the interaction part of the Casimir energy can be derived from those obtained before simply by using the conformal relation that relates the two problems. The fields are connected by the formula  $\bar{\varphi}(x) = \Omega^{(1-D)/2} \varphi(x)$ . Making use of this relation, from Eqs. (3) and (28) we obtain the following relations between the Robin coefficients:

$$\beta_j = \left[ \Omega_j + (-1)^j \frac{D-1}{2\Omega_j} \bar{\beta}_j \Omega_j' \right]^{-1} \bar{\beta}_j, \quad (29)$$

where  $\Omega_j' = \Omega_j'(x_j^{D_1})$ . We conclude that for a conformally coupled massless scalar field with boundary conditions (28), the interaction part of the vacuum energy in the region between the branes is given by (22), where the coefficients  $\beta_j$  are defined by the relations (29). In particular, for the case of Neumann boundary conditions ( $1/\bar{\beta}_j = 0$ ), one has  $\beta_j = 2\Omega_j(-1)^j / [(D-1)\Omega_j']$ . In the special case of the AdS bulk used in the Randall-Sundrum braneworld model

[15] (note that in this model only the inside region between the branes is considered) we have  $\Omega(x^{D_1}) = r_D/x^{D_1}$ , being  $r_D$  the AdS radius. The corresponding Robin coefficients for an untwisted scalar are given by the relations [18,24,27,36]

$$\bar{\beta}_j^{-1} = (-1)^j c_j / 2 - 2D\zeta / r_D, \quad (30)$$

where  $c_1$  and  $c_2$  are the mass parameters in the surface action of the scalar field for the left and right branes, respectively. For a twisted scalar field, Dirichlet boundary conditions are obtained on both branes.

To summarize, as we see, in the case of the warped geometry the corresponding vacuum energy is not, in general, a monotonic function of the interbrane distance and can display a minimum, corresponding to the stable equilibrium point. This property can be used in braneworld models for the stabilization of the radion field. An important difference between the warped geometry and the one discussed before is that now the single brane contributions to the vacuum energy depend on the location of the brane and, hence, give additional contributions to the force acting on the brane. The divergences in the single brane components are absorbed by adding the respective counterterms to the brane action. The coefficients of these counterterms are not computable within the framework of the low-energy effective theory and should be considered as parameters which are fixed by imposing renormalization conditions on the corresponding effective potential (see also the discussions in Refs. [16,18,19,25,26,28,45]).

#### IV. THE CASIMIR FORCE

The vacuum energy corresponding to the region  $0 \leq x^l \leq c_l$ ,  $l = 1, \dots, D_1 - 1$ ,  $a_1 \leq x^{D_1} \leq a_2$  will be denoted  $E_{[a_1, a_2]} c_1 \cdots c_{D_1-1}$ . The volume of this region is  $V = V_{\Sigma} c_1 \cdots c_{D_1-1} a$ , being  $V_{\Sigma}$  the volume of the internal space. The corresponding vacuum stress at  $x^{D_1} = a_1 +$  is given by

$$P(a_1+) = -\frac{\partial}{\partial V} E_{[a_1, a_2]} c_1 \cdots c_{D_1-1} = P_0 + \Delta P(a_1+), \quad (31)$$

where

$$P_0 = -\frac{E_{R^{(D_1,1)} \times \Sigma}}{V_{\Sigma}}, \quad \Delta P(a_1+) = -\frac{1}{V_{\Sigma}} \frac{\partial}{\partial a} \Delta E_{[a_1, a_2]}. \quad (32)$$

Using Eq. (22), we find

$$\begin{aligned} \Delta P(a_1+) &= -\frac{2(4\pi)^{-D_1/2}}{V_{\Sigma} \Gamma(D_1/2)} \sum_{\beta} \int_{m_{\beta}}^{\infty} dx x^2 \\ &\quad \times \frac{(x^2 - m_{\beta}^2)^{D_1/2-1}}{\frac{(\beta_1 x - 1)(\beta_2 x - 1)}{(\beta_1 x + 1)(\beta_2 x + 1)} e^{2ax} - 1}. \end{aligned} \quad (33)$$

As can be easily seen, the vacuum stress at  $x^{D_1} = a_2 -$  is given by the same expression:  $\Delta P(a_2-) = \Delta P(a_1+)$ .

For the geometry of two parallel plates, the total vacuum energy is the sum of the contributions from the regions  $x^{D_1} \leq a_1$ ,  $a_1 \leq x^{D_1} \leq a_2$  and  $a_2 \leq x^{D_1}$ . When investigating the resulting force on the plate at  $x^{D_1} = a_1$ , in order to deal with finite regions from both sides, we will consider a pistonlike geometry (with large transverse dimensions, for a piston with finite cross section see below), assuming the presence of an additional plate located at  $x^{D_1} = a_0 < a_1$ . For the corresponding vacuum stress at  $x^{D_1} = a_1 -$ , one has

$$\begin{aligned} P(a_1-) &= P_0 + \Delta P(a_1-), \\ \Delta P(a_1-) &= -\frac{1}{V_\Sigma} \frac{\partial}{\partial b} \Delta E_{[b_1, a_1]}, \end{aligned} \quad (34)$$

