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# New scalar fields in noncommutative geometry

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In this publication we present an extension of the standard model within the framework of Connes' noncommutative geometry. The model presented here is based on a minimal spectral triple which contains the standard model particles, new vectorlike fermions, and a new $U(1)$ gauge subgroup. Additionally a new complex scalar field appears that couples to the right-handed neutrino, the new fermions, and the standard Higgs particle. The bosonic part of the action is given by the spectral action which also determines relations among the gauge couplings, the quartic scalar couplings, and the Yukawa couplings at a cutoff energy of $\sim 10^{17}$ GeV. We investigate the renormalization group flow of these relations. The low energy behavior allows to constrain the Higgs mass, the mass of the new scalar, and the mixing between these two scalar fields.

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## I. INTRODUCTION

We present an extension of the standard model in its noncommutative formulation [1]. This model is based on the classification of finite spectral triples [2–7]. It extends a minimal model found in [7] which contains the first family of standard model fermions as well as a new family of particles we will call X particles. These X particles are assumed to exist in three generations, just like the standard model particles. The formulation is done in the recent variant where the $KO$ dimension of the internal part of the spectral triple is taken to be six [8,9].

We add to the minimal model right-handed neutrinos together with their Majorana masses. It turns out that these right-handed neutrinos open the possibility to add Dirac mass terms connecting the right-handed neutrinos and the left-handed X particle. These Dirac mass terms induce through the fluctuations of the Dirac operator a new scalar field. This new field and its interaction with the Higgs field will be one of the main concerns of this publication.

Our model has as gauge group $G = U(1)_Y \times SU(2) \times SU(3) \times U(1)_X$ where the standard model subgroup $G_{\rm SM} = U(1)_Y \times SU(2) \times SU(3)$ is broken by the usual Higgs mechanism to $U(1)_{\rm em} \times SU(3)$. The fate of the new subgroup $U(1)_X$ turns out to be closely related to the mass of the X particles. Depending on this mass, the vacuum expectation value of the new scalar is either zero or nonzero, thus $U(1)_X$ can be broken or remain unbroken. Both models permit a considerable modification of the Higgs phenomenology for certain mass regions of the new scalar particle. We will explore some of the consequences. We will focus on the masses of the Higgs boson and the new scalar as well as the possible mixing of the two particles.

Previous attempts to extend the standard model within the framework of noncommutative geometry proved to be extremely difficult. Most of the early attempts unfortunately failed to produce physically interesting models [10]. The only known extension which appears to have an interesting phenomenology just adds new fermions to the standard model [11,12] and possibly new gauge bosons [13]. At least one of these models, the AC model, provides for an interesting dark matter candidate [14]. But the scalar sector has remained so far the usual Higgs sector of the standard model.

It would of course also be desirable to understand the origin of the internal space, i.e. the source of the matrix algebra. There are hints that a connection to loop quantum gravity exists [15]. Also double Fell bundles seem a plausible structure in noncommutative geometry [16]. They could provide a deep connection to category theory and give better insights into the mathematical structure of almost-commutative geometries such as the standard model.

Another open problem is the mass mechanism for neutrinos. In $KO$ dimension zero the masses are of Dirac type [17–19], while $KO$ dimension six also allows for Majorana masses [8,9] and the SeeSaw mechanism, although minor problems concerning an axiom of noncommutative geometry may occur [20]. Another possibility lies in the modification of the spectral action [21].

For a numerical analysis of the standard model with SeeSaw mechanism we refer to [9,22,23] for the models with three and four summands in the matrix algebra.

This paper is organized as follows: In Sec. II we give the construction of the internal space based on a minimal Krajewski diagram found in the classification of finite spectral triples in [7]. This diagram contains the first family of the standard model fermions and additionally a new family fermions, the X particles. We calculate the lift of the gauge group and the fluctuated Dirac operator. This fluctuation leads to the standard model Higgs and a new scalar field.

In the third section we calculate the relevant parts of the spectral action. This calculation provides the potential for the Higgs and the scalar field, as well as constraints on the

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quartic couplings, the Yukawa couplings, and the gauge couplings of the non-Abelian subgroup of the gauge group.

The necessary $\beta$ functions needed to evolve the couplings down to lower energies are given in Sec. IV.

In Sec. V we analyze the running of the couplings and the consequences for the masses of the Higgs boson and the new scalar. Here we assume that the mass of the new scalar is roughly of the same order of magnitude as the Higgs boson mass. In this analysis we neglect the Dirac mass connecting the right-handed Neutrino and the left-handed X particle and we also ignore the implications of the SeeSaw mechanism.

## II. THE INTERNAL SPACE

Internal spaces of almost-commutative geometries are conveniently encoded in Krajewski diagrams [24]. Here we will follow the minimal approach that led to a classification of the internal spaces of almost-commutative geometries [2–7] with respect to the number of summands in the matrix algebra. In [7] essentially one model beyond the standard model results from the classification. Picking one of the diagrams leading to this specific model and extending it as minimally as possible leads to the model presented here.

To construct the internal space of the model we begin by enlarging the minimal Krajewski diagram 2 found in [7]. In its minimal version this diagram encodes the first family of the standard model (without a right-handed neutrino) and a new fermion with Dirac mass term. We will call this new particle the X particle. In principle the X particle may appear in each family. We add to this diagram a right-handed neutrino, its Dirac mass term with the lepton doublet, its Majorana mass term, and a new Dirac mass term coupling the right-handed neutrino to the left-handed X particle. No further mass terms are permitted by the axioms of noncommutative geometry.

The Krajewski diagram for this model is depicted in Fig. 1. Note that the Majorana mass term does not appear explicitly since we have left out the antiparticles to keep the diagram simple. We will not go into the details of the SeeSaw mechanism following from the Majorana mass; details can be found in [8,9].

The model presented here has a minor mathematical shortcoming which it shares with the almost-commutative standard model if right-handed neutrinos with Majorana mass are added. It turns out [20] that the representation of the right-handed neutrino is incompatible with the axiom of orientability. But it seems to be necessary to have such a representation for a working SeeSaw mechanism, which in turn is required by the constraints put on the Yukawa couplings by the spectral action (see Sec. III). A model with no such shortcomings would of course be desirable.

As was shown in [7] the model represented by the Krajewski diagram in Fig. 1 allows an anomaly free lift of the gauge group if the internal algebra is $\mathcal{A} = \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$. Here the first four summands are the well-known algebra of the standard model as found in [4–6].

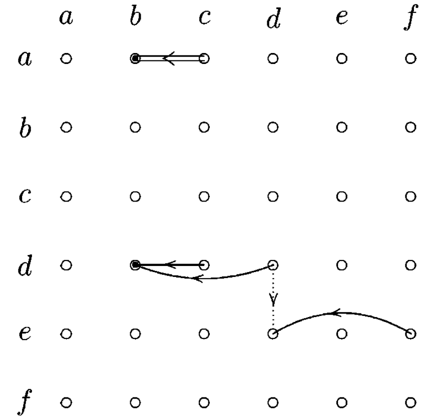


FIG. 1. Krajewski diagram of the extended standard model. The dotted line indicates the Dirac mass term leading to the new scalar field $\varphi$.

