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Kaluza-Klein models as pistons

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We consider the influence of extra dimensions on the force in Casimir pistons. Suitable analytical expressions are provided for the Casimir force in the range where the plate distance is small, and where it is large, compared to the size of the extra dimensions. We show that the Casimir force tends to move the center plate toward the closer wall; this result is true independently of the cross section of the piston and the geometry or topology of the additional Kaluza-Klein dimensions. The statement also remains true at finite temperature. In the limit where one wall of the piston is moved to infinity, the result for parallel plates is recovered. If only one chamber is considered, a criterion for the occurrence of Lukosz-type repulsion, as opposed to the occurrence of renormalization ambiguities, is given; we comment on why no repulsion has been noted in some previous cosmological calculations that consider only two plates.

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I. INTRODUCTION

In recent years Casimir pistons have received an increasing amount of interest because they allow the unambiguous prediction of forces, free of the divergences that often plague Casimir calculations. In their modern form they were introduced by Cavalcanti [1] in a two-dimensional setting. Namely, in his paper a Casimir piston consists of a rectangular box divided by a movable partition into two compartments, A and B, of dimensions $a \times b$ and $(L - a) \times b$, respectively. Imposing Dirichlet boundary conditions, as $L \rightarrow \infty$, it is shown that the piston is attracted to the nearest end of the box. Higher-dimensional pistons have been considered with various boundary conditions [2–9]. Hertzberg *et al.* showed that in three dimensions for perfect metallic boundary conditions the rectangular piston is attracted to the closest base [4,5]; for pistons with rectangular cross sections and Dirichlet or Neumann boundary conditions see also [2,10]. The same conclusion was reached in [3] for perfect magnetic conductor (infinitely permeable) boundary conditions in a rectangular piston of arbitrary dimension. Finally, a unified treatment reached the same conclusion for a scalar field with periodic, Dirichlet, or Neumann boundary conditions and an electromagnetic field with perfect electric conductor or perfect magnetic conductor boundary conditions [6]. However, with the judicious choice of a perfectly conducting piston inside a closed cylinder of arbitrary cross section with infinitely permeable walls, or a Dirichlet piston with Neumann walls, etc., a repulsive force is found [7,11,12]; those results generalize the famous observation of Boyer [13] for parallel plates of unlike nature. This work mostly considered pistons of rectangular cross section, where closed answers can be obtained. An exception is [8] where it was shown that in three dimensions a piston of arbitrary cross section, with all surfaces perfectly conducting, is attracted to the closest wall.

In the limit where the transversal dimensions as well as one of the walls are sent to infinity the configuration of two parallel plates is obtained. This is the procedure used in early calculations of the Casimir force between parallel plates [14–16]. But although in principle pistons were studied at that time, auxiliary plates at finite but large distances were introduced for mere mathematical convenience in contrast to the modern pistons which are studied in their own right. The scalar Casimir force between parallel plates in the presence of compactified extra dimensions has been used to put restrictions on the size of the extra dimensions [17–22]; for more recent discussions of perfectly conducting parallel plates affecting the five-component electromagnetic field in five dimensions (reaching different conclusions) see [23,24]. The Casimir effect was also used to argue against the possibility that vacuum energy plays the role of a cosmological constant responsible for the observed dark energy [25–27]; cutoff scales that could produce the needed dark energy led to the prediction of repulsive Casimir forces at distances between the plates where attraction is verified experimentally [28–30].

In this article we consider three-dimensional pistons of arbitrary cross section in the context of Kaluza-Klein models. We show that in a scalar field theory with Dirichlet or Neumann boundary conditions the piston is attracted to the closest wall and that this statement holds independently of the cross section of the piston, of the

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geometry and topology of the additional Kaluza-Klein dimensions (with one minor caveat), and of the temperature. We use zeta function techniques [31–36] to find closed answers for the Casimir force and to show these results. In the Appendix indications are made how to use path sums (or images) and an ultraviolet cutoff to reach the same conclusions. It should be noted that mathematically the Kaluza-Klein dimensions and the transverse dimensions of the macroscopic piston play very similar roles, only their respective magnitudes relative to the plate separation being quantitatively significant.

The correction to the force due to the additional dimensions is exponentially damped as long as the distance between the plates is large compared to the size of the extra dimensions. However, if the distance is comparable to or smaller than the size of the extra dimensions the standard Casimir force in the space of the full dimension is found. The crossover between the two regimes is complicated, and different representations clearly showing the different behavior are provided. The expansion in terms of the ratio of distance between the plates over size of other dimensions clearly shows how the geometry of the cross section and of the Kaluza-Klein dimensions enters.

Furthermore, it is shown that a Lukosz-type repulsive force [37] can appear only in a naive calculation where just one chamber of the piston is taken into account; for related remarks see [4,5]. Moreover, an unambiguous prediction in such a case is possible only if a particular geometric invariant of the extra dimensions vanishes. A crucial point is that this issue arises in the presence of Kaluza-Klein dimensions even when the large transverse dimensions are infinite; indeed, it is even more cogent there than for a macroscopic box. The present paper developed from a commentary [38] on these points as they arose in the papers of Cheng [17,18], and it fulfills our pledge there to publish the details of our calculations. There is some overlap with an article by Teo [39] that appeared in the meantime and gives special attention to the finite-temperature theory. The recent cosmological papers [19–21,24] do not explicitly consider the outer chamber of a piston, but nevertheless they do not report a repulsive force; we investigate the reason for that apparent discrepancy.

The article is organized as follows. We start by considering parallel plates in the presence of extra dimensions. We first find the Casimir force resulting from the space between the plates only; the features just outlined are derived. We then add the contributions from the exterior space to obtain the generically attractive force between the plates. Section IV generalizes the results to an arbitrary cross section of the piston. Results for plate distances large, respectively, small, compared to the size of the extra dimensions are given. For the case of a torus as Kaluza-Klein manifold, more explicit results are provided. In Sec. VI we summarize the main results of the article and add pertinent remarks about the finite-temperature case.

II. PARALLEL PLATES IN KALUZA-KLEIN MODELS: CONTRIBUTIONS FROM BETWEEN THE PLATES

Let $M = \mathbb{R}^3 \times N$. We want to consider a piston geometry that lives in the three-dimensional real space, and where there are additional dimensions described by the smooth Riemannian manifold N of dimension d. We realize the parallel plates as obtained from a piston geometry with appropriate dimensions sent to infinity: Considering one chamber of the piston with two dimensions already sent to infinity, only two parallel plates a distance D apart remain. The correct answer for the parallel plates is obtained by adding up answers for $D = a$ and $D = L - a$ sending $L \to \infty$. In this section we deliberately consider only one chamber to highlight the serious flaws of this procedure.

