

Computation of the radiation amplitude of oscillonsGyula Fodor,¹ Péter Forgács,^{1,3} Zalán Horváth,² and Márk Mezei²¹MTA RMKI, H-1525 Budapest 114, P.O.Box 49, Hungary²Institute for Theoretical Physics, Eötvös University, H-1117 Budapest, Pázmány Péter sétány 1/A, Hungary³LMPT, CNRS-UMR 6083, Université de Tours, Parc de Grandmont, 37200 Tours, France

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The radiation loss of small-amplitude oscillons (very long-living, spatially localized, time-dependent solutions) in one-dimensional scalar field theories is computed in the small-amplitude expansion analytically using matched asymptotic series expansions and Borel summation. The amplitude of the radiation is beyond all orders in perturbation theory and the method used has been developed by Segur and Kruskal in Phys. Rev. Lett. **58**, 747 (1987). Our results are in good agreement with those of long-time numerical simulations of oscillons.

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I. INTRODUCTION

Time-dependent spatially localized solutions in various field theories, which are long-living in the sense of staying localized much longer than the light crossing time, had been already found in the 1970s [1–4] and ever since they are still attracting considerable interest [5–8]. An obvious reason for interest is the unexpected longevity of these objects, all of which exhibit nearly periodic oscillations in time. The presence of at least one real massive scalar field seems to be necessary in order that such long-living, spatially localized field oscillations—*oscillons*—could form. Since oscillons appear in the course of time evolution starting from rather generic initial data, this provides another reason to consider them of physical importance. Oscillons have been found to form in physical processes, e.g., as a result of vortex-antivortex annihilation in a two-dimensional Abelian Higgs model [9], domain wall collapse in ϕ^4 theory [10], QCD phase transition [11], or during symmetry breaking in a three-dimensional Abelian Higgs model [12]. Therefore there is some reason to believe that oscillons (or configurations close to them) influence the dynamics, and they play a role in phase transitions, cosmology, and the dynamics of extended objects (cosmic strings, domain walls, etc.), see, e.g., [13–16]. Importantly oscillons have been also found in the bosonic sector of the standard model [17–19]. There are attempts for including fermion fields in the study of oscillons [20]. Oscillons resemble breathers of the one-dimensional sine-Gordon (SG) theory, with the important difference that unlike true breathers they are continuously losing energy by slow radiation. An oscillon just like a breather possesses a localized “core,” but differs significantly in its “radiative” region outside of the core. Oscillons have been observed in various spatial dimensions, from $d = 1$ up to $d = 6$ [21,22], there is, however, a marked difference between oscillons in $d \leq 2$ and in $d > 2$. In dimensions $d = 1, 2$ oscillons can be well described by an adiabatic time evolution of breatherlike configura-

tions with an ever-decreasing amplitude and with an increasing frequency tending towards a limit determined by the mass threshold [23–25]. In higher dimensions there are various types of oscillons; they exhibit instabilities and their behavior is more complex [7,8,26]. Oscillons have been studied on a $1 + 1$ dimensional expanding background in Refs. [27,28].

In this paper we consider oscillons in scalar field theories with a general self-interaction potential in one spatial dimension. So far most work on oscillons has been either purely numerical or based on various approximations. For example, oscillon energy and lifetime have been recently estimated in Ref. [29]. Our starting point is the small-amplitude expansion [30,31] which yields breatherlike configurations with spatially localized cores. The small-amplitude expansion yields an asymptotic series for the core. Solutions of the field equations are either periodic in time or are radiating. In the first case standing wave tails are present outside of the breather core, whereas in the radiative case there are outgoing waves. The latter correspond to oscillons. Following Segur and Kruskal [32] and adapting the approach developed in Refs. [33,34] we compute analytically one of the most important physical characteristics of oscillons, the amplitude of the outgoing wave responsible for their eventual demise. Although strictly speaking our results are only valid for small-amplitude oscillons, this is not a major drawback, since in $d = 1$ the long-time behavior of any oscillon (which is what we are interested in) is determined by that of the small-amplitude one. The energy loss of a small-amplitude oscillon in $d = 1$ to leading order can be written as

$$\frac{dE}{dt} = -Ae^{-B/E}, \quad (1)$$

where E is the energy of the oscillon core, and A and B are constants determined by the theory. The corresponding equation for the ϕ^4 theory has been found first in Ref. [32]. Geicke [35] has verified numerically the asymptotic energy loss following from Eq. (1), i.e., $E(t) \sim B/\ln t$.

Since this is not a completely straightforward numerical exercise (one has to prepare very good initial data, one needs long-time simulations to high precision, etc.) we have also made a detailed numerical investigation. For small enough amplitudes, ε , one has $E \propto \varepsilon$ and it is clear that the energy loss is beyond all orders in ε . In this work we compute the constants A, B analytically, using matched asymptotic expansion and Borel summation techniques, as well as by an independent numerical method. Also we have been able to give convincing numerical evidence for the validity of the radiation law (1) by preparing good initial data and performing long-time simulations. We have also checked the validity of our analytical and numerical results on the example of the sine-Gordon theory where A is exactly zero.

The plan of this paper is the following: First, in Sec. II A we present the essential points of the small-amplitude expansion in a one-dimensional scalar field theory and relate it to the Fourier expansion of time-periodic solutions. Then we analytically extend the solution of the mode equations to the complex plane. In Sec. II B we relate the radiative tail of oscillons to an exponentially small correction to the asymptotic expansion. In Sec. II C the radiation law, Eq. (1), is derived. In Sec. II D we determine the amplitude of the exponential correction by solving numerically the Fourier mode equations in the complex plane. In Sec. II E the Borel summation procedure is used to calculate this correction analytically. In the second half of the paper numerical simulations are presented supporting the earlier analytical results. In Secs. III A and III B we introduce the methods used in numerical simulations and for generating good initial data starting from the small-amplitude expansion and employing a tuning, respectively. In Sec. III C we estimate the lattice effects by putting the sine-Gordon breather on the lattice and measuring its energy loss. In Sec. III D we investigate ϕ^4 oscillons evolving from initial data extracted from the small-amplitude expansion. We found that the functional form of the theoretical radiation law, Eq. (1), fits our data points; however, the fitted value of A differs from the theoretical one. One reason for the disagreement is the exceptionally small value of A in the ϕ^4 theory. Furthermore we have performed simulations starting with the initial data of Ref. [35]. In Sec. III E we investigate oscillon radiation in a specific ϕ^6 theory where the value A is maximal. In this case our numerical results agree with the theoretical one with satisfactory precision.

II. ANALYTIC APPROACH TO OSCILLON RADIATION

A. Small-amplitude expansion

We consider a real scalar theory in a $1 + 1$ dimensional Minkowski space-time, with a general self-interaction potential, $U(\phi)$. The equation of motion is just a nonlinear wave equation (NLWE) given by

$$-\phi_{,tt} + \phi_{,xx} = U'(\phi) = \phi + \sum_{k=2}^{\infty} g_k \phi^k, \quad (2)$$

where ϕ is a real scalar field. In Eq. (2) the mass of the field is chosen to be 1, and it has been assumed that the potential, $U(\phi)$, can be written as a power series in ϕ where the g_k are real constants.

As explained in detail in Refs. [30,31] small-amplitude solutions of Eq. (2) can be represented by the series:

$$\phi = \sum_{k=1}^{\infty} \varepsilon^k \phi_k(\tau, \zeta), \quad (3)$$

where ε is a small parameter and the coordinates have been rescaled as $\tau = \omega t$ and $\zeta = \varepsilon x$, with $\omega = \sqrt{1 - \varepsilon^2}$. The time dependence of the functions $\phi_k(\tau, \zeta)$ is found to be determined recursively by a set of (forced) oscillator equations. For example the first few ϕ_k can be written as

$$\begin{aligned} \phi_1 &= p_1(\zeta) \cos \tau & \phi_2 &= \frac{1}{6} g_2 p_1^2(\zeta) (\cos(2\tau) - 3) \\ \phi_3 &= p_3(\zeta) \cos \tau + \frac{1}{72} (4g_2^2 - 3\lambda) p_1^3(\zeta) \cos(3\tau), \end{aligned} \quad (4)$$

where $p_1(\zeta), p_3(\zeta)$ are given in terms of the parameters of the potential and of a single function, $S(\zeta)$ as

$$\begin{aligned} p_1(\zeta) &= \frac{S(\zeta)}{\sqrt{\lambda}}, \\ p_3(\zeta) &= \frac{1}{\lambda^{5/2}} \left[\left(\frac{1}{24} \lambda^2 - \frac{1}{6} \lambda g_2^2 + \frac{5}{8} g_5 - \frac{7}{4} g_2 g_4 + \frac{35}{27} g_4^2 \right) \right. \\ &\quad \left. \times Z(\zeta) - \frac{1}{54} \lambda g_2^2 S(\zeta) (32 + 19S^2(\zeta)) \right], \end{aligned} \quad (5)$$

together with $Z = S(4 - S^2)/3$, and $\lambda = 5g_2^2/6 - 3g_3/4 > 0$. The function $S(\zeta)$ is a globally regular solution of the following nonlinear equation:

$$\frac{d^2 S}{d\zeta^2} - S + S^3 = 0. \quad (6)$$

By fixing the center of symmetry at $\zeta = 0$ the regular solution of Eq. (6) is given simply by

$$S(\zeta) = \sqrt{2} \operatorname{sech}(\zeta). \quad (7)$$

The only condition on the potential ensuring the existence of an exponentially decreasing solution to Eq. (6) is the positivity of the parameter λ . It is clear that each term in the series Eq. (3) is exponentially decaying in space and is periodic in time, hence if it would converge it would yield exponentially localized breathers. It has been demonstrated by Segur and Kruskal [32] for the example of the ϕ^4 theory with $U(\phi) = \phi^2(\phi - 2)^2/8$ that the small-amplitude expansion (3) does not converge, actually it is an asymptotic series. Nevertheless for sufficiently small values of ε this asymptotic series constitutes an excellent approximation for oscillons of frequency $\omega(\varepsilon)$ over a large interval in ζ

[31]. The physical reason for the absence of spatially localized, exactly time-periodic breathers is simply that for generic potentials, time-dependent solutions of the NLWE (2) radiate. This makes it very plausible that the asymptoticity of the series (3) in general models is due to radiative phenomena. An interesting prototype exception, i.e., when (3) converges, is the celebrated breather solution of the sine-Gordon theory, where $U(\phi) = 1 - \cos(\phi)$. The SG breather can be written as

$$\begin{aligned}\phi(x, t)_B &= 4 \arctan \left[\frac{\varepsilon \sin(\omega(\varepsilon)t)}{\omega(\varepsilon) \cosh(\varepsilon x)} \right] \\ &= 4 \arctan \left[\frac{\varepsilon}{\sqrt{1 - \varepsilon^2}} \frac{S(\zeta) \sin \tau}{\sqrt{2}} \right].\end{aligned}\quad (8)$$