From the Krajewski diagram we read off the representation for $\mathcal{A} \ni (a, b, c, d, e, f)$:

$$\begin{aligned}
\rho_L &= \begin{pmatrix} b \otimes 1_3 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \bar{d} \end{pmatrix}, \\
\rho_R &= \begin{pmatrix} c \otimes 1_3 & 0 & 0 & 0 & 0 \\ 0 & \bar{c} \otimes 1_3 & 0 & 0 & 0 \\ 0 & 0 & \bar{c} & 0 & 0 \\ 0 & 0 & 0 & \bar{d} & 0 \\ 0 & 0 & 0 & 0 & f \end{pmatrix}, \\
\rho_L^c &= \begin{pmatrix} 1_2 \otimes a & 0 & 0 \\ 0 & d 1_2 & 0 \\ 0 & 0 & e \end{pmatrix}, \\
\rho_R^c &= \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & e \end{pmatrix},
\end{aligned} \tag{1}$$

and the Dirac mass matrix:

$$\mathcal{M} = \begin{pmatrix} M_u \otimes 1_3 & M_d \otimes 1_3 & 0 & 0 & 0 \\ 0 & 0 & M_e & M_\nu & 0 \\ 0 & 0 & 0 & M_{\nu X} & M_X \end{pmatrix}, \tag{2}$$

where $M_u$, $M_d$, $M_e$, $M_\nu \in M_{2\times1}(\mathbb{C})$ are the usual mass matrices of the quarks and leptons while $M_{\nu X} \in \mathbb{C}$ represents the Dirac mass connecting the right-handed neutrino and the left-handed X particle and $M_X \in \mathbb{C}$ is the Dirac mass term of the X particle.

The internal part $\mathcal{D}$ of the Dirac operator can be decomposed as follows:

$$\mathcal{D} = \begin{pmatrix} \Delta & M \\ M & \bar{\Delta} \end{pmatrix}, \quad \text{with} \quad \Delta = \begin{pmatrix} 0 & \mathcal{M} \\ \mathcal{M}^* & 0 \end{pmatrix}. \tag{3}$$

The Majorana mass matrix of the right-handed neutrino is

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & M_M & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad M_M \in \mathbb{R}. \tag{4}$$

The non-Abelian subgroup of unitaries of the matrix algebra $\mathcal{A}$ is $\mathcal{U}^{nc} = U(2) \times U(3)$. It contains two $U(1)$ subgroups via the determinant that may be lifted to the fermionic Hilbert space [25]. We will call these two subgroups suggestively $U(1)_Y$ and $U(1)_X$, since the first one is nothing else but the standard model hypercharge subgroup and the second one is associated with the X particles. The X particles are neutral with respect to the non-Abelian part of the standard model gauge group, i.e. the X particles are $SU(2) \times SU(3)$ singlets.

For simplicity we will assume that the hypercharge $U(1)_Y$ couples only to the sStandard model sector of the model, while the $U(1)_X$ couples only to X particles and the newly emerging scalar field which we will call $\varphi$. This choice is natural since the anomaly cancellation forces the standard model particles to couple proportionally to each possible $U(1)$ subgroup of the gauge group. Therefore the standard model only ''sees'' one linear combination of the $U(1)$'s while the X particles may see another linear combination. So what we essentially do by our choice is set the electrical charge of the X particles to zero.

The anomaly free lift $L$ then decomposes into the usual standard model lift $L_{\mathrm{SM}}$ which can be found in [25] and the lift $L_X$ acting on the X particles. This can be written as

$$L(\det(u), \det(v), u, v) = L_{\mathrm{SM}}(\det(v), \tilde{u}, \tilde{v}) \oplus L_X(\det(u)), \tag{5}$$

where $u \in U(2)$, $v \in U(3)$, $\tilde{u} \in SU(2)$, and $\tilde{v} \in SU(3)$. For $L_{\mathrm{SM}}$ we find the standard lift [25] and for the new part of the lift $L_X$ we find

$$L_X(\det(u)) = \mathrm{diag}(\det(u)^{Q_X}, \det(u)^{Q_X}; \det(u)^{-Q_X}, \det(u)^{-Q_X}). \tag{6}$$

Here $Q_X$ is the charge of the X particles under $U(1)_X$ and the semicolon divides the particles from the antiparticles. One notices that the X particles couple vectorially to $U(1)_X$ and therefore their Dirac mass $M_X$ is gauge invariant. It follows that the gauge group of our model is $G = U(1)_Y \times SU(2) \times SU(3) \times U(1)_X$.

Next we need to fluctuate the Dirac operator [1] to obtain the gauge bosons as well as the Higgs field $\phi$ and the new scalar field $\varphi$. We define the fluctuated Dirac operator ${}^f\mathcal{D}$ according to [2]:

$${}^f\mathcal{D} = \sum_i r_i L(\det(u_i), \det(v_i), u_i, v_i)\mathcal{D}L(\det(u_i), \det(v_i), u_i, v_i)^{-1}, \; r_i \in \mathbb{R}. \tag{7}$$

One obtains the standard Higgs doublet $\phi$ embedded into a quaternion and a new complex scalar field. For definiteness we put the $U(1)_X$ charge of the X particles to $Q_X = 1$ and therefore the charge of $\varphi$ under $U(1)_X$ is $Q_X = -1$:

$${}^f\mathcal{D}|_{\mathrm{rest}} = M_{\nu X} \sum_i r_i \det(u_i)^{-1} = M_{\nu X}\varphi, \tag{8}$$

where ${}^f\mathcal{D}|_{\mathrm{rest}}$ denotes the part of the fluctuated Dirac operator restricted to the mass matrix that does not commute with the fluctuation of $L_X$.

The Majorana mass matrix of the neutrino commutes with the fluctuation. So we find for the fluctuated mass matrix

$${}^f\mathcal{M} = \begin{pmatrix} \phi M_u \otimes 1_3 & \phi M_d \otimes 1_3 & 0 & 0 & 0 \\ 0 & 0 & \phi M_e & \phi M_\nu & 0 \\ 0 & 0 & 0 & \varphi M_{\nu X} & M_X \end{pmatrix}. \tag{9}$$

From this mass matrix we can now calculate the spectral action which will give us the kinetic term of the scalars as well as the potential for the Higgs field and the new scalar.

## III. SPECTRAL ACTION AND CONSTRAINTS ON THE COUPLINGS

According to [1] the spectral action $S_{CC}$ is given by the number of eigenvalues of the Dirac operator D up to a cutoff energy $\Lambda$. This can be written approximately with the help of a positive cutoff function $f$ and then be calculated explicitly via a heat-kernel expansion:

$$S_{CC} = \mathrm{tr}\left(f\left(\frac{\mathrm{D}^2}{\Lambda^2}\right)\right) = \frac{1}{16\pi^2}\int \mathrm{dV}(a_4 f_4 \Lambda^4 + a_2 f_2 \Lambda^2 + a_0 f_0 + o(\Lambda^{-2})). \tag{10}$$

Here $f_i$ are the first moments of the cutoff function $f$. They enter as free parameters into the model. The heat-kernel coefficients $a_i$ are well known [26] and for the present calculation only $a_2$ and $a_0$ will be of concern. Note that we use the numerating convention of [9], where the number of the coefficient $a_i$ corresponds to the power of $\Lambda$.

