Galileon as a local modification of gravity

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In the Dvali-Gabadadze-Porrati (DGP) model, the ''self-accelerating'' solution is plagued by a ghost instability, which makes the solution untenable. This fact, as well as all interesting departures from general relativity (GR), are fully captured by a four-dimensional effective Lagrangian, valid at distances smaller than the present Hubble scale. The 4D effective theory involves a relativistic scalar π, universally coupled to matter and with peculiar derivative self-interactions. In this paper, we study the connection between self-acceleration and the presence of ghosts for a quite generic class of theories that modify gravity in the infrared. These theories are defined as those that at distances shorter than cosmological, reduce to a certain generalization of the DGP 4D effective theory. We argue that for infrared modifications of GR locally due to a universally coupled scalar, our generalization is the only one that allows for a robust implementation of the Vainshtein effect—the decoupling of the scalar from matter in gravitationally bound systems—necessary to recover agreement with solar-system tests. Our generalization involves an internal Galilean invariance, under which π's *gradient* shifts by a constant. This symmetry constrains the structure of the π Lagrangian so much so that in 4D there exist only five terms that can yield sizable nonlinearities without introducing ghosts. We show that for such theories in fact there are ''self-accelerating'' de Sitter solutions with no ghostlike instabilities. In the presence of compact sources, these solutions can support spherically symmetric, Vainshtein-like nonlinear perturbations that are also stable against small fluctuations. We investigate a possible *infrared* completion of these theories at scales of order of the Hubble horizon, and larger. There are however some features of our theories that may constitute a problem at the theoretical or phenomenological level: the presence of superluminal excitations; the extreme *subluminality* of other excitations, which makes the quasistatic approximation for certain solar-system observables unreliable due to Cherenkov emission; the very low strong-interaction scale for $\pi\pi$ scatterings.

I. INTRODUCTION

The beauty of general relativity (GR) lies in the simplicity of its principles and in its empirical adequacy in a broad range of phenomena. The modern viewpoint is that GR is an effective theory valid below some ultraviolet cutoff, presumably of order $M_P \sim 10^{19}$ GeV, and that its validity becomes increasingly better the lower the energy, that is the longer the distance scales. In particular, this viewpoint explains the adequacy of GR at macroscopic and solar-system scales and implies also its perfect adequacy to describe cosmology at late enough times (and at present). However the undeniable beauty of GR has been stained, from almost the beginning, by the mystery of the cosmological constant. The inability to provide, to date, a plausible field theory mechanism for the absence or smallness of the cosmological constant, perhaps indicates that something is missing in the simple infrared picture of GR. Indeed, the strong evidence accumulated in recent years that the Universe is undergoing a phase of accelerated expansion [1] provides extra motivation to consider a modification of gravitational dynamics on cosmological scales. Could late time acceleration follow from the very

dynamical mechanism that screens the cosmological constant? At the moment, we have no concrete example in that sense. On the other hand, there is instead a plausible, and even remarkable, picture where GR is the right infrared description and where the size of the cosmological constant is just explained by anthropic selection [2]. Nonetheless the big effort of recent years in trying to come up with, and to understand, modifications of gravity has produced an interesting know-how. We have now several examples of theories that modify gravity [3–6]. In a broad technical sense we may define a modification of gravity as a field theory possessing solutions over which new degrees of freedom affect the propagation of gravity while the background is not producing any sizable energy momentum tensor [7]. In other words, in these theories the graviton propagates differently even in flat space. Examples of this type based on purely four-dimensional field theory are the theories of massive gravity, like Fierz-Pauli or more potentially interesting models based on the mechanism of ghost condensation [3]. A perhaps more remarkable example is, however, represented by the Dvali-Gabadadze-Porrati (DGP) model [6], based on five-dimensional space-time. The DGP model is interest-

ing for several reasons. It cannot truly be assimilated to a fluid, as the infrared dynamics involves a continuum of states, corresponding to space-time being five-dimensional, not just a few phonons. It potentially represents a stepping stone towards the realization of the old idea of diluting the cosmological constant into extra dimensions [8,9]. It provides an alternative explanation for the observed late time acceleration of the Universe [10]. Indeed it nicely illustrates that in order to curve space-time at long distances it is not necessary to add a zero derivative term, that is the cosmological constant, while it is "enough" of a one-derivative term. The latter is in DGP formally represented by the five-dimensional kinetic term, which is more "relevant" in the technical sense than the 4D Einstein term localized at the boundary. Yet another reason of interest in the DGP model lies in its successful implementation of the Vainshtein mechanism [11] for decoupling the additional modes from gravitational dynamics at subcosmological distances. The problem at hand, which was first pointed out by van Dam, Veltman, and Zakharov [12], is tied to the fact that there is no such thing as a purely IR modification of gravity: degrees of freedom with mass of order of the Hubble parameter, can expectedly play a role at shorter distances. Indeed, in both Fierz-Pauli and DGP these short-distance effects are accurately described just by one additional scalar degree of freedom [13,14], normally indicated by π. The π modifies at $O(1)$ the linearized gravitational potential produced by any energy momentum source. Tests of GR within the solar system rule out such $O(1)$ effects, and seemingly the models that produce them. As first remarked by Vainshtein long ago, nonlinear effects in the dynamics of π are however important. Now, it turns out that in the case of Fierz-Pauli this does not lead to any benefits, as the nonlinearities always imply the presence of a light ghost state and the consequent breakdown of effective field theory [15–17]. However, in the case of DGP, under an interesting range of conditions on the energy momentum tensor, the Vainshtein effect screens the contribution of π below the experimentally acceptable level [18–20] without introducing extra ghost states. Unfortunately, the interesting self-accelerating solution is not in this class, in that the π is a ghost already on unperturbed de Sitter (dS) space [14,21,22]. Now, one crucial property of the π Lagrangian $\mathcal{L}_\pi$ in DGP, the one that gives the Vainshtein effect a chance to work, is that, while involving higher-derivative interactions, it nonetheless gives equations of motion of second order in the derivatives [21]. Indeed all of the DGP physics at distances shorter than H^{-1}, including the existence of the self-accelerating solution, is correctly reproduced by just the π field coupled to 4D gravity, without any reference to its 5D origin. These facts motivate a classification and study of all the purely 4D Lagrangians involving just one scalar field π coupled to gravity and to itself in the same peculiar way as in DGP. In particular, this approach will be enough

to study the possible correlation between self-acceleration and the existence of ghosts, or else to find self-accelerating solutions that lack ghosts. This will be the main goal of this paper.

II. LOCAL ANALYSIS

A. Cosmology as a weak gravitational field

A Friedmann-Robertson-Walker (FRW) universe is a highly nonlinear solution of Einstein's equations. Still, at short scales—shorter than the Hubble radius—a cosmological solution can certainly be thought of as a small perturbation of Minkowski space. For instance, given a comoving observer sitting at the origin, we can choose coordinates that for all times make any deviation from Minkowski metric weighted by $\sim H^2|\vec{x}|^2$, where $|\vec{x}|$ is the physical distance away from the observer. Such a coordinate choice is not unique. A particularly convenient one is Newtonian gauge, for which g_{0i} vanishes and the spatial metric is isotropic,

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Psi)d\vec{x}^2. \quad (1)$$

More explicitly, if τ and $\vec{y}$ are FRW comoving coordinates,

$$ds^2 = -d\tau^2 + a^2(\tau)d\vec{y}^2, \quad (2)$$

we perform the coordinate change

$$\tau = t - \frac{1}{2}H|\vec{x}|^2, \qquad \vec{y} = \frac{\vec{x}}{a}\left[1 + \frac{1}{4}H^2|\vec{x}|^2\right], \quad (3)$$

where a and H are evaluated at t, rather than at τ. Then the metric locally (i.e., for $|\vec{x}| \ll H^{-1}$) becomes

$$ds^2 \simeq -[1 - (\dot{H} + H^2)|\vec{x}|^2]dt^2 + [1 - \tfrac{1}{2}H^2|\vec{x}|^2]d\vec{x}^2, \quad (4)$$

which corresponds to Newtonian gauge with potentials

$$\Phi = -\tfrac{1}{2}(\dot{H} + H^2)|\vec{x}|^2, \qquad \Psi = \tfrac{1}{4}H^2|\vec{x}|^2. \quad (5)$$

Corrections to the above metric are of order H^4x^4. Notice that the Newtonian gauge is a complete gauge fixing only for $\vec{k} \neq 0$ (see, e.g., Ref. [23]); on the other hand our potentials are supported at $\vec{k} = 0$ exclusively, $|\vec{x}|^2 \sim \nabla_{\vec{k}}^2 \delta^{(3)}(k)$. We will come back to the resulting extra gauge-freedom later. The above form of the metric, Eq. (4), is convenient because of it being locally a small deviation from Minkowski space for all times. That is all along the worldline of the observer sitting at $\vec{x} = 0$, the metric is $\eta_{\mu\nu}$ and the connection vanishes. Therefore our gauge fixing corresponds to Fermi normal coordinates.

Despite the name, Newtonian gauge does not imply any Newtonian limit, and is still fully relativistic. However, for a matter-dominated universe, all the nontrivial cosmological dynamics is captured in our gauge (4) by Newtonian local physics. Indeed, the Newtonian potential Φ above obeys Poisson's equation

$$\nabla^2\Phi = 4\pi G\rho, \quad (6)$$

provided that H^2 and $\dot{H}$ satisfy Friedmann equations. The local fluid velocity field is given by the Hubble flow,

$$\vec{v} = H\vec{x}, \tag{7}$$

which solves Newton's law,

$$\dot{\vec{v}} = -\vec{\nabla}\Phi = \frac{\ddot{a}}{a}\vec{x}. \tag{8}$$

The covariant conservation of the stress-energy tensor follows from the Newtonian continuity equation,

$$\dot{\rho} + \vec{\nabla} \cdot (\rho\vec{v}) = \dot{\rho} + 3H\rho = 0. \tag{9}$$

In fact, although we deduced the local form of a cosmological solution from its exact, global FRW geometry, for a matter-dominated universe one can *derive* the correct FRW dynamics just from local Newtonian physics. Indeed, assuming isotropy around the origin, Poisson's equation with constant ρ yields $\Phi = \frac{2\pi G}{3}\rho|\vec{x}|^2$. The Hubble flow $\vec{v} \propto \vec{x}$ is the only velocity field that is homogeneous—it is invariant under the combined action of a translation and a (time-dependent) Galilean boost.[1] Newton's law (or Euler's equation) $\dot{\vec{v}} = -\vec{\nabla}\Phi$ relates the proportionality factor [i.e., the Hubble parameter $H(t)$] to ρ, via a differential equation. The continuity equation (9) then gives the correct time dependence of ρ. The fact that this Newtonian derivation gives the correct GR results is no accident: for nonrelativistic sources and at distances much shorter than the curvature radius, the dynamics *must* reduce to Newtonian gravitational physics. On the other hand, for a relativistic fluid with sizable pressure, the local interpretation is subtler because of relativistic effects, and the coupling to gravity is not fully captured by the Newtonian potential.