It is a simple matter now to show that the small-amplitude expansion of the SG breather yields a convergent series, indeed. Let us note here that in Eq. (8) $|\varepsilon| < 1$, and it is not necessarily small. By performing the ε expansion of (8) we

$$\begin{aligned}\left[\varepsilon^2 \frac{d^2}{d\zeta^2} - 1 \right] \Phi_0 &= g_2 \Phi_0^2 + g_3 \Phi_0^3 + \left(\frac{3g_3}{2} \Phi_0 + \frac{g_2}{2} \right) \sum_{m=1}^{\infty} \Phi_m^2 + \frac{g_3}{4} \sum_{m,p,q=1}^{\infty} \Phi_m \Phi_p \Phi_q \delta_{m,\pm p \pm q} + \dots, \\ \left[\varepsilon^2 \frac{d^2}{d\zeta^2} + (n^2 \omega^2 - 1) \right] \Phi_n &= (3g_3 \Phi_0^2 + 2g_2 \Phi_0) \Phi_n + \left(\frac{3g_3}{2} \Phi_0 + \frac{g_2}{2} \right) \sum_{m,p=1}^{\infty} \Phi_m \Phi_p \delta_{n,\pm m \pm p} \\ &\quad + \frac{g_3}{4} \sum_{m,p,q=1}^{\infty} \Phi_m \Phi_p \Phi_q \delta_{n,\pm m \pm p \pm q} + \dots, \quad n = 1, 2, \dots,\end{aligned}\quad (11)$$

where $\delta_{m,\pm p \pm q} = \delta_{m,p+q} + \delta_{m,p-q} + \delta_{m,-p+q} + \delta_{m,-p-q}$. A remarkable simplification takes place, if the potential is symmetric around zero, i.e., $g_{2k} = 0$ for $k = 1, 2, \dots$. In this case only the odd Fourier coefficients are nonzero in the Fourier expansion and the mode equations take the form:

$$\begin{aligned}\left[\varepsilon^2 \frac{d^2}{d\zeta^2} + (n^2 \omega^2 - 1) \right] \Phi_n \\ = \frac{g_3}{4} \sum_{m,p,q=1}^{\infty} \Phi_m \Phi_p \Phi_q \delta_{n,\pm m \pm p \pm q} + \dots,\end{aligned}\quad (12)$$

$n, m, p, q = 1, 3, 5, \dots$

Equations (11) admit solutions with a spatially well-localized core and an oscillating (standing wave) tail whose amplitude tends to a constant for $|x| \rightarrow \infty$. The asymptotic tail can be approximated as consisting of a superposition of standing waves of frequencies $n\omega$, $n = 2, 3, \dots$. We note that for bounded solutions all modes Φ_n for $n > 1$ contain two parameters which can be interpreted as an amplitude and a phase of the corresponding frequency standing wave. We are interested in solutions for which the amplitudes of the tails are much smaller than that of the core, and these are the ones related to oscillons. Because of the existence of the asymptotic standing wave

obtain

$$\begin{aligned}\phi(\tau, \zeta)_B &= \sqrt{2} \varepsilon S(\zeta) [2 + \varepsilon^2 - \varepsilon^2 S^2(\zeta)/4] \sin \tau \\ &\quad + \sqrt{2} \varepsilon^3 S^3(\zeta) \sin(3\tau)/6 + \mathcal{O}(\varepsilon^5)\end{aligned}\quad (9)$$

and it is easy to see that this series converges for $|\varepsilon S| < 1$, thus for $\varepsilon < 1/\sqrt{2}$ the small-amplitude expansion of the SG breather converges.

For a given theory the series (3) is unique. Since all of its terms are time-periodic, it represents a family of breather-like solutions in the sense of an asymptotic series. At this point it is natural to look for time-periodic solutions of the NLWE Eq. (2) by expanding the field ϕ in a Fourier series:

$$\phi(\zeta, \tau) = \sum_{k=0}^{\infty} \Phi_k(\zeta) \cos(k\tau),\quad (10)$$

leading to an infinite set of mode equations for the Φ_k

tail these solutions are not unique. Intuitively it is clear that for a given frequency, solutions with the smallest possible amplitude standing wave tail should be close to the “inner part” of oscillons. It is quite plausible that for a fixed value of ε , the solution with a minimal amplitude tail and being symmetric with respect to the origin is actually unique. This hypothesis is supported by the results of Ref. [26], where such breather-type solutions have been named quasibreathers. Another type of quasibreather is the (most likely unique) solution, for which all modes, $\Phi_n \rightarrow 0$ exponentially for say $x \rightarrow -\infty$, with a small-amplitude oscillating tail in the other direction. We shall refer to such objects as asymmetric quasibreathers.

For sufficiently small values of ε the tail amplitudes become exponentially small in ε . In this case the core can be treated separately from the tail, and a linear superposition of the two is a very good approximation to the solution. One can verify that the small-amplitude expansion of (11) reproduces the terms of the asymptotic expansion (4). We remark that the small-amplitude expansion represents an essentially unique core. As already mentioned the amplitude of the standing wave tail is $\mathcal{O}(\exp(-1/\varepsilon))$, i.e., it is beyond all orders in perturbation theory. Segur and Kruskal (SK) [32] has worked out a method to compute this “transcendentally small” amplitude on the example of the ϕ^4 theory using matched asymptotic series expansion. In the

following we shall use the SK method to find the amplitude of the standing wave tail of quasibreathers when $\varepsilon \rightarrow 0$. The main idea of the SK method is to define an “inner” problem in the complex ζ -plane in the neighborhood of the singularity of S closest to the real axis. Clearly it is located at $\zeta = \pm i\pi/2$ where the function S has a simple pole. In fact close to the pole at $i\pi/2$

$$S(y) = -\frac{i\sqrt{2}}{\varepsilon y} + \frac{i\sqrt{2}\varepsilon y}{6} + \mathcal{O}((\varepsilon y)^3), \quad (13)$$

where the rescaled variable y is defined as

$$\zeta = i\pi/2 + \varepsilon y. \quad (14)$$

The inner problem is defined by the rescaled variables near the singularity and keeping only the leading terms in ε . For example, for symmetric potentials the inner equations are

$$\left[\frac{d^2}{dy^2} + (n^2 - 1) \right] \Phi_n = \frac{g_3}{4} \sum_{m,p,q=1}^{+\infty} \Phi_m \Phi_p \Phi_q \delta_{n,\pm m \pm p \pm q} + \dots \quad (15)$$

where it has been also used that $\omega^2 = 1$ to leading order in ε . We look for such solutions of the mode equations of the inner problem, which can be matched to the solution of the outer problem. In our case the outer problem is defined by the analytic continuation of the small-amplitude expansion from the real axis. The matching region is parametrized in the following way:

$$\left\{ |\varepsilon y| \ll 1 (\varepsilon y \rightarrow 0), |y| \gg 1 (|y| \rightarrow \infty), \right. \\ \left. -\pi \leq \arg(y) \leq -\frac{\pi}{2} \right\}. \quad (16)$$

Making use of the fact that the leading ε order of Φ_n on the real axis is proportional to $\varepsilon^n S^n$ it follows that in the matching region

$$\Phi_0 = \sum_{k=2}^{+\infty} \frac{a_k^{(0)}}{y^k} + \mathcal{O}(\varepsilon^2) \quad (17)$$

$$\Phi_n = \sum_{k=n}^{+\infty} \frac{a_k^{(n)}}{y^k} + \mathcal{O}(\varepsilon^2) \quad (18)$$

where the coefficients $a_k^{(n)}$ are uniquely determined by the small-amplitude expansion on the real axis. The first few terms in the expansion (17) and (18), up to order $1/y^4$, can be written as

$$\Phi_0 = \frac{g_2}{\lambda} \frac{1}{y^2} + \dots$$

$$\Phi_1 = -\frac{i\sqrt{2}}{\sqrt{\lambda}} \frac{1}{y} - \frac{i2\sqrt{2}}{3\lambda^2\sqrt{\lambda}} \left(\frac{1}{24} \lambda^2 + \frac{8}{9} \lambda g_2^2 + \frac{5}{8} g_5 \right. \\ \left. - \frac{7}{4} g_2 g_4 + \frac{35}{27} g_2^4 \right) \frac{1}{y^3} + \dots$$

$$\Phi_2 = -\frac{g_2}{3\lambda} \frac{1}{y^2} + \dots \quad \Phi_3 = \frac{i\sqrt{2}(4g_2^2 - 3\lambda)}{36\lambda^{3/2}} \frac{1}{y^3} + \dots \quad (19)$$

B. Correction beyond all orders

In this subsection we will construct an exponential correction to the asymptotic series, Eq. (19), which after matching to the outer region determines the amplitude of the standing wave tail. A correction beyond all orders to a divergent series might seem meaningless at first glance, however, following Ref. [32], we can give meaning to the method by finding a place where at least the imaginary part of the original series converges. This region is the imaginary axis $\text{Re} y = 0$, where the algebraic asymptotic series (19) is real, so $\text{Im} \Phi_n = 0$, and the imaginary part of the series converges trivially. We divide Φ_n into real and imaginary parts,

$$\Phi_n = \Psi_n + i\Omega_n, \quad (20)$$

and decompose the inner version of Eq. (11) (with $\varepsilon = 0$ and using the variable y) into real and imaginary parts. Then one can linearize the imaginary parts of the mode equations, and obtain coupled linear differential equations for Ω_n along the imaginary axis. These equations contain Ω_m terms multiplied by various powers of Ψ_k . As a first approximation, in the matching region these can be neglected, and one gets decoupled homogeneous linear differential equations with constant coefficients for Ω_n . For Ω_0 the solutions are $\Omega_0 = \exp(\pm y)$, which are oscillating, nondecreasing functions in the direction of the imaginary axis and have to be omitted, as they cannot be matched to the ε expansion on the real axis. The solutions for Ω_1 are linear in y and the matching conditions forbid them as in the previous case. The solutions Ω_n for $n > 1$ which are tending to zero as $\text{Im} y \rightarrow -\infty$ are

$$\Omega_n = \nu_n \exp(-i\sqrt{n^2 - 1}y). \quad (21)$$

Because of linearization one cannot determine the amplitudes ν_n ; methods to calculate them will be presented in the next subsections.