We consider a scalar field model with Dirichlet boundary conditions on the plates. The relevant eigenvalue spectrum of the Laplacian on M then is

$$\omega^2 = k_1^2 + k_2^2 + \left(\frac{n\pi}{D}\right)^2 + \lambda_i^2,$$

where n and i are positive integers, $k_1^2 + k_2^2$ comes from the two free transversal dimensions in $\mathbb{R}^3$, $(n\pi/D)^2$ results from the Dirichlet plates ($D = a$ for the left chamber and $D = L - a$ for the right chamber), and λ_i^2 are the eigenfrequencies in the additional dimensions,

$$-\Delta_N \varphi_i = \lambda_i^2 \varphi_i. \tag{2.1}$$

The zeta function (density) associated with this spectrum is

$$\zeta(s) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 \times \sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\left[k_1^2 + k_2^2 + \left(\frac{n\pi}{D}\right)^2 + \lambda_i^2\right]^{-s}. \tag{2.2}$$

Performing the k_1 and k_2 integration we find

$$\zeta(s) = \frac{1}{4\pi(s-1)}\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\left[\left(\frac{n\pi}{D}\right)^2 + \lambda_i^2\right]^{-s+1}. \tag{2.3}$$

In order to write down the necessary analytical continuation of this expression, as is standard, a resummation of the n summation is applied. In that process, the zero modes $\lambda_j = 0$ need separate treatment. Letting g_0 be the degeneracy of the zero eigenstate and assuming that $\lambda_i \geq 0$ we can write

$$\zeta(s) = \frac{g_0}{4\pi(s-1)}\left(\frac{D}{\pi}\right)^{2s-2}\zeta_R(2s-2) + \frac{1}{4\pi(s-1)}\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}{}'\left[\left(\frac{n\pi}{D}\right)^2 + \lambda_i^2\right]^{-s+1}, \tag{2.4}$$

where the prime at the i summation indicates that the modes with $\lambda_j = 0$ are to be omitted from the summation.

We rewrite the n summation as

$$\sum_{n=1}^{\infty} = \frac{1}{2}\left(\sum_{n=-\infty}^{\infty} - (n=0)\right),$$

and the $n = 0$ term causes the occurrence of the zeta function related to the eigenvalue problem (2.1) on N,

$$\zeta_N(s) = \sum_{i=1}^{\infty}{}' \lambda_i^{-2s}. \tag{2.5}$$

Using a Mellin transform this allows the rewriting of (2.4) as

$$\begin{aligned}\zeta(s) = {} & \frac{g_0}{4\pi(s-1)}\left(\frac{D}{\pi}\right)^{2s-2}\zeta_R(2s-2) \\ & - \frac{1}{8\pi(s-1)}\zeta_N(s-1) + \frac{1}{8\pi\Gamma(s)} \\ & \times \int_0^\infty dt t^{s-2} \sum_{n=-\infty}^{\infty}\sum_{i=1}^{\infty}{}' e^{-[(n\pi/D)^2+\lambda_i^2]t}. \end{aligned} \tag{2.6}$$

The last term is suitably manipulated employing for $\alpha \in \mathbb{R}_+$ [40]

$$\sum_{n=-\infty}^{\infty} e^{-\alpha n^2} = \sqrt{\frac{\pi}{\alpha}}\sum_{n=-\infty}^{\infty} e^{-\pi^2 n^2/\alpha}. \tag{2.7}$$

As a result, for $n \neq 0$ we encounter the integral representation of modified Bessel functions [41]

$$K_\nu(zx) = \frac{z^\nu}{2}\int_0^\infty \exp\left\{-\frac{x}{2}\left(t + \frac{z^2}{t}\right)\right\} t^{-\nu-1} dt.$$

This allows us to obtain

$$\begin{aligned}\zeta(s) = {} & \frac{g_0}{4\pi(s-1)}\left(\frac{D}{\pi}\right)^{2s-2}\zeta_R(2s-2) - \frac{1}{8\pi(s-1)}\zeta_N(s-1) \\ & + \frac{D\Gamma(s-\frac{3}{2})}{8\pi^{3/2}\Gamma(s)}\zeta_N\left(s-\frac{3}{2}\right) + \frac{D^{s-1/2}}{2\pi^{3/2}\Gamma(s)} \\ & \times \sum_{n=1}^{\infty}\sum_{i=1}^{\infty}{}' \left(\frac{n^2}{\lambda_i^2}\right)^{(1/2)(s-(3/2))} K_{(3/2)-s}(2Dn\lambda_i). \end{aligned} \tag{2.8}$$

In order to find the Casimir energy, and then the force, for this setting, we need to evaluate this expression about the value $s = -1/2$. Whereas the first and last term are well-defined at $s = -1/2$, and thus $s = -1/2$ can simply be substituted there, more care is needed for the second and third term. From general theory, see e.g. [36,42,43], it is known that $\zeta_N(s-1)$ will have a pole at $s = -1/2$ and that $\zeta_N(s - 3/2)$ will not vanish at $s = -1/2$. With $s = -1/2 + \epsilon$ and expanding about $\epsilon = 0$ we therefore write

$$\begin{aligned}\zeta_N(s-1) &= \zeta_N\left(-\frac{3}{2}+\epsilon\right) \\ &= \frac{1}{\epsilon}\,\mathrm{Res}\zeta_N\left(-\frac{3}{2}\right) + \mathrm{FP}\zeta_N\left(-\frac{3}{2}\right) + \mathcal{O}(\epsilon), \\ \zeta_N\left(s-\frac{3}{2}\right) &= \zeta_N(-2+\epsilon) \\ &= \zeta_N(-2) + \epsilon\zeta_N'(-2) + \mathcal{O}(\epsilon^2), \end{aligned}$$

and so

$$\begin{aligned}\frac{1}{s-1}\zeta_N(s-1) = {} & -\frac{2}{3}\left(\left[\frac{1}{\epsilon}+\frac{2}{3}\right]\mathrm{Res}\zeta_N\left(-\frac{3}{2}\right)\right. \\ & \left. + \mathrm{FP}\left(-\frac{3}{2}\right)\right) + \mathcal{O}(\epsilon), \\ \frac{\Gamma(s-\frac{3}{2})}{\Gamma(s)}\zeta_N\left(s-\frac{3}{2}\right) = {} & -\frac{1}{4\sqrt{\pi}}\left(\zeta_N(-2)\left[\frac{1}{\epsilon}-\frac{1}{2}+2\ln 2\right]\right. \\ & \left. + \zeta_N'(-2)\right) + \mathcal{O}(\epsilon). \end{aligned}$$

This allows us to obtain

$$\begin{aligned}\zeta\left(-\frac{1}{2}+\epsilon\right) = {} & -\frac{\pi^2 g_0}{720}\frac{1}{D^3} + \frac{1}{12\pi}\left(\mathrm{Res}\zeta_N\left(-\frac{3}{2}\right)\left[\frac{1}{\epsilon}+\frac{2}{3}\right]\right. \\ & \left. + \mathrm{FP}\zeta_N\left(-\frac{3}{2}\right)\right) - \frac{D}{32\pi^2}\left(\zeta_N(-2)\left[\frac{1}{\epsilon}-\frac{1}{2}\right.\right. \\ & \left.\left. + 2\ln 2\right] + \zeta_N'(-2)\right) - \frac{1}{4\pi^2 D} \\ & \times \sum_{n=1}^{\infty}\sum_{i=1}^{\infty}{}' \frac{\lambda_i^2}{n^2}K_2(2Dn\lambda_i). \end{aligned} \tag{2.9}$$