with  $b = a_1 - a_0$ . The resulting pressure on the plate at  $x^{D_1} = a_1$  is given by the difference

$$P(a_1) = \Delta P(a_1+) - \Delta P(a_1-). \quad (35)$$

As we see, the contributions to the vacuum force coming from the term  $P_0$  are the same from the left and from the right sides of the plate, so that there is no netto contribution to the effective force. In the limit  $a_0 \rightarrow -\infty$  one has  $\Delta P(a_1-) \rightarrow 0$  and the Casimir force acting on the plate at  $x^{D_1} = a_j$ ,  $j = 1, 2$ , in the original two-plate geometry is given by the expression

$$P = -\frac{2(4\pi)^{-D_1/2}}{V_\Sigma \Gamma(D_1/2) a^{D_1+1}} \sum_\beta \int_{am_\beta}^\infty dx \frac{x^2 (x^2 - a^2 m_\beta^2)^{D_1/2-1}}{\frac{(b_1 x - 1)(b_2 x - 1)}{(b_1 x + 1)(b_2 x + 1)} e^{2x} - 1}. \quad (36)$$

This force is attractive when  $P < 0$  and repulsive when  $P > 0$ . If one does not take into account the contributions from the exterior regions  $x^{D_1} \leq a_1$  and  $x^{D_1} \geq a_2$ , the effective pressure is given by (31) where the renormalized value for  $P_0$  does not depend on the separation of the plates. In the case  $\Sigma = S^1$  and for periodic boundary conditions along the compactified dimension, the renormalized value  $P_0$  is positive (see next section, Eq. (47)) which would correspond to the repulsive force between the plates observed in the first paper of [6] (see also the discussion in Ref. [7]).

As is clearly seen from Eq. (36), the sign of the vacuum stress on the plate is determined by the sign of the integral in this formula. In Fig. 1 we have plotted the location of the zeros for this integral on the  $(b_1, b_2)$ -plane in the case  $D_1 = 3$  and for different values of  $am_\beta$  (figures on the curves). For a given  $am_\beta$ , these zeros are located on two curves symmetric with respect to the line  $b_1 = b_2$ . The integral is positive in the region containing this line and it is negative outside. In particular, the Casimir force between the plates is always attractive for symmetric boundary conditions with  $\beta_1 = \beta_2 < 0$ . This result is a special case of the general theorem [46], which dictates an attrac-

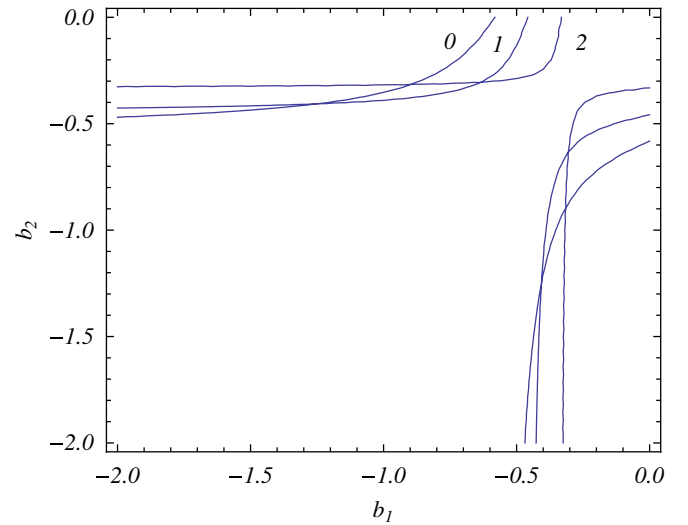


FIG. 1 (color online). The location of the zeros for the integral in (36) in the case  $D_1 = 3$  and for different values of the parameter  $am_\beta$  (numbers near the curves).

tion between bodies with the *same* properties. Note that the curves in Fig. 1 display the locations of the zeros for the Casimir force in the geometry of two parallel plates on the background of a four-dimensional Minkowski spacetime.

For the special cases of Dirichlet and Neumann boundary conditions, making use of the recurrence relations for the function  $K_\nu(z)$ , the Casimir forces can be written as

$$\begin{aligned} P^{(J,J)} &= -\frac{2(4\pi)^{-D_1/2}}{V_\Sigma \Gamma(D_1/2)} \sum_\beta \int_{m_\beta}^\infty dx x^2 \frac{(x^2 - m_\beta^2)^{D_1/2-1}}{e^{2ax} - 1} \\ &= \frac{2a^{-D_1-1}}{(8\pi)^{(D_1+1)/2} V_\Sigma} \sum_\beta \sum_{n=1}^\infty \frac{1}{n^{D_1+1}} [f_{(D_1+1)/2}(2nam_\beta) \\ &\quad - f_{(D_1+3)/2}(2nam_\beta)], \end{aligned} \quad (37)$$

with  $J = D, N$ . Again, in the case  $D_1 = 3$  and for a massless scalar field this result reduces to the one obtained in [8]. The forces described by Eq. (37) are attractive for all distances between the plates, irrespective of the geometry of the internal subspace. For Dirichlet boundary conditions on one plate and Neumann boundary conditions on the other, the expression for the Casimir force is obtained from Eq. (37) after introducing an additional factor  $(-1)^{n+1}$  in the summation over  $n$ . In this case, Casimir forces are repulsive. In the general case of Robin boundary conditions the Casimir forces can be either attractive or repulsive, depending on the coefficients present in the definition of the boundary conditions, and on the distance between the plates. For the special case of the topology  $R^{(D-1,1)} \times S^1$  this issue will be illustrated in the next section.