For our purposes, it will be more convenient to employ the residual gauge freedom of Newtonian gauge at $\vec{k} = 0$ and work with slightly different coordinates, where the metric is locally a small *conformal* deformation of Minkowski space. The downside is that in such a form the metric is not near-Minkowskian for all times, i.e., deviations from Minkowski space now will only be small locally in space *and* time. Around $\vec{x} = 0$, $t = 0$ the needed infinitesimal gauge transformation is

$$\begin{aligned}\xi_0 &= -\tfrac{1}{4}(2\dot{H} + H^2)(|\vec{x}|^2 t + \tfrac{1}{3}t^3),\\ \xi_i &= \tfrac{1}{4}(2\dot{H} + H^2)x_i t^2,\end{aligned} \tag{10}$$

under which the metric (4) transforms to

$$ds^2 \simeq [1 - \tfrac{1}{2}H^2|\vec{x}|^2 + \tfrac{1}{2}(2\dot{H} + H^2)t^2](-dt^2 + d\vec{x}^2). \tag{11}$$

At the order we are working at H^2 and $\dot{H}$ can be evaluated at $t = 0$ and treated as constants. The Newtonian potentials now are

$$\Phi = -\tfrac{1}{4}H^2|\vec{x}|^2 + \tfrac{1}{4}(2\dot{H} + H^2)t^2, \qquad \Psi = -\Phi. \tag{12}$$

B. Mixing with π

Let us consider a long-distance modification of GR where, at least locally and for weak gravitational fields, deviations from GR are due to a light scalar degree of freedom π kinetically mixed with the metric. At quadratic order, this amounts to replacing the Einstein action $\sqrt{-g}R$ with $\sqrt{-g}(1 - 2\pi)R$ plus whatever dynamics π may have on its own. For reasons that will become clear in the following, even for weak gravitational fields and small π values we allow for generic nonlinearities in the π sector, whereas we treat the gravitational field and π's contribution to it at linear order [in the equation of motion (EOM)]. In this conformal frame we do not couple π directly to matter, which we assume to be minimally coupled to the metric $g_{\mu\nu}$. This ensures that introducing π does not impair the universality of gravitational interactions. If we demix π and $g_{\mu\nu}$ by performing the Weyl rescaling

$$h_{\mu\nu} = \hat{h}_{\mu\nu} + 2\pi\eta_{\mu\nu}, \tag{13}$$

the action becomes

$$S = \int d^4x \left[\frac{1}{2}M_{\text{Pl}}^2\sqrt{-\hat{g}}\hat{R} + \frac{1}{2}\hat{h}_{\mu\nu}T^{\mu\nu} + \mathcal{L}_\pi + \pi T^\mu{}_\mu\right], \tag{14}$$

where once again we are working at quadratic order in $\hat{h}_{\mu\nu}$, and $\mathcal{L}_\pi$ encodes the so far unspecified dynamics of π. Now, according to the discussion in the Introduction, if we want the modification of gravity mediated by π to be an interesting one, π's effect on the geometry perceived by matter should be dominated by the coupling $\pi T^\mu{}_\mu$ rather than by π's contribution to the stress-energy tensor sourcing $\hat{h}_{\mu\nu}$. For this reason, in this conformal frame we neglect the gravitational backreaction associated with π (we will come back to the validity of this approximation in Sec. IV B). In this case, the field equations for $\hat{h}_{\mu\nu}$ are the same as in pure GR, and so for given sources and boundary conditions, their solution will be the GR one. This means that if we know the "physical" metric $h_{\mu\nu}$ for given $T_{\mu\nu}$ in our modified theory, we can get the π configuration associated with it by subtracting the corresponding GR solution $\hat{h}_{\mu\nu}$ with the same $T_{\mu\nu}$, via Eq. (13). However in comparing the two metrics we must allow for gauge transformations, because the Weyl rescaling Eq. (13) does not preserve a generic gauge fixing. The first

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[1]Notice that, by performing a time-dependent boost we are in fact going to an accelerated reference frame. Since in Newtonian physics acceleration is physical, strictly speaking the velocity field is not homogenous—the only inertial comoving observer is that sitting at the origin. However, even in Newtonian physics the only observable accelerations are the *relative* ones—that is the essence of the (Newtonian) equivalence principle. From this viewpoint, all our Newtonian comoving observers are equivalent.

step is to fix gauges for $h_{\mu\nu}$ and $\hat{h}_{\mu\nu}$ such that their difference is conformally flat (if this is not possible, it means that the modification of gravity under study is *not* due to a scalar kinetically mixed with the metric). Then

$$\pi\eta_{\mu\nu} = \tfrac{1}{2}(h_{\mu\nu} - \hat{h}_{\mu\nu})_{\text{conformal}}. \qquad (15)$$

However such a gauge choice is not unique. Namely, we are still free to perform infinitesimal conformal transformations, under which the metric changes by a conformal factor. Infinitesimal dilations shift π by a constant, whereas infinitesimal special conformal transformations add a linear piece to it,

$$\pi \to \pi + c + b_\mu x^\mu. \qquad (16)$$

This means that without further knowledge of the π dynamics, π is unambiguously determined by the geometry only up to a constant and a linear piece. Alternatively, we could try to determine π by comparing the curvature tensors, which characterize the geometry in a gauge-invariant fashion. However at linear order we have

$$R_{\mu\nu} = \hat{R}_{\mu\nu} - 2\partial_\mu\partial_\nu\pi - \eta_{\mu\nu}\Box\pi, \qquad (17)$$

which is only sensitive to the second derivatives of π. Once again, the constant and linear pieces in π are left undetermined.

Consider now a cosmological solution in such a modified-gravity theory. According to our discussion of Sec. II A, given the solution $a(\tau)$ we can choose coordinates such that locally around $\vec{x} = 0$, $t = 0$ the metric is a small conformal perturbation of Minkowski space, Eq. (11). If we were to do the same in GR, at the same value of ρ and p we would get a different conformal factor, with H_0 and $\dot{H}_0$ in place of H and $\dot{H}$. Then, to extract the π configuration that is locally responsible for the modified cosmological behavior, we just have to use Eqs. (11) and (15). In terms of the Newtonian potential in conformal gauge

$$\pi(\vec{x}, t) = \Phi - \hat{\Phi} \qquad (18)$$

$$= -\tfrac{1}{4}(H^2 - H_0^2)|\vec{x}|^2 + \tfrac{1}{4}[2(\dot{H} - \dot{H}_0) + (H^2 - H_0^2)]t^2$$
$$+ c + bt, \qquad (19)$$

where the GR quantities H_0 and $\dot{H}_0$ are to be computed at the same values of ρ and p as H and $\dot{H}$. We have included the undetermined constant and linear pieces discussed above. In particular, given isotropy, the linear piece can only be proportional to t.

C. An example: DGP cosmology

As an example of our scenario and of the effectiveness of our method, we consider cosmological solutions in the DGP model. We will see that locally the modifications to the cosmological dynamics are *fully* captured by a 4D scalar. The four-dimensional modified Friedmann equation reads [10]

$$H^2 = \frac{1}{3M_{\text{Pl}}^2}\rho + \epsilon mH = H_0^2 + \epsilon mH, \qquad (20)$$

where $H_0^2 = \frac{1}{3M_{\text{Pl}}^2}\rho$ is the Hubble rate we would have in GR at the same value of ρ and p, $m = 1/L_{\text{DGP}}$ is the critical scale, and ϵ is the sign that distinguishes between the two branches of solutions. We can absorb ϵ into the sign of m. Then, the self-accelerated branch corresponds to positive m. Taking the time derivative of the above equation, and using the covariant conservation of energy, $\dot{\rho} = -3H(\rho + p)$, we get the second Friedmann equation,

$$\dot{H} = -\frac{1}{2M_{\text{Pl}}^2}(\rho + p) \cdot (1 - m/2H)^{-1}$$

$$= \dot{H}_0 + \frac{1}{2}m(\dot{H}/H), \qquad (21)$$

where once again $\dot{H}_0 = -\frac{1}{2M_{\text{Pl}}^2}(\rho + p)$ is the corresponding GR value for the given sources. Now we want to interpret the deviations from GR as due, at least locally in space and time, to a scalar field π kinetically mixed with the metric. Then Eq. (19) tells us the solution for π,

$$\pi(\vec{x}, t) = -\tfrac{1}{4}mH \cdot |\vec{x}|^2 + \tfrac{1}{4}m(\dot{H}/H + H) \cdot t^2 + c + bt. \qquad (22)$$

We claim that this is the brane-bending mode identified in Ref. [14]. To see this, consider the EOM that comes from varying the 4D effective action of Ref. [14] with respect to π,[2]

$$6\Box\pi - \frac{4}{m^2}(\partial_\mu\partial_\nu\pi)^2 + \frac{4}{m^2}(\Box\pi)^2 = -T^\mu{}_\mu = \rho - 3p. \qquad (24)$$

If we plug the π configuration (22) into this EOM, we get a linear combination of the two modified Friedmann equations above, namely, $12 \times (20) + 6 \times (21)$. This shows that, despite being intrinsically a five-dimensional model, DGP is well described at short distances by a local 4D theory of a scalar kinetically mixed with the metric [14,21]. This description is locally *exact*, with no need of subtle "decoupling" limits—it yields the correct modification to the Friedmann equation as implied by the full 5D theory.

D. The structure of $\mathcal{L}_\pi$

We should now characterize the general structure of $\mathcal{L}_\pi$ that corresponds to a potentially realistic modification of gravity. On one side, we want π to give rise to an $O(1)$ modification of the Hubble flow, meaning that π exchange

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[2]The present π has a different normalization from that of Refs. [14,21]. For instance, with respect to Ref. [21]

$$\frac{\hat{\pi}}{M_4}\Big|_{\text{there}} = 2 \cdot \pi_{\text{here}}, \qquad M_4^{\text{there}} = \frac{1}{2}M_{\text{Pl}}^{\text{here}}. \qquad (23)$$

has gravitational strength on cosmological scales. On the other, we cannot tolerate deviations larger than $O(10^{-3})$ from Einsteinian gravity at solar-system distances. It is obviously impossible to satisfy these requests when the π dynamics is linear throughout the Universe. However if nonlinearities are important, then the extrapolation between different length scales is nontrivial and it would be in principle possible to have a viable theory. The DGP model with nonlinearities corresponding to the Vainshtein effect represents an interesting example in that sense. So a first necessary request we want to make is that, even in the assumed regime where π can be treated as a small perturbation of the geometry, its dynamics is nonetheless nonlinear. Consider then the general cosmological solution for π. At subhorizon distances the leading behavior is captured by a quadratic approximation

$$\pi_c = C + B_\mu x^\mu + A_{\mu\nu} x^\mu x^\nu + O(x^3 H^3). \quad (25)$$

We remark that the quadratic approximation to the solution is invariant under the combined action of a space-time translation and a shift $\pi(x) \rightarrow \pi(x) + b_\mu x^\mu + c$. More precisely

$$\pi_c(x) = \pi_c(x + \delta) - 2A_{\mu\nu}\delta^\mu x^\nu - B_\mu \delta^\mu, \quad (26)$$

that is $b_\mu = -2A_{\mu\nu}\delta^\nu$ and $c = -B_\mu \delta^\mu$. Barring accidents, a symmetry in a solution follows from an invariance of the equations of motion and thus constrains the form of the Lagrangian. In the case at hand, taking for granted translational invariance, we would deduce that the Lagrangian we want must be invariant under the shift[3]

$$\pi(x) \rightarrow \pi(x) + b_\mu x^\mu + c. \quad (27)$$

However before taking this deductive step, we have to be more careful, as our symmetry just trivially follows from our quadratic approximation: the solution to any scalar field theory can be approximated by its Taylor expansion at second order in the coordinates around one point. In fact, normally, approximating the solution to quadratic order in the coordinates corresponds to linearizing the equations of motion in the field. This is easily seen for an ordinary two-derivative Lagrangian $-\frac{1}{2}(\partial\pi)^2 - V(\pi)$ where the equation of motion linearized in $\varphi = \pi(x) - \pi(0)$ is

$$\Box\varphi = V'(\pi(0)) = \text{const.} \quad (28)$$

Notice indeed that the above equation admits Eq. (25) among its solutions. Moreover Eq. (28) is, as it should be expected, invariant under the shift $\varphi(x) \rightarrow \varphi(x) + b_\mu x^\mu +$