If $\nu_2 \neq 0$ the dominant among the exponential corrections is Ω_2 which will yield the leading term in the radiation. It is possible to get correction terms to ν_n by taking into account terms proportional to Ψ_k in the differential equations and substituting them by the leading-order terms of the asymptotic expansion. Also considering the coupling

between different Ω_n , we get

$$\Omega_2 = \nu_2 \exp(-i\sqrt{3}y) \left[1 + \frac{2i\sqrt{3}(g_2^2 - 3\lambda)}{9\lambda y} + \mathcal{O}\left(\frac{1}{y^2}\right) \right] + \mathcal{O}\left(\frac{1}{y} \exp(-i\sqrt{8}y)\right). \quad (22)$$

In the case of symmetric potentials Φ_{2k} s are absent, so the dominant contribution comes from

$$\Omega_3 = \nu_3 \exp(-i\sqrt{8}y) \left[1 - \frac{i}{y\sqrt{2}} + \mathcal{O}\left(\frac{1}{y^2}\right) \right] + \mathcal{O}\left(\frac{1}{y^2} \exp(-i\sqrt{24}y)\right). \quad (23)$$

Similar exponential correction appears in the neighborhood of the singularity $-i\pi/2$. We should still match the correction to the imaginary part, Eq. (22), to the solution on the real axis. Hence we linearize the equation of Φ_2 , (11) about the quasibreather core and get the following solution:

$$\delta\Phi_2 = C \sin(\sqrt{3}x + \alpha), \quad (24)$$

where C and α are arbitrary constants. We analytically continue $\delta\Phi_2$ to the complex plane. We match it to the exponential correction (22) obtained in the inner region around the pole $i\pi/(2\varepsilon)$, and to the corresponding expression around $-i\pi/(2\varepsilon)$. This determines C and α

$$C = 2\nu_2 \exp\left(-\frac{\sqrt{3}\pi}{2\varepsilon}\right), \quad \alpha = 0. \quad (25)$$

Neglecting contributions from higher Fourier modes the derivative of the field ϕ in the origin oscillates as

$$\partial_x \phi|_{x=0} = 2\nu_2 \exp\left(-\frac{\sqrt{3}\pi}{2\varepsilon}\right) \sqrt{3} \cos(2t). \quad (26)$$

For symmetric potentials the above matching procedure works in a completely analogous way. The original problem of determining a periodic solution of the field equation, Eq. (2), is well posed if we impose boundary conditions. We use the boundary conditions introduced by Segur and Kruskal in Ref. [32], namely, we require the field to vanish at $x \rightarrow -\infty$ and that the solution remains bounded. These requirements provide sufficient conditions to make the solution unique. We obtained a time-periodic solution of the field equation which is asymmetric with respect to $x = 0$. It is clearly not a breather, since it has a standing wave tail in $x \rightarrow +\infty$ and in this sense it is weakly localized.

We can add a transcendentally small standing wave to Φ_2 as it solves the corresponding linearized mode equation about the asymmetric quasibreather (AQB) denoted by ϕ_{AQB} , and it will give the leading-order transcendental term in the new solution. Therefore one can write a symmetric configuration ϕ_S in the following form:

$$\phi_S = \phi_{\text{AQB}} + \phi_{\text{st}}, \quad (27)$$

where ϕ_{st} denotes a transcendentally small standing wave.

ϕ_{st} has to make ϕ_S symmetric about the origin. This requirement is enough to uniquely determine it. The derivative of ϕ_{AQB} at $x = 0$ is given by (26), hence

$$\phi_{\text{st}} = 2\nu_2 \exp\left(-\frac{\sqrt{3}\pi}{2\varepsilon}\right) \frac{\cos(\sqrt{3}x + \xi)}{\sin\xi} \cos(2t), \quad (28)$$

where ξ is an arbitrary constant. The symmetric configuration with the minimal amplitude tail, corresponding to $\xi = \pi/2$, has been named quasibreather in Ref. [31]. This result is consistent with our knowledge about symmetric solutions, as when we take into account the standing wave tail in one mode, one of the two free parameters is required for symmetrization. From Eq. (28) one immediately gets the standing wave tail of the asymmetric quasibreather:

$$\phi_{\text{AQB}} = 4\nu_2 \exp\left(-\frac{\sqrt{3}\pi}{2\varepsilon}\right) \sin(\sqrt{3}x) \cos(2t) \quad \text{for } x \rightarrow +\infty. \quad (29)$$

In conclusion we were able to determine the asymptotic field of the asymmetric and symmetric quasibreaters up to one parameter ν_2 (ν_3). In Secs. IID and IIE we will determine this parameter.

C. Radiation law for small-amplitude oscillons

We can construct a symmetric time-dependent solution of the field equation by repeating the same steps as in the time-periodic case and replacing the standing wave ϕ_{st} with a moving wave ϕ_{rad} . Let us denote the oscillon field with ϕ_{osc} . Then from $\phi_{\text{osc}} = \phi_{\text{AQB}} + \phi_{\text{rad}}$ it follows that

$$\phi_{\text{rad}} = -2\nu_2 \exp\left(-\frac{\sqrt{3}\pi}{2\varepsilon}\right) \sin(\sqrt{3}x + 2t). \quad (30)$$

We find that the amplitude of the tail of the quasibreather determined by $\xi = \pi/2$ is equal to the amplitude of the outgoing radiation from the corresponding oscillon. This property has been already noted in Ref. [26].

After the little digression on quasibreather tails and the determination of the oscillon radiation field we focus on the radiation law. The asymptotic oscillon field from Eq. (30) is

$$\phi_{\text{osc}} = 2\nu_n \exp\left(-\frac{\pi\sqrt{n^2-1}}{2\varepsilon}\right) \sin(\sqrt{n^2-1}x - nt) \quad \text{for } x \rightarrow +\infty, \quad (31)$$

$$\phi_{\text{osc}} = -2\nu_n \exp\left(-\frac{\pi\sqrt{n^2-1}}{2\varepsilon}\right) \sin(\sqrt{n^2-1}x + nt) \quad \text{for } x \rightarrow +\infty, \quad (32)$$

where $n = 2$ for asymmetric potentials and $n = 3$ for symmetric potentials.

The energy carried away by these oscillating tails determines the time-averaged radiation power W of the oscillon:

$$\frac{dE}{dt} = W = -4n\sqrt{n^2 - 1}\nu_n^2 \exp\left(-\frac{\pi\sqrt{n^2 - 1}}{\varepsilon}\right). \quad (33)$$

Since the radiation field is transcendentally small, it is reasonable to assume that during its time evolution the core of the oscillon goes through undistorted quasibreather states. This statement will be referred to as the adiabatic hypothesis. The energy content E of the core of the quasibreathers as a function of ε (or equivalently as a function of its frequency ω) is easily determined, it is given by

$$E = \frac{2\varepsilon}{\lambda} + \mathcal{O}(\varepsilon^3). \quad (34)$$

Now the equation determining the change of the core energy with time (energy loss) for oscillons can be seen to be of the form given by Eq. (1). From Eq. (1) one can easily deduce that the leading-order late-time behavior of the energy is given as

$$E(t) \approx \frac{B}{\ln t} \left(1 - \frac{2 \ln \ln t}{\ln t}\right). \quad (35)$$

In this subsection the radiation law for small-amplitude oscillons has been determined up to a single unknown parameter, A . The problem of finding A or what is equivalent the parameters ν_2 or ν_3 will be done in the following two subsections.

D. Determining the radiation amplitude by solving the complex mode equations numerically

In this subsection we numerically determine the leading radiation amplitude coefficients, namely ν_2 for asymmetric potentials and ν_3 in the case of symmetric potentials. First, we consider the ϕ^4 theory, in which case the only non-vanishing coefficients in the expansion of the potential are $g_2 = -\frac{3}{2}$ and $g_3 = \frac{1}{2}$. We consider various order truncations of the Fourier mode Eqs. (11) in the region close to the singularity. To illustrate our method we present in more details the calculations for the simplest truncated system that radiates, i.e., that for which only up to $\cos(2\tau)$ modes are kept. For the inner problem, in the $\varepsilon \rightarrow 0$ limit, the mode Eqs. (11) are

$$\begin{aligned} \frac{\partial^2 \Phi_0}{\partial y^2} - \Phi_0 &= \frac{1}{2} \Phi_0^3 - \frac{3}{2} \Phi_0^2 + \frac{3}{4} (\Phi_1^2 + \Phi_2^2) (\Phi_0 - 1) \\ &\quad + \frac{3}{8} \Phi_1^2 \Phi_2, \\ \frac{\partial^2 \Phi_1}{\partial y^2} &= \frac{3}{2} \Phi_0^2 \Phi_1 + \frac{3}{4} \Phi_1 (\Phi_2 - 2) (2\Phi_0 + \Phi_2) \\ &\quad + \frac{3}{8} \Phi_1^3, \\ \frac{\partial^2 \Phi_2}{\partial y^2} + 3\Phi_2 &= \frac{3}{2} \Phi_0^2 \Phi_2 + \frac{3}{4} \Phi_1^2 (\Phi_0 + \Phi_2 - 1) - 3\Phi_0 \Phi_2 \\ &\quad + \frac{3}{8} \Phi_2^3. \end{aligned} \quad (36)$$

Expanding the mode equations into powers of $1/y$

$$\Phi_i = \sum_{j=1}^{\infty} a_j^{(i)} \frac{1}{y^j}, \quad (37)$$

we find that all coefficients are fixed after choosing $a_2^{(1)}$ and the sign in $a_1^{(1)} = \pm 2i/\sqrt{3}$. Furthermore, $a_j^{(0)} = 0$, for $j < 2$ and $a_i^{(j)} = 0$ for $j < i$. In order to agree as well as possible with (19) obtained from the ε expansion evaluated at the singularity region we set $a_2^{(1)} = 0$ and $a_1^{(1)} = -2i/\sqrt{3}$. Then it turns out that $a_{i+1+2j}^{(i)} = 0$ for all integer j . Raising the truncation order high enough by adding more Fourier modes to the system all coefficients will necessarily agree with (19). The expansion consistent with (36) is

$$\begin{aligned} \Phi_0 &= -\frac{1}{y^2} - \frac{131}{12y^4} - \frac{461381}{1728y^6} - \frac{631478123}{51840y^8} + \mathcal{O}\left(\frac{1}{y^{10}}\right), \\ \Phi_1 &= -\frac{i}{\sqrt{3}} \left[\frac{2}{y} + \frac{17}{3y^3} + \frac{103877}{1728y^5} + \frac{8278867}{5760y^7} + \mathcal{O}\left(\frac{1}{y^9}\right) \right], \\ \Phi_2 &= \frac{1}{3y^2} + \frac{16}{9y^4} + \frac{107597}{5184y^6} + \frac{19189237}{38880y^8} + \mathcal{O}\left(\frac{1}{y^{10}}\right). \end{aligned} \quad (38)$$

According to expansion (38) the imaginary part $\Omega_2 = \text{Im}\Phi_2$ vanishes to all orders on the imaginary axis. However, this is not a convergent but an asymptotic series, and the actual solution of (36) may include an exponentially small correction to Ω_2 on the imaginary axis, in accordance with (22). Of course, the value of Ω_2 will depend on the chosen boundary conditions. The method introduced by Kruskal and Segur [32] is to integrate the differential equations numerically along a constant $\text{Im}y = y_i$ line from a large negative $\text{Re}y = y_r$ value to the axis $\text{Re}y = 0$. The boundary conditions at $y = y_r + iy_i$ are given by the expansion (38) and its derivative, truncated to an appropriate order in $1/y$. This works well for modes Φ_i with $i \geq 1$, but Φ_0 has the tendency to exponentially blow up along constant $\text{Im}y$ lines. This numerically problematic issue can be avoided by treating the equation for Φ_0 as a two-point boundary value problem, setting $\Phi_0 = 0$ at the axis point $y = iy_i$ and using the asymptotic series as the boundary value at $y = y_r + iy_i$. The other modes are treated as initial value problems by specifying their value and first derivatives at $y = y_r + iy_i$. We note that in the case of symmetric potentials $\Phi_{2i} = 0$ everywhere, and the integration procedure simplifies to the pure initial value problem. In that case the first radiating mode is Φ_3 .