Let us now consider the asymptotic behavior of the Casimir force as a function of the size of the internal space. Note that if the size of the internal space is of the order  $L$ , then for nonzero modes one has  $\lambda_\beta \sim 1/L$ . For small

values of  $L$  and for the nonzero modes,  $\lambda_\beta$  is large. The contribution of these modes is exponentially suppressed and the main contribution comes from the zero mode. In this case, from (33) we recover the Casimir force for two parallel plates in  $(D_1 + 1)$ -dimensional Minkowskian spacetime, namely

$$V_\Sigma P \approx -\frac{2(4\pi)^{-D_1/2}}{\Gamma(D_1/2)} \int_m^\infty dx \frac{x^2(x^2 - m^2)^{D_1/2-1}}{(\beta_1 x - 1)(\beta_2 x - 1) e^{2ax} - 1}. \quad (38)$$

For the case of a degenerated zero eigenstate the corresponding degeneracy factor must be included on the right-hand side. In some models of compactification the zero mode is absent (for example, in models with twisted boundary conditions along the compactified dimensions, see below). In such cases, for small values of  $L$  the main contribution to the Casimir force comes from the lowest mode  $\lambda_\beta = \lambda_0$  and, to leading order, one gets

$$V_\Sigma P \approx -\frac{m_0^{D_1+1}}{(4\pi)^{D_1/2}} \frac{(\beta_1 m_0 x + 1)(\beta_2 m_0 x + 1)}{(\beta_1 m_0 x - 1)(\beta_2 m_0 x - 1)} \frac{e^{-2am_0}}{(am_0)^{D_1/2}}, \quad (39)$$

where  $m_0 = \sqrt{\lambda_0^2 + m^2}$ . Hence, here the Casimir forces are exponentially suppressed for small sizes of the internal space.

For small values of the interplate distance,  $a/|\beta_j| \ll 1$ , the main contribution into the integral in Eq. (33) comes from larger values of  $x$  and, to leading order, one has

$$P \approx -\frac{2(4\pi)^{-D_1/2}}{V_\Sigma \Gamma(D_1/2)} \sum_\beta \int_{m_\beta}^\infty dx x^2 \frac{(x^2 - m_\beta^2)^{D_1/2-1}}{e^{2ax} - 1}, \quad (40)$$

except for the case of Dirichlet boundary conditions on one plate and non-Dirichlet boundary conditions on the other. We see that in this limit (40) the Casimir force is attractive. However, in the case of Dirichlet boundary condition on one plate and non-Dirichlet boundary condition on the other the Casimir force is repulsive at small distances, what is indeed a remarkable result.

## V. PARTICULAR CASES

As a simple example of a particular application of the general results obtained above, we will first consider the special case where  $\Sigma = S^1$ , with the size of the internal space being  $2\pi L$ . For the compact dimension we assume a general periodicity condition of the form

$$\psi_\beta(X + 2\pi L) = e^{2\pi i \alpha} \psi_\beta(X), \quad (41)$$

with constant  $\alpha$ ,  $0 \leq \alpha \leq 1$ . The specific cases  $\alpha = 0$  and  $\alpha = 1/2$  correspond to untwisted and to twisted fields, respectively. The corresponding part of the scalar eigen-

functions is

$$\psi_\beta(X) = \frac{e^{iKX}}{\sqrt{2\pi L}}, \quad K = (\beta + \alpha)/L, \quad (42)$$

$$\beta = 0, \pm 1, \pm 2, \dots$$

The formulas for the topology  $R^{(D-1,1)} \times S^1$  are obtained from the results given in the previous sections, by taking

$$\sum_\beta = \sum_{\beta=-\infty}^{+\infty}, \quad \lambda_\beta = \frac{|\beta + \alpha|}{L}, \quad (43)$$

$$m_\beta = \sqrt{(\beta + \alpha)^2/L^2 + m^2}, \quad D_1 = D - 1.$$

In particular, for the Casimir force one has

$$P = -\frac{(4\pi)^{-(D-1)/2}}{\pi \Gamma((D-1)/2)L} \sum_{\beta=-\infty}^{+\infty} \int_{m_\beta}^\infty dx$$

$$\times \frac{x^2(x^2 - m_\beta^2)^{(D-3)/2}}{(\beta_1 x - 1)(\beta_2 x - 1) e^{2ax} - 1}. \quad (44)$$

For large values of  $L$  the main contribution to the series in (44) comes from large values of  $\beta$  and one can replace the summation over  $\beta$  by an integration. After some transformations, to leading order we find the result

$$P \approx -\frac{2(4\pi)^{-D/2}}{\Gamma(D/2)} \int_0^\infty dx \frac{x^2(x^2 - m^2)^{D/2-1}}{(\beta_1 x - 1)(\beta_2 x - 1) e^{2ax} - 1}, \quad (45)$$

which is, in fact, the Casimir force for two parallel plates in  $(D + 1)$ -dimensional Minkowskian spacetime.