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[3]Strictly speaking, for a given solution, characterized by $A_{\mu\nu}$ and B_μ, combining the transformation in Eq. (26) with a translation we obtain a four-parameter family of transformations where b_μ and c are functions of a four-vector δ. However the commutator of these transformations with translations is a constant shift $\pi \rightarrow \pi + c$, so we recover the full five-parameter group.

c. Therefore in ordinary cases the symmetry $\pi(x) \rightarrow \pi(x) + b_\mu x^\mu + c$ of the approximated solution corresponds trivially to an accidental symmetry of the linearized equations of motion. However, as explained above, here we are interested in the cases where the π dynamics, even at local quadratic order in the coordinates, is described by a nonlinear equation of motion. As follows from our discussion, we are thus lead to make a second nontrivial assumption, namely, that the equations of motion of π be invariant under the shift symmetry (27). This is equivalent to demanding that each π in the equation of motion be acted upon by at least two derivatives. In fact in order to avoid trouble with ghost degrees of freedom, a third final request we should make is that the π equation of motion be just of second order. Again the need for this request is exemplified by comparing the DGP model to Fierz-Pauli massive gravity. In the latter, unlike the first, the analogue of the π dynamics leads to fourth-order equations [15], with a resulting ghost at the onset of nonlinearity. Our requests imply the equation of motion to take the form

$$\frac{\delta\mathcal{L}_\pi}{\delta\pi} = F(\partial_\mu \partial_\nu \pi) \quad (29)$$

where F is a nonlinear Lorentz-invariant function of the tensor $\partial_\mu \partial_\nu \pi$. In the next paragraphs we shall classify the Lorentz invariants satisfying Eq. (29). It turns out that there is only a handful of invariants (more precisely 5 of them in 4D) that do so. Such a very constrained class of Lagrangians represents a generalization of the π Lagrangian describing modifications of gravity at sub-Hubble distances in the DGP model [14]. This is indeed not surprising, since our three constraints correspond to the three main properties that guarantee viability of the DGP model at solar-system distances. To conclude, the shift corresponding to c makes π a Goldstone boson. However the vectorial parameter b_μ corresponds to constant shift of the gradient $\partial_\mu \pi \rightarrow \partial_\mu \pi + b_\mu$. That is the space-time generalization of the Galilean symmetry $\dot{x} \rightarrow \dot{x} + v$ of nonrelativistic mechanics $(0 + 1$ field theory). We are thus naturally led to term $\pi \rightarrow \pi + bx$ Galilean transformation and to call π *Galileon*.

Before proceeding, we would like to mention that Galilean invariance of $\mathcal{L}_\pi$ can be motivated in an alternative and perhaps more direct way. Indeed as discussed in Sec. II B, the modification brought by π to the geometry is invariant under the change in Eq. (16). That is to say that the correction to the Ricci (and Riemann) tensor due to π, $\Delta R_{\mu\nu} = -2\partial_\mu \partial_\nu \pi - \eta_{\mu\nu}\Box\pi$, is unaffected by Eq. (16). Now, a potentially realistic modification of gravity can be defined as one that corrects curvature at $O(H^2)$ at cosmological scales but which is suppressed with respect to ordinary GR at shorter scales where the curvature due to local sources is bigger than H^2. As we said, such a behavior may in principle arise provided the π dynamics is nonlinear. Then if we further assume the π equations of motion

only depend on $\Delta R_{\mu\nu}$

$$F(\Delta R_{\mu\nu}) = -T^\mu{}_\mu, \tag{30}$$

we have that corrections to the geometry are directly determined by the distribution of mass in the Universe (and by boundary conditions). In this situation, we have a better chance to have a robust implementation of the Vainshtein effect where $\Delta R_{\mu\nu}$ is screened sufficiently near localized sources. Thus one viewpoint on Galilean invariance is that it is needed to have a robust and economic implementation of the Vainshtein effect.

We are thus led to classify the Lorentz-invariant operators satisfying Eq. (29). These Lagrangian terms must necessarily involve n powers of π and $2n - 2$ space-time derivatives. Of course the generic term of this form is not Galilean invariant and correspondingly its contribution to the EOM also involves first derivatives of π. It thus requires some work to find these peculiar invariants at each order in n. The simplest possibility is given by $n = 1$, where we have the "tadpole" $\mathcal{L}_1 = \pi$. Notice that the tadpole is invariant under Galilean transformations only up to a total derivative $\delta\mathcal{L}_1 = b_\rho x^\rho + c = \partial_\mu(b_\rho x^\rho x^\mu/5 + cx^\mu/4)$. But this ensures anyway the invariance of the equations of motion and conservation of a local current. The next to simplest example is the standard kinetic energy, $(\partial\pi)^2$, whose variation under a Galileo boost is

$$\delta(\partial\pi)^2 = -4c^\mu\partial_\mu\pi, \tag{31}$$

which is a total derivative. A less trivial example is the DGP cubic interaction of Ref. [14], $\Box\pi(\partial\pi)^2$, whose variation is

$$\delta(\Box\pi(\partial\pi)^2) = -4c^\mu\partial_\mu\pi\Box\pi$$
$$= -4c^\mu\partial^\alpha[\partial_\alpha\pi\partial_\mu\pi - \tfrac{1}{2}\eta_{\alpha\mu}(\partial\pi)^2]. \tag{32}$$

Again, as follows from Galilean invariance, the variation of the above term with respect to π only depends on second derivatives of π. We can go on and study higher-order invariants. In the Appendix, we show that formally there exists one and only one such Galilean invariant at each order in π. However, since each derivative Lagrangian term we will construct will be associated with one Cayley invariant of the matrix $\partial_\mu\partial_\nu\pi$, we will have as many nontrivial derivative Galilean invariants as the rank of $\partial_\mu\partial_\nu\pi$, that is the number of space-time dimensions. In particular, in 4D we only have five invariants: from the tadpole term, to a quintic derivative interaction.[4]

At the nth order in π, the generic structure of a Galileo-invariant Lagrangian term is $(\partial^2\pi)^{n-2}\partial\pi\partial\pi$. For notational convenience, let us denote by Π the matrix of second derivatives of π, $\Pi^\mu{}_\nu \equiv \partial^\mu\partial_\nu\pi$. Also the brackets $[\dots]$

stand for the trace operator, and the "·" stands for the standard Lorentz-invariant contraction of indices. So, for instance

$$[\Pi]\partial\pi \cdot \partial\pi \equiv \Box\pi\partial_\mu\pi\partial^\mu\pi. \tag{33}$$

Then, up to fifth order in π the Galileo-invariant terms are

$$\mathcal{L}_1 = \pi \tag{34}$$

$$\mathcal{L}_2 = -\tfrac{1}{2}\partial\pi \cdot \partial\pi \tag{35}$$

$$\mathcal{L}_3 = -\tfrac{1}{2}[\Pi]\partial\pi \cdot \partial\pi \tag{36}$$

$$\mathcal{L}_4 = -\tfrac{1}{4}([\Pi]^2\partial\pi \cdot \partial\pi - 2[\Pi]\partial\pi \cdot \Pi \cdot \partial\pi$$
$$- [\Pi^2]\partial\pi \cdot \partial\pi + 2\partial\pi \cdot \Pi^2 \cdot \partial\pi) \tag{37}$$

$$\mathcal{L}_5 = -\tfrac{1}{5}([\Pi]^3\partial\pi \cdot \partial\pi - 3[\Pi]^2\partial\pi \cdot \Pi \cdot \partial\pi$$
$$- 3[\Pi][\Pi^2]\partial\pi \cdot \partial\pi + 6[\Pi]\partial\pi \cdot \Pi^2 \cdot \partial\pi$$
$$+ 2[\Pi^3]\partial\pi \cdot \partial\pi + 3[\Pi^2]\partial\pi \cdot \Pi \cdot \partial\pi$$
$$- 6\partial\pi \cdot \Pi^3 \cdot \partial\pi). \tag{38}$$

The overall normalizations have been chosen to have simple normalizations in the equations of motion; see below. Higher-order Galileo invariants are trivial in 4D, being just total derivatives.

For our purposes it is more convenient to work directly at the level of the equations of motion, where there are no integration-by-parts ambiguities, and only second derivatives appear. Defining $\mathcal{E}_i \equiv \frac{\delta\mathcal{L}_i}{\delta\pi}$, we get

$$\mathcal{E}_1 = 1 \tag{39}$$

$$\mathcal{E}_2 = \Box\pi \tag{40}$$

$$\mathcal{E}_3 = (\Box\pi)^2 - (\partial_\mu\partial_\nu\pi)^2 \tag{41}$$

$$\mathcal{E}_4 = (\Box\pi)^3 - 3\Box\pi(\partial_\mu\partial_\nu\pi)^2 + 2(\partial_\mu\partial_\nu\pi)^3 \tag{42}$$

$$\mathcal{E}_5 = (\Box\pi)^4 - 6(\Box\pi)^2(\partial_\mu\partial_\nu\pi)^2 + 8\Box\pi(\partial_\mu\partial_\nu\pi)^3$$
$$+ 3[(\partial_\mu\partial_\nu\pi)^2]^2 - 6(\partial_\mu\partial_\nu\pi)^4 \tag{43}$$

where by $(\partial_\mu\partial_\nu\pi)^n$ we denote the cyclic contraction, $(\partial_\mu\partial_\nu\pi)^n \equiv [\Pi^n]$.