The coefficient $a_2$ will give us the mass terms of the potential for the scalar fields while $a_4$ will provide for the kinetic terms for the scalar fields, the quartic couplings of the potential, and also mass terms. All the following relations hold at the cutoff energy $\Lambda$. They are not stable under the renormalization group flow but they provide for the values of the quartic and the Yukawa couplings at the cutoff energy. From there they have to be evolved down into the low energy regime using the renormalization group equations.

To calculate the relevant parts of $a_2$ and $a_4$ we need the traces of ${}^f\mathcal{D}^2$ and ${}^f\mathcal{D}^4$. Since these calculations get easily quite confusing we will greatly simplify the matter by putting at this point all negligible mass terms to zero. This will be all quark masses, apart from the top mass $m_t$, and all lepton masses apart from the tau-neutrino mass $m_\nu$. We also only keep the largest Majorana mass $m_M$ of the neutrinos, the mass $m_X$ of the heaviest X-particle family, and the Dirac mass $m_{\nu X}$ connecting the right-handed tau neutrino to the heaviest X particle.

For the traces ${}^f\mathcal{D}^2$ and ${}^f\mathcal{D}^4$ we find

$$\mathrm{tr}\,{}^f\mathcal{D}^2 = 4[|\phi|^2(3m_t^2 + m_\nu^2) + |\varphi|^2 m_{\nu X}^2 + m_X^2 + \tfrac{1}{2}m_M^2] \tag{11}$$

and

$$\begin{aligned}\mathrm{tr}\,{}^f\mathcal{D}^4 = 4[&|\phi|^4(3m_t^4 + m_\nu^4) + |\varphi|^4 m_{\nu X}^4 + 2m_\nu^2 m_M^2|\phi|^2 \\ &+ 2m_\nu^2 m_{\nu X}^2|\phi|^2|\varphi|^2 + 2(m_M^2 + m_X^2)m_{\nu X}^2|\varphi|^2 \\ &+ m_X^4 + \tfrac{1}{2}m_M^4],\end{aligned} \tag{12}$$

where $|\cdot|$ is the absolute value including the appropriate trace for the quaternionic realization of the Higgs.

From $a_0$ we find the kinetic term for the scalar fields [1]:

$$\frac{f_0}{8\pi^2}(3m_t^2 + m_\nu^2)\,\mathrm{tr}((D_\mu\phi)^*(D^\mu\phi)) + \frac{f_0}{4\pi^2}m_{\nu X}^2(D_\mu\varphi)^*(D^\mu\varphi). \tag{13}$$

One observes that the scalar fields have mass dimension zero. Therefore we have to normalize the scalar fields, $\phi \to \tilde{\phi}$ and $\varphi \to \tilde{\varphi}$, to obtain the standard kinetic terms of the Lagrangian

$$\mathcal{L}_{\mathrm{kin}} = \frac{f_0}{8\pi^2}(3m_t^2 + m_\nu^2)\,\mathrm{tr}((D_\mu\phi)^*(D^\mu\phi)) + \frac{f_0}{4\pi^2}m_{\nu X}^2(D_\mu\varphi)^*(D^\mu\varphi) \tag{14}$$

$$\stackrel{!}{=}(D_\mu\tilde{\phi})^*(D^\mu\tilde{\phi}) + (D_\mu\tilde{\varphi})^*(D^\mu\tilde{\varphi}). \tag{15}$$

From this we deduce the normalization

$$|\phi|^2 = \frac{4\pi^2}{f_0(3m_t^2 + m_\nu^2)}|\tilde{\phi}|^2 \quad \text{and} \quad |\varphi|^2 = \frac{4\pi^2}{f_0 m_{\nu X}^2}|\tilde{\varphi}|^2, \tag{16}$$

which coincides with the standard normalization

$$\tilde{\phi} = \tfrac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + i\phi_2 \\ \phi_3 + i\phi_4\end{pmatrix} \quad \text{and} \quad \tilde{\varphi} = \tfrac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2) \tag{17}$$

for the real scalar fields $\phi_i$ and $\varphi_j$.

Now all the terms that are quadratic in the scalar fields are collected from $a_2$ and $a_0$ to calculate the mass terms.

$$\begin{aligned}\mathcal{L}_{\mathrm{quad}} = &-\frac{f_2}{2\pi^2}(3m_t^2 + m_\nu^2)\Lambda^2|\phi|^2 - \frac{f_2}{2\pi^2}m_{\nu x}^2\Lambda^2|\varphi|^2 \\ &+ \frac{f_2}{4\pi^2}m_\nu^2 m_M^2|\phi|^2 \\ &+ \frac{f_2}{2\pi^2}m_{\nu X}^2 m_X^2|\varphi|^2 \stackrel{!}{=} -\mu_1^2|\tilde{\phi}|^2 - \mu_2^2|\tilde{\varphi}|^2\end{aligned} \tag{18}$$

leads us with the normalization (16) to

$$\begin{aligned}\mu_1^2 &= 2\frac{f_2}{f_0}\Lambda^2 - \frac{g_\nu^2}{3g_t^2 + g_\nu^2}m_M^2 \quad \text{and} \\ \mu_2^2 &= 2\frac{f_2}{f_0}\Lambda^2 - 2m_X^2,\end{aligned} \tag{19}$$

where $g_t$ is the Yukawa coupling of the top quark and $g_\nu$ is the Yukawa coupling of the tau neutrino. Here we have used the fact that $m_i/m_j = g_i/g_j$ where $m_i$ would be the masses whereas $g_i$ are the corresponding Yukawa couplings.

At this stage we encounter an interesting new phenomenon. It turns out that the constraints, which will be determined later, enforce the cutoff energy to be $\Lambda \sim 10^{17}$ GeV and the Majorana mass has been determined to be $m_M \sim 10^{14}$ GeV [22,23]. It follows that $\mu_1^2$ is positive and therefore the minimum of the Higgs potential is nonzero. For $\mu_2^2$ the situation is different. Since the mass of the X particles is gauge invariant one would expect it to be of the order of cutoff energy. So depending on the exact value of $m_X$ the sign of $\mu_2^2$ can be positive or negative and thus allowing for a nonzero or a zero minimum of the potential. We will explore these two cases in detail later.