As we already noted before, depending on the values of the coefficients  $\beta_j$  and of the distance between the plates, the Casimir force (44) can be either attractive or repulsive. In Fig. 2, and corresponding to the model with  $D = 4$  and for a massless scalar field with Dirichlet boundary conditions, we have plotted the ratio  $2\pi LP/P_C$  as a function of  $a/L$ , where  $P_C = -\pi^2/(480a^4)$  is the standard Casimir force. The values on each of the curves correspond to those of the parameter  $\alpha$ . As we have explained before, for  $\alpha \neq 0$  the zero mode is absent and for large values of  $a/L$  the Casimir force is exponentially suppressed.

In Fig. 3 the Casimir force is plotted for the topology  $R^{(3,1)} \times S^1$  and an untwisted ( $\alpha = 0$ ) massless scalar field with Robin coefficients  $\beta_1/a_0 = -0.1$ ,  $\beta_2/a_0 = -0.5$  ( $a_0$  a fixed length scale) as a function of  $a/a_0$  for  $L/a_0 = 1$  (full curve) and  $L/a_0 = 0.5$  (dashed curve). The thick curve corresponds to the Casimir force for two parallel plates in Minkowski spacetime  $R^{(3,1)}$  with the same Robin coefficients. As is seen, the corresponding Casimir forces are attractive for small and large distances between the plates while they are repulsive for intermediate distances. There are two equilibrium points corresponding to the zeros of the function  $P$ . The leftmost point is unstable whereas the rightmost one is stable. Hence, in this case

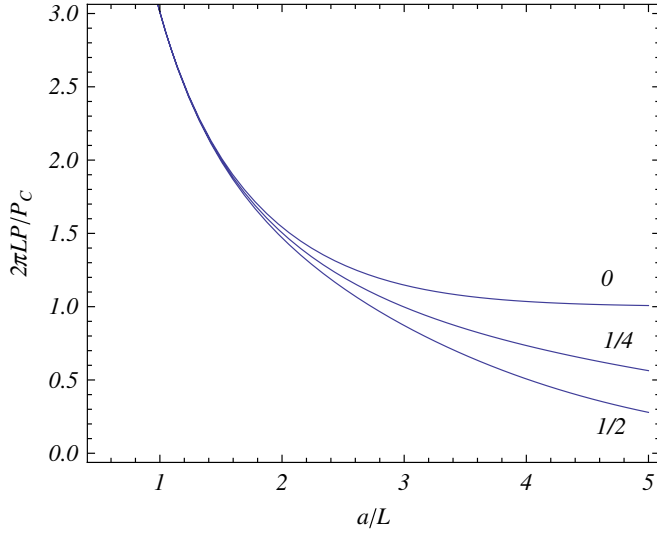


FIG. 2 (color online). Ratio of the Casimir force for two parallel plates in the spacetime with topology  $R^{(3,1)} \times S^1$  to the standard Casimir force in  $R^{(3,1)}$ , for a massless Dirichlet scalar, as a function of  $a/L$ . The values on each of the curves correspond to those of the parameter  $\alpha$ .

the Casimir force *stabilizes* the distance between the plates. This feature can be used in braneworld models for the stabilization of the radion field. We see from Fig. 3 that the height of the barrier between the stable and the unstable equilibrium points is increased by the presence of the internal space. As a consequence, an enhancement of the repulsive Casimir effect, coming from the extra dimension, occurs.

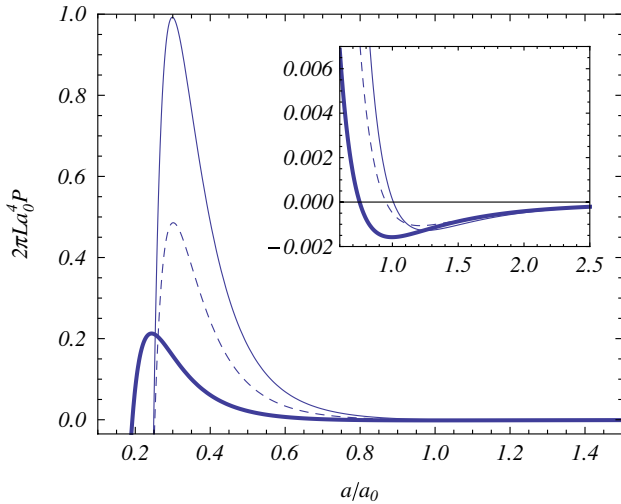


FIG. 3 (color online). Casimir force for two parallel plates in the spacetime with topology  $R^{(3,1)} \times S^1$ , for an untwisted massless scalar field with Robin coefficients  $\beta_1/a_0 = -0.1$ ,  $\beta_2/a_0 = -0.5$ , as a function of the distance between the plates. The full (dashed) curve corresponds to a size of the internal space with  $L/a_0 = 1$  ( $L/a_0 = 0.5$ ). The thick curve corresponds to the Casimir force for two parallel plates in Minkowski spacetime  $R^{(3,1)}$  with the same Robin coefficients.