The complete Lagrangian for π is a linear combination of the above invariants

$$\mathcal{L}_\pi = \sum_{i=1}^5 c_i\mathcal{L}_i, \tag{44}$$

where the c_i's are generic coefficients. Likewise, the equation of motion is

---

[4]In 1D, i.e., mechanics, we just have two: the Galilean kinetic energy $\tfrac{1}{2}m\dot{x}^2$ and the linear potential x.

$$\mathcal{E} \equiv \frac{\delta \mathcal{L}_\pi}{\delta \pi} = \sum_{i=1}^{5} c_i \mathcal{E}_i = -T^\mu_{\ \mu}. \qquad (45)$$

E. Technical analysis

We now want to investigate the existence of well-behaved self-accelerating solutions. By "self-accelerating" we mean solutions for π that correspond to a de Sitter geometry even for the vanishing stress-energy tensor. Equation (19) gives us the relevant π configuration,

$$\pi_{\mathrm{dS}}(x) = -\tfrac{1}{4}H^2 x_\mu x^\mu, \qquad \partial_\mu \partial_\nu \pi_{\mathrm{dS}} = -\tfrac{1}{2}H^2 \eta_{\mu\nu}, \quad (46)$$

where we used $H_0 = \dot{H}_0 = 0$, because the stress-energy tensor vanishes, and $\dot{H} = 0$, because we are looking for a de Sitter solution. The curvature H^2 is determined by the equation of motion (45), with $T^\mu_{\ \mu} = 0$. Indeed plugging the de Sitter ansatz into the EOM, we get an algebraic equation for H^2,

$$c_1 - 2c_2 H^2 + 3c_3 H^4 - 3c_4 H^6 + \tfrac{3}{2}c_5 H^8 = 0. \qquad (47)$$

Then, assuming we have a de Sitter solution, we are interested in studying its stability and in general the dynamics of local perturbations that do not change the asymptotics. With an abuse of notation, we will refer to these perturbations by π. That is, we perform the replacement $\pi \rightarrow \pi_{\mathrm{dS}} + \pi$ everywhere in the Lagrangian and in the EOM The background de Sitter solution (46) is Lorentz invariant. Then Lorentz symmetry is unbroken, and it must be an invariance of the action for the perturbations. Furthermore, the background is invariant under the combined action of a space-time translation and of an internal Galileo boost,

$$x^\mu \rightarrow x^\mu + a^\mu, \qquad \pi_{\mathrm{dS}} \rightarrow \pi_{\mathrm{dS}} + \tfrac{1}{2}H^2 a_\mu x^\mu, \qquad (48)$$

which must also be a symmetry of the perturbations' dynamics. But in the EOM, only second derivatives appear, and they are automatically invariant under internal Galileo boosts. Also, the background's second derivatives are translationally invariant. This means that the EOM for the perturbations is separately invariant under space-time translations and under internal Galileo boosts, as well as under space-time Lorentz boosts. That is, apart from boundary terms in the action, the space-time Poincaré symmetry and the internal Galileo boosts are unbroken. The EOM for the perturbations then must be a linear combination of our Galileo invariants (40)–(43),

$$\frac{\delta \mathcal{L}_\pi}{\delta \pi}\bigg|_{\mathrm{dS}} = \sum_{i=2}^{5} d_i \mathcal{E}_i, \qquad (49)$$

but now with different parameters d_i. The latter are linear combinations of the original parameters c_i with H^2-dependent coefficients. The constant term $\mathcal{E}_1$ does not appear in (49) because it is precisely the EOM evaluated on the de Sitter background (47), which vanishes. Explicit

expressions for the mapping $\{c_i\} \leftrightarrow \{d_i\}$ $(i = 2, \ldots, 5)$ are given in the Appendix. The exact relation will not be important for us. Suffice it to say that it is invertible.

In conclusion, as long as we are interested in solutions with de Sitter asymptotics, we can forget about the original Lagrangian coefficients c_i, pretend to be in flat space, and parametrize the dynamics of perturbations with arbitrary coefficients d_i. The determination of the de Sitter Hubble parameter does not impose any constraint on the coefficients d_i: for a given set $d_2, \ldots, d_5$ we can compute the corresponding $c_2, \ldots, c_5$ and by choosing appropriately c_1 there always exists a solution of (47) with positive H^2.

In order for our Galilean model to be a fairly realistic description of our Universe, it should pass a number of zeroth order tests. It should admit a de Sitter solution that is stable against small perturbations. Furthermore, about this de Sitter configuration there should exist spherically symmetric solutions that describe the π field generated by compact sources. These configurations should also be ghost-free. Finally, we will require that small pertubations about these solutions do not propogate superluminally. As we will see our model fails this last test. We will briefly comment on this in Sec. V.

For spherically symmetric configurations, $\pi = \pi(r)$, the EOM drastically simplifies. In particular, like in DGP, it becomes an *algebraic* equation for $\pi'(r)$. This is because the π EOM is the divergence of the Noether current associated with the shift symmetry $\pi \rightarrow \pi + c$. For generic π configurations the current J^μ involves first as well as second derivatives of π. Once we take the divergence, all third derivatives cancel, thus yielding two-derivative EOM However for a radial configuration $\pi(r)$ the current cannot depend on $\pi''(r)$—otherwise the third derivatives could not cancel in the EOM:

$$\partial_\mu J^\mu(\pi', \pi''; r) = \frac{1}{r^2} \frac{d}{dr}(r^2 J^r)$$
$$= \frac{2}{r}J^r + \frac{\partial J^r}{\partial \pi'}\pi''(r) + \frac{\partial J^r}{\partial \pi''}\pi'''(r) + \frac{\partial J^r}{\partial r}. \qquad (50)$$

Then the current only depends on $\pi'(r)$ (and on r), and the EOM is easily integrated via Gauss's theorem, thus yielding an algebraic equation for $\pi'(r)$. Moreover, since each Galilean term in the EOM is of the form $\mathcal{E}_{n+1} \sim (\partial \partial \pi)^n$, the corresponding piece in the current is schematically $J_{n+1} \sim (\partial \partial \pi)^{n-1} \partial \pi$. For radial configurations $\partial \pi \sim \pi'$ and $\partial \partial \pi \sim \pi''$, $\frac{\pi'}{r}$, and since the current cannot contain π'', we have

$$J_{n+1} \sim r \cdot (\pi'/r)^n. \qquad (51)$$

This means that the EOM reduces in fact to an algebraic equation for (π'/r). Explicitly, we have

$$\mathcal{E}_2 \rightarrow \frac{1}{r^2} \frac{d}{dr}(r^2 \pi') \qquad (52)$$

$$\mathcal{E}_3 \rightarrow \frac{2}{r^2}\frac{d}{dr}(r\pi'^2) \qquad (53)$$

$$\mathcal{E}_4 \rightarrow \frac{2}{r^2}\frac{d}{dr}(\pi'^3) \qquad (54)$$

$$\mathcal{E}_5 \rightarrow 0. \qquad (55)$$

The vanishing of the last contribution is due to the fact that for a time-independent π the matrix $\partial_\mu\partial_\nu\pi$ has rank three, thus the fourth-order invariant identically vanishes. For a source localized at the origin, $\rho = M\delta^3(\vec{r})$,

$$
\begin{aligned}
\frac{\delta\mathcal{L}_\pi}{\delta\pi} &= \sum_i d_i\mathcal{E}_i \\
&= \frac{1}{r^2}\frac{d}{dr}r^3[d_2(\pi'/r) + 2d_3(\pi'/r)^2 + 2d_4(\pi'/r)^3] \\
&= M\delta^3(\vec{r}),
\end{aligned}
\qquad (56)
$$

we get

$$d_2(\pi'/r) + 2d_3(\pi'/r)^2 + 2d_4(\pi'/r)^3 = \frac{M}{4\pi r^3}. \qquad (57)$$

1. Existence

We see that the existence of a radial solution for all r's constrains the coefficients d_i to some extent. Indeed varying r from zero to infinity makes the right-hand side of Eq. (57) span all positive real numbers. Let us call $P(y)$ the polynomial of $y = \pi'/r$ on the left-hand side,

$$P(y) \equiv d_2 y + 2d_3 y^2 + 2d_4 y^3. \qquad (58)$$

Then, for a solution to exist $P(y)$ must be invertible when it is positive. We already know that d_2 must be strictly positive for the background (de Sitter) solution to be stable against small perturbations: d_2 is the coefficient of $-\frac{1}{2} \times (\partial\pi)^2$ in the perturbations' Lagrangian. Now, far from the source $y = \pi'/r$ is small, hence the linear term in $P(y)$ dominates. Then, at small y's, positive P implies positive y. P is invertible if $P'(y)$ does not cross zero. Thus, in moving to larger y's (smaller r's) P has to increase. As a consequence, P is positive when y is positive. This means that the requirement that $P(y)$ be invertible whenever it is positive is equivalent to ask that

$$P'(y) = d_2 + 4d_3 y + 6d_4 y^2 > 0 \quad \forall\, y > 0. \qquad (59)$$

A radial solution exists if this condition is obeyed. At large y (small r), the quadratic term in $P'(y)$ dominates, thus d_4 has to be non-negative. Then Eq. (59) is satisfied for every $y > 0$ if we also have $d_3 > -\sqrt{\frac{3}{2}d_2 d_4}$. In conclusion, there exist spherically symmetric solutions with de Sitter asymptotics iff

$$d_2 > 0 \qquad d_4 \geq 0 \qquad d_3 > -\sqrt{\frac{3}{2}d_2 d_4}. \qquad (60)$$

2. Stability

We now move to study the stability of the radial solution $\pi_0(r)$ against small perturbations, $\pi = \pi_0(r) + \varphi$. Given the symmetries of the solution, the quadratic Lagrangian for small perturbations will have the form

$$
\begin{aligned}
S_\varphi = \frac{1}{2}\int d^4x[&K_t(r)(\partial_t\varphi)^2 - K_r(r)(\partial_r\varphi)^2 \\
&- K_\Omega(r)(\partial_\Omega\varphi)^2],
\end{aligned}
\qquad (61)
$$

where $(\partial_\Omega\varphi)^2$ is the angular part of $(\vec{\nabla}\varphi)^2$, and the kinetic coefficients K_i depend on r through the background radial solution $\pi_0(r)$ and its derivatives. That fact that the full EOM, Eq. (49), only contains second derivatives guarantees that small perturbations about any solution will have a two-derivative quadratic Lagrangian, despite the formal appearance of many high-derivative terms in the original Lagrangian for π. For the solution to be stable against short wavelength perturbations, all the K_i's must be positive (see a related discussion in Ref. [21]).

Rather than obtaining the K_i's by expanding the full π Lagrangian at quadratic order in φ, it is faster to compute them by expanding the EOM at linear order,

$$
\begin{aligned}
\mathcal{E}[\pi_0 + \varphi] &\rightarrow \frac{\delta S_\varphi}{\delta\varphi} \\
&= -K_t(r)\ddot\varphi + \frac{1}{r^2}\partial_r(r^2 K_r(r)\partial_r\varphi) \\
&\quad + K_\Omega(r)\partial_\Omega^2\varphi,
\end{aligned}
\qquad (62)
$$

where ∂_Ω^2 is the angular part of the Laplacian. In particular, we can get the radial kinetic coefficient from the radial EOM Evidently, comparing Eq. (56) with the radial part of Eq. (62),

$$K_r(r) = P'(y)|_{y=\pi_0'/r}, \qquad (63)$$

where $P(y)$ is the polynomial defined in Eq. (58). The positivity of K_r is thus guaranteed by the existence of the radial solution itself, Eq. (59).