For the actual numerical integration of the ϕ^4 system we have chosen various y_i values in the interval $[-5, -13]$, y_r in $[-20, -500]$, and the expansion in the initial data was truncated to orders from 4 to 20. The equations were generated by Maple, and its default boundary value problem differential equation solver was used for the numerical integration. For $y_r < -50$ the obtained values for Ω_2 were only changing in the fourth digits on varying y_r . The choice

TABLE I. Dependence of the exponential correction Ω_2 on the location of the integration line $\text{Im}y = y_i$ for the ϕ^4 theory truncated at $\cos(2\tau)$. The corresponding radiation amplitudes $\nu_2^{(n)}$ are also given by taking into account n -th order corrections in $1/y$.

y_i	Ω_2	$\nu_2^{(0)}$	$\nu_2^{(1)}, \nu_2^{(2)}$	$\nu_2^{(4)}$	$\nu_2^{(6)}$
-4	$9.58187 \cdot 10^{-6}$	0.00977982	0.00854627	0.00885621	0.00858961
-6	$2.86915 \cdot 10^{-7}$	0.00935565	0.00853442	0.00863027	0.00859166
-8	$8.74658 \cdot 10^{-9}$	0.00911169	0.00849838	0.00853980	0.00853022
-10	$2.69068 \cdot 10^{-10}$	0.00895494	0.00846614	0.00848769	0.00848446
-12	$8.31980 \cdot 10^{-12}$	0.00884615	0.00844008	0.00845283	0.00845151

of truncation order in $1/y$ in the initial data was not changing the results in their less than fourth digits if the order was chosen larger than 8. In Table I. we present the obtained y_i dependence of the imaginary part Ω_2 , when $y_r = -300$ and the initial data is of order 15. According to (22) we approximate ν_2 to leading order as $\nu_2^{(0)} = \Omega_2 \exp(\sqrt{3}y_i)$. The $1/y$ correction gives a more precise result $\nu_2^{(1)} = \Omega_2 \exp(\sqrt{3}y_i)/(1 - 1/(y_i\sqrt{3}))$. We note that for the special case of the ϕ^4 theory the coefficient of the $1/y^2$ correction in Ω_2 vanishes, and hence $\nu_2^{(1)}$ is also valid to second order. For larger values of $|y_i|$ it is possible to improve the precision by adding even higher-order corrections.

From Table I. we may give a first estimate on the actual value of the radiation amplitude as $\nu_2 = (8.45 \pm 0.03) \cdot 10^{-3}$. However this value turns out to change drastically when adding higher Fourier modes to the system (36). In Table II. we give the calculated values for ν_2 when keeping Fourier modes up to order $\cos(n\tau)$. It is somewhat surprising that the addition of the $\cos(3\tau)$ mode changes the sign of ν_2 , while the magnitude is quite close to the proper value. The addition of higher than Φ_6 modes does not make any significant change in the value of ν_2 . As a conclusion, we can state that $\nu_2 = (-8.454 \pm 0.01) \cdot 10^{-3}$. The numerical value obtained by Kruskal and Segur differs by a factor of 2 due to their use of complex notations for the Fourier modes. In our units their result is $(-9.0 \pm 2.0) \cdot 10^{-3}$.

In the case of symmetric potentials $\Phi_i = 0$ for even i , and the numerical integration method simplifies considerably. A specific example we have looked at is a specific ϕ^6 theory, with $U'(\phi) = \phi - \phi^3 + \phi^5$. When using the mode equations only for Φ_1 and Φ_3 we get $\nu_3 =$

-0.91026 . Adding the fifth and seventh modes changes the result to -0.90982 and -0.90977 , respectively. Since the addition of even higher modes does not make a significant change, we can state that for the ϕ^6 theory

$$\nu_3 = -0.9098 \pm 0.0001. \quad (39)$$

An important check of the reliability of our method is to calculate ν_3 for the sine-Gordon potential $U(\phi) = 1 - \cos(\phi)$. In this case, when taking into account all the mode equations and all expansion coefficients in the expansion of the potential, the exact result is known to be zero. Instead of changing these independently, when we solved mode equations up to order Φ_i we assumed that $g_j = 0$ for $j > i$. As it can be seen from Table III., the results appear to tend to zero fast as i increases.

E. Determining the radiation amplitude by Borel summation of the algebraic asymptotic series

In this subsection we will solve Eq. (15) using the algebraic asymptotic series ansatz (18) in the neighborhood of the singularity. Our considerations will be applicable to the case of symmetric potentials. At the end of this subsection we will briefly discuss the problem of asymmetric potentials. The solution is unique as we have to match it to the original asymmetric quasibreather continued analytically from the real axis. We truncate both the Taylor expansion of the potential and the Fourier expansion in order to have a finite set of equations and will work until we reach convergence as in Sec. IID.

We will demonstrate how the determination of $a_k^{(n)}$ coefficients works for big k in leading order of k in the case of the minimal system Φ_1 and Φ_3 with a cubic nonlinearity.

TABLE II. Dependence of the radiation amplitude on the order of the used Fourier expansions in the ϕ^4 theory.

n	ν_2
2	$8.45 \cdot 10^{-3}$
3	$-7.115 \cdot 10^{-3}$
4	$-8.431 \cdot 10^{-3}$
5	$-8.454 \cdot 10^{-3}$
6	$-8.454 \cdot 10^{-3}$

TABLE III. Radiation amplitude ν_3 for the sine-Gordon theory truncated to order i in both the mode equations and potential expansion.

i	ν_3
4	2.32
6	-0.2316
8	$8.35 \cdot 10^{-3}$
10	$1.14 \cdot 10^{-5}$
12	$7.2 \cdot 10^{-7}$

Since $a_k^{(1)}$ and $a_k^{(3)}$ are vanishing for even k we redefine the coefficients in order to get a more convenient form:

$$\Phi_1 = i \sum_{k=1}^{\infty} \frac{A_k}{y^{2k-1}} \quad (40)$$

$$\Phi_3 = i \sum_{k=2}^{\infty} \frac{B_k}{y^{2k-1}}. \quad (41)$$

We will show in the following that the behavior consistent with Eq. (15) to leading order in k is

$$A_k \sim \frac{B_k}{2k^2} \quad (42)$$

$$B_k \sim K(-1)^k \frac{(2k-2)!}{8^{k-1/2}}. \quad (43)$$

The constant K can be determined by solving the equations up to some large order k and matching the gained coefficients to the determined asymptotic behavior. To do so, we write up the mode equations:

$$\frac{d^2}{dy^2} \Phi_1 = \frac{3g_3}{4} \Phi_1 (\Phi_1^2 + \Phi_1 \Phi_3 + 2\Phi_3^2) \quad (44)$$

$$\left(\frac{d^2}{dy^2} + 8 \right) \Phi_3 = \frac{g_3}{4} (\Phi_1^3 + 6\Phi_1^2 \Phi_3 + 3\Phi_3^3), \quad (45)$$

and then determine the equations for the coefficients of $1/y^{2k-1}$ keeping only terms of order $(2k-4)!$:

$$(2k-3)(2k-2)A_{k-1} + \frac{3g_3}{4} A_1^2 B_{k-1} = 0 \quad (46)$$

$$(2k-3)(2k-2)B_{k-1} + 8B_k + \frac{3g_3}{2} A_1^2 B_{k-1} = 0. \quad (47)$$

It is easy to figure out the value of A_1 from the matching conditions (19), which gives $A_1 = -\sqrt{-8/(3g_3)}$. (Obviously we get the same result with an indeterminate sign by solving Eq. (44) for the coefficient of $1/y^3$.) Using the value of A_1 we get the asymptotic behavior of A_k from Eq. (46) already given in Eq. (42). From Eq. (47) we can even determine the $\mathcal{O}(1/k^2)$ corrections to the asymptotic behavior of B_k for large k as

$$B_k \sim K(-1)^k \frac{(2k-2)!}{8^{k-1/2}} \left(1 + \frac{1}{k} + \frac{5}{4k^2} \right). \quad (48)$$

This formula enables us to determine the numerical value of K very precisely from B_k s with moderate k value.

By examining the structure of the equations and making use of the fact that the algebraic asymptotic series of Φ_n starts only at $1/y^n$ it is easy to prove that the above asymptotics do not change. Even the $\mathcal{O}(\frac{1}{k^2})$ corrections to B_k are not affected by the involvement of further modes or higher-order nonlinearities. The only effect of the intro-

duction of further Fourier modes and higher-order nonlinearities which originate in the self-interaction potential is the changing of the value K .

On the one hand, this result gives the proof of the asymptoticity of the series, i.e., the coefficients are not more gravely divergent than $(2k-2)!$. On the other hand, this property allows us to Borel-sum the series. It will turn out that the behavior of the Borel-summed series in the vicinity of its singularity gives us the dominant radiation field configuration, i.e., the exponentially small imaginary correction to Φ_3 on the imaginary axis, in the matching region.