As already explained before, in the case of Dirichlet boundary conditions on one plate and non-Dirichlet boundary conditions on the other, the Casimir force is repulsive at small separations. In Fig. 4 we illustrate this feature for the topology  $R^{(3,1)} \times S^1$  in the case of an untwisted massless scalar field with  $\beta_1 = 0$ ,  $\beta_2/a_0 = -0.5$ . As in Fig. 3, the full (dashed) curve stands for  $L/a_0 = 1$  ( $L/a_0 = 0.5$ ) and the thick curve corresponds to the Casimir force for two parallel plates in Minkowski spacetime  $R^{(3,1)}$  with the same Robin coefficients.

For the topology under consideration, the Casimir energy for the bulk without boundaries,

$$E_{R^{(D-1,1)} \times S^1} = \frac{1}{2} \int \frac{d\mathbf{k}_{D-1}}{(2\pi)^{D-1}} \times \sum_{\beta=-\infty}^{+\infty} \sqrt{k_{D-1}^2 + (\beta + \alpha)^2/L^2 + m^2}$$

can be further simplified through the Abel-Plana summation formula, in the form [44,47]

$$\sum_{\beta=-\infty}^{+\infty} f(|\beta + \alpha|) = 2 \int_0^\infty dx f(x) + i \int_0^\infty dx \sum_{\lambda=\pm 1} \frac{f(ix) - f(-ix)}{e^{2\pi(x+i\lambda\alpha)} - 1}, \quad (46)$$

what leads to the result

$$E_{R^{(D-1,1)} \times S^1} = \frac{2\pi L}{2} \int \frac{d^D k}{(2\pi)^D} \sqrt{k_D^2 + m^2} - \frac{2(Lm)^{(D+1)/2}}{(2\pi)^D L^D} \times \sum_{n=1}^\infty \frac{\cos(2\pi n\alpha)}{n^{(D+1)/2}} K_{(D+1)/2}(2\pi nLm). \quad (47)$$

The second term on the right-hand side of this expression is

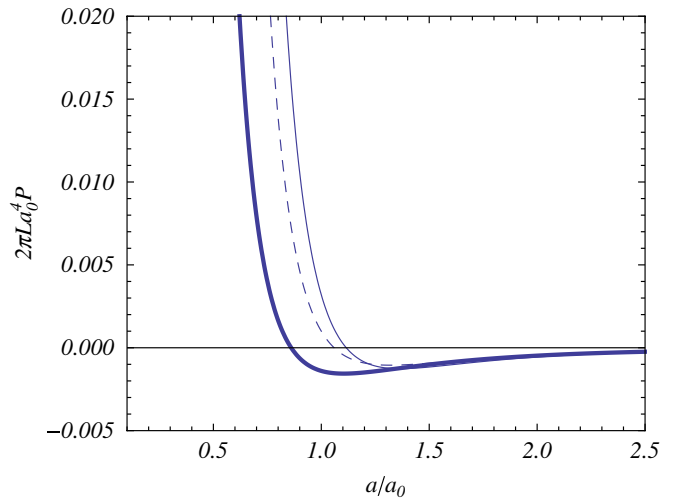


FIG. 4 (color online). Same as in Fig. 3 for a scalar field with Robin coefficients  $\beta_1 = 0$ ,  $\beta_2/a_0 = -0.5$ .



finite and introduction of a cutoff function is necessary for the first term only. Note that the latter is the vacuum energy density for the spatial topology  $R^D$  and, hence, the second term on the right-hand side of Eq. (47) is the contribution to the vacuum energy induced by the compactness of the  $x^D$  dimension. In particular, the topological part of the vacuum energy is always negative (positive) for untwisted (twisted) scalars.

In a similar way, we can also extract the topological contributions in the single plate terms of the vacuum energy. After applying the summation formula (46) and after integration, these contributions yield

$$E_j = 2\pi L E_j^{(M)} + \frac{(Lm)^{D/2}}{2(2\pi L)^{D-1}} \sum_{n=1}^{\infty} \frac{\cos(2\pi n\alpha)}{n^{D/2}} K_{D/2}(2\pi n L m) + \frac{(4\pi)^{-(D-1)/2} L \beta_j}{\Gamma((D+1)/2)} \sum_{\lambda=\pm 1} \int_m^{\infty} dy \frac{(y^2 - m^2)^{D/2}}{e^{2\pi Ly + 2\pi i \lambda \alpha} - 1} \times \int_0^1 dx \frac{(1-x^2)^{(D-1)/2}}{1 + \beta_j^2 (y^2 - m^2) x^2}, \quad (48)$$

where

$$E_j^{(M)} = -\frac{1}{8} \int \frac{d\mathbf{k}_{D-1}}{(2\pi)^{D-1}} \sqrt{k_{D-1}^2 + m^2} - \frac{\beta_j}{2\pi} \int \frac{d\mathbf{k}_{D-1}}{(2\pi)^{D-1}} \int_0^{\infty} dx \frac{\sqrt{k_{D-1}^2 + x^2 + m^2}}{1 + \beta_j^2 x^2} \quad (49)$$

is the vacuum energy (per unit volume along the coordinates  $x^1, \dots, x^{D-1}$ ) for a single plate in Minkowski space-time with trivial topology  $R^{(D,1)}$ . Hence, the last two terms on the right-hand side of (48) are the terms in the vacuum energy corresponding to a single plate and due to the compactness of the dimension  $x^D$ . These terms are finite and renormalization is needed for the Minowskian part  $E_j^{(M)}$  only. Note that for Dirichlet and Neumann boundary conditions the last term on the right of Eq. (48) vanishes.