For the other K_i's, we get

$$K_t(r) = \frac{1}{3r^2}\frac{d}{dr}[r^3(d_2 + 6d_3 y + 18d_4 y^2 + 24d_5 y^3)_{y=\pi_0'/r}] \qquad (64)$$

$$K_\Omega(r) = \frac{1}{2r}\frac{d}{dr}[r^2(d_2 + 4d_3 y + 6d_4 y^2)_{y=\pi_0'/r}]. \qquad (65)$$

These depend on y as well as on its derivative with respect to r. Also they explicitly involve r. Thus it appears that the question of their positivity is now a differential problem rather than an algebraic one. However we can eliminate

$y'(r)$ in favor of y and r through the implicit function theorem: we know that the background solution satisfies

$$F(r, y) \equiv P(y) - \frac{M}{4\pi r^3} = 0. \qquad (66)$$

Then, on the solution

$$y'(r) = -\frac{\partial_r F}{\partial_y F}. \qquad (67)$$

This yields

$$K_t = \frac{d_2^2 + 4d_2 d_3 y + 12(d_3^2 - d_2 d_4)y^2 + 24(d_3 d_4 - 2d_2 d_5)y^3 + 12(3d_4^2 - 4d_3 d_5)y^4}{d_2 + 4d_3 y + 6d_4 y^2} \qquad (68)$$

$$K_\Omega = \frac{d_2^2 + 2d_2 d_3 y + (4d_3^2 - 6d_2 d_4)y^2}{d_2 + 4d_3 y + 6d_4 y^2}. \qquad (69)$$

Notice that the explicit r dependence canceled out. This is because both the background EOM and the kinetic coefficients K_i for the fluctuation only involve the matrix of second derivatives of π. For radial configurations we have $\partial\partial\pi \sim \pi''$, $\frac{\pi'}{r}$. Moreover the background EOM are *linear* in π'', being the divergence of a current only involving π'. As a consequence on the solution π'' is a *rational* function of π'/r, which makes the kinetic coefficients K_i rational functions of $\pi'/r \equiv y$.

Recall that the solution spans all positive values of y. Thus, we have to require that K_t and K_Ω are positive for all positive y's. Given the constraints (60), the denominators are automatically positive. The numerator in K_Ω then is positive for $d_3 \geq \sqrt{\frac{3}{2}d_2 d_4}$ which makes the numerator in K_t also positive provided that $d_5 \leq \frac{3}{4}\frac{d_4^2}{d_3}$. In summary, the radial solution with de Sitter asymptotics exists and is stable against small perturbations iff

$$d_2 > 0 \qquad d_4 \geq 0 \qquad d_3 \geq \sqrt{\frac{3}{2}d_2 d_4} \qquad d_5 \leq \frac{3}{4}\frac{d_4^2}{d_3}. \qquad (70)$$

3. Subluminality

Increasing further our level of pickiness, we can ask whether small fluctuations of our solution propagate at subluminal speeds. The speed of fluctuations propagating radially is simply

$$c_r^2 = \frac{K_r}{K_t}. \qquad (71)$$

Given the constraints imposed by existence and stability, Eq. (70), this is forced to be *superluminal* at large distances from the source (small y). Indeed

$$c_r^2 = 1 + 4\frac{d_3}{d_2}y + \mathcal{O}(y^2) > 1. \qquad (72)$$

Notice that the sign of the deviation from the speed of light is positive because of the requirement of existence and stability of spherical solutions, independently from the

presence or not of the self-accelerating background. In moving to smaller distances (larger y) the propagation speed can be made subluminal. This can be easily seen from the difference

$$K_r - K_t = 4d_2 d_3 y + 4(d_3^2 + 6d_2 d_4)y^2 \\ + 24(d_3 d_4 + 2d_2 d_5)y^3 + 48d_3 d_5 y^4, \qquad (73)$$

which becomes negative at large y if d_5 is negative.

On the other hand, angular excitations are always subluminal: their propagation speed is

$$c_\Omega^2 = \frac{K_\Omega}{K_t}. \qquad (74)$$

The difference between the numerator and the denominator is, apart from an overall positive factor,

$$K_\Omega - K_t \propto -2d_2 d_3 y - (8d_3^2 - 6d_2 d_4)y^2 \\ - 24(d_3 d_4 - 2d_2 d_5)y^3 - 12(3d_4^2 - 4d_3 d_5)y^4, \qquad (75)$$

which is negative, as each power of y has a negative coefficient. We end up with the following constraints:

$$d_2 > 0 \qquad d_4 \geq 0 \qquad d_3 \geq \sqrt{\frac{3}{2}d_2 d_4} \qquad d_5 < 0. \qquad (76)$$

III. SOME PHENOMENOLOGY

In this section we will broadly discuss the scales associated with classical nonlinearities and with quantum effects. The analysis can be straightforwardly applied also to the fully conformally-invariant case that we discuss in Sec. IV.

A. The coupling to matter

The Galileon Lagrangian has the general form

$$\mathcal{L} = f^2 \partial\pi\partial\pi F(\partial\partial\pi/H_0^2). \qquad (77)$$

We have directly chosen the scale controlling higher orders

in $\partial\partial\pi$ to be the present Hubble parameter H_0, so that the π contribution affects at $O(1)$ the present visible expansion. f^2 is an overall coefficient that gives the π Lagrangian the correct dimensions. We can think of f as measuring the coupling between π quanta and matter. Indeed in terms of the canonically normalized π_c, upon demixing at linear order, the coupling to matter is $\sim \frac{1}{f}\pi_c T^\mu{}_\mu$. The scale f is bounded by considering the effects, direct and indirect, of π on the geometry at scales shorter than or comparable to $1/H_0$. An upper bound is determined by considering the gravitational backreaction to the π energy momentum tensor. At subhorizon scales we have $\pi \sim H_0^2 x^2$, so that Eq. (77) gives $\Delta T_{\mu\nu} = O(f^2 H_0^4 x^2)$. Requesting that $\Delta T_{\mu\nu}$ does not lead to a curvature $\gg H_0^2$ at scales just below H_0^{-1} implies $f \lesssim M_{\rm Pl}$. A comparable lower bound is instead determined by considering the direct contribution of π to the geometry perceived by matter. A bound is thus provided by those gravitationally bound systems where the adequacy of GR has been directly tested to work at least at $O(1)$. To derive a constraint, a qualitative estimate of the effect of π is sufficient. To do so we must figure out which regime, linear or nonlinear, characterizes the dynamics of the perturbation π over the cosmological background $\pi_{\rm ds}$. Very much like in the DGP model, and as discussed in Sec. II E, π can be treated as linear whenever $\partial\partial\pi \lesssim O(H_0^2)$, which is normally far enough from a localized gravitational source; while $\partial\partial\pi \gtrsim O(H_0^2)$ corresponds to the nonlinear Vainshtein regime, that normally sets in near enough to sources. To get a qualitative idea, consider a source of total mass M corresponding to a Newtonian potential $h_N \sim \frac{1}{M_P^2}M/r = R_S/r$. At large enough distance where the linear regime applies the Galileon contribution to the metric goes like $\pi \sim \frac{1}{f^2}M/r = (M_P/f)^2 R_S/r$ so we have

$$\frac{\partial\partial\pi}{H_0^2} \sim \frac{M_P^2}{f^2}\frac{R_S}{r^3}\frac{1}{H_0^2} = \frac{M_P^2}{f^2}\frac{\mathcal{R}}{H_0^2} \qquad (78)$$

where by $\mathcal{R}$ we indicate the Riemann curvature induced by the source. Now, a system behaves as gravitationally bound when the local gravitational attraction wins over the tidal force associated with Hubble expansion, that is when $\mathcal{R} \gtrsim H_0^2$. This equation shows that when $f \lesssim M_P$, for any gravitationally bound system π is in the Vainshtein regime. To obtain a bound on the case $f \lesssim M_P$ we thus have to estimate the effects of π by using the behavior in the Vainshtein region. From Eq. (57) we see that at short distances the quartic coupling dominates the equation of motion. The coefficients d_n according to Eq. (77) scale like

$$d_n = f^2 \frac{a_n}{H_0^{2n-4}} \qquad (79)$$

with $a_n \sim \mathcal{O}(1)$, then the asymptotic behavior in the Vainshtein region is $\pi \sim (MH_0^4/f^2)^{1/3}r$. The relative size of the π potential and the Newtonian one is

$$\frac{\pi}{h_N} \sim \left(\frac{M_P}{f}\right)^{2/3}\frac{H_0^{4/3}r^2}{R_S^{2/3}} = \left(\frac{M_P}{f}\frac{H_0^2}{\mathcal{R}}\right)^{2/3}. \qquad (80)$$

Although the strongest bound on π/h_N comes from tests of gravity within the solar system, that bound does not translate into the strongest bound on f. This is because $\mathcal{R}$ within the solar system is much bigger than H_0^2. The strongest bound on f comes instead from more loosely bound systems like galaxy superclusters, at whose scales $\mathcal{R}$ are just a bit bigger than H_0^2, and for which general relativity seems valid at $O(1)$ or better. This translates into a lower bound, roughly

$$f > \frac{M_P}{\text{a few}}. \qquad (81)$$

In the end we basically have no choice other than $f \sim M_P$. The fact that both the upper and lower bound on f almost coincide is yet another illustration of how difficult it is to come up with consistent modifications of gravity. Notice on the other hand that in the DGP model, where π originates from the 5D geometry, one automatically obtains $f = M_P$.

B. Strong-interaction scales and retardation effects

With the scale f basically constrained to be M_P we can now discuss the issue of quantum corrections to the classical picture we developed so far. Our discussion essentially follows Sec. 7 of Ref. [21], where the same issue was studied in the case of the DGP model. However, as we shall see, some important differences arise between the two cases. Using $f = M_P$ in Eq. (77) and going to the canonical field $\hat{\pi} \equiv \pi/M_P$, we have

$$\mathcal{L} = \partial\hat{\pi}\partial\hat{\pi}F(\partial\partial\hat{\pi}/\Lambda^3) \qquad \Lambda^3 = M_{\rm Pl}H_0^2, \qquad (82)$$

corresponding to a "naive" strong-interaction scale $\Lambda \sim (1000 \text{ km})^{-1}$ precisely like in DGP. Here we say "naive" because, while this result applies around flat space, we expect modifications around nontrivial solutions. This is because in the Vainshtein region the propagation of the excitations $\varphi = \pi - \pi_0$ is dominated by the higher-order terms in the Lagrangian, thus invalidating the above estimate. In order to understand what happens we shall focus on the spherical solution that was discussed in Sec. II E. In the case of DGP, it was shown in Ref. [21] that, for a spherical solution, inside the Vainshtein region $\varphi = \pi - \pi_0$ acquires a large kinetic term. This damps φ's quantum fluctuations and raises the effective quantum cutoff to $\tilde{\Lambda} = \Lambda(R_V/r)^{3/4}$, where the Vainshtein radius of a spherical source is $R_V^3 = R_S/H_0^2$. For instance, on the surface of earth one has $\tilde{\Lambda}^{-1} \sim 1$ cm. From the discussion in Sec. III A we have

$$\pi_0 \sim \frac{R_S}{R_V^2}r \rightarrow \tilde{y} \equiv \frac{1}{H_0^2}\frac{\pi_0'}{r} \sim \frac{R_V}{r} \gg 1, \qquad (83)$$