The resulting force *from one chamber* therefore reads

$$\begin{aligned}F &= -\frac{1}{2}\frac{\partial}{\partial D}\zeta\left(-\frac{1}{2}+\epsilon\right) \\ &= -\frac{\pi^2 g_0}{480 D^4} + \frac{1}{64\pi^2}\left(\zeta_N(-2)\left[\frac{1}{\epsilon}-\frac{1}{2}+2\ln 2\right]\right. \\ &\quad \left. + \zeta_N'(-2)\right) + \frac{1}{8\pi^2}\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}{}' \frac{\lambda_i^2}{n^2}\frac{\partial}{\partial D}\frac{1}{D}K_2(2Dn\lambda_i). \end{aligned} \tag{2.10}$$

Notice that the second term on the right-hand side represents minus the Casimir energy of the manifold $\mathbb{R}^3 \times N$. Because of the pole at $\epsilon = 0$ in (2.10), for $\zeta_N(-2) \neq 0$ the zeta method leaves a finite renormalization ambiguity proportional to $\zeta_N(-2)$. From a theoretical point of view no prediction about the sign of the force can be made. In case $\zeta_N(-2) = 0$ the force appears to be finite. (In the setting of an ultraviolet cutoff [see the Appendix] there are additional divergences, but we shall not discuss them here.) This can happen under certain restrictions on the geometry of the manifold N. If we define $K_N(t)$ to be the heat kernel associated with the eigenvalue problem (2.1), its asymptotic expansion reads

$$K_N(t) = \sum_{i=1}^{\infty} e^{-\lambda_i^2 t} \sim \sum_{l=0,1/2,1,\ldots}^{\infty} b_l t^{l-(d/2)}. \qquad (2.11)$$

The heat-kernel coefficients are determined in terms of geometric tensors of N and its boundary, if present; for a collection of known results see [36,42,44,45]. Using

$$\zeta_N(-2) = 2b_{(d+4)/2} = 0$$

[43], we thus have a geometric condition on when the force becomes finite. When this vanishing occurs, $\zeta_N'(-2) < 0$ indicates that the force is definitely negative (attractive). If $\zeta_N'(-2) > 0$, asymptotically for $D \gg 1$ the force seems to be positive and turns negative at some critical distance $D_{\rm crit}$. In other words, if the Casimir energy of $\mathbb{R}^3 \times N$ is positive, a Lukosz-type repulsion cannot occur, whereas for a negative Casimir energy it can. We come back to this discussion in Sec. V when N is chosen to be a torus and where indeed $\zeta_N(-2) = 0$.

III. PARALLEL PLATES IN KALUZA-KLEIN MODELS: ADDING EXTERIOR CONTRIBUTIONS

Let us now take into account the second chamber of the piston. (In the present context that merely means adding a third plate at a large distance.) Denoting the plate separations by a and $L - a$ and the associated zeta functions by $\zeta_a(s)$ and $\zeta_{L-a}(s)$, from (2.9) one has immediately

$$\begin{aligned}
&\zeta_a\left(-\frac{1}{2}+\epsilon\right) + \zeta_{L-a}\left(-\frac{1}{2}+\epsilon\right) \\
&\quad = -\frac{\pi^2 g_0}{720}\frac{1}{a^3} - \frac{\pi^2 g_0}{720}\frac{1}{(L-a)^3} + \frac{1}{6\pi}\left(\text{Res}\,\zeta_N\left(-\frac{3}{2}\right)\right. \\
&\qquad \times\left[\frac{1}{\epsilon}+\frac{2}{3}\right] + \left.\text{FP}\,\zeta_N\left(-\frac{3}{2}\right)\right) - \frac{L}{32\pi^2}\left(\zeta_N(-2)\left[\frac{1}{\epsilon}-\frac{1}{2}\right.\right. \\
&\qquad \left.\left.+ 2\ln 2\right] + \zeta_N'(-2)\right) - \frac{1}{4\pi^2 a}\sum_{n=1}^{\infty}{\sum_{i=1}^{\infty}}' \frac{\lambda_i^2}{n^2} K_2(2an\lambda_i) \\
&\qquad - \frac{1}{4\pi^2(L-a)}\sum_{n=1}^{\infty}{\sum_{i=1}^{\infty}}' \frac{\lambda_i^2}{n^2} K_2(2(L-a)n\lambda_i). \qquad (3.1)
\end{aligned}$$

Despite the fact that the Casimir *energy* in general needs renormalization, the *force* this time is always well-defined no matter what the geometry or topology of N looks like. In particular,

$$\begin{aligned}
F &= -\frac{1}{2}\frac{\partial}{\partial a}\left[\zeta_a\left(-\frac{1}{2}+\epsilon\right) + \zeta_{L-a}\left(-\frac{1}{2}+\epsilon\right)\right] \\
&= -\frac{\pi^2 g_0}{480 a^4} + \frac{\pi^2 g_0}{480(L-a)^4} + \frac{1}{8\pi^2}\sum_{n=1}^{\infty}{\sum_{i=1}^{\infty}}' \frac{\lambda_i^2}{n^2}\frac{\partial}{\partial a} \\
&\qquad \times \frac{1}{a}K_2(2an\lambda_i) + \frac{1}{8\pi^2}\sum_{n=1}^{\infty}{\sum_{i=1}^{\infty}}' \frac{\lambda_i^2}{n^2}\frac{\partial}{\partial a}\frac{1}{L-a} \\
&\qquad \times K_2(2(L-a)n\lambda_i). \qquad (3.2)
\end{aligned}$$

The force vanishes for $a = L/2$, is negative for $a < L/2$, and is positive for $a > L/2$; that is, the plate at a is always attracted to the closer wall. As $L \to \infty$ the very simple result

$$F = -\frac{\pi^2 g_0}{480 a^4} + \frac{1}{8\pi^2}\sum_{n=1}^{\infty}{\sum_{i=1}^{\infty}}' \frac{\lambda_i^2}{n^2}\frac{\partial}{\partial a}\frac{1}{a}K_2(2an\lambda_i) \qquad (3.3)$$

is obtained. The force is negative independently of any details of the topology or geometry of the extra dimensions (within the confine $\lambda_i \geq 0$).

IV. PISTONS WITH FINITE CROSS SECTION

In this section we show that a negative force is guaranteed whenever the boundary conditions on the plates are both Dirichlet (or both Neumann), no matter what the cross section $\mathcal{C}$ and the manifold N are. Assume two parallel plates of some arbitrary shape within a cylinder of that same cross section, along with general Kaluza-Klein dimensions. With Dirichlet boundary conditions on the plates (at separation D), this gives rise to a spectrum of the form

$$\omega^2 = \left(\frac{n\pi}{D}\right)^2 + \mu_i^2. \qquad (4.1)$$

The part μ_i^2 comes from the manifold $T = \mathcal{C} \times N$. Proceeding exactly as before, denoting

$$\zeta_T(s) = {\sum_{i=1}^{\infty}}' \mu_i^{-2s},$$

we obtain

$$\begin{aligned}
\zeta(s) &= \pi^{-2s} D^{2s} g_0 \zeta_R(2s) - \frac{1}{2}\zeta_T(s) \\
&\quad + \frac{D\Gamma(s-\frac{1}{2})}{2\sqrt{\pi}\Gamma(s)}\zeta_T\left(s-\frac{1}{2}\right) + \frac{2D^{s+1/2}}{\sqrt{\pi}\Gamma(s)} \\
&\quad \times \sum_{n=1}^{\infty}{\sum_{i=1}^{\infty}}' \left(\frac{n^2}{\mu_i^2}\right)^{1/2(s-1/2)} K_{1/2-s}(2Dn\mu_i). \qquad (4.2)
\end{aligned}$$