The case of a  $D_2$ -dimensional torus as internal space,  $\Sigma = (S^1)^{D_2}$ , can be considered in a similar way. For a scalar field with the periodicity condition  $\psi_{\beta}(X^l + 2\pi L_l) = e^{2\pi i \alpha_l} \psi_{\beta}(X^l)$  along the coordinate  $X^l$ ,  $0 \leq X^l \leq 2\pi L_l$ , the formulas for the Casimir energy and force are obtained from the general expressions in Secs. III and IV with the substitutions

$$\sum_{\beta} = \sum_{j_1=-\infty}^{+\infty} \cdots \sum_{j_{D_2}=-\infty}^{+\infty}, \quad (50)$$

$$m_{\beta}^2 = \sum_{l=1}^{D_2} (j_l + \alpha_l)^2 / L_l^2 + m^2.$$

Concerning the issues of embedding the model in string theory and of the discussions of the holographic principle there, the case of the internal space  $\Sigma = S^{D_2}$  is of very special interest. The corresponding eigenfunctions  $\psi_{\beta}(X)$

are expressed in terms of spherical harmonics of degree  $l$ ,  $l = 0, 1, 2, \dots$ . For the internal space with radius  $L$  the expressions for the Casimir energy and Casimir force are quite easily obtained from the general formulas given in the above sections, just by replacing

$$\sum_{\beta} \rightarrow \sum_{l=0}^{\infty} (2l + D_2 - 1) \frac{\Gamma(l + D_2 - 1)}{l! \Gamma(D_2)}, \quad (51)$$

$$\lambda_{\beta} \rightarrow \frac{1}{L} \sqrt{l(l + D_2 - 1) + \zeta D_2 (D_2 - 1)}.$$

Here the factor under the summation sign is the degeneracy of the angular mode with a given  $l$ .

## VI. GENERALIZED PISTON GEOMETRY

In a way very much similar to the procedure described in the preceding sections, we are able to treat the more general case when a part of the dimensions  $x^1, \dots, x^{D_1-1}$  are still constrained by boundary conditions. This corresponds to considering a generalized piston geometry, a quite fashionable situation nowadays, in particular, in the quest for negative Casimir forces (for the investigation of the Casimir effect in a piston geometry see [48] and references therein). Here we have obtained those in the configurations above, but would like to see now the differences introduced in our results by the consideration of piston geometries.

We will denote by  $d_1$  the number of unconstrained dimensions (coordinates  $x^1, \dots, x^{d_1}$ ) and by  $\gamma_i^2$ , with a collective index  $i$ , the eigenvalues of the Laplacian along the constrained directions:

$$\Delta_{D_1-1-d_1} \varphi_{\alpha}(x^M) = -\gamma_i^2 \varphi_{\alpha}(x^M). \quad (52)$$

The eigenfrequencies in the region between the plates are here given by

$$\omega(z_n) = \sqrt{k_{d_1}^2 + z_n^2/a^2 + \gamma_i^2 + m_{\beta}^2}, \quad (53)$$

with the vacuum energy being

$$E_{[a_1, a_2]} = \frac{1}{2} \int \frac{d\mathbf{k}_{d_1}}{(2\pi)^{d_1}} \sum_{i, \beta} \sum_{n=1}^{\infty} \sqrt{k_{d_1}^2 + z_n^2/a^2 + \gamma_i^2 + m_{\beta}^2}. \quad (54)$$

After applying the summation formula (12) to the sum over  $n$ , we can write the energy in the decomposed form

$$E_{[a_1, a_2]} = \frac{a}{2} \int \frac{d\mathbf{k}_{d_1+1}}{(2\pi)^{d_1+1}} \sum_{i, \beta} \sqrt{k_{d_1+1}^2 + \gamma_i^2 + m_{\beta}^2} + \sum_{j=1,2} E_j + \Delta E_{[a_1, a_2]}. \quad (55)$$

Here the expressions for the terms  $E_j$  and  $\Delta E_{[a_1, a_2]}$  are obtained from Eqs. (16) and (17) by the replacements

$$\int \frac{d\mathbf{k}}{(2\pi)^{D_1-1}} \rightarrow \int \frac{d\mathbf{k}_{d_1}}{(2\pi)^{d_1}}, \quad k^2 \rightarrow k_{d_1}^2, \quad (56)$$

$$\sum_{\beta} \rightarrow \sum_{i,\beta}, \quad D_1 \rightarrow d_1 + 1, \quad m_{\beta}^2 \rightarrow \gamma_i^2 + m_{\beta}^2. \quad (57)$$