$\tilde{y}$ is the parameter controlling the magnitude of nonlinear effects inside the Vainshtein region. For instance, from the discussion after Eq. (61), we have that the kinetic coefficients in the radial and temporal directions K_r and K_t generically grow like $\tilde{y}^2$. On the other hand K_Ω generically goes like a constant, indicating that angular fluctuations are not generically damped when approaching the source. Indeed from Eq. (69) we see that this constant behavior arises when $d_4 \neq 0$, while in DGP $d_4 = d_5 = 0$ and K_r, $K_t, K_\Omega \sim \tilde{y} \gg 1$. As we shall see in a moment, the lack of damping for angular fluctuations is an unpleasant novelty compared to DGP. In order to study how the fluctuation φ self-interacts, it is useful to write the effective action by distinguishing $x_\perp$, the spacial coordinates tranverse to $\vec{\nabla}\pi_0$ (the angular coordinates for our solution) and $x_\parallel = (t, r)$. Since strong coupling in our derivatively coupled theory is a short-distance phenomenon, in order to assess what the effective strong scale is at a given point, it is enough to zoom in on that point and "linearize" our coordinates and treat them as Cartesian. Moreover, precisely as we did with the quadratic Lagrangian for φ, we can work out the interaction terms working directly on the equations of motion. Qualitatively (that is without paying attention to the different contractions of indices) the structure of the effective action is

$$S_{\text{eff}} = \int d^2x_\parallel d^2x_\perp \Big[\tilde{y}^2(\partial_\parallel \varphi)^2 + (\partial_\perp \varphi)^2$$
$$+ a_3 \frac{1}{\Lambda^3} (\partial_\perp \varphi)^2 \partial_\perp^2 \varphi + a_4 \frac{\tilde{y}}{\Lambda^3} (\partial_\perp \varphi)^2 \partial_\parallel^2 \varphi$$
$$+ a_5 \frac{\tilde{y}^2}{\Lambda^3} (\partial_\parallel \varphi)^2 \partial_\parallel^2 \varphi + \dots \Big], \tag{84}$$

where the dots indicate less important interactions of cubic, and higher, order. Notice that the interactions involving only transverse derivatives are not enhanced by $\tilde{y}$ factors. This arises because for a linear solution $\pi_0 \propto r$ one has $\partial_i \partial_j \pi_0 \propto \delta_{ij}^\perp$ with $\delta_{ij}^\perp$ representing the metric in the transverse directions. Then if we focus on terms involving only transverse derivatives we have that both the background $\partial_i \partial_j \pi_0$ and the fluctuation are rank two matrices living in the transverse space. But such rank two objects vanish beyond cubic order because of the rank condition on our Galilean invariants. To estimate the strong scale at a given point, we can treat $\tilde{y}$ as constant. The difficulty with the above Lagrangian is the lack of isotropy. One way to proceed is to change variables and make the kinetic term canonical. This can be achieved by working with rescaled coordinates along the transverse direction

$$x_\perp = \tilde{x}_\perp / \tilde{y} \rightarrow \partial_\perp = \tilde{y} \tilde{\partial}_\perp, \qquad d^2x_\perp = \frac{d^2\tilde{x}_\perp}{\tilde{y}^2}, \tag{85}$$

so that the action is rewritten as

$$S_{\text{eff}} = \int d^2x_\parallel d^2\tilde{x}_\perp \Big[(\partial_\parallel \varphi)^2 + (\tilde{\partial}_\perp \varphi)^2$$
$$+ a_3 \frac{\tilde{y}^2}{\Lambda^3} (\tilde{\partial}_\perp \varphi)^2 \tilde{\partial}_\perp^2 \varphi + a_4 \frac{\tilde{y}}{\Lambda^3} (\tilde{\partial}_\perp \varphi)^2 \partial_\parallel^2 \varphi$$
$$+ a_5 \frac{1}{\Lambda^3} (\partial_\parallel \varphi)^2 \partial_\parallel^2 \varphi + \dots \Big]. \tag{86}$$

In this form we can easily estimate the scattering amplitudes $2 \rightarrow 2$ that arise working at second order in the cubic interaction terms. The most dangerous terms are the first two. The third term gives the same cutoff as around flat space, that is $\sim \Lambda$. The first cubic interaction, the one involving only transverse derivatives, gives an amplitude that roughly scales like

$$\mathcal{A}_{2 \rightarrow 2} \sim \frac{\tilde{y}^4}{\Lambda^6} \frac{\tilde{p}_\perp^8}{\max(\tilde{p}_\perp^2, p_\parallel^2)} = \frac{1}{\tilde{y}^4 \Lambda^6} \frac{p_\perp^8}{\max(p_\perp^2/\tilde{y}^2, p_\parallel^2)}, \tag{87}$$

where the $1/p^2$ factor is an estimate of the propagator and where we have rewritten the amplitude also in terms of the unrescaled momentum along the transverse directions. Notice that we have the dispersion relation $\omega \equiv p_\parallel^0 = \sqrt{(p_\parallel^1)^2 + p_\perp^2/\tilde{y}^2} \gtrsim p_\perp/\tilde{y}$. The above amplitude is clearly maximized by scattering waves moving along the transverse directions $p_\parallel \sim p_\perp/\tilde{y}$. In that case we deduce an effective momentum cutoff

$$\Lambda_p \sim \tilde{y}^{1/3} \Lambda = \left(\frac{R_V}{r} \right)^{1/3} \Lambda. \tag{88}$$

However because of the anisotropic dispersion relation, the corresponding energy cutoff is much lower

$$\Lambda_E \sim \tilde{y}^{-2/3} \Lambda = \left(\frac{R_V}{r} \right)^{-2/3} \Lambda. \tag{89}$$

One similarly estimates the amplitude determined by just the second cubic interaction term, the one involving both transverse and longitudinal derivatives

$$\mathcal{A}_{2 \rightarrow 2} \sim \frac{\tilde{y}^2}{\Lambda^6} \frac{\tilde{p}_\perp^4 \tilde{p}_\parallel^4}{\max(\tilde{p}_\perp^2, p_\parallel^2)} = \frac{1}{\tilde{y}^2 \Lambda^6} \frac{p_\perp^4 p_\parallel^4}{\max(p_\perp^2/\tilde{y}^2, p_\parallel^2)}. \tag{90}$$

With the above amplitude it is convenient to consider two regions: (i) $p_\parallel \sim p_\perp/\tilde{y}$ and (ii) $p_\parallel > p_\perp/\tilde{y}$. In the first region we find the same $\Lambda_p \sim \tilde{y}^{2/3} \Lambda$ and $\Lambda_E \sim \tilde{y}^{-1/3}\Lambda$, while in the second region we find that the cutoff is $\sim \tilde{y}^{1/3}\Lambda$ for both momentum and energy. Finally the third cubic interaction term, the one involving only parallel derivatives, implies a cutoff $\sim \Lambda$ in both momentum and energy. The combined result of this discussion is therefore that the space cutoff remains the same $\sim \Lambda$ but the energy cutoff is significantly lowered due to the scattering of the slow moving modes along the transverse directions. To get a

rough idea, we can for instance apply the above results to the Earth-Moon system, neglecting all other sources of π other than the Earth. For the Earth we have $R_V \sim 10^{19}$ cm and the radius of the Moon orbit around the Earth is $R_M \sim 4 \times 10^{10}$ cm and thus on length scales of the order of R_M the energy cutoff in time units is

$$T_E = \Lambda_E^{-1} = \Lambda^{-1}\left(\frac{R_V}{R_M}\right)^{2/3} \sim 10^3 \text{ s.} \qquad (91)$$

The domain of applicability of our theory is therefore rather limited. Anyway, if the one we just illustrated was the only limitation we would still, at least in principle, be able to defend the estimate of the effects of π on the Moon orbit, which was one of the interesting features in DGP. After all in that case the time scale, a month, is much bigger than T_E. However there are worse limitations (or better complications) in computing the effects of π within the solar system already at the classical level. This is because of the big retardation effects associated with the slow motion of the π excitations. In order to compute the π field generated by the Earth we should first find the π solution due to the Sun, and then solve the nonlinear problem around the Earth. Now, given the Vainshtein radius $R_{V\odot}$ associated with the Sun mass and the Earth orbit radius R_E we have that angular fluctuations propagate with a speed

$$v \sim \frac{R_E}{R_{V\odot}} \times c \sim 10^{-8} c, \qquad (92)$$

which is orders of magnitude smaller that the velocity of the Earth on its orbit. The computation of the π field around the Earth could then not be performed in the static limit. Correspondingly Cerenkov radiation in the transverse direction would instead be emitted due to the motion of the Earth. The basic conclusion is thus that the lack of enhancement of transverse kinetic terms around a spherical source causes trouble. On one side we have a technical trouble at the classical level, as huge retardation effects prevent a simple static computation of the background. On the other side they have a conceptual trouble at the quantum level, due to a rather low cutoff, worse in energy than in momentum, again because of retardation. Maybe our conclusions for the solar system are too drastic in that we have not considered other effects, like that due to the Galaxy, which will enhance the transverse kinetic terms above their $O(1)$ value we used for our study. In principle one could try and estimate these effects. For instance by studying the π background around two separated spherical sources. The lack of spherical symmetry makes this a much tougher problem that the spherical one we studied, and is beyond the scope of the present paper.

C. Special cases

So far we have discussed the most general Lagrangian $\mathcal{L}_\pi$ in Eq. (44), with all the coefficients c_i different from

zero. In the following we examine if restricting to a subclass of these theories can alleviate some of the difficulties found in the previous sections.

From the qualitative analysis of Sec. III B we can see that the presence of the higher-order Galilean terms $\mathcal{L}_{4,5}$ is responsible for the slow motion of the π excitations that in turn restricts the domain of applicability of the theory as one goes to shorter distances. It is interesting then to consider what happens if we choose $d_4 = d_5 = 0$, or equivalent $c_4 = c_5 = 0$. In this case the only interaction left is the cubic one of the DGP model and the kinetic terms K_r, K_t, and K_Ω in the region where classical nonlinearities are important and have an isotropic enhancement $\propto \tilde{y}$. This large kinetic energy suppresses quantum fluctuations φ with the result that the position dependent cutoff increases, for example, on the surface of the Earth we have $\Lambda_\oplus^{-1} \sim 1$ cm [21]. The difference however is that—contrary to DGP—there can still be a self-accelerating solution without ghosts because of the presence in the theory of the tadpole $c_1\pi$. Moreover the conditions (70) can be satisfied when $d_4 = d_5 = 0$.[5]

The case $c_1 = 0$ could be regarded as theoretically preferable, as long as both the Minkowski and de Sitter solutions are stable. For instance in applications to early cosmology this would potentially allow one to describe inflation and big bang cosmology within the same effective theory. Moreover for $c_1 = 0$ the de Sitter solution would arise in the absence of a potential, which is one possible definition of self-acceleration. However one should keep in mind that also the case $c_1 \neq 0$ gives a quite novel way of generating a de Sitter solution: there is, sure, a potential for the scalar π but, unlike the case of a cosmological constant it does not have any, even approximately, stationary points.

Anyway, with the above caveat in mind it makes sense to better analyze the case $c_1 = 0$ to find out if our zeroth order requirements can be satisfied. Can this case be made compatible also with the existence of a spherically symmetric solution? Contrary to the analysis carried out in Sec. II E, when $c_1 = 0$ Eq. (47) gives a constraint on the coefficients d_i: we can rewrite it factoring out an overall H^2 and defining $Q(\frac{1}{2}H^2)$ the resulting polynomial,

$$Q(y) \equiv 4(d_2 + 3d_3y + 6d_4y^2 + 6d_5y^3). \qquad (93)$$

Then, for the background de Sitter solution to exist at all, $Q(y)$ must have a zero at positive y,

$$\exists y_0 > 0 \,|\, Q(y_0) = 0. \qquad (94)$$

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[5]Notice in passing that the possibility to choose also $d_3 = 0$ seems to be compatible with the constraints (70). What goes wrong in this case is that without any nonlinear effects in the dynamics of π we can have de Sitter asymptotics but there are $O(1)$ deviation from GR already at solar-system scales. As we stressed throughout the paper, the implementation of the Vainshtein effect is a necessary condition to reconcile theories with a Lagrangian of the form (14) with present astronomical observations.