The zeta function for the piston is the sum of two such contributions with D replaced by a and $L - a$ respectively. For the force this gives

$$\begin{aligned}
F &= -\frac{\pi g_0}{24 a^2} + \frac{\pi g_0}{24(L-a)^2} \\
&\quad + \frac{1}{2\pi}\sum_{n=1}^{\infty}{\sum_{i=1}^{\infty}}' \frac{\mu_i}{n}\frac{\partial}{\partial a}K_1(2an\mu_i) \\
&\quad + \frac{1}{2\pi}\sum_{n=1}^{\infty}{\sum_{i=1}^{\infty}}' \frac{\mu_i}{n}\frac{\partial}{\partial a}K_1(2(L-a)n\mu_i). \qquad (4.3)
\end{aligned}$$

Again it is clearly seen that even in this generalized scenario the piston is attracted to the closest wall. In the limit $L \to \infty$, the force reduces to

$$F = -\frac{\pi g_0}{24a^2} + \frac{1}{2\pi}\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}{}' \frac{\mu_i}{n}\frac{\partial}{\partial a}K_1(2an\mu_i), \quad (4.4)$$

which again is manifestly negative.

The essential difference between this calculation and that of Sec. III is that we have only one infinite dimension instead of three; up to this point the transverse and the Kaluza-Klein dimensions have played identical roles. If we had carried out the analog of Sec. II, we would have encountered similar results. In particular, the famous repulsive force of Lukosz [37] corresponds to rectangular cross section and $N = \varnothing$.

Although the representation (4.3) is most suitable for reading off the sign of the force, it is not useful numerically unless the plate separation is sufficiently large compared to other scales. An expression suitable for reading off the small-a behavior is found by following formally a procedure used for large-mass expansions [46], where the role of the mass is played by the large parameter π/a. We first rewrite the zeta function associated with the spectrum (4.1) as

$$\zeta(s) = \frac{1}{\Gamma(s)}\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\int_0^{\infty} dt t^{s-1} e^{-[(n\pi/D)^2+\mu_i^2]t}. \quad (4.5)$$

We note that the small-D expansion follows from the small-t behavior of the heat-kernel,

$$K(t) = \sum_{i=1}^{\infty} e^{-\mu_i^2 t} \sim \sum_{l=0,1/2,1,\ldots}^{\infty} a_l t^{l-(d+2)/2};$$

note that the spectrum μ_i^2 results from a second-order partial differential operator in dimension $(d+2)$.

Substituting this expansion into (4.5), asymptotically as $D \to 0$ we find

$$\zeta(s) = \frac{1}{\Gamma(s)}\sum_{l=0,1/2,1,\ldots}^{\infty} a_l \Gamma\left(s + l - \frac{d+2}{2}\right) \times \left(\frac{D}{\pi}\right)^{2s+2l-d-2}\zeta_R(2s+2l-d-2).$$

We evaluate this about $s = -1/2$ using well-known properties of the Γ function and of the Riemann zeta function [41]. The answers for d even and odd look slightly different; we denote by $\sum_l^d$ the summation over $l = 0, 1, \ldots, (d+2)/2, l > (d+4)/2$ for d even, but over $l = 1/2, 3/2, \ldots, (d+2)/2, l > (d+4)/2$ for d odd. Furthermore, $\lfloor x \rfloor$ denotes the greatest integer not larger than x. With $s = -1/2 + \epsilon$, we get

$$\begin{aligned}\zeta\left(-\frac{1}{2}+\epsilon\right) &= -\frac{1}{2\sqrt{\pi}}\sum_l^d a_l \Gamma\left(l - \frac{d+3}{2}\right)\left(\frac{D}{\pi}\right)^{2l-d-3} \\ &\quad \times \zeta_R(2l-d-3) + \sum_{j=1}^{\lfloor (d+3)/2 \rfloor} a_{((d+3)/2)-j} \\ &\quad \times \frac{(-1)^{j+1}}{j!\sqrt{\pi}}\left(\frac{\pi}{D}\right)^{2j}\zeta_R'(-2j) \\ &\quad + a_{(d+3)/2}\left(\frac{1}{4\sqrt{\pi}\epsilon} + \frac{\ln(4D)-1}{2\sqrt{\pi}}\right) \\ &\quad + a_{(d+4)/2}\left(-\frac{D}{4\pi\epsilon} + \frac{D}{2\pi}\left(1 - \gamma + \ln\left(\frac{\pi}{D}\right)\right)\right) \\ &\quad + \mathcal{O}(\epsilon).\end{aligned}$$

This result is used to find the force from the left chamber. The contribution to the force from the right chamber, where $D = L - a$ with $L \to \infty$, follows easily to be

$$\begin{aligned}F_2 &= \frac{1}{8\pi\epsilon}\zeta_T(-1) + \frac{1}{8\pi}(\zeta_T'(-1) + \zeta_T(-1)[-1 + \ln 4]) \\ &= -\frac{1}{8\pi}a_{(d+4)/2}\left(\frac{1}{\epsilon} - 1 + \ln 4\right) + \frac{1}{8\pi}\zeta_T'(-1).\end{aligned}$$

Here we used $\zeta_T(-1) = -a_{(d+4)/2}$. As before, when the forces are added, the terms singular as $\epsilon \to 0$ cancel and, asymptotically as $a \to 0$, the unambiguous answer for the force is found,

$$\begin{aligned}F &= \frac{1}{2\pi^{3/2}}\sum_l^d a_l \Gamma\left(l - \frac{d+1}{2}\right)\left(\frac{a}{\pi}\right)^{2l-d-4}\zeta_R(2l-d-3) \\ &\quad + \frac{1}{\pi^{3/2}}\sum_{j=1}^{\lfloor (d+3)/2 \rfloor} a_{((d+3)/2)-j}\frac{(-1)^{j+1}}{(j-1)!}\left(\frac{\pi}{a}\right)^{2j+1}\zeta_R'(-2j) \\ &\quad - \frac{1}{4\sqrt{\pi}a}a_{(d+3)/2} + \frac{1}{8\pi}\zeta_T'(-1) \\ &\quad - \frac{1}{4\pi}a_{(d+4)/2}\left(\ln\left(\frac{2\pi}{a}\right) - \gamma - \frac{1}{2}\right). \quad (4.6)\end{aligned}$$

Explicit expressions for given cross sections $\mathcal{C}$ and Kaluza-Klein manifolds N are easily obtained from known expressions for the heat-kernel coefficients [36,42,45].

In particular, if there are no additional dimensions, $N = \varnothing$, and Neumann boundary conditions are imposed on the cylinder walls, then

$$a_0 = (4\pi)^{-1}\mathrm{vol}(\mathcal{C}), \qquad a_{1/2} = \frac{1}{4}(4\pi)^{-1/2}\mathrm{vol}(\partial\mathcal{C}),$$

and the first few terms of the expansion reproduce the results in [4,5].