The last term needed is the quartic term of the potential. For the Lagrangian we find

$$\begin{aligned}\mathcal{L}_{\mathrm{quart}} = &\frac{f_0}{4\pi^2}(3m_t^4 + m_\nu^4)|\phi|^4 + \frac{f_0}{4\pi^2}m_{\nu X}^4|\varphi|^4 \\ &+ \frac{f_0}{2\pi}m_\nu^2 m_{\nu X}^2|\phi|^2|\varphi|^2 \\ &\stackrel{!}{=} \frac{\lambda_1}{6}|\tilde{\phi}|^4 + \frac{\lambda_2}{6}|\tilde{\varphi}|^4 + \frac{\lambda_3}{3}|\tilde{\phi}|^2|\tilde{\varphi}|^2.\end{aligned} \tag{20}$$

Comparing the coefficients and using the normaliztion (16) we obtain the following relations for the quartic couplings:

$$\begin{aligned}\lambda_1 = 24\frac{\pi^2}{4}\frac{3g_t^4 + g_\nu^4}{(3g_t^2 + g_\nu^2)^2}, \qquad \lambda_2 = 24\frac{\pi^2}{4}, \\ 24\frac{\pi^2}{4}\frac{g_\nu^2}{(3g_t^2 + g_\nu^2)}.\end{aligned} \tag{21}$$

The last set of relations to be determined has its origin in the fermionic part of the action $S_{\mathrm{ferm}} = (\Phi, \mathrm{D}\Phi)$. In this case the normalization (16) leads to the identification

$$(3m_t^2 + m_\nu^2)|\phi|^2 + m_{\nu X}^2|\varphi|^2 \stackrel{!}{=} (3g_t^2 + g_\nu^2)|\tilde{\phi}|^2 + g_{\nu X}^2|\tilde{\varphi}|^2 \tag{22}$$

which gives

$$g_{\nu X}^2 = 3g_t^2 + g_\nu^2 = 4\frac{\pi^2}{f_0}. \tag{23}$$

At last the cutoff energy is fixed by the relation for the $SU(2)$ gauge coupling $g_2$ and the $SU(3)$ gauge coupling $g_3$. At $\Lambda$ the equation

$$g_2^2 = g_3^2 = \frac{\pi^2}{f_0} \tag{24}$$

has to hold [1]. This allows to eliminate $f_0$ from the constraints and to combine the previously obtained relations.

Collecting the conditions for the quartic couplings (20), the Yukawa couplings (23), and the gauge couplings (24) we obtain the final relations

$$\begin{aligned} g_2^2 = g_3^2 &= \frac{\lambda_1}{24}\frac{(3g_t^2 + g_\nu^2)^2}{3g_t^4 + g_\nu^4} = \frac{\lambda_2}{24} = \frac{\lambda_3}{24}\frac{3g_t^2 + g_\nu^2}{g_\nu^2} \\ &= \frac{1}{4}g_{\nu X}^2 = \frac{1}{4}(3g_t^2 + g_\nu^2) \end{aligned} \tag{25}$$

which are to hold at the cutoff energy $\Lambda$.

## IV. THE RENORMALIZATION GROUP EQUATIONS

We will now give the one-loop $\beta$ functions of the standard model with $N = 3$ generations with X particles and new scalar field $\tilde{\varphi}$ to evolve the constraints (25) from $E = \Lambda$ down to the low energy regime at $E = m_Z$. We set: $t := \ln(E/m_Z)$, $\mathrm{d}g/\mathrm{d}t =: \beta_g$, $\kappa := (4\pi)^{-2}$.

As mentioned above all fermion masses below the top mass will be neglected. We will also neglect threshold effects. A Dirac mass $m_\nu$ for the $\tau$ neutrino induced by spontaneous symmetry breaking is admitted and is taken to be of the order of the top mass. The Majorana mass $m_M$ is fixed to be $\sim 10^{14}$ GeV to obtain the SeeSaw mechanism [27]. The effect of the running of these Majorana masses on the other couplings was shown to be tiny [22,23], so we will neglect it. Furthermore the mass $m_X$ of the X particle will be taken to be of the order of $\Lambda$.

Since the Dirac mass $m_{\nu X}$ couples the ultra heavy right-handed neutrino and left-handed X particle we will also neglect this coupling by virtue of the Appelquist-Carazzone decoupling theorem [28].

The gauge couplings for the subgroups of the gauge group $G = U(1)_Y \times SU(2) \times SU(3) \times U(1)_X$ are denoted $g_1$, $g_2$, $g_3$, and $g_4$. The $\beta$ functions are [29,30]

$$\begin{aligned} \beta_{g_i} &= \kappa b_i g_i^3, \\ b_i &= (\tfrac{20}{9}N + \tfrac{1}{6}, -\tfrac{22}{3} + \tfrac{4}{3}N + \tfrac{1}{6}, -11 + \tfrac{4}{3}N, \tfrac{1}{3}), \end{aligned} \tag{26}$$

$$\beta_t = \kappa\left[-\sum_i c_i^u g_i^2 + \frac{9}{2}g_t^2\right]g_t, \qquad c_i^t = \left(\frac{17}{12}, \frac{9}{4}, 8, 0\right), \tag{27}$$

$$\begin{aligned} \beta_{\lambda_1} = \kappa[&\tfrac{9}{4}(g_1^4 + 2g_1^2g_2^2 + 3g_2^4) - (3g_1^2 + 9g_2^2)\lambda_1 \\ &+ 12g_t^2\lambda_1 - 36g_t^4 + 4\lambda_1^2 + \tfrac{2}{3}\lambda_3^2], \end{aligned} \tag{28}$$

$$\beta_{\lambda_2} = \kappa[36g_4^4 - g_4^2\lambda_2 + \tfrac{10}{3}\lambda_2^2 + \tfrac{4}{3}\lambda_3^2], \tag{29}$$

$$\begin{aligned} \beta_{\lambda_3} = \kappa[&-(\tfrac{3}{2}g_1^2 + \tfrac{9}{2}g_2^2 + 6g_4^2)\lambda_3 + 6g_t^2\lambda_3 + \tfrac{4}{3}\lambda_3^2 \\ &+ 2\lambda_1\lambda_3 + \tfrac{4}{3}\lambda_2\lambda_3]. \end{aligned} \tag{30}$$

The four gauge couplings decouple from the other equations

$$g_i(t) = g_{i0}/\sqrt{1 - 2\kappa b_i g_{i0}^2 t}. \tag{31}$$

The initial conditions are taken from experiment [31]:

$$g_{10} = 0.3575, \qquad g_{20} = 0.6514, \qquad g_{30} = 1.221. \tag{32}$$

Since $g_1$ is unconstrained the unification scale $\Lambda$ is the solution of $g_2(\ln(\Lambda/m_Z)) = g_3(\ln(\Lambda/m_Z))$,

$$\Lambda = m_Z \exp\frac{g_{20}^{-2} - g_{30}^{-2}}{2\kappa(b_2 - b_3)} = 1.1 \times 10^{17}\ \text{GeV}, \tag{33}$$

and is independent of the number of generations. Next we choose $g_\nu = Rg_t$ at $E = \Lambda$ in order to recover the correct top quark mass. Then we solve numerically the evolution equations for $\lambda_1$, $\lambda_2$, $\lambda_3$, and $g_t$ with initial conditions at $E = \Lambda$ from the noncommutative constraints (25):

$$g_2^2 = \frac{\lambda_1}{24}\frac{(3 + R^2)^2}{3 + R^4} = \frac{\lambda_2}{24} = \frac{\lambda_3}{24}\frac{3 + R^2}{R^2} = \frac{3 + R^2}{4}g_t^2. \tag{34}$$

We note that these constraints imply that all couplings remain perturbative and at our energies we obtain the pole masses of the Higgs, the new scalar field, and the top quark. The top quark mass is then given by

$$m_t = \sqrt{2}\frac{g_t(m_t)}{g_2(m_t)}m_W, \tag{35}$$

while the Higgs mass and the mass of the new particle are obtained by diagonalizing the possibly nondiagonal mass matrix generated by the $\lambda_3$ coupling term. The parameter $R$ will be of no further interest to us here. It will be fixed to $R \sim 1.5$ and allows to recover the top mass $m_t \sim 170$ GeV. It turns out to be rather insensitive to the running of the non-standard model couplings.