If $d_4 = d_5 = 0$, we can see immediately that (94) cannot be satisfied: it is clearly incompatible with the simultaneous requirement of stable fluctuations around de Sitter ($d_2 > 0$) and existence of a radial solution for all r [$d_3 \geq 0$ (60)]. This is the well known lesson of DGP.

However even when d_4, $d_5 \neq 0$ there is some tension between the two algebraic Eqs. (59) and (94). In spite of that, it is easy too see that they can both be satisfied if we add to the conditions (60) the requirement $d_5 < 0$. Therefore when $c_1 = 0$ the presence of the quintic term with a negative coefficient in the Lagrangian becomes crucial to make the de Sitter background and spherically symmetric solutions coexist. Notice that $d_5 < 0$ is also compatible with the requirements of stability and subluminality of the fluctuations (76), at least close to the source.

We can still take $d_4 = 0$ trying to avoid the problem related to the speed of the angular fluctuations. In this situation we have K_r, $K_\Omega \sim \tilde{y}$ while $K_t \sim \tilde{y}^3$. Even before paying attention to the UV cutoff, this case looks right away uninteresting since the retardation effects are even worse than the general case discussed in the previous section. Indeed all modes have a speed $v \sim c/\tilde{y}$. Now, from Eq. (57) at $d_4 = 0$, at a distance r from a spherical source with Vainshtein radius R_V we have $\tilde{y} \sim (R_V/r)^{3/2}$. So in the case of the Earth orbiting the Sun $v \sim 10^{-12}c$.

IV. THE IR COMPLETION

A. Self-acceleration from symmetry

We now consider the issue of whether there exists a "global completion" of the local π dynamics we have been discussing so far. It may be that at large distances the description in terms of π breaks down, like in DGP, where at horizon scales the four-dimensional description is no longer applicable. Here we want to entertain the possibility that instead, the physics at large scales is well described by a four-dimensional Lagrangian for π. To this purpose, it is instructive to characterize the acceleration of the Universe in terms of symmetries. If the Universe asymptotes to a de Sitter phase at late times, it is in fact approaching an enhanced symmetry configuration. Indeed a generic FRW solution is invariant under spatial translations and rotations, whereas de Sitter space is maximally symmetric, with as many isometries as Minkowski space. The cosmological constant is perhaps the most elegant way of realizing such symmetries: having a stress-energy tensor proportional to the metric, it does not introduce any new tensor thus being consistent with the highest possible number of symmetries a metric may have. It is certainly the most economic way, in that it does not introduce any new degree of freedom. *A priori*, a possible alternative is the usual way in which symmetries are realized in field theory: at the level of the Lagrangian, *off shell*. That is, if a solution to the field equations has some symmetries, barring accidents these must be shared by the Lagrangian that

gives rise to such a solution. The most economic possibility in this class is to realize the desired symmetries introducing the least possible number of new degrees of freedom. As we will see shortly, for the de Sitter symmetry group one scalar suffices.

Let us start by considering the limit in which gravity is decoupled $M_{\text{Pl}} \to \infty$. Notice that the full Poincaré-like symmetry group of four-dimensional de Sitter space—the generalization of translations, rotations, and boosts—is $SO(4, 1)$, that is, the group of five-dimensional Lorentz transformations. Indeed D-dimensional de Sitter space can be embedded into $(D + 1)$-dimensional Minkowski space as an hyperboloid, which is a Lorentz-invariant submanifold. This suggests a quick way of constructing 4D theories that realize the de Sitter symmetry group as an internal symmetry: write down the most generic effective Lagrangian for a 4D brane living in 5D Minkowksi space. The presence of the brane itself will necessarily break spontaneously some combinations of 5D translations, rotations, and boosts. The two largest possible residual groups are the four-dimensional de Sitter group $SO(4, 1)$—when the brane is a hyperboloid—and its flat limit, the four-dimensional Poincaré group. The existence of de Sitter solutions will ultimately depend on the Lagrangian, but may be generically expected given the symmetry arguments above. From this geometrical perspective, it is clear that we are introducing just one new 4D scalar—the position of the brane. Notice that in this approach the fifth "extra-dimension" is fictitious—no field propagates in it. It is simply a useful geometric picture of the new degree of freedom. However, if we take it more seriously, we introduce gravity in the 5D ambient space, and we also gauge gravity on the brane, what we get is nothing but the DGP model.[6] From this viewpoint it is not surprising that DGP admits "self-accelerating" de Sitter solutions.

Another possibility, which is the one we will analyze in the following, is to make the 4D Lagrangian conformal invariant. The 4D conformal group is $SO(4, 2)$, whose maximal subgroups are $SO(3, 2)$, four-dimensional Poincaré, and $SO(4, 1)$—the isometry groups of four-dimensional anti-de Sitter, Minkowski, and de Sitter spaces, respectively. It is well known that the spontaneous breakdown of the conformal group to any of these subgroups can be achieved by a single scalar, the dilaton [24]. Indeed the 4D conformal group is the isometry group of 5D anti-de Sitter (AdS) space, and the dilaton can be thought of as the position of a 4D brane in 5D AdS. Then each of the above breaking patterns corresponds to a maximally symmetric brane geometry.

Let us stress again that for the moment we are decoupling gravity. We will see in Sec. IV B that reintroducing dynamical gravity drastically changes the picture at large

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[6]For DGP an orbifold projection is also necessary.

distances. However for the local analysis we are carrying out, at distances much smaller than the Hubble scale, the effects of dynamical gravity are negligible. Let us therefore consider the case in which matter is minimally coupled to the metric

$$g_{\mu\nu} = e^{2\pi}\eta_{\mu\nu}, \qquad (95)$$

where the exponential is just a convenient nonlinear completion of Eq. (13), and we ''froze'' dynamical gravity, $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$. Then π nonlinearly realizes the conformal group:

(i) Dilations

$$D: \pi(x) \rightarrow \pi'(x) \equiv \pi(\lambda x) + \log\lambda; \qquad (96)$$

(ii) Infinitesimal special conformal transformations

$$K_\mu: \pi(x) \rightarrow \pi'(x)$$
$$\equiv \pi(x + (cx^2 - 2(c \cdot x)x)) - 2c_\mu x^\mu; \qquad (97)$$

(iii) Translations

$$P_\mu: \pi(x) \rightarrow \pi'(x) \equiv \pi(x + a); \qquad (98)$$

(iv) Boosts

$$M_{\mu\nu}: \pi(x) \rightarrow \pi'(x) \equiv \pi(\Lambda \cdot x). \qquad (99)$$

The three possible maximally symmetric solutions (AdS, Minkowski, dS) correspond to different unbroken combinations of the above generators. Clearly, Minkowski space corresponds to a trivial (constant) configuration for π, for which translations and boosts are unbroken, whereas dilations and special conformal transformations are spontaneously broken. More interesting are the other solutions. The de Sitter solution is [24]

$$e^{\pi_{\text{dS}}} = \frac{1}{1 + \frac{1}{4}H^2 x^2}, \qquad (100)$$

which manifestly leaves boosts unbroken. It is also invariant under the combined (infinitesimal) action of K_μ and P_μ,

$$S_\mu = P_\mu - \frac{1}{4}H^2 K_\mu: \pi(x) \rightarrow \pi'(x)$$
$$\equiv \pi(x + a - \frac{1}{4}H^2(ax^2 - 2(a \cdot x)x)) + \frac{1}{2}H^2 a_\mu x^\mu, \qquad (101)$$

as can be straightforwardly checked. Indeed the four generators S_μ together with the boost generators $M_{\mu\nu}$ form the full algebra of $SO(4, 1)$, the de Sitter symmetry group. Likewise, the AdS configuration corresponds to

$$e^{\pi_{\text{AdS}}} = \frac{1}{1 - \frac{1}{4}k^2 x^2}, \qquad (102)$$

which preserves boosts as well as the combination $R_\mu = P_\mu + \frac{1}{4}k^2 K_\mu$. The unbroken symmetries make up the AdS symmetry group, $SO(3, 2)$ [24].

If we start from a conformally-invariant Lagrangian for π, we may generically expect one of these configurations to describe the vacuum. Which one will be a solution to the dynamics, and with what curvature scale, clearly depends on the Lagrangian. However, we just have to ask whether we can generate all the Galilean-invariant terms we need for a local de Sitter solution to exist, be stable, and admit stable spherically symmetric perturbations, starting from a conformally-invariant Lagrangian. If we can, then by symmetry the de Sitter solution will exist globally, and will have the same local properties (stability, etc.) everywhere.

To construct conformally-invariant Lagrangians, we simply write down all possible diff-invariant Lagrangian terms involving the fictitious metric (95). In fact, the conformal group is a subgroup of diffeomorphisms—the subgroup that leaves the metric conformally flat. The simplest term is obviously

$$\int d^4x \sqrt{-g} = \int d^4x e^{4\pi}, \qquad (103)$$

which corresponds, upon linearization in π to the term proportional to π in the Galileon Lagrangian of the previous sections. It is clearly conformal invariant, upon changing the integration variable. For instance, under special conformal transformations, Eq. (97), we get

$$\int d^4x e^{4\pi(x)} \rightarrow \int d^4x e^{4\pi(x+(cx^2-2(c\cdot x)x))}e^{-8c_\mu x^\mu}$$
$$= \int d^4x' e^{4\pi(x')}. \qquad (104)$$

The next terms, in a derivative expansion, are the curvature invariants R, R^2, $R_{\mu\nu}R^{\mu\nu}$, $R^\mu{}_{\nu\rho\sigma}R_\mu{}^{\nu\rho\sigma}$, and so on. Let us look more closely at their structure in terms of π. The Riemann tensor for the conformally flat metric Eq. (95) is schematically

$$R^\mu{}_{\nu\rho\sigma} \sim \partial\pi\partial\pi + \partial^2\pi. \qquad (105)$$

Thus a generic nth order curvature invariant is of the form

$$\int d^4x \sqrt{-g} R^n \sim \int d^4x e^{(4-2n)\pi}(\partial\pi\partial\pi + \partial^2\pi)^n$$
$$\sim \int d^4x e^{(4-2n)\pi}[(\partial\pi\partial\pi)^n + \dots$$
$$+ \partial\pi\partial\pi(\partial^2\pi)^{n-1} + (\partial^2\pi)^n]. \qquad (106)$$

The Galilean invariants we constructed in the previous sections correspond to the second-last term in Eq. (106). The terms with more π's and fewer derivatives per π—those on the left of the Galilean one in Eq. (106)—are less important at short distances and small values of π. For instance on the de Sitter configuration $\pi \simeq -\frac{1}{4}H^2 x^2$, the relative importance of one such term with respect to the

Galilean one is

$$\frac{(\partial\pi\partial\pi)^m(\partial^2\pi)^{n-m}}{\partial\pi\partial\pi(\partial^2\pi)^{n-1}} \sim (H^2x^2)^{m-1}, \quad (107)$$

which is negligible at subhorizon scales. The same conclusion obviously holds for the exponential prefactor in Eq. (106), which at lowest order we can set to one. In fact at short distances and at small π values, conformal invariance reduces to Galilean invariance. This is manifest from expanding dilations [Eq. (96)] and special conformal transformations [Eq. (97)] at lowest order in π and x.