The first step in the Borel summation is

$$\begin{aligned} V(z) &= \sum_{k=1}^{\infty} \frac{iB_k}{(2k-1)!} z^{2k-1} \sim \sum_{k=1}^{\infty} iK \frac{(-1)^k}{2k-1} \left(\frac{z}{\sqrt{8}} \right)^{2k-1} \\ &= -\frac{K}{2} \ln \left[\frac{1 + iz/\sqrt{8}}{1 - iz/\sqrt{8}} \right]. \end{aligned} \quad (49)$$

This Borel-summed series has logarithmic singularities at $z = \pm i\sqrt{8}$. The Laplace transform of $V(z)$ will give us the Borel-summed series of Φ_3 which we denote by $\tilde{\Phi}_3$

$$\tilde{\Phi}_3(y) = \int_0^{\infty} dt e^{-t} V\left(\frac{t}{y}\right). \quad (50)$$

The integrand of Eq. (50) has logarithmic singularities at $t/y = \pm i\sqrt{8}$. It has been explained in Ref. [34] how to compute the integral. We only take into account the singularity $t/y = i\sqrt{8}$ and not $t/y = -i\sqrt{8}$, because we would like to get the correction in the imaginary part of Φ_3 for y points with negative imaginary parts, as we aim to approach the real axis. For $\text{Im}y < 0$ the other singularity stays away from the integration path, while the singularity from $t/y = i\sqrt{8}$ appears for $t = i\sqrt{8}y$. In order to define the integral for $\text{Re}y = 0$ we use the analyticity of $\tilde{\Phi}_3$. When y is in the matching region determined by Eq. (16) the singularity in t is in the lower half-plane, i.e., the contour in Eq. (50) is above the singularity. When $\text{Re}y \rightarrow 0$ the singularity tends to the real axis and the contour must stay above the singularity. The logarithmic singularity of $V(t/y)$ does not contribute to the integral and integrating on the branch cut starting from it yields the imaginary part

$$\text{Im} \tilde{\Phi}_3(y) = \int_{i\sqrt{8}y}^{\infty} dt e^{-t} \frac{K\pi}{2} = \frac{K\pi}{2} \exp[-i\sqrt{8}y]. \quad (51)$$

This result agrees with that of Eq. (23), hence we were able to determine ν_3 analytically:

$$\nu_3 = \frac{K\pi}{2}. \quad (52)$$

We already gave the method for determining the value of K by solving linear equations recursively for the coefficients of the algebraic asymptotic series up to some large k values. In the first step, we will show that our method is consistent with the fact that the sine-Gordon breather does

not radiate. We will truncate the SG potential in various orders and determine the value of K by solving the complex mode equations with the algebraic asymptotic series ansatz. We will experience a monotonous decrease in the value of K as we increase the order of the Taylor expansion. This can be interpreted in the following way. K determines the increase in the coefficients of the algebraic asymptotic series which correspond to the coefficients appearing in the small-amplitude expansion. As we get closer to the SG theory the asymptotic series for small-amplitude quasibreathers in the truncated SG theory grows less and less. On the other hand, this means that the corresponding oscillons would radiate more slowly, as the radiation amplitude is proportional to K . In the limit of SG theory we should get a convergent series in the small-amplitude expansion and thus a nonradiating breather. Our numerical experiences show that we can even use a smaller number of mode equations than the biggest power in the Taylor expansion to reach satisfactory convergence for the value of K for the given theory. We collected the results for K in the truncated SG theory in Table IV.

In the second step we will be looking for a symmetric ϕ^6 theory in which oscillon radiation is the fastest, i.e., K takes the biggest value. We will use this theory in numerical simulations for the verification of the theoretical radiation law for oscillons because we hope to most accurately measure the radiation rate in this theory. We will make use of the fact that g_3 can be defined into the fields, thus the only essential parameter in a symmetric ϕ^6 theory is g_5 . ($g_3 < 0$ in order to have a localized solution of Eq. (6).) To do so, we solved the theory for Φ_1 - Φ_7 up to $k = 30$ with the coefficient g_5 :

$$U(\phi) = \frac{1}{2}\phi^2 - \frac{1}{4}\phi^4 + \frac{g_5}{6}\phi^6 \quad (53)$$

$$U'(\phi) = \phi - \phi^3 + g_5\phi^5. \quad (54)$$

The value of K as a function of g_5 can be found in Fig. 1. We calculated K from B_{30} in the figure, however the exact value of k only matters in the sixth digit. We only deal with positive g_5 s as they are the ones with a stable vacuum. We

TABLE IV. The value of K in truncated SG theories from the solution of the same number of mode equation as the order of truncation. K denotes the value obtained by Borel summation, while K_{num} is the result of the numerical solution of the mode equations on the complex plane.

Order of truncation	K	K_{num}
4	1.486	1.48
6	-0.1472	-0.147
8	$5.306 \cdot 10^{-3}$	$5.31 \cdot 10^{-3}$
10	$7.306 \cdot 10^{-5}$	$7.2 \cdot 10^{-5}$
12	$4.686 \cdot 10^{-7}$	$4.6 \cdot 10^{-7}$

see two zeros in this domain. This does not mean that we found breathers in the corresponding theories; these points represent oscillons, the dominant radiation field of which is in fifth mode, Φ_5 . These configurations are extremely long-living objects. The first zero is very close to the SG theory for which $g_5 = 3/10$ with the $g_3 = -1$ normalization. Upon introducing further nonlinearities this zero would move exactly to the g_k point representing the SG theory. From Fig. 1 $g_5 = 1$ seems to be a very comfortable choice for numerical simulation. We find the precise value of K for $g_5 = 1$ to be

$$K = -0.57915, \quad K_{\text{num}} = -0.5792, \quad (55)$$

where K_{num} has been extracted from Eq. (39).

In the end we will briefly discuss the class of asymmetric potentials, to which the ϕ^4 potential belongs. In these theories the presence of the Φ_0 mode results in the following asymptotics for large ks in the Φ_0 mode:

$$a_k^{(0)} \sim (2k - 2)!. \quad (56)$$

This behavior dominates the asymptotics of all other modes as well. Because we do not have an alternating sign in this dominant behavior, the Borel-summed series will have a singularity on the real axis, hence we do not get an imaginary correction to the asymptotic series from this calculation. Thus, we cannot determine the radiation this way. The asymptotics which would determine the radiation in the case of asymmetric theories is $a_k^{(2)} \sim (-1)^k(2k - 2)!/3^k$. This is not the leading behavior of Φ_2 's coefficients and is significantly suppressed. We have not yet succeeded in determining the hidden alternating part of the coefficients, therefore we cannot determine the radiation amplitude by this analytic method.

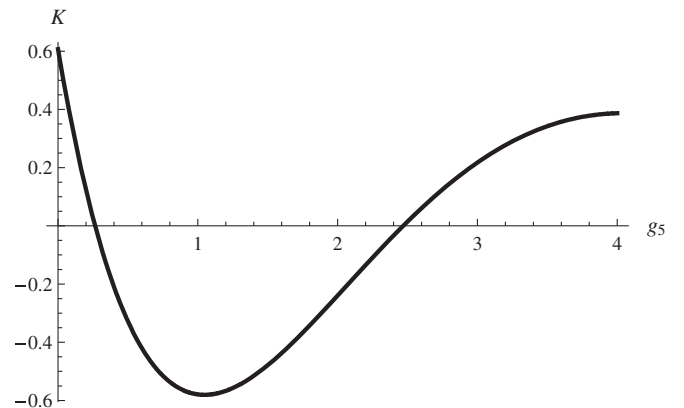


FIG. 1. The value of K as a function of g_5 for symmetric ϕ^6 theories.

III. NUMERICAL SIMULATIONS

A. Aspects of the numerical simulations

The numerical simulation of oscillons and their decay was performed with a slightly modified version of the fourth order method of line code developed and used in Refs. [26,36,37]. The spatial grid was chosen to be uniform in the compactified radial coordinate R defined by

$$x = \frac{2R}{\kappa(1 - R^2)}, \quad (57)$$

where κ is a constant. By this choice the whole range $0 \leq x < \infty$ of the physical radial coordinate x is mapped to the interval $0 \leq R < 1$, avoiding the need for explicitly describing boundary conditions at some large but finite radius. In most of our simulations we worked with $\kappa = 0.05$, which proved to be ideal for oscillons for which radiation was numerically observable.

The frequency of oscillons is determined by measuring the time elapsed between two subsequent maximums of the field configurations in the origin. By integrating the energy density in an $R = 0.873$ ($x = 146.7$) sphere at every time slice we determine the energy E of the oscillon core and trace the energy loss $W = dE/dt$. Although this way of calculating the energy contains an arbitrariness, for small-amplitude oscillons the corresponding asymptotic field is exponentially small and the energy density of the radiation tail can be neglected compared to that of the core. We measure the radiation in a relatively short time interval from the slope of the energy as a function of time.

Starting from initial data obtained from the small-amplitude expansion we simulate the time evolution and radiation of oscillons. Although our analytic calculations are only valid for infinitesimal ε , we find that they approximate oscillons quite well after a one-parameter tuning.

Studying the slow energy loss of oscillons we determine a semiempirical radiation law. This means that we keep the functional form (33) but fit the parameters appearing in it. We find satisfactory agreement with the theoretical value of the parameters obtained in the previous sections. We also follow the evolution of a single oscillon through a very long time interval. This process will be precisely described by the radiation obtained by the fit law proving the assumption made when making analytic considerations: the system evolves through undistorted quasibreather states adiabatically.

B. Initial data

We used the asymptotic series (3) truncated to order N , at the moment of time-reflection symmetry as initial data for the numerical evolution code,

$$\phi^{(\tau=0)} = \sum_{k=1}^N \varepsilon^k \phi_k^{(\tau=0)}. \quad (58)$$

The aim of our numerical analysis is to obtain oscillon states which are as periodic as possible, which means that their basic oscillation frequency and amplitude changes very slowly, uniformly, and monotonically. The time evolution of these “clean” oscillons can be approximated by adiabatic evolution through corresponding frequency quasibreather states. However, since the expansion (3) is not convergent, the initial data (58) differs from the intended quasibreather configuration. In general, using it as initial data, first a small portion of the energy is quickly emitted by radiation, and then a very long-living localized oscillating configuration remains. However, generally, the frequency and amplitude of this “unclean” oscillon state also possesses a lower frequency modulation. We observed that the amplitude of this modulation can be significantly decreased by multiplying the initial data by an appropriate constant. By making a one-parameter tuning code we were able to obtain oscillon states clean enough for studying their basic energy loss rate. Otherwise one could not distinguish between the energy emitted by the oscillon and the energy released by the decay of the low-frequency modulation.

Through the whole domain of simulation the sum (58) with $N = 3$ proved to yield the cleanest oscillon states. It is worth mentioning, that even for quite large values of ε , after the tuning the asymptotic series yields appropriate initial data, although the ε value of the initial data and the one calculated from the frequency by $\varepsilon = \sqrt{1 - \omega^2}$ during the time evolution may differ.