If the manifold N is nonempty and has no boundary, this time imposing Dirichlet boundary conditions on the cylinder walls, one easily finds

$$a_0 = (4\pi)^{-(d+2)/2}\mathrm{vol}(\mathcal{C})\mathrm{vol}(N),$$

$$a_{1/2} = -\frac{1}{4}(4\pi)^{-(d+1)/2}\mathrm{vol}(\partial\mathcal{C})\mathrm{vol}(N),$$

and the leading two terms in the asymptotic expansion of the force read

$$F \sim 2^{-d-3}\pi^{(d+3)/2}\mathrm{vol}(\mathcal{C})\mathrm{vol}(N)a^{-d-4}\begin{cases}\Gamma(-\frac{d+1}{2})\zeta_R(-d-3) & d \text{ even}\\ \frac{2(-1)^{(d+1)/2}}{(\frac{d+1}{2})!}\zeta_R'(-d-3) & d \text{ odd}\end{cases}$$
$$-2^{-d-3}\pi^{(d+2)/2}\mathrm{vol}(\partial\mathcal{C})\mathrm{vol}(N)a^{-d-3}\begin{cases}\frac{(-1)^{d/2}}{(d/2)!}\zeta_R'(-d-2) & d \text{ even}\\ \frac{1}{2}\Gamma(-\frac{d}{2})\zeta_R(-d-2) & d \text{ odd}\end{cases}. \quad (4.7)$$

Higher orders would involve the extrinsic curvature of $\partial\mathcal{C}$ and the curvature of N. It is clearly seen that as soon as the plate separation gets small compared to the sizes of other dimensions, the Casimir force between the plates is significantly modified, revealing information about the volume, and at higher order the curvature, of the Kaluza-Klein dimensions.

V. TORUS AS KALUZA-KLEIN MANIFOLD

The series over the Bessel functions in Eqs. (2.10), (3.3), and (4.4) are numerically suitable as long as the argument of the Bessel function grows sufficiently fast with the eigenvalues. If that is the case, taking into account only a few eigenvalues will be enough as contributions are exponentially damped. However, in order to analyze how the Casimir force behaves when the distance between the plates is smaller than the size of the extra dimensions, a different procedure is necessary, as we have seen, and in general only asymptotic answers can be obtained.

For the case where $N = T^d$ it is possible to obtain closed answers that allow consideration of several limits exactly. For simplicity let us assume an equilateral torus of radius R, with the two macroscopic transverse dimensions effectively infinite. The relevant eigenvalue spectrum then reads

$$\omega^2 = k_1^2 + k_2^2 + \left(\frac{n\pi}{D}\right)^2 + \frac{1}{R^2}\sum_{i=1}^d n_i^2. \quad (5.1)$$

For reasons explained above, the previously obtained representations involving the Bessel functions cannot be used easily to analyze the range where $D \ll R$. If only the asymptotic behavior as $D \to 0$ is wanted, the use of (4.6) is sufficient. For the torus, however, it is possible to recover also the exponentially damped contributions as $D \to 0$. In fact, all technical tools have been provided to find closed expressions in that regime. In particular it is again the resummation (2.7) that is relevant, but it should be applied to the toroidal dimensions and not to the dimension in which Dirichlet conditions are applied. Applying resummation to all d sums originating from the torus, the result for the left chamber corresponding to Eq. (2.8) reads

$$\zeta(s) = \frac{\pi^{(3d/2)-2s+1}\Gamma(s-\frac{d}{2}-1)}{4\Gamma(s)}\left(\frac{R}{D}\right)^d D^{2s-2}\zeta_R(2s-d-2) + \frac{\pi^{(d/2)-1}}{2\Gamma(s)}\left(\frac{R}{D}\right)^{d/2}(RD)^{s-1}\sum_{n=1}^\infty \sideset{}{'}\sum_{n_1,\ldots,n_d=-\infty}^\infty \times\left(\frac{n^2}{n_1^2+\cdots+n_d^2}\right)^{1/2(1-s+(d/2))} \times K_{1+(d/2)-s}\left(\frac{2\pi^2 R}{D}n\sqrt{n_1^2+\cdots+n_d^2}\right). \quad (5.2)$$

For the right chamber we are mostly interested in the limit $D \to \infty$, and so Eq. (2.8) is the appropriate form. Because

$$\lambda_i^2 = \frac{1}{R^2}\sum_{j=1}^d n_j^2,$$

the zeta function $\zeta_N(s)$ turns out to be the Epstein function [47,48]

$$Z_d(s;R) = R^{2s}\sideset{}{'}\sum_{n_1,\ldots,n_d=-\infty}^\infty (n_1^2+\cdots+n_d^2)^{-s}. \quad (5.3)$$

Its analytical continuation is very well understood [33,49–52] and we get

$$\zeta(s) = \frac{\pi^{1-2s}D^{2s-2}}{4(s-1)}\zeta_R(2s-2) - \frac{1}{8\pi(s-1)}Z_d(s-1;R) + \frac{D}{8\pi^{3/2}}\frac{\Gamma(s-\frac{3}{2})}{\Gamma(s)}Z_d\left(s-\frac{3}{2};R\right) + \frac{D^{s-1/2}R^{s-3/2}}{2\pi^{3/2}\Gamma(s)}\sum_{n=1}^\infty\sideset{}{'}\sum_{n_1,\ldots,n_d=-\infty}^\infty \times\left(\frac{n^2}{n_1^2+\cdots+n_d^2}\right)^{1/2(s-(3/2))} \times K_{(3/2)-s}\left(\frac{2Dn}{R}\sqrt{n_1^2+\cdots+n_d^2}\right). \quad (5.4)$$

Using this representation (5.4) for both chambers, and noting $g_0 = 1$, we get immediately from Eq. (3.3) the force

$$F = -\frac{\pi^4}{480a^4} + \frac{1}{8\pi^2R^2}\sum_{n=1}^\infty\sideset{}{'}\sum_{n_1,\ldots,n_d=-\infty}^\infty \frac{n_1^2+\cdots+n_d^2}{n^2} \times\frac{\partial}{\partial a}\frac{1}{a}K_2\left(\frac{2an}{R}\sqrt{n_1^2+\cdots+n_d^2}\right). \quad (5.5)$$

This representation is particularly suitable for $R < a$ because the contributions from K_2 are exponentially damped. It shows that as long as the size of the extra dimensions is small compared to the separation of the plates, the correction to the well-known Casimir force between parallel plates is very small.