However by the same token, at short distances and for small π the Galilean term in Eq. (106) is negligible with respect to the one on its right, which has one less π and two derivatives acting on every π. For instance their ratio on the de Sitter solution is of order H^2x^2. The terms with two derivatives per π are exactly the terms we did not want in Sec. II D, because they lead to fourth-order equations of motion, and thus to ghosts. Therefore, we need to find combinations of curvature invariants for which these unwanted higher-derivative contributions vanish. For such combinations, the most important Lagrangian terms at short distances will be the Galilean ones, and our local analysis of the previous sections will apply. Since the tuning that sets to zero the $(\partial^2\pi)^n$ terms correspond to removing a degree of freedom, terms of this type (by definition of perturbation theory) will consistently be generated at the quantum level only with a coefficient with a small enough size that the ghost will be at the cutoff. This issue is discussed in Ref. [21].

The first curvature invariant is

$$\int d^4x\sqrt{-g}R = \int d^4x e^{2\pi}[-6(\partial\pi)^2 - 6\square\pi]. \quad (108)$$

Notice that here we do not need (and we could not either) cancel the $\square\pi$ term. Indeed integrating by parts

$$e^{2\pi}\square\pi \longrightarrow -2e^{2\pi}(\partial\pi)^2, \quad (109)$$

and so Eq. (108) is nothing but the π kinetic term.

The next invariant is

$$\int d^4x\sqrt{-g}[aR^2 + bR_{\mu\nu}R^{\mu\nu} + cR^\mu{}_{\nu\rho\sigma}R_\mu{}^{\nu\rho\sigma}]. \quad (110)$$

Here we have a complication. Upon integration by parts, the three terms above yield all the same structure in terms of π

$$\sqrt{-g} \times R^2, (R_{\mu\nu})^2, (R^\mu{}_{\nu\rho\sigma})^2$$
$$\propto [(\partial\pi)^4 + 2(\partial\pi)^2\square\pi + (\square\pi)^2], \quad (111)$$

where $(\partial\pi)^4 \equiv ((\partial\pi)^2)^2$. Hence, there is no way of tuning the coefficients in Eq. (110) as to make the $(\square\pi)^2$ term vanish while keeping a nontrivial Galilean term. However this is an accident peculiar to 4D. We can ask whether there is a suitable combination of invariants in d dimensions, for

which the $(\square\pi)^2$ vanishes and which is regular and nontrivial in the $d \to 4$ limit—a sort of derivative with respect to d. In d dimensions the Ricci tensor and scalar read

$$R_{\mu\nu} = (d - 2)\partial_\mu\pi\partial_\nu\pi - (d - 2)\eta_{\mu\nu}(\partial\pi)^2$$
$$- (d - 2)\partial_\mu\partial_\nu\pi - \eta_{\mu\nu}\square\pi \quad (112)$$

$$R = -e^{-2\pi}[(d - 1)(d - 2)(\partial\pi)^2 + 2(d - 1)\square\pi] \quad (113)$$

(we will not need the Riemann tensor). Then for instance the combination

$$\int d^dx\sqrt{-g_d}\frac{1}{d - 4}\left[\frac{R_{\mu\nu}R^{\mu\nu}}{d - 1} - \frac{R^2}{(d - 1)^2}\right] = \quad (114)$$

$$= \int d^dx e^{(d-4)\pi}\left[\frac{(d - 2)(3d - 4)}{2(d - 1)}(\partial\pi)^2\square\pi\right.$$
$$\left. + \frac{(d - 2)^3}{2(d - 1)}(\partial\pi)^4 + (\square\pi)^2\right] \quad (115)$$

is regular for $d \to 4$, and linearly independent of Eq. (111). By combining Eqs. (111) and (114) we can tune to zero the $(\square\pi)^2$ while retaining the Galilean interaction [as well as the subleading $(\partial\pi)^4$ term].

Alternatively, we can shelve for a second the auxiliary metric $g_{\mu\nu} = e^{2\pi}\eta_{\mu\nu}$, and look for invariants in 4D directly in terms of π. At the four-derivative level (which corresponds to R^2 etc.), invariance under dilations [Eq. (96)] tells us that there is no $e^{n\pi}$ overall factor. Then, under special conformal transformations [Eq. (97)] the individual terms transform as

$$\int d^4x(\partial\pi)^4 \to \int d^4x[(\partial\pi)^4 - 8c^\mu\partial_\mu\pi(\partial\pi)^2] \quad (116)$$

$$\int d^4x(\partial\pi)^2\square\pi \to \int d^4x[(\partial\pi)^2\square\pi + 4c^\mu\partial_\mu\pi(\partial\pi)^2$$
$$- 4c^\mu\partial_\mu\pi\square\pi] \quad (117)$$

$$\int d^4x(\square\pi)^2 \to \int d^4x[(\square\pi)^2 + 8c^\mu\partial_\mu\pi\square\pi] \quad (118)$$

where on the right-hand sides we changed the integration variable, $x^\mu \to x^\mu + (c^\mu x^2 - 2(c \cdot x)x^\mu)$. We thus see that at the Lagrangian level the only invariant combination is precisely Eq. (111). However, if we require that the EOM be invariant, the Lagrangian need only be invariant up to a total derivative. Then the $(\square\pi)^2$ term is invariant by itself, since $\partial_\mu\pi\square\pi$ is a total derivative; see, e.g., Eq. (32). This means that we can set it to zero while preserving conformal invariance.

Notice in passing that Lagrangian terms of the form $e^{(4-2n)\pi}(\partial^2\pi)^m(\partial\pi)^{2n-2m}$ are automatically invariant under dilations, Eq. (96), whereas as we saw only some very specific combinations of them are invariant under special conformal transformations as well. This goes

against the common expectation that scale invariance alone in enough to guarantee full conformal invariance. In fact, the standard arguments do not apply for higher-derivative terms; see, e.g., Ref. [25].

At the next orders, R^3 and R^4, there are no complications. Suitable combinations of curvature invariants involving just the Ricci tensor and scalar give the Galilean terms we want, while tuning the unwanted $(\Box\pi)^3$ and $(\Box\pi)^4$ terms to zero. At order R^3 we get (after integration by parts)

$$\sqrt{-g}\left[\frac{7}{36}R^3 + R(R_{\mu\nu})^2 - (R_{\mu\nu})^3\right] = e^{-2\pi}[24\mathcal{L}_4 + \ldots],$$

(119)

where $\mathcal{L}_4$ is the fourth-order Galilean invariant, Eq. (37), and the dots stand for terms with fewer derivatives per π, which are negligible at short distances. Likewise, at order R^4

$$\sqrt{-g}\left[\frac{93}{2\cdot 6^4}R^4 - \frac{39}{4\cdot 6^2}R^2(R_{\mu\nu})^2 + \frac{5}{12}R(R_{\mu\nu})^3\right.$$
$$\left. + \frac{3}{16}((R_{\mu\nu})^2)^2 - \frac{3}{8}(R_{\mu\nu})^4\right] = e^{-4\pi}[16\mathcal{L}_5 + \ldots].$$

(120)

In conclusion, we can reproduce any Galilean-invariant Lagrangian of the form Eq. (44) as the short-distance limit of a globally defined, conformally-invariant Lagrangian for π. Then, the existence of a de Sitter solution as well its properties (stability, etc.) can be studied locally, about any point, as we did in the previous section. The symmetries of the full Lagrangian guarantee that the de Sitter solution we find locally can be promoted to a global one, and the isometries of de Sitter space guarantee that the stability and the other local properties of the solution will be the same throughout space.

B. Gravitational backreaction

We now consider the (phenomenologically interesting) case where gravity is dynamical, $M_{\rm Pl} < \infty$. In our notation, this amounts to changing the metric seen by matter, Eq. (95) into

$$g_{\mu\nu} = e^{2\pi}\hat{g}_{\mu\nu},$$

(121)

of which Eq. (13) is the small field limit, and making $\hat{g}_{\mu\nu}$ dynamical. Also we should covariantize the π action, which corresponds to coupling π to $\hat{g}_{\mu\nu}$. As usual there are many inequivalent ways to do so, parametrized by nonminimal couplings of π to curvature invariants. Now, if we want our de Sitter solution to survive after the coupling to gravity is taken into account, we have to require that gravitational interactions do not spoil the conformal invariance of the π action—so that the existence of a de Sitter solution is guaranteed by symmetry. But the only way to couple a conformal theory to gravity while

preserving conformal invariance is imposing that the action be invariant under Weyl rescalings of the metric:

$$\hat{g}_{\mu\nu}(x) \rightarrow \hat{g}'_{\mu\nu}(x) = e^{2\omega(x)}\hat{g}_{\mu\nu}(x).$$

(122)

If we do so, the dilaton of the conformal field theories and the metric have to appear precisely in the combination (121), so that a conformal transformation on π may be reabsorbed into a Weyl transformation of $\hat{g}_{\mu\nu}$. This corresponds to gauging the conformal group. In fact, this is how we constructed the conformally-invariant action for π in the first place—by making the conformal group that acts on π a subgroup of diffeomorphisms acting on $g_{\mu\nu}$. The full action is then the sum of the manifestly covariant operators of the last section, where now $R^\mu{}_{\nu\rho\sigma}$ stands for the Riemann tensor of the metric (121).[7] As a result, our dilaton π disappears from the action—nowhere do π and $\hat{g}_{\mu\nu}$ appear separately from each other. In fact, via a Weyl transformation we can choose "unitary gauge,"

$$\pi = 0, \qquad g_{\mu\nu} = \hat{g}_{\mu\nu}.$$

(124)

The action now is just a higher-derivative action for the metric, coupled to matter. The problem with this action, is that the energy scale that "suppresses" the higher-derivative terms is minuscule. Indeed, in the presence of an Einstein-Hilbert term $M_{\rm Pl}^2 R$, in order to have a de Sitter solution with curvature $\sim H_0^2$, we need either a nonderivative operator

$$\sqrt{-g}\Lambda \sim \sqrt{-g}M_{\rm Pl}^2 H_0^2,$$

(125)

which is not surprising news, or higher-order curvature invariants weighed by inverse powers of H_0^2,

$$\sqrt{-g}\frac{M_{\rm Pl}^2}{H_0^{2n-2}}R^n.$$

(126)

The latter possibility, which is the nonconventional one, corresponds to a *ultraviolet* modification of GR at curvatures of order H_0^2, which, even at the classical level, is clearly ruled out by experiments.

We are thus led to the conclusion that viable gravitational interactions of π cannot preserve conformal invariance—a conclusion that is also strengthened by Weyl invariances being generically anomalous at the quantum

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[7]The combination in Eq. (114) is nonsingular in $d = 4$ only for conformally flat metrics, as manifest from its left-hand side. This means that the DGP-like term

$$2(\partial\pi)^2\Box\pi + (\partial