We illustrate the main steps by the example of the ϕ^4 theory. The first few terms of the asymptotic series are

$$\begin{aligned} \phi_1^{(\tau=0)} &= \sqrt{\frac{2}{3}} S & \phi_2^{(\tau=0)} &= \frac{1}{3} S^2 \\ \phi_3^{(\tau=0)} &= \frac{1}{9} \sqrt{\frac{2}{3}} \left(-\frac{25}{2} S^3 + \frac{49}{2} S \right) & (59) \\ \phi_4^{(\tau=0)} &= \frac{1}{9} \left(-\frac{125}{6} S^4 + \frac{103}{3} S^2 \right). \end{aligned}$$

The most naive estimate for an ideal truncation of an asymptotic series is that we should find the order where the terms of the series are equally big and higher order terms are starting to grow from this threshold. From (59) for $0.3 < \varepsilon < 0.6$ truncation of the series at third order appears to be appropriate. This is also supported by the results of the numerical simulations. In the ϕ^4 theory we found that $\varepsilon = 0.6$ oscillons are the biggest ones which we can clean from the noise and for which the adiabatic hypothesis works. Figure 2 shows the obtained unclean oscillon states from various order initial data. It is apparent that the order $N = 3$ gives the state with the smallest amplitude modulation. As already noted, multiplying the initial data with a constant close to 1 decreases the ampli-

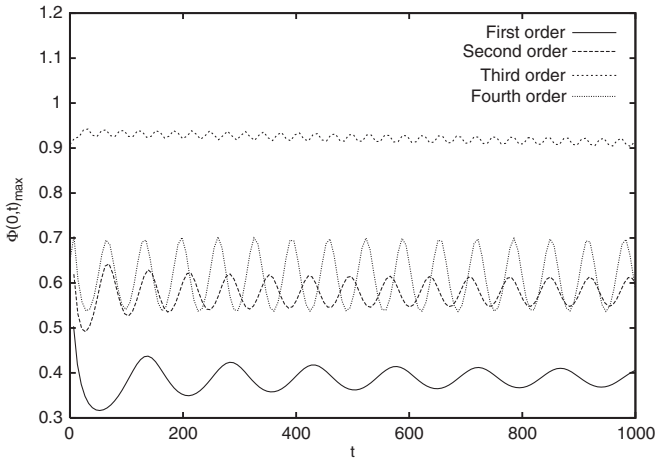


FIG. 2. Evolution from initial data in various orders for $\varepsilon = 0.6$ in the ϕ^4 theory; field value maximums at the center are plotted.

tude of this modulation even more. Figure 3 shows how effectively this method smooths the oscillon for $\varepsilon = 0.5$.

C. Reliability of the numerical results concerning oscillon radiation

The reliability of the subsequent results concerning oscillon radiation can be estimated by checking how well our code simulates the exactly periodic sine-Gordon breather. We use the breather field configuration (8) as initial data for the time evolution. We note that using the series (9) truncated at third order and employing the tuning method gives the same order of radiation. Table V. contains for different resolutions the radiation power and W/E_B where E_B is the total energy of the breather ($E_B = 16\varepsilon$). The number of lattice sites is given by the formula $N_{\text{lat}} = \text{RES} \times 128$. It can be observed that the duplication of the lattice points results in the decrease of radiation due to lattice effects with a factor of 30. From now on we will keep ourselves to the following rule: we only consider oscillons at a certain resolution if the numerically calculated W/E is at least one magnitude bigger than the value given in Table V. If this condition fails, that means one has to use a higher number of lattice sites to measure the

TABLE V. Resolution dependence of radiation power W in the sine-Gordon theory. The third column gives the ratio compared to the energy of the breather E_B . The number of lattice sites are $N_{\text{lat}} = \text{RES} \times 128$.

RES	W	$ \frac{W}{E_B} $
4	$-1.18062 \cdot 10^{-6}$	$1.48 \cdot 10^{-7}$
8	$-3.66652 \cdot 10^{-8}$	$4.58 \cdot 10^{-9}$
16	$-1.14045 \cdot 10^{-9}$	$1.42 \cdot 10^{-10}$
32	$-3.5542 \cdot 10^{-11}$	$4.44 \cdot 10^{-12}$
64	$-1.10946 \cdot 10^{-12}$	$1.38 \cdot 10^{-13}$

radiation rate reliably, which, however, can make the simulation time impracticably long. This is the reason why the energy loss of oscillons for very small ε cannot be obtained by our method.

D. Oscillons in the ϕ^4 theory

1. Confirmation of the adiabatic hypothesis and the radiation law

In Table VI we give the energy loss for the ϕ^4 theory for the biggest possible interval where we are able to use our numerical approach. The lower bound is limited by the need of time to perform high-resolution simulations; we can see only lattice effects below the lowest ε values in the table. Above the upper bound the oscillon decay is very fast and we cannot determine W and ε reliably. We will see the radiation law fail for large ε , in these cases the interaction with the radiation field may become essential, and the system does not evolve through undistorted quasi-breather states. The ν_2 values in Table VI are calculated from the measured radiation power and ε by using the theoretical radiation law (33).

We intend to confirm the radiation law, Eq. (33), numerically. We performed two fits with two free parameters on the logarithm of the data shown in Fig. 4(a). In the first fit we took into account all the data points, while in the

TABLE VI. Radiation power W and ε in the ϕ^4 theory from various initial data. For the oscillons ε is measured from the frequency during time evolution. The $|W/E|$ column determines the resolution RES required for the simulation, while ν_2 is calculated from Eq. (33).

ε	W	$ \frac{W}{E} $	ν_2
0.5858	$-1.05 \cdot 10^{-5}$	$1.34 \cdot 10^{-5}$	0.0905
0.58255	$-9.58 \cdot 10^{-6}$	$1.23 \cdot 10^{-5}$	0.0887
0.5778	$-8.45 \cdot 10^{-6}$	$1.10 \cdot 10^{-5}$	0.0866
0.57123	$-7.12 \cdot 10^{-6}$	$9.35 \cdot 10^{-6}$	0.0839
0.56261	$-5.69 \cdot 10^{-6}$	$7.58 \cdot 10^{-6}$	0.0807
0.55183	$-4.27 \cdot 10^{-6}$	$5.80 \cdot 10^{-6}$	0.0768
0.53894	$-3.01 \cdot 10^{-6}$	$4.18 \cdot 10^{-6}$	0.0725
0.52418	$-1.98 \cdot 10^{-6}$	$2.84 \cdot 10^{-6}$	0.0679
0.49217	$-7.61 \cdot 10^{-7}$	$1.16 \cdot 10^{-6}$	0.059
0.48278	$-5.64 \cdot 10^{-7}$	$8.77 \cdot 10^{-7}$	0.0565
0.47334	$-4.14 \cdot 10^{-7}$	$6.57 \cdot 10^{-7}$	0.0542
0.46387	$-3.01 \cdot 10^{-7}$	$4.87 \cdot 10^{-7}$	0.0520
0.45437	$-2.17 \cdot 10^{-7}$	$3.58 \cdot 10^{-7}$	0.0498
0.44485	$-1.54 \cdot 10^{-7}$	$2.60 \cdot 10^{-7}$	0.0478
0.43533	$-1.09 \cdot 10^{-7}$	$1.87 \cdot 10^{-7}$	0.0459
0.42581	$-7.58 \cdot 10^{-8}$	$1.34 \cdot 10^{-7}$	0.0441
0.41629	$-5.23 \cdot 10^{-8}$	$9.43 \cdot 10^{-8}$	0.0424
0.40679	$-3.58 \cdot 10^{-8}$	$6.59 \cdot 10^{-8}$	0.0408
0.3973	$-2.42 \cdot 10^{-8}$	$4.57 \cdot 10^{-8}$	0.0394
0.35954	$-4.13 \cdot 10^{-9}$	$8.61 \cdot 10^{-9}$	0.0334
0.32017	$-4.45 \cdot 10^{-10}$	$1.04 \cdot 10^{-9}$	0.0278
0.28404	$-4.40 \cdot 10^{-11}$	$1.16 \cdot 10^{-10}$	0.0258

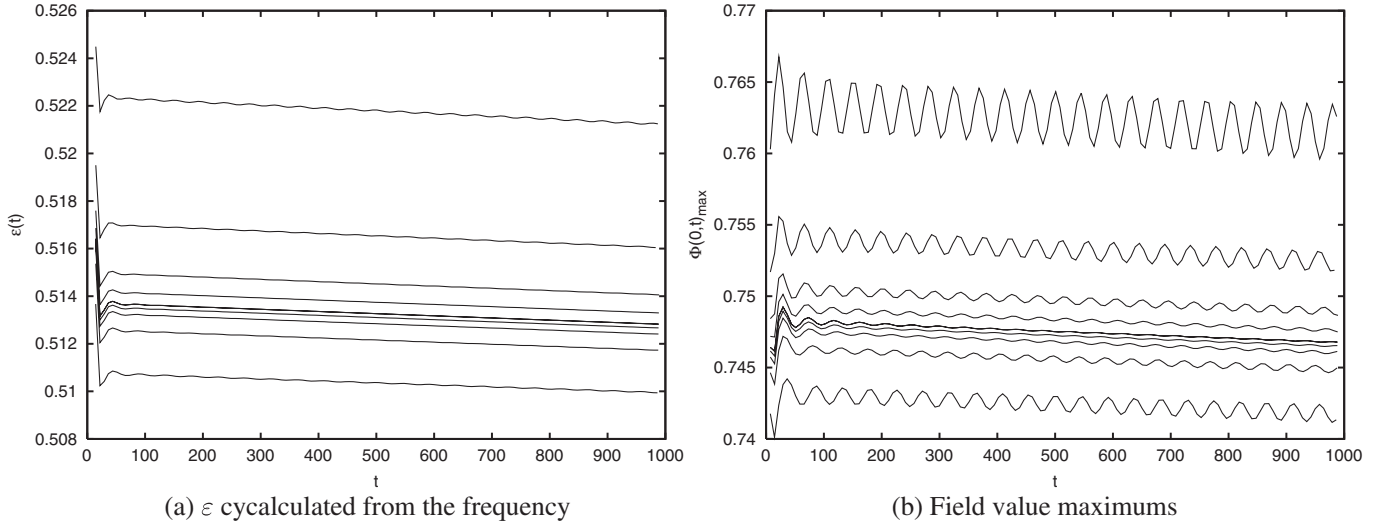


FIG. 3. Time evolution of ε and the central amplitude for tuned $\varepsilon = 0.5$ initial data in the ϕ^4 theory. Multiplication factors are in the range $[0.9978, 1.0169]$, and the smoothest evolution is with 1.00263.

second we fitted for data points with $\varepsilon < 0.42$ to get closer to the theoretical pole term. We define the semiempirical radiation law:

$$\frac{dE}{dt} = W = -8\sqrt{3}\nu_2^2 \cdot \exp\left[-\frac{\sqrt{3}\pi b}{\varepsilon}\right], \quad (60)$$

where ν_2 and b are parameters to be fitted. Equation (60) is to be interpreted as follows. For finite values of ε there are various higher-order corrections to the radiation law which are now encoded in two effective parameters, ν_2 and b . These corrections originate in Eq. (22). We remark that the value of b_{theory} will increase accordingly. When fitting for all the data points the difference between b and $b_{\text{theory}} = 1$

is bigger, than in the case of the fit for $\varepsilon < 0.42$ points, because for smaller ε values the correction due to other pole terms are smaller.

If we set $b = b_{\text{theory}}$ in Eq. (60) we get back the theoretical radiation law, Eq. (33). On Fig. 4(b) we plotted the ν_2 values from Table VI. We see that we are getting closer to the theoretical ν_2 as ε decreases. In Eqs. (63) and (64) we give the results of the fits of ν_2 with $b = b_{\text{theory}}$ on all the data points and for data points with $\varepsilon < 0.42$, respectively.