## V. PHYSICAL CONSEQUENCES

In the following we will replace $\tilde{\phi}$ and $\tilde{\varphi}$ by $\phi$ and $\varphi$ to obtain a simpler notation. This is not to be confused with the $\phi$ and $\varphi$ which had been normalized in (16).

We will now examine the basic physical features of the model presented above. Since the mass term $\mu_2^2$ of the new scalar field $\varphi$ in the quadratic part of the Lagrangian (18) may have either positive or negative sign, depending on the X-particles mass term $m_X$, we treat these cases separately.

The new scalar field $\varphi$ and the new gauge coupling $g_4$ associated to the gauge subgroup $U(1)_X$ do not influence the running of the non-Abelian gauge couplings $g_2$ and $g_3$. Therefore the cutoff energy $\Lambda$ which is determined by the constraint $g_2 = g_3$, valid at $\Lambda$, remains unaltered compared to the pure standard model value of $\Lambda = 1.1 \times 10^{17}$ GeV [9].

Our main focus will be on the masses of the Higgs particle $\phi$ and the new scalar field $\varphi$. Putting together the relevant Lagrangians (18) and (20) we get the potential

$$V(\phi, \varphi) = -\mu_1^2|\tilde{\phi}|^2 - \mu_2^2|\tilde{\varphi}|^2 + \frac{\lambda_1}{6}|\tilde{\phi}|^4 + \frac{\lambda_2}{6}|\tilde{\varphi}|^4 + \frac{\lambda_3}{3}|\tilde{\phi}|^2|\tilde{\varphi}|^2. \tag{36}$$

From the constraints (25) we get that $\lambda_i(\Lambda) > 0$ for $i = 1 \ldots 3$. To ensure that the potential is bounded from below, we will require that the quartic couplings remain positive under the renormalization group flow. This requirement will put a limit on the possible values of $g_4$.

### A. The case $\mu_2 < 0$

We remind the reader that the quadratic coupling $\mu_2^2$ of the scalar field $\varphi$ is given by

$$\mu_2^2 = 2\frac{f_2}{f_0}\Lambda - 2m_X^2. \tag{37}$$

Since the Majorana mass $m_M$ of the neutrino is of the order of $\sim 10^{14}$ GeV, the quadratic coupling $\mu_1^2$ of the Higgs field $\phi$ has always positive sign. Following [9] we can estimate $f_2/f_0 \sim \tau^2/\rho^2$. Here $\tau \sim 5.1$ and $\rho^2 = G(\Lambda)/G(M_Z)$ is a measure for the running of Newton's constant $G$. Assuming a moderate running of $G$ at high energies and $m_X \gtrsim (f_2/f_0)\Lambda$ it is possible to achieve $\mu_2 < 0$.

As a consequence of $\mu_2 < 0$ the potential (36) of the scalar fields implies a zero vacuum expectation value (vev) $|\langle\hat{\varphi}\rangle| = 0$ for the new scalar field. Therefore the vev of the Higgs field is $|\langle\hat{\phi}\rangle| = v_1/\sqrt{2} = \sqrt{6\mu_1^2/\lambda_1}/\sqrt{2}$.

As a further consequence of the zero vev of $\varphi$ the gauge group $U(1)_X$ remains unbroken and the combined scalar sector breaks the whole gauge group as

$$U(1)_Y \times SU(2) \times SU(3) \times U(1)_X \to U(1)_{\mathrm{em}} \times SU(3) \times U(1)_X. \tag{38}$$

Furthermore the $\varphi$ is not charged under the standard model gauge subgroup while the Higgs field $\phi$ is uncharged under $U(1)_X$. It follows that the gauge bosons do not mix, i.e. the mass of the $W^\pm$ boson is still given by $m_{W^\pm} = (g_2/2)v_1$. Therefore $v_1$ takes its experimental value $v_1 = 246$ GeV. This will remain true even if the vev of the new scalar field is nonzero. In the case of $|\langle\hat{\varphi}\rangle| = 0$ the $U(1)_X$ gauge boson remains massless and appears as a second photon $\gamma'$.

If the vev of $\varphi$ is zero we have for the Higgs mass

$$m_H^2 = \frac{4}{3}\frac{\lambda_1(m_H)}{g_2(m_Z)^2}m_W^2, \tag{39}$$

while the mass of the new scalar field is given by

$$m_\varphi^2 = \frac{2}{3}\frac{\lambda_3(m_\varphi)}{g_2(m_Z)^2}m_W^2 + \mu_2(m_\varphi)^2. \tag{40}$$

The parameters which determine the Higgs mass is $\lambda_1$. The free parameters which determine the mass of the new scalar are $\mu_2$ and implicitly through the $\beta$ functions of the renormalization group equations, $g_4$.

We pursue now the following general strategy: First we evolve $g_2$ to the cutoff energy $\Lambda = 1.1 \times 10^{17}$ GeV. Using the constraints (25) the quartic couplings $\lambda_1$, $\lambda_2$, $\lambda_3$ as well as the top quark Yukawa coupling $g_t$ and the parameter $R$ for the right-handed neutrino are fixed. Then $g_4$ and $\mu_2$ are chosen at $m_Z$ as a free parameters.

Having fixed the free couplings we use the renormalization group equations (30) to evolve the couplings down to low energies. When the pole masses have been reached we calculate the mass of the physical Higgs boson using (39) and the mass of the new scalar using (40). For simplicity we will only consider the region where $m_H/2 \le m_\varphi \le 500$ GeV.

The initial conditions are taken from experiment [31]:

$$g_{10} = 0.3575, \qquad g_{20} = 0.6514, \qquad g_{30} = 1.221. \tag{41}$$

For the top quark mass we take $m_t = 170$ GeV and for the $W^\pm$ boson mass $m_W = 80.4$ GeV. We will ignore all the uncertainties on these values since we are only interested in the general behavior of the model. A detailed investigation using latest data will follow in a later publication.

To ensure that $\lambda_2$ remains positive throughout the running of the couplings we have to take $g_4(m_Z) \le 0.845$. The quartic Higgs coupling turns out to be almost unaltered by any choice of $\mu_2$ and $g_4$ within the range specified above.

Let us first study the most interesting case where $\mu_2(m_\varphi) \ll m_W$. Assuming this we obtain

$$m_\varphi^2 \approx \frac{2}{3}\frac{\lambda_3(m_\varphi)}{g_2(m_Z)^2} m_W^2. \tag{42}$$

In Fig. 2 we have plotted $\lambda_2(m_\phi)$ and $\lambda_3(m_\phi)$ with respect to $g_4(m_Z)$. One observes the steep drop of $\lambda_2$ as $g_4$ reaches its critical value of 0.845.