The above representation is not suitable for the range with plate separation smaller than R, because $K_2(z) \sim 2/z^2$ as $z \to 0$. As we will see, the leading contribution as $a \to 0$ will then come from the series. A better suited representation is obtained by rewriting Eq. (5.5) using the fact that Eqs. (5.2) and (5.4) equal each other. This first shows

$$\frac{D^{s-1/2}R^{s-3/2}}{2\pi^{3/2}\Gamma(s)}\sum_{n=1}^{\infty}\sum_{n_1,\ldots,n_d=-\infty}^{\infty}{}'\left(\frac{n^2}{n_1^2+\cdots+n_d^2}\right)^{1/2(s-(3/2))}K_{(3/2)-s}\left(\frac{2Dn}{R}\sqrt{n_1^2+\cdots+n_d^2}\right)$$
$$=\frac{\pi^{(3d/2)-2s+1}\Gamma(s-\frac{d}{2}-1)}{4\Gamma(s)}\left(\frac{R}{D}\right)^d D^{2s-2}\zeta_R(2s-d-2)+\frac{\pi^{(d/2)-1}}{2\Gamma(s)}\left(\frac{R}{D}\right)^{d/2}(RD)^{s-1}$$
$$\times\sum_{n=1}^{\infty}\sum_{n_1,\ldots,n_d=-\infty}^{\infty}{}'\left(\frac{n^2}{n_1^2+\cdots+n_d^2}\right)^{1/2(1-s+(d/2))}K_{1+(d/2)-s}\left(\frac{2\pi^2R}{D}n\sqrt{n_1^2+\cdots+n_d^2}\right)-\frac{\pi^{1-2s}D^{2s-2}}{4(s-1)}\zeta_R(2s-2)$$
$$+\frac{1}{8\pi(s-1)}Z_d(s-1;R)-\frac{D}{8\pi^{3/2}}\frac{\Gamma(s-\frac{3}{2})}{\Gamma(s)}Z_d\left(s-\frac{3}{2};R\right).$$

Analytically continuing this to $s=-1/2$, one obtains

$$\frac{1}{8\pi^2R^2}\sum_{n=1}^{\infty}\sum_{n_1,\ldots,n_d=-\infty}^{\infty}{}'\frac{n_1^2+\ldots+n_d^2}{n^2}\frac{\partial}{\partial a}\frac{1}{a}K_2\left(\frac{2an}{R}\sqrt{n_1^2+\cdots+n_d^2}\right)$$
$$=-(d+3)\frac{\pi^{3/2(d+1)}}{16}\left(\frac{R}{a}\right)^d a^{-4}\begin{cases}\Gamma(-\frac{d+3}{2})\zeta_R(-d-3), & d \text{ even}\\ \frac{2(-1)^{(d-1)/2}}{(\frac{d+3}{2})!}\zeta_R'(-d-3), & d \text{ odd}\end{cases}+\frac{\pi^2}{480a^4}-\frac{1}{64\pi^2}Z_d'(-2;R)+\frac{\pi^{(d-3)/2}R^{(d-3)/2}}{8}$$
$$\times\sum_{n=1}^{\infty}\sum_{n_1,\ldots,n_d=-\infty}^{\infty}{}'\left(\frac{n^2}{n_1^2+\cdots+n_d^2}\right)^{(3+d)/4}\frac{\partial}{\partial a}a^{-(3+d)/2}K_{(d+3)/2}\left(\frac{2\pi^2R}{a}n\sqrt{n_1^2+\cdots+n_d^2}\right).$$

Using this in Eq. (5.5) then gives the force in the form

$$F=\frac{\pi^{3/2(d+1)}}{8}\left(\frac{R}{a}\right)^d a^{-4}\begin{cases}\Gamma(-\frac{d+1}{2})\zeta_R(-d-3), & d \text{ even}\\ \frac{2(-1)^{(d+1)/2}}{(\frac{d+1}{2})!}\zeta_R'(-d-3), & d \text{ odd}\end{cases}-\frac{1}{64\pi^2}Z_d'(-2;R)+\frac{\pi^{(d-3)/2}R^{(d-3)/2}}{8}$$
$$\times\sum_{n=1}^{\infty}\sum_{n_1,\ldots,n_d=-\infty}^{\infty}{}'\left(\frac{n^2}{n_1^2+\cdots+n_d^2}\right)^{(3+d)/4}\frac{\partial}{\partial a}a^{-(3+d)/2}K_{(d+3)/2}\left(\frac{2\pi^2R}{a}n\sqrt{n_1^2+\cdots+n_d^2}\right). \quad (5.6)$$

This result shows that if $a \ll R$, the compactness of the extra dimensions can be ignored, and the force is the standard Casimir force, but in the space of the full dimension, namely, of dimension $3+d$. The first term agrees with the general result (4.7) when specialized to $N = T^d$.

For this example it is clear that $\zeta_N(-2) = Z_d(-2;R) = 0$, because the torus is a flat manifold without boundary. Therefore the force resulting from one chamber only, as given in (2.10), is finite. Using the reflection formula for the Epstein function [49] it is obtained as

$$F=-\frac{\pi^2}{480a^4}+\frac{\Gamma(\frac{d}{2}+2)}{32\pi^{6+d/2}R^4}Z_d\left(\frac{d}{2}+2;1\right)$$
$$+\frac{1}{8\pi^2R^2}\sum_{n=1}^{\infty}\sum_{n_1,\ldots,n_d=-\infty}^{\infty}{}'\frac{n_1^2+\cdots+n_d^2}{n^2}$$
$$\times\frac{\partial}{\partial a}\frac{1}{a}K_2\left(\frac{2an}{R}\sqrt{n_1^2+\cdots+n_d^2}\right). \quad (5.7)$$

For $a \gg R$ this force, obtained by neglecting the second chamber, is positive and asymptotically constant. Considering N to be rectangular with Neumann boundary conditions leads to the same result [17]. If for the rectangular parallelepiped Dirichlet conditions are imposed instead, the asymptotic nature of the force depends on the dimension of N. From [49] it follows, for example, that for $d=1$ the force (5.7) is asymptotically repulsive, whereas for $d=2$ it is asymptotically attractive. Whatever may be the case for a macroscopic conducting box (where there may not be an external piston shaft), in a Kaluza-Klein cosmology the extra dimensions are indisputably present outside the parallel plates as well as inside. Therefore, formula (5.7) surely must be rejected as spurious.

However, the papers [19–21,24] did not take the outer chamber of the Kaluza-Klein piston into account, but nevertheless they did not find a repulsive force. Closer examination (see, for example, Eqs. (24) and (25) of

[24], or p. 5 of [19]) shows that all those authors have indeed subtracted the term (linear in a) that we here consider to be the piston correction. Their reasoning is to subtract the Casimir energy (caused by the small compact dimensions) that would exist in the region between the plates if the plates were not there. In the Cavalcanti piston, the analogous reasoning would be to make the piston shaft infinite in both directions, ignore the outer chambers, remove the plates, and subtract the energy in the inner chamber. We believe that our analysis is more convincing.

VI. CONCLUSIONS

In this article we have analyzed forces occurring in pistons of arbitrary cross section in a cosmological Kaluza-Klein setting. We have shown that irrespective of the details of the cross section and of the geometry and topology of the Kaluza-Klein manifold, the piston is always attracted to the closest wall. This implies that parallel plates always attract no matter what the properties of the additional dimensions are (except for the physically mild restriction that the eigenvalues λ_i^2 or μ_i^2 all be nonnegative). Repulsive forces between Dirichlet plates can occur only in a naive calculation that takes into account only one of the chambers; see the explanations at the ends of Secs. II and V. Furthermore, we have derived an asymptotic expansion of the force for small distances between the piston and the wall, Eq. (4.7). It is clearly seen how the geometries of the cross section and the Kaluza-Klein manifold enter the answer. In this limit the plates notice the dimensions of all space and the force obtained is the standard Casimir force in the space of the full dimension.

All results for the force remain valid if Neumann boundary conditions on both plates instead of Dirichlet boundary conditions are considered, because the Neumann and Dirichlet spectra differ only by a-independent eigenvalues.