The result of the first fit supports the functional dependence gotten from the theoretical arguments, while the second fit leads us closer to the theoretical values of the parameters:

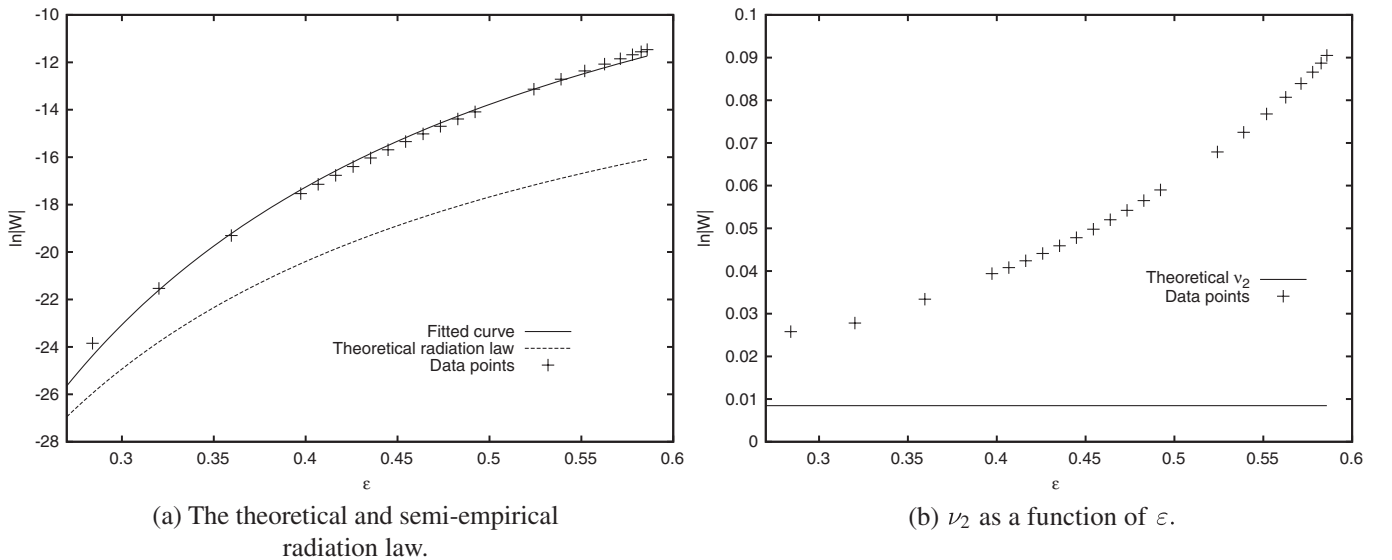


FIG. 4. The measured data points, the theoretical radiation law and the results of the first fit with parameters to be found in Eq. (61) in the case of the ϕ^4 theory. We see the points are approaching the theoretical curve as ε decreases.

$$\nu_{2,\text{fit}} = 0.29 \pm 0.03 \quad b_{\text{fit}} = 1.28 \pm 0.2 \quad \text{for all data points,} \quad (61)$$

$$\nu_{2,\text{fit}} = 0.12 \pm 0.02 \quad b_{\text{fit}} = 1.17 \pm 0.2 \quad \text{for data points with } \varepsilon < 0.42, \quad (62)$$

$$\nu_{2,\text{fit}} = 0.054 \pm 0.004 \quad b = 1 \quad \text{for all data points,} \quad (63)$$

$$\nu_{2,\text{fit}} = 0.034 \pm 0.004 \quad b = 1 \quad \text{for data points with } \varepsilon < 0.42, \quad (64)$$

$$\nu_{2,\text{theory}} = 0.00845 \quad b_{\text{theory}} = 1. \quad (65)$$

Unfortunately the theoretical ν_2 value of the ϕ^4 theory is exceptionally small. This might be the reason why our data are consistent with the theoretical pole terms and why there is a difference in the value of ν_2 . It should be noted that we are fitting over 7 orders of magnitude. An other independent source of discrepancy could be the tuning method and that the superfluous energy from the oscillon core goes out in shells [38]. These shells are observed to be present even for a large time, and their movement could increase the radiation power measured by our method.

Let us turn our attention to the verification of the semiempirical radiation law determined from the short-time evolution of various oscillon states. We shall confirm from a long-time simulation, that the evolution from an initial oscillon state is driven by our radiation law. We claim two states to be identical if they possess the same ε value with the same energy for a relatively long time. This should mean that the field configurations are very close to each other, and the interaction with the radiation field is negligible. By testing these properties in two different simulations we can match these data and get a longer process.

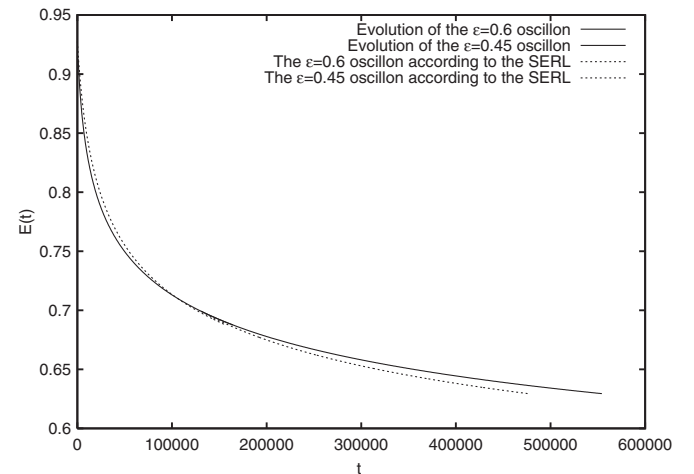
To compare the time dependence of the measured energy with the predicted one, we need the function $E(\varepsilon)$. This should not be a problem, as we can measure this function from the short simulations. The results of simulations are in very good agreement with the semiempirical radiation law. We would like to emphasize that the radiation law is not a simple fit, it has been determined independently; we only use the initial ε value and the function $E(\varepsilon)$ from the simulation to get the predicted curve.

In Fig. 5 we see an $\varepsilon = 0.6$ oscillon evolving. We used two simultaneously performed simulations for this graph (starting from $\varepsilon = 0.6$ and $\varepsilon = 0.45$), which perfectly fit together; another fact backing the hypothesis of adiabatic evolution. The solid lines follow the evolution observed in numerical simulations, while the dashed lines are the predictions of the semiempirical radiation law (SERL). We end both curves at the same ε value.

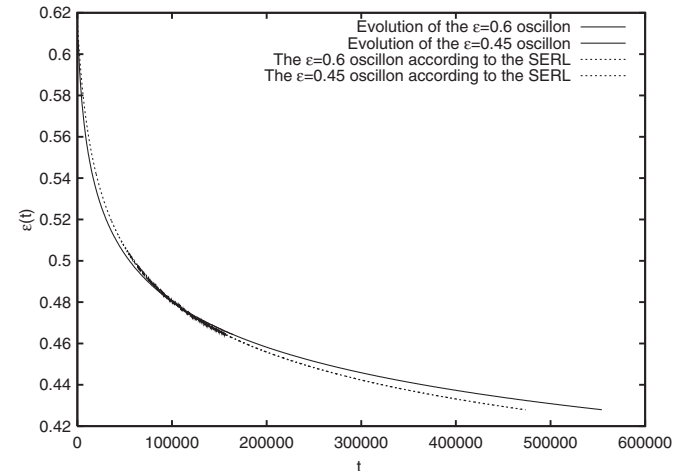
2. Oscillons from kink-antikink initial data

After being able to create and examine clean oscillon states we aim to identify, what the logarithmically decaying object is in our terminology that Geicke found in

Refs. [35,39]. It turns out that the objects he found are composite oscillon states and there is a continuum of them, without any essential difference between these configurations in contradiction with what he conjectured. These objects do not obey the semiempirical radiation law set for clean oscillon states in their early but very long period



(a) The energy



(b) ε versus time

FIG. 5. The evolution of an $\varepsilon = 0.6$ and an $\varepsilon = 0.45$ oscillon in the ϕ^4 theory matched together; both the numerical data and the prediction of the SERL are plotted. The time evolutions of the two oscillons are overlapping in a long-time interval around $t = 100000$.

(≈ 150000) of life, they radiate more rapidly, because the energy stored in the modulation modes of frequency has different radiation properties. However, their lifetime is in the same magnitude of oscillons with approximately the same frequency and energy. After their initial stage of evolution they begin to obey the semiempirical radiation law, their modulation, however, does not disappear. We shall confirm the logarithmic fits of Ref. [35], and in this stage of observation we can explain qualitatively what he found and how his results are related to the analytic considerations.

Following Ref. [35], we examine a kink-antikink pair initially at rest:

$$\phi(x, t = 0) = \tanh\left[\frac{x - a}{2}\right] - \tanh\left[\frac{x + a}{2}\right] + 1 \quad (66)$$

$$\partial_t \phi(x, t = 0) = 0. \quad (67)$$

For different values of a we examined the evolution of the initial state. The resolution was selected low ($\text{RES} = 4$) because we were not interested in the exact decay rate, and we aimed to draw qualitative consequences. The results are collected in Fig. 6. For bigger values of a ($a > 1.0$) we find that after an early period of formation approximately the same state arises. For moderate values of a we see different states arising in a relatively short period of time; these states radiate very slowly. Below a certain value of a ($a \lesssim 0.45$) we do not observe any definite oscillon core. This behavior is remarkable: we found the same pattern in the collision of a sine-Gordon soliton-antisoliton pair initially at rest; the same state arises for $a > 1.4$ and for the region $a \lesssim 0.8$ we see no definite oscillon state.

To find out how these configurations radiate we ran a simulation with $\text{RES} = 8$, $a = 0.8$; this is the initial data of Geicke in Ref. [35]. In Fig. 7 it is depicted what the semiempirical radiation law would predict and what really happens. The lumps emerging from kink-antikink initial

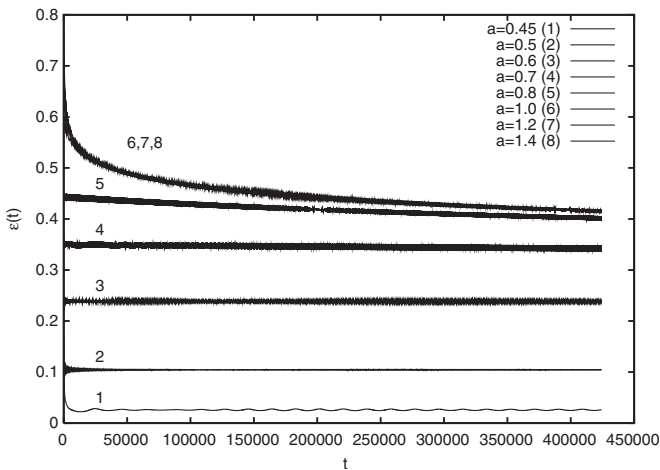


FIG. 6. The evolution of the kink-antikink initial data in the ϕ^4 theory for various a values; ε versus time plotted.