In Fig. 3 we have plotted the Higgs mass $m_H$ and the mass of the new scalar $m_\varphi$ with respect to $g_4(m_Z)$. Indeed the Higgs mass is almost independent of $g_4$ and takes its pole mass at $m_H \approx 163$ GeV. Compared to the standard model value of ~168 GeV this is only a minor decrease. In contrast to that, the mass of the new scalar field depends rather strongly on $g_4$. It reaches from $m_\varphi \approx 73$ GeV for $g_4 < 0.1$ to $m_\varphi \approx 107$ GeV for $g_4 \approx 0.81$.

If $\mu_2$ is increased and becomes comparable to $m_W$ it will merely shift the mass of $m_\varphi$ upwards as can be seen from (40).

Physically the most interesting case is certainly $2m_\varphi \leq m_H$. In this case the Higgs boson may decay into the new scalar fields. But these scalar fields do not couple to standard model fermions and would thus be unobservable in particle detectors used at Tevatron or LHC. The decay width of the standard model Higgs into $W^\pm$ bosons will therefore be decreased. This fact could reconcile the predicted Higgs mass of $m_H \approx 163$ GeV with recent Tevatron data [32].

### B. The case $\mu_2 > 0$

Let us now turn to the case $\mu_2 > 0$, i.e. $m_X < (f_2/f_0)\Lambda$. Now the potential $V(\phi, \varphi)$ (36) requires both scalar fields to have nonzero vevs, $|\langle\hat{\phi}\rangle| = v_1/\sqrt{2} \neq 0$ and $|\langle\hat{\varphi}\rangle| = v_2/\sqrt{2} \neq 0$. It is still possible to determine the vev $v_1$ of the Higgs field since the relation for the $W^\pm$ boson mass, $m_{W^\pm} = (g_2/2)v_1$, continues to hold. The vev of $\varphi$ in contrast is a free parameter, essentially determined by $\mu_2$.

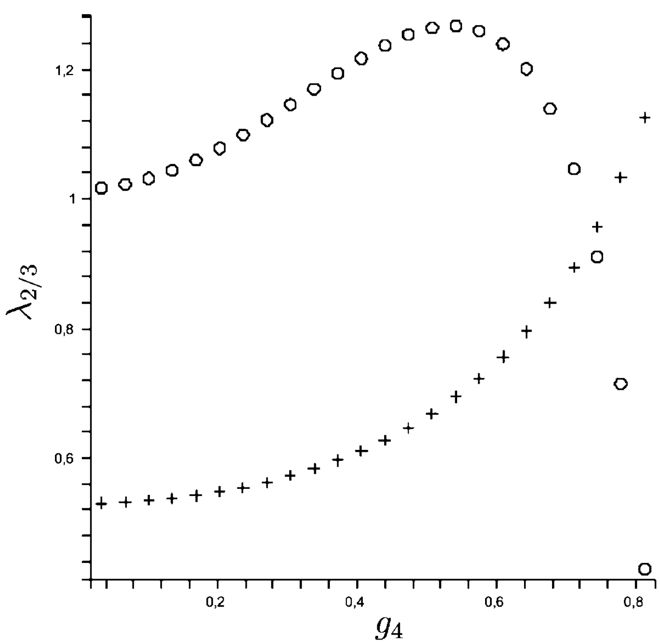


FIG. 2. Dependence of $\lambda_2(m_\varphi)$ (circles) and $\lambda_3(m_\varphi)$ (crosses) on $g_4(m_Z)$ for $\mu_2(m_\varphi) \ll m_W$.

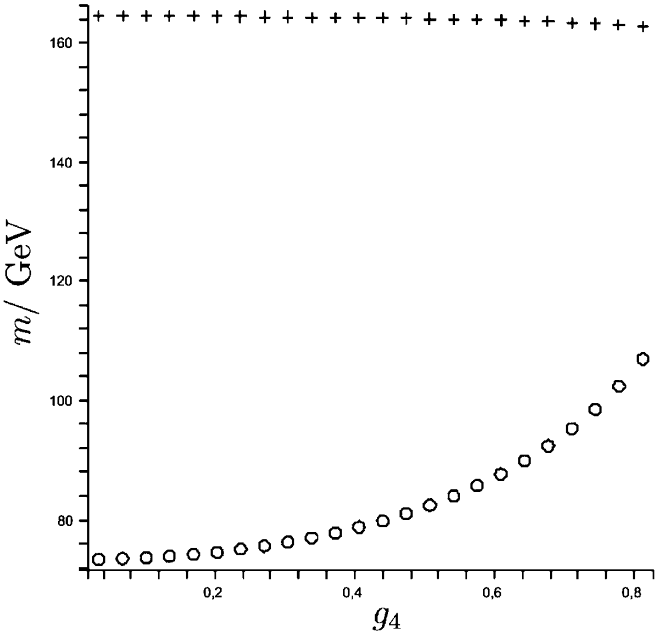


FIG. 3. Dependence of the Higgs mass $m_H$ (circles) and the mass of the new scalar $m_\varphi$ (crosses) on $g_4(m_Z)$ for $\mu_2(m_\varphi) \ll m_W$.

We obtain the physical real Higgs $h_1$ and real scalar field $h_2$ in the standard notation

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ h_1 + v_1 \end{pmatrix} \quad \text{and} \quad \varphi = \frac{1}{\sqrt{2}}(h_2 + v_2). \tag{43}$$

But now the Higgs field and the new scalar mix through the $\lambda_3|\phi|^2|\varphi|^2$ term in the potential and the two nonzero vevs produce a nondiagonal mass matrix. The physical scalar fields correspond therefore to the mass eigenvalues which are easily calculated to be [33]

$$m_{H_1,H_2} = \frac{\lambda_1}{6}v_1 + \frac{\lambda_2}{6}v_2 \mp \sqrt{\left(\frac{\lambda_1}{6}v_1 - \frac{\lambda_2}{6}v_2\right)^2 + \frac{\lambda_3^2}{9}v_1 v_2}, \tag{44}$$

where the real mass eigenstates $H_1$ and $H_2$ are given by

$$\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \tag{45}$$

and

$$\tan(2\theta) = \frac{2\lambda_3 v_1 v_2}{\lambda_1 v_1^2 - \lambda_2 v_2^2}. \tag{46}$$

The mass eigenvalues as well as the mixing angles of $H_1$ and $H_2$ depend on the $U(1)_X$ gauge coupling $g_4$. For comparison we have plotted the two mass eigenvalues