The attraction of the piston to the closest wall is further enhanced by finite-temperature contributions. Assuming a piston with arbitrary cross section, the relevant finite-temperature spectrum reads

$$\omega^2 = \left(\frac{2\pi j}{\beta}\right)^2 + \left(\frac{n\pi}{D}\right)^2 + \mu_i^2, \qquad j \in \mathbb{Z},$$

where β is the inverse temperature. The energy associated with the system is defined to be [32,35]

$$E = -\frac{1}{2}\frac{\partial}{\partial\beta}[\zeta_\beta'(0) + \zeta_\beta(0)\ln\mu^2],$$

where $\zeta_\beta(s)$ is the zeta function arising from this spectrum and μ is a renormalization scale. Using as is standard a resummation of the Matsubara sum, the following form can be obtained [53]:

$$E = \frac{1}{2}\mathrm{FP}\zeta\left(-\frac{1}{2}\right) + \mathrm{Res}\zeta\left(-\frac{1}{2}\right)\ln\left(\frac{\mu e}{2}\right) + \sum_{n=1}^{\infty}\sum_{i=1}^{\infty}\frac{\lambda_{i,n}}{(e^{\beta\lambda_{i,n}} - 1)},$$

where $\lambda_{i,n}^2 = (\frac{n\pi}{D})^2 + \mu_i^2$ and $\zeta(s)$ is the zeta function analyzed in Sec. IV. The temperature contribution is a decreasing function of D [39]. Thus, when the contributions of the two chambers are added, the finite-temperature part, just like the zero-temperature part discussed previously, tends to move the piston toward the closer wall, as has been observed long ago for infinite parallel conducting plates [54,55].

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APPENDIX: THE METHOD OF IMAGES

As in Sec. V, consider two infinite transverse dimensions; one Dirichlet plate separation, D; and d periodic Kaluza-Klein dimensions, all of circumference $2\pi R$. The system can easily be treated by the methods of [56] (for example).

The free cylinder kernel (a certain Green function for the Laplacian in $\mathbb{R}^{d+4}$) is

$$T_0 = C_{d+3}t[t^2 + \| \mathbf{x} - \mathbf{x}' \|^2]^{-(d+4)/2}, \quad C_{d+3} = \frac{\Gamma(\frac{d+4}{2})}{\pi^{(d+4)/2}}. \tag{A1}$$

Thus

$$-\frac{1}{2}\frac{\partial T_0}{\partial t} = \frac{1}{2}C_{d+3}\{-[t^2 + \| \mathbf{x} - \mathbf{x}' \|^2]^{-(d+4)/2} + (d+4)t^2[t^2 + \| \mathbf{x} - \mathbf{x}' \|^2]^{-(d+6)/2}\}. \tag{A2}$$

The corresponding Green function $T(t, \mathbf{x}, \mathbf{x}')$ in a rectangular geometry is given exactly by a sum over all classical paths from $\mathbf{x}'$ to $\mathbf{x}$ or, equivalently, by a sum over T_0 displaced to appropriate "image charges." In the limit $t \downarrow 0$, formally the energy density is

$$T_{00} = -\frac{1}{2}\frac{\partial T}{\partial t}\bigg|_{t=0} = -\frac{1}{2}\sum_{\text{images}} C_{d+3} \| \mathbf{x} - \mathbf{x}' \|^{-(d+4)}. \tag{A3}$$

For a careful study of divergences one would maintain the last factor in the form $[t^2 + \| \mathbf{x} - \mathbf{x}' \|^2]^{-(d+4)/2}$. Let $B_d = -\frac{1}{2} C_{d+3}$.

Let $\mathbf{N} = (N_1, \ldots, N_d) \in \mathbb{Z}^d$, and let $\hat{\mathbf{k}}$ be the unit vector perpendicular to the plates in the physical space (the z direction). Periodic boundary conditions are imposed by summing displacements of formula (A3) over a periodic lattice, and the lattice of Dirichlet images is a difference of two periodic lattices:

$$\begin{aligned} T_{00} &= B_d \sum_{N_1=-\infty}^{\infty} \cdots \sum_{N_d=-\infty}^{\infty} \sum_{N=-\infty}^{\infty} \{\| \mathbf{x} - (\mathbf{x} + 2D\hat{\mathbf{k}}N \\ &\quad + 2\pi R\mathbf{N}) \|^{-(d+4)} \\ &\quad - \| \mathbf{x} - (\mathbf{x} - 2z\hat{\mathbf{k}} - 2D\hat{\mathbf{k}}N - 2\pi R\mathbf{N}) \|^{-(d+4)}\} \\ &= B_d \sum_{N_1=-\infty}^{\infty} \cdots \sum_{N_d=-\infty}^{\infty} \sum_{N=-\infty}^{\infty} \{\| 2D\hat{\mathbf{k}}N + 2\pi R\mathbf{N} \|^{-(d+4)} \\ &\quad - \| 2z\hat{\mathbf{k}} + 2D\hat{\mathbf{k}}N + 2\pi R\mathbf{N} \|^{-(d+4)}\}. \end{aligned} \tag{A4}$$

To get the energy per unit area one should integrate over z and over the periodic coordinates. The latter amounts to multiplying by $(2\pi R)^d$.

We now sketch the process of discarding divergent terms (which appear in the present ultraviolet cutoff method as negative powers of t, but were automatically eliminated by the zeta function regularization). The term $N = 0$, $\mathbf{N} = 0$ is the free vacuum energy. The other periodic terms with $N = 0$ (the analog of terms called PV in [56]) are (after the integration)

$$B_d (2\pi R)^{-4} D \sum_{\mathbf{N}\neq 0} \| \mathbf{N} \|^{-(d+4)} . \tag{A5}$$

This expression (with $D = a$) will add to the corresponding term from the piston shaft (with $D = L - a$) to give an energy per area independent of a, hence no pressure. Without the shaft, however, (A5) (with $D = a$) gives a repulsive Lukosz pressure independent of a. The remaining terms in the periodic orbit sum [the first term in the final version of (A4)] are

$$B_d (\pi R)^d 2^{-4} D \sum_{N\neq 0} \sum_{\mathbf{N}} \| \pi R\mathbf{N} + D\hat{\mathbf{k}}N \|^{-(d+4)} . \tag{A6}$$

These are the PD and PH terms in [56]; (A6) (with $D = a$) is the main Casimir energy. With more effort it could be shown to yield the same forces as found in Sec. V.