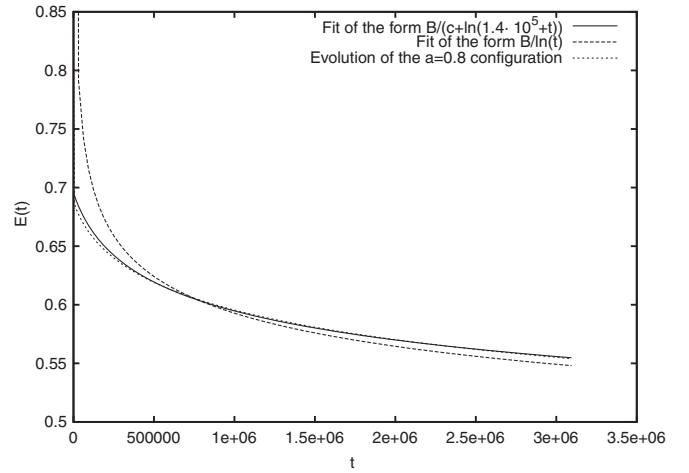


FIG. 7. The evolution of the kink-antikink initial data in the ϕ^4 theory with the $a = 0.8$ parameter illustrated by the energy in the entire simulation.

data radiate faster than a clean oscillon due to its frequency modulation. The following asymptotic logarithmic fit considered in Ref. [35] works well:

$$E(t) = \frac{B}{c + \ln(t + \sqrt{2} \cdot 10^5)} \quad (68)$$

$$c = 0.52 \pm 0.12 \quad B = 8.60 \pm 0.07. \quad (69)$$

Reference [35] uses $m = \sqrt{2}$; its result for the B parameter in the units we use in this paper is $B_g = 9.05$; the reason for the difference between B and B_g might be the use of the Sommerfeld boundary condition in its simulation instead of compactification. We fitted the function which we get by extrapolating the theoretical result for infinitesimal, Eq. (35), oscillons to our case as well:

$$E(t) \approx \frac{B}{\ln t} \quad (70)$$

$$B_{\text{theory}} = \frac{4\pi}{\sqrt{3}} \approx 7.255 \quad (71)$$

$$B_{\text{fit}} = 8.190 \pm 0.007. \quad (72)$$

We see that the numerical values from the fit and from theory are satisfactory close to each other. We note that such a logarithmic fit is not influenced by the value of ν_2 in the theoretical radiation law, Eq. (33). (Ref. [35] claims that the results of its fit, $B = 14.395$, agree with the theoretical result $B = 14.503$ in its units; however, the correct B value from the theory with $m = \sqrt{2}$ mass scale is $B = 10.260$.)

We conclude that the initial state observed by Ref. [35] evolves as a complex oscillonlike object. After the “early” period of $t \approx 150000$ it loses a large part of its energy and an oscillon with modulated frequency is created. Through

its evolution the ε value changes from 0.45 to 0.38. From the solutions of the semiempirical radiation (SERL) law it can be clearly seen that the initial configuration decays much faster than an oscillon; however, if we regard the $t \approx 150000$ as initial data, it obeys the radiation law with high precision despite the presence of modulation. (We did not plot the prediction of the SERL with $t_0 = 150000$ in Fig. 7, as the difference between the prediction and the numerical simulation results does not differ visibly.) It seems to us that the modulation degree of freedom is an adiabatically decaying mode.

E. Verification of the theoretical radiation law

We will examine oscillons in the specific symmetric ϕ^6 theory in which oscillon radiation is the largest, i.e. when K is maximal. We determined the g_5 value for this theory in Sec. II E. Thus, contrary to the ϕ^4 theory the value of K is maximal and the next-to-leading-order corrections are significantly smaller compared to the leading-order term, than in the case of the ϕ^4 theory because the ϕ^6 potential is symmetric. These corrections originate in (23). It should be kept in mind that because of the smaller pole term we cannot go as low in ε as in the ϕ^4 case. We collect the results of the simulations in Table VII in analogy to Table VI:

We performed two fits: in the first we fitted both ν_3 and b , in the second we used $b = b_{\text{theory}}$ and fitted ν_3 as we were interested in how accurately we could determine the value of ν_3 from numerical simulations.

$$\frac{dE}{dt} = -24\sqrt{2}\nu_3^2 \exp\left[-\frac{\sqrt{8}\pi b}{\varepsilon}\right] \quad (73)$$

$$\nu_{3,\text{fit}}^{(1)} = 0.4 \pm 1.2 \quad b_{\text{fit}}^{(1)} = 0.89 \pm 0.1 \quad (74)$$

$$\nu_{3,\text{fit}}^{(2)} = 1.35 \pm 0.1 \quad (75)$$

$$\nu_{3,\text{theory}} = 0.9098 \quad b_{\text{theory}} = 1. \quad (76)$$

One can notice an anomalous point among the data points in Fig. 8. This anomalous configuration was further examined by us, but no visible anomaly in the field con-

TABLE VII. Radiation power for oscillons with different ε values in the symmetric ϕ^6 theory.

ε	W	$ \frac{W}{E} $	ν_3
0.391886	$-1.256 \cdot 10^{-8}$	$1.20 \cdot 10^{-8}$	1.614
0.389854	$-7.983 \cdot 10^{-9}$	$7.68 \cdot 10^{-9}$	1.365
0.366842	$-5.692 \cdot 10^{-10}$	$5.82 \cdot 10^{-10}$	0.745
0.352178	$-5.483 \cdot 10^{-10}$	$5.84 \cdot 10^{-10}$	1.211
0.339333	$-3.291 \cdot 10^{-10}$	$3.64 \cdot 10^{-10}$	1.512
0.333239	$-2.225 \cdot 10^{-10}$	$2.50 \cdot 10^{-10}$	1.579
0.304623	$-2.293 \cdot 10^{-11}$	$2.82 \cdot 10^{-11}$	1.774

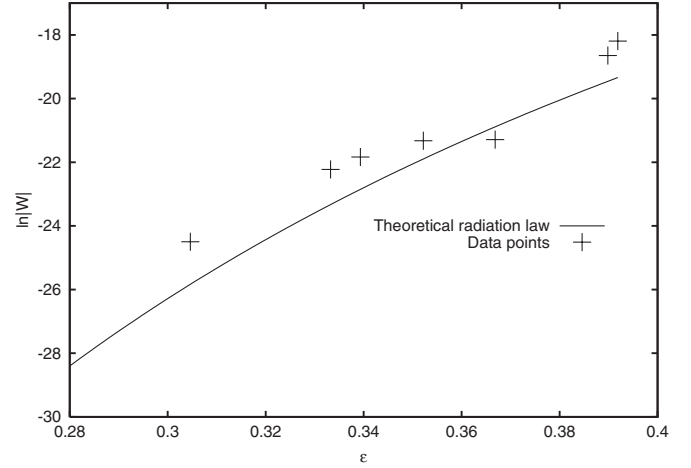


FIG. 8. The relation between the numerical data and the theoretical radiation law in the ϕ^6 theory. We plotted the data points and the theoretically predicted curve only, as the results of the fits would only make the graph less understandable.

figuration was observed. The oscillon has a proper oscillating tail and a smooth core. Its radiation power was measured for different resolutions and also by using different values for the conformal factor κ , but all these had no substantial effects. As it is, we do not know whether this anomaly is an awkward lattice effect or genuine one, i.e., present in the continuum limit as well.

We would like to investigate the problem of oscillon radiation from the perspective of field configuration as well. In Fig. 9 we show the field values as a function of time at $x = 49.6$ for the $\varepsilon = 0.352178$ oscillon. We plot the theoretical prediction of this oscillating tail, so that the frequency of the wave is set to 3ω , the amplitude is determined by theory and only the phase of the wave is fitted. We find satisfactory agreement in the case of wave amplitude and precise agreement in the case of frequency.

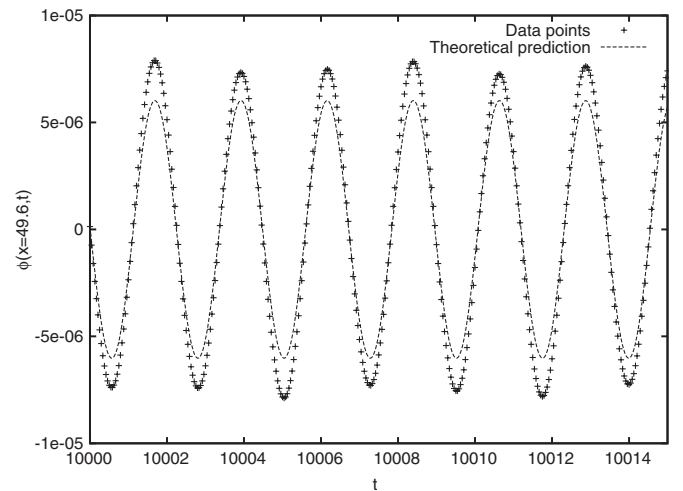


FIG. 9. The field configuration observed in numerical simulations and the theoretically predicted wave amplitude for the $\varepsilon = 0.352178$ oscillon in the ϕ^6 theory.

If we compare the radiation power calculated from this plane wave and the one in Table VII we find good agreement. Hence we conclude that the oscillons on the lattice lose energy via radiation and that our assumptions were correct when deriving the radiation law for small-amplitude oscillons.

There are various possible explanations for the discrepancy between theoretical and numerical results. First, we could argue that the ε values used in numerical simulations are too big and the theoretical calculation of the radiation amplitude only works for infinitesimal ε values. For finite ε values we can only expect an approximate agreement. Second, we cannot be sure about the initial data. Although the frequency of our objects is very stable, there is no way to decide whether we work with undistorted oscillons. The configuration with anomalous radiation emphasizes these problems. Lattice effects are less likely to play a role, as we see no major resolution and κ dependence of the radiation powers.

We conclude that we can accurately determine the b parameter of the radiation law from the numerical simulation of oscillon decay in the case of both the ϕ^4 and ϕ^6 case. The parameter ν_2 can be less accurately determined in the case of the ϕ^4 theory. There the next-to-leading corrections play a role because of the finite ε oscillons and the value of ν_2 is too small to be seen. In the case of the

ϕ^6 theory the next-to-leading-order corrections are smaller and the value of ν_3 is bigger, hence we can get the value of ν_3 from the simulations with satisfactory precision. The investigation of the radiation field of the oscillons shows equally good agreement with the theoretical formulae giving a firm basis for the theoretical calculations from another perspective.

IV. CONCLUSIONS

In a general class of one-dimensional scalar field theories we have computed the magnitude of the radiative tail of oscillons, determining their energy loss, in the small-amplitude limit. The magnitude of the tail is nonperturbatively small in the amplitude. We have used the Segur-Kruskal method of matched asymptotic expansion together with Borel summation techniques to calculate it. These results have also been verified numerically. We have also performed numerical simulations to compute directly the energy loss of oscillons, as well as the radiative tail. The numerical results are in a satisfactory agreement with the theoretical predictions.

ACKNOWLEDGMENTS

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