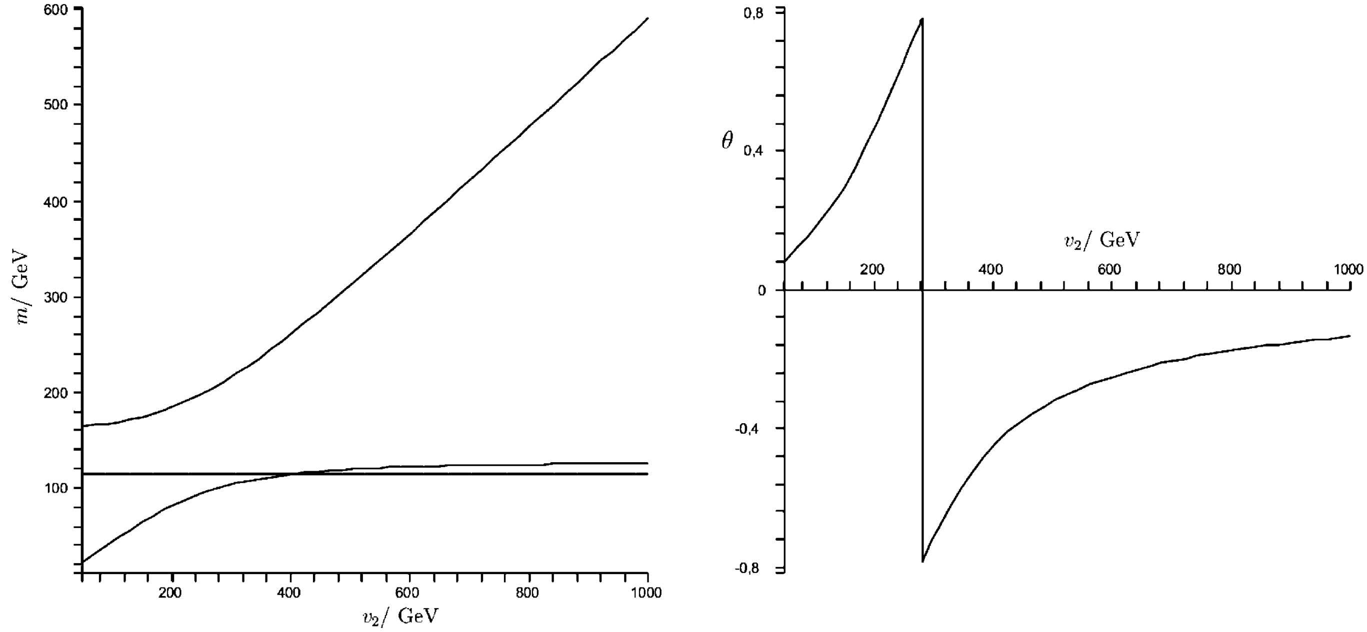


FIG. 4. $g_4(m_Z) = 0.01$: The left figure shows the dependence of $m_{H_1}$ and $m_{H_2}$ on $v_2$. The constant line is the 114 GeV experimental LEP threshold for the standard model Higgs mass. The right figure shows the dependence of the mixing angle $\theta$ on $v_2$.

$m_{H_1}$ and $m_{H_2}$ and the mixing angle $\theta$ in dependence of $v_2$ for the three values $g_4(m_Z) = 0.01$ (Fig. 4), $g_4(m_Z) = 0.3$ (Fig. 5), and $g_4(m_Z) = 0.7$ (Fig. 6). The smaller mass eigenvalue approaches in each case a maximal value as $v_2$ becomes big. This maximal mass crosses the $\sim$114 GeV line for $g_4 \lesssim 0.64$. But throughout the whole range of $v_2$ the mixing angles remain strictly nonzero. Therefore neither mass eigenstate corresponds exactly to the standard model model Higgs.

Since the vev of the new scalar field is now nonzero the scalar sector breaks the gauge group as follows:

$$U(1)_Y \times SU(2) \times SU(3) \times U(1)_X \to U(1)_{\rm em} \times SU(3), \tag{47}$$

where the gauge group $U(1)_X$ is broken into the discrete group $\mathbb{Z}_2$. As a consequence of the breaking of the gauge group the $U(1)_X$ gauge boson acquires a mass which is determined by the vev $v_2$ of the new scalar $\varphi$ and the gauge

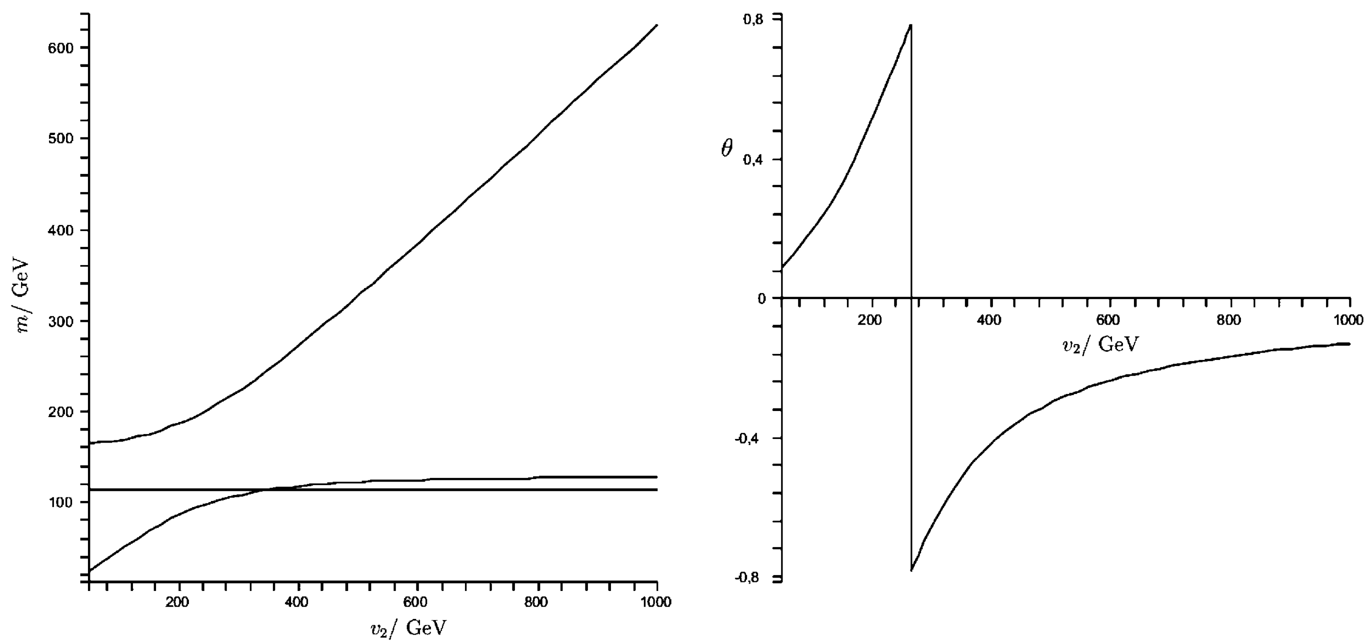


FIG. 5. $g_4(m_Z) = 0.3$: The left figure shows the dependence of $m_{H_1}$ and $m_{H_2}$ on $v_2$. The constant line is the 114 GeV experimental LEP threshold for the standard model Higgs mass. The right figure shows the dependence of the mixing angle $\theta$ on $v_2$.

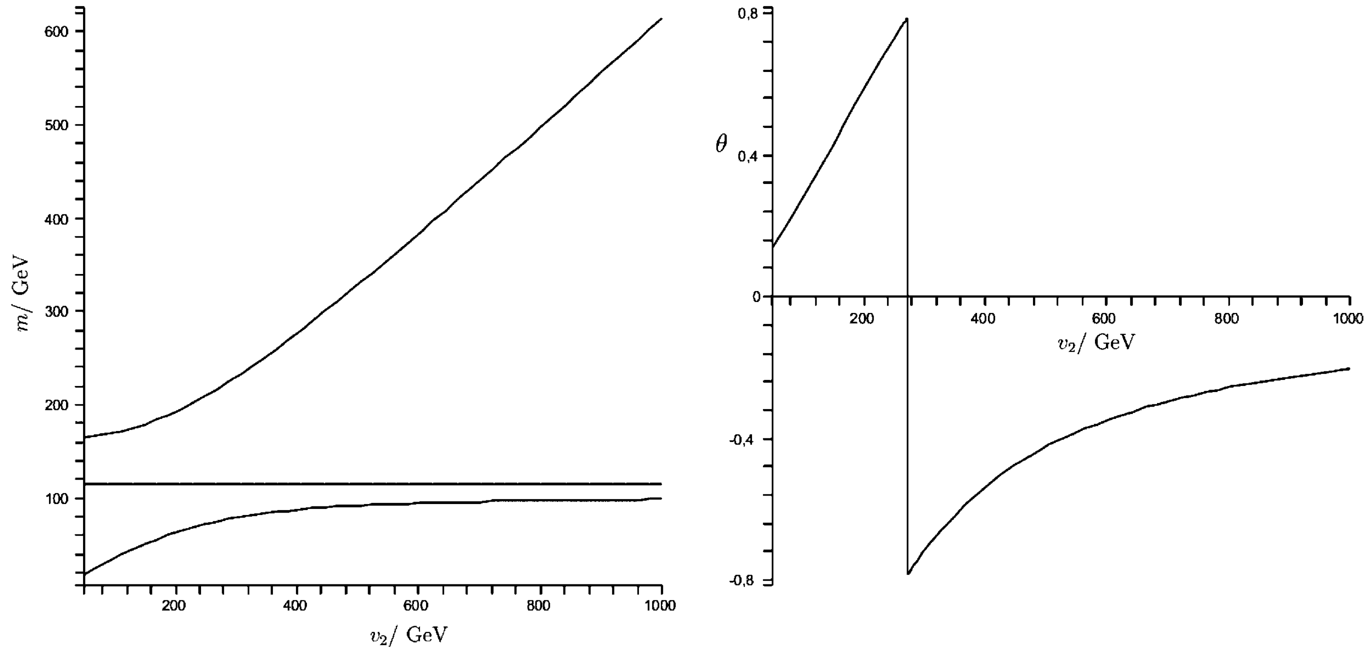


FIG. 6. $g_4(m_Z) = 0.7$: The left figure shows the dependence of $m_{H_1}$ and $m_{H_2}$ on $v_2$. The constant line is the 114 GeV experimental LEP threshold for the standard model Higgs mass. The right figure shows the dependence of the mixing angle $\theta$ on $v_2$.

coupling $g_4$. One finds for the mass of the $Z'$ boson

$$m_{Z'}^2 = g_4 v_2^2. \tag{48}$$

## VI. CONCLUSION AND OUTLOOK

In this publication we presented an extension of the standard model within the framework of Connes' noncommutative geometry [1]. To obtain the model we slightly extended an extension of the standard model found in the classification of minimal spectral triples [7]. The fermionic sector of this minimal spectral triple contains the first family of the standard model fermions and an extra particle which we call the X particle. This minimal model allows for an anomaly free charge assignment under the enlarged standard model gauge group $G = U(1)_Y \times SU(2) \times SU(3) \times U(1)_X$. We assumed that the standard model particles, including the Higgs doublet, are neutral to the new $U(1)_X$ gauge group, while the X particles are neutral to the standard model gauge group but couple vectorially to $U(1)_X$. Consequently their masses are gauge invariant and are therefore assumed to be of the order of the cutoff energy $\Lambda \sim 10^{17}$ GeV.

To this basic model we add right-handed neutrinos, together with their Majorana mass terms. At this stage something interesting happens. The axioms of noncommutative geometry, which can be encoded in Krajewski diagrams [24], permit an additional Dirac mass term. This new mass term connects the right-handed neutrinos and the left-handed X particles. Fluctuating the Dirac operator with the lifted group of unitaries of the internal matrix algebra $\mathcal{A} = \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ then produces the standard Higgs field and a new scaler field. Calculating the spectral action for this model results in a term mixing these two fields, thus altering the standard model Higgs sector considerably.

An intriguing fact of the spectral action principle is that it allows to fix the quartic couplings of the model at a cutoff energy. This property has been exploited to calculate the value of the coupling constants at low energies. From these values the masses of the Higgs field and its coupling to the new scalar can be calculated.

It turns out that the sign of the quadratic coupling of the new scalar field is determined by the mass of the X particles. If at least one family of X particles is sufficiently heavy compared to the cutoff energy, the sign is negative and we have a mass term. If the mass is small compared to the cutoff energy we obtain a positive sign and the new scalar field acquires a nonzero vacuum expectation value.

These two cases have been studied and the renormalization group technique has been applied to the quartic couplings. The results have been presented in Sec. V. We found that the numerical values depend on the $U(1)_X$ gauge coupling $g_4$ as well as the mass of the X particles which enters implicitly through the quadratic coupling of the new scalar.

The phenomenology of this model seems intriguing. Since the classical prediction of the Higgs mass of $m_H \approx 170$ GeV [1,9] from the spectral action is almost certainly excluded by the Tevatron [32] the model presented here may open a new window.

For the case of a zero vacuum expectation value the mass of the Higgs particle remains almost unchanged compared to the standard model value of $m_H \approx 170$ GeV. But the mass of the new particle can be as low as $m_\varphi \approx 73$ GeV which is less than half the Higgs mass. Therefore the Higgs may decay into the new scalar thus changing its decay width. This could perhaps evade the restrictions posed by the Tevatron [32].

For the case of a nonzero vacuum expectation value the new scalar and the Higgs mix considerably. The mass eigenstates will in general consist of a rather light scalar particle $m_{H_1} \sim 120$ GeV and a heavy particle $m_{H_2} \geq 170$ GeV.

Similar models with additional real and complex scalar fields have been studied before. For example the so-called stealth model [34] where the new scalar field can hide the Higgs field completely from detection. This model might also provide an interesting candidate for dark matter [35]. See also [36] where a closely related model has been studied, the main difference to our model being that the new $U(1)$ group is assumed to be a global symmetry. Models with gauged new $U(1)$ group have also been considered, see [33] for a $(B-L)$-type extension of the standard model.

A detailed study of the phenomenology of the model presented here is in progress and will be published soon. Open issues are the compatibility with LEP data and Tevatron data, the existence of viable dark matter candidates, and perhaps a mechanism to obtain the baryon asymmetry of the Universe. If these problems could be (partially) solved by the model presented here, this would be a rather strong case for the spectral action principle. It is intriguing that despite the new degrees of freedom like the gauge coupling $g_4$ and the mass of the X particles, the resulting models are still extremely constrained by the relations among the couplings (25) at the cutoff energy.

## ACKNOWLEDGMENTS

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