In the terminology of [56] there are no VP, VD, or C terms in this problem, because there are no reflections in the periodic directions. The last term in (A4) consists of HP terms ($\mathbf{N} = 0$) and HD terms as they are called in [56]. The HP terms formally give the energy

$$\begin{aligned} &-(\text{constant}) \int_0^D dz \sum_{N=-\infty}^{\infty} |z + DN|^{-(d+4)} \\ &\quad = -(\text{constant}) \int_{-\infty}^{\infty} |z|^{-(d+4)} dz. \end{aligned} \tag{A7}$$

The integral (which would converge if we had kept $t > 0$) is independent of D and hence gives no force. This is the surface energy of the plates (renormalizes their masses). Finally, the HD terms, before integration over z, are

$$\begin{aligned} &-B_d (\pi R)^d 2^{-4} \sum_{\mathbf{N}\neq 0} \sum_{N=-\infty}^{\infty} \| (z + DN)\hat{\mathbf{k}} + \pi R\mathbf{N} \|^{-(d+4)} \\ &\quad = -B_d (\pi R)^d 2^{-4} \sum_{\mathbf{N}\neq 0} \sum_{N=-\infty}^{\infty} [(z + DN)^2 \\ &\qquad + (\pi R)^2 \| \mathbf{N} \|^2]^{-(d+4)/2}. \end{aligned}$$

Upon integration the N sum again telescopes:

$$-B_d (\pi R)^d 2^{-4} \sum_{\mathbf{N}\neq 0} \int_{-\infty}^{\infty} [z^2 + (\pi R)^2 \| \mathbf{N} \|^2]^{-(d+4)/2} dz. \tag{A8}$$

Again this contribution is independent of D (being a surface effect, albeit dependent on the geometry of the extra dimensions). The HD energy does not contribute to the force, just as in the original Cavalcanti piston (or single box).

[1] R. M. Cavalcanti, Phys. Rev. D **69**, 065015 (2004).
[2] A. Edery, Phys. Rev. D **75** 105012 (2007).
[3] A. Edery and V. Marachevsky, Phys. Rev. D **78**, 025021 (2008).
[4] M. P. Hertzberg, R. L. Jaffe, M. Kardar, and A. Scardicchio, Phys. Rev. Lett. **95**, 250402 (2005).
[5] M. P. Hertzberg, R. L. Jaffe, M. Kardar, and A. Scardicchio, Phys. Rev. D **76**, 045016 (2007).
[6] S. C. Lim and L. P. Teo, arXiv:0808.0047 [Eur. Phys. J. C (to be published)].
[7] S. C. Lim and L. P. Teo, New J. Phys. **11**, 013055 (2009).
[8] V. Marachevsky, Phys. Rev. D **75**, 085019 (2007).
[9] N. F. Svaiter and B. F. Svaiter, J. Phys. A **25**, 979 (1992).
[10] A. Edery and I. MacDonald, J. High Energy Phys. 09 (2007) 005.
[11] X.-Z. Li, H.-B. Cheng, J.-M. Li, and X.-H. Zhai, Phys. Rev. D **56**, 2155 (1997).
[12] X.-H. Zhai and X.-Z. Li, Phys. Rev. D **76**, 047704 (2007).
[13] T. H. Boyer, Phys. Rev. A **9**, 2078 (1974).
[14] T. H. Boyer, Ann. Phys. (N.Y.) **56**, 474 (1970).
[15] M. Fierz, Helv. Phys. Acta **33**, 855 (1960).
[16] E. A. Power, *Introductory Quantum Electrodynamics* (Elsevier, New York, 1964), Appendix I.
[17] H. Cheng, Phys. Lett. B **643**, 311 (2006).

[18] H. Cheng, Phys. Lett. B **668**, 72 (2008).
[19] M. Frank, I. Turan, and L. Ziegler, Phys. Rev. D **76**, 015008 (2007).
[20] M. Frank, N. Saad, and I. Turan, Phys. Rev. D **78**, 055014 (2008).
[21] S. Hofmann, K. Poppenhaeger, S. Hossenfelder, and M. Bleicher, Phys. Lett. B **582**, 1 (2004).
[22] R. Obousy and G. Cleaver, arXiv:0810.1096.
[23] A. Edery and V. Marachevsky, J. High Energy Phys. 12 (2008) 035.
[24] F. Pascoal, L. F. A. Oliveira, F. S. S. Rosa, and C. Farina, Braz. J. Phys. **38**, 581 (2008).
[25] M. Doran and J. Jaeckel, J. Cosmol. Astropart. Phys. 08 (2006) 010.
[26] G. Mahajan, T. Padmanabhan, and S. Sarkar, Phys. Lett. B **641**, 6 (2006).
[27] L. Perivolaropoulos, Phys. Rev. D **77**, 107301 (2008).
[28] G. Bressi, G. Carugno, R. Onofrio, and G. Ruoso, Phys. Rev. Lett. **88**, 041804 (2002).
[29] S. K. Lamoreaux, Phys. Rev. Lett. **78**, 5 (1997).
[30] U. Mohideen and A. Roy, Phys. Rev. Lett. **81**, 4549 (1998).
[31] A. A. Bytsenko, G. Cognola, L. Vanzo, and S. Zerbini, Phys. Rep. **266**, 1 (1996).
[32] J. S. Dowker and R. Critchley, Phys. Rev. D **13**, 3224 (1976).
[33] E. Elizalde, *Ten Physical Applications of Spectral Zeta Functions*, Lecture Notes in Physics, New Series m Vol. 35 (Springer-Verlag, Berlin, 1995).
[34] E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko, and S. Zerbini, *Zeta Regularization Techniques with Applications* (World Scientific, Singapore, 1994).
[35] S. W. Hawking, Commun. Math. Phys. **55**, 133 (1977).
[36] K. Kirsten, *Spectral Functions in Mathematics and Physics* (Chapman and Hall, London, 2002).
[37] W. Lukosz, Physica (Amsterdam) **56**, 109 (1971).
[38] S. A. Fulling and K. Kirsten, Phys. Lett. B **671**, 179 (2009).
[39] L. P. Teo, Phys. Lett. B **672**, 190 (2009).
[40] E. Hille, *Analytic Function Theory* (Ginn, Boston, 1962), Vol. 2.
[41] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1965).
[42] P. B. Gilkey, *Invariance Theory, the Heat Equation and the Atiyah-Singer Index Theorem* (CRC Press, Boca Raton, 1995).
[43] R. T. Seeley, in *Singular Integrals: Proceedings of the Symposium in Pure Mathematics, Chicago, 1966*, edited by A. P. Calderon (American Mathematical Society, Providence, RI, 1968), Vol. 10, p. 288.
[44] P. B. Gilkey, *Asymptotic Formulae in Spectral Geometry* (Chapman and Hall, London, 2004).
[45] D. V. Vassilevich, Phys. Rep. **388**, 279 (2003).
[46] K. Kirsten, G. Cognola, and L. Vanzo, Phys. Rev. D **48**, 2813 (1993).
[47] P. Epstein, Math. Ann. **56**, 615 (1903).
[48] P. Epstein, Math. Ann. **63**, 205 (1907).
[49] J. Ambjørn and S. Wolfram, Ann. Phys. (N.Y.) **147**, 1 (1983).
[50] E. Elizalde, J. Math. Phys. (N.Y.) **35**, 6100 (1994).
[51] K. Kirsten, J. Phys. A **24**, 3281 (1991).
[52] K. Kirsten, J. Phys. A **26**, 2421 (1993).
[53] J. S. Dowker and G. Kennedy, J. Phys. A **11**, 895 (1978).
[54] L. S. Brown and G. J. Maclay, Phys. Rev. **184**, 1272 (1969).
[55] J. Mehra, Physica (Amsterdam) **37**, 145 (1967).
[56] S. A. Fulling, L. Kaplan, K. Kirsten, Z. H. Liu, and K. A. Milton, arXiv:0806.2468 [J. Phys. A (to be published)].