These formulas are further simplified after integrating over the angular part of the vector  $\mathbf{k}_{d_1}$ . The corresponding

$$\Delta E_{[a_1, a_2]} = \frac{(4\pi)^{-(d_1+1)/2}}{\Gamma((d_1+1)/2)} \sum_{i,\beta} \int_{\sqrt{\gamma_i^2 + m_{\beta}^2}}^{\infty} dx x (x^2 - \gamma_i^2 - m_{\beta}^2)^{(d_1-1)/2} \ln \left[ 1 - \frac{(\beta_1 x + 1)(\beta_2 x + 1)}{(\beta_1 x - 1)(\beta_2 x - 1)} e^{-2ax} \right]. \quad (58)$$

The expression for the Casimir pressure takes the form

$$P(a, \beta_1, \beta_2) = - \frac{2(4\pi)^{-(d_1+1)/2}}{V_{cs} V_{\Sigma} \Gamma((d_1+1)/2)} \sum_{i,\beta} \int_{\sqrt{\gamma_i^2 + m_{\beta}^2}}^{\infty} dx \frac{x^2 (x^2 - \gamma_i^2 - m_{\beta}^2)^{(d_1-1)/2}}{\frac{(\beta_1 x - 1)(\beta_2 x - 1)}{(\beta_1 x + 1)(\beta_2 x + 1)} e^{2ax} - 1}, \quad (59)$$

where  $V_{cs}$  is the volume of the piston cross section along the coordinates  $x^{d_1+1}, \dots, x^{D_1-1}$ . In particular, for Dirichlet and Neumann boundary conditions on the plates, we find

$$P^{(J,J)}(a) = \frac{2a^{-d_1-2}}{(8\pi)^{d_1/2+1} V_{cs} V_{\Sigma}} \sum_{i,\beta} \sum_{n=1}^{\infty} \frac{1}{n^{d_1+2}} \times [f_{d_1/2+1}(z) - f_{d_1/2+2}(z)]_{z=2na\sqrt{\gamma_i^2 + m_{\beta}^2}}, \quad (60)$$

$J = D, N$ , the function  $f_{\nu}(z)$  being defined in Eq. (26). The corresponding force remains attractive independently of the form of the cross section. In the case of Dirichlet boundary condition on one plate and Neumann boundary condition on the other the expression for the Casimir force is obtained from (60) after introducing an additional factor  $(-1)^{n+1}$ , and the resulting force is always repulsive. In the special case  $d_1 = 0$  and for a massless scalar field, Eq. (60) reduces to the formula for the Casimir force in Ref. [8], as it should.

On the base of Eqs. (58) and (59) we can analyze the geometry of a generalized piston with two chambers, assuming that the plates are located at  $x^{D_1} = a_0, a_1, a_2$ , with the Robin coefficient  $\beta_0$  for the left plate (for the Casimir effect in a piston geometry see, for example, Refs. [48]). For an arbitrary cross section, the effective pressure on the plate at  $x^{D_1} = a_1$  is given by

$$P_{a_1} = P(a, \beta_1, \beta_2) - P(b, \beta_0, \beta_1), \quad (61)$$

with  $b = a_1 - a_0$ . For  $P_{a_1} < 0$  ( $P_{a_1} > 0$ ) the resulting force on the plate is directed towards the right (left) plate. For Dirichlet and Neumann boundary conditions the Casimir stress given by Eq. (60) is a monotonic function of the plate separation and, hence, in the piston geometry with two chambers the resulting force (61) is directed toward the closer plate.

In the special case of the geometry of a piston with circular cross section in the plane ( $x^{D_1-2}, x^{D_1-1}$ ), the corresponding part in the eigenfunctions has the form

expressions are obtained from the results of Secs. III and IV, with the replacements (57). In particular, for the interaction part of the Casimir energy (per unit volume along the direction  $x^1, \dots, x^{d_1}$ ) one has

$J_{|q|}(\eta r) e^{iq\phi}$ ,  $q = 0, \pm 1, \pm 2$ , where  $r$  and  $\phi$  are polar coordinates on this plane and  $J_q(z)$  is the Bessel function. The eigenvalues for the quantum number  $\eta$  are quantized by the boundary conditions on the cylindrical surface  $r = r_0$ , with  $r_0$  being the piston radius. For example, in the case of a Dirichlet boundary condition one has  $\eta = j_{|q|,p}/r_0$ , where  $j_{\nu,p}$  is the  $p$ -th positive zero of the function  $J_{\nu}(z)$ . The corresponding formulas for the Casimir energy and forces are obtained from the general results (36) and (58) with the substitutions

$$d_1 = D_1 - 3, \quad \sum_i = \sum_{q=-\infty}^{+\infty} \sum_{p=1}^{\infty}, \quad \gamma_i = j_{|q|,p}/r_0. \quad (62)$$

Other types of boundary conditions on the cylindrical boundary can be considered in a similar way. Moreover, the generalization of our procedure to more than three plates is also easy to carry out.

## VII. CONCLUSION

We have investigated in this paper the influence of extra dimensions on the Casimir energy and on the Casimir force for a massive scalar field with an arbitrary curvature coupling parameter, in the usual geometry of two parallel plates. We have assumed that on the plates the field obeys Robin boundary conditions with, in general, different coefficients for the two different plates. The corresponding eigenfrequencies are expressed in terms of solutions of a transcendental Eq. (9), thus they are known implicitly only. By applying the summation formula (12) to the corresponding series in the mode-sum for the vacuum energy in the region between the plates, we have explicitly extracted, in a cutoff independent way, the boundary-free (topological) part and the contributions induced by the single plates (when the other plate is absent). The remaining interaction part is finite for all nonzero interplate distances and is cutoff independent. The surface divergences in the Casimir energy are contained in the single plate

components only. But the latter do not depend on the location of the plate and do not contribute to the Casimir force. For an arbitrary internal space, the interaction part of the Casimir energy is given by Eq. (22). In the special cases of Dirichlet and Neumann boundary conditions on both plates this formula leads to the result (25) and the corresponding energy is always negative. For Dirichlet boundary conditions on one plate and Neumann boundary conditions on the other, the interaction component of the vacuum energy is given by Eq. (27), and it is positive for all values of the interplate distance. In the case of a conformally coupled massless field on the background of a space-time conformally related to the one described by the line element (2) with the conformal factor  $\Omega^2(x^{D_1})$ , the interaction part of the Casimir energy is given by Eq. (22), with the coefficients  $\beta_j$  being related to the specific coefficients of the Robin boundary conditions (28) and to the conformal factor by Eqs. (29). In the Randall-Sundrum two brane model with a compact internal space, the corresponding Robin coefficients are given by Eq. (30) and the corresponding vacuum energy can have a minimum, corresponding to the stable equilibrium point. This feature is useful in braneworld models for the stabilization of the radion field.

The interaction forces between the plates for the most general case of internal space have been considered in Sec. IV. In order to obtain the resulting force, the contributions from both sides of the plates must be taken into account. Then, the forces coming from the topological parts of the vacuum energy cancel out and only the interaction terms contribute to the Casimir force. In order to show this important fact explicitly, we have considered a pistonlike geometry, by introducing a third plate. At the end of the calculation this plate is sent to infinity. The resulting Casimir force is given by Eq. (36). With independence of the geometry of the internal space, the force is attractive for Dirichlet or Neumann boundary conditions on both plates (formula (37)) and it is repulsive for Dirichlet boundary conditions on one plate and Neumann boundary conditions on the other. In both cases the force is a monotonic function of the distance. For general Robin boundary conditions the Casimir force can be either attractive (corresponding to negative values of  $P$ ) or repulsive (positive values of  $P$ ), depending on the particular Robin coefficients and on the distance between the plates. For small values of the size of the internal space and in models where the zero modes along the internal space are present, the main contribution to the Casimir force comes from the zero modes and the contributions of the nonzero modes are exponentially suppressed. In this limit, to leading order we recover the standard result for the Casimir force between two plates in  $(D_1 + 1)$ -dimensional Minkowski spacetime. When the zero mode is absent (for example, in the case of twisted boundary conditions along the compactified dimensions), the Casimir forces are exponentially sup-

pressed in the limit of small size of the internal space. For small values of the interplate distance the Casimir forces are attractive, independently of the values of the Robin coefficients, except for the case of Dirichlet boundary conditions on one plate and non-Dirichlet boundary conditions on the other. In this latter case, the Casimir force is repulsive at small distances. It is interesting to remark that this property could be used in the proposal of a Casimir experiment with the purpose to carry out an explicit detailed observation of “large” extra dimensions as allowed by some models of particle physics.

As an illustration of the general results, in Sec. V we have considered a special model for the internal space  $\Sigma = S^1$ , with the periodicity condition (41) along the compactified dimension. For the specific values  $\alpha = 0, 1/2$  this condition corresponds to untwisted and twisted scalar fields, respectively. In Fig. 2, for Dirichlet boundary conditions we depicted the dependence of the Casimir force on the distance between the plates for different values of the parameter  $\alpha$ . In Fig. 3 a plot of the Casimir force in the case of Robin boundary conditions on both plates as a function of the interplate distance has been provided. In the example considered, the Casimir force is attractive both for large and for small distances, while it is repulsive at intermediate distances. The Casimir force vanishes at two values of the interplate distance, which correspond to equilibrium points. The leftmost point is unstable and the rightmost one is locally stable. As shown in the plot, the stability of the rightmost equilibrium point is enhanced by the presence of the internal space. Formulas for the Casimir energy and force for the more general internal spaces  $(S^1)^{D_2}$  ( $D_2$ -torus) and  $S^{D_2}$  have been obtained from the general results of Secs. III and IV.

In the last Sec. VI, we have extended the results from the previous ones to the case of a piston geometry with finite cross section of arbitrary form along some subset of the dimensions. The corresponding expressions for the Casimir energy and Casimir force between the plates are given by Eqs. (58) and (59). We have checked that the qualitative features described above remain basically unaltered. In particular, the possibility of a repulsive Casimir effect is again observed. In the special case of a piston geometry of circular cross section on the plane  $(x^{D_1-2}, x^{D_1-1})$ , the corresponding formulas have been specified in (62).

The search for specific applications of this study to practical situations in braneworld models, nanophysics and particle physics will keep us busy for some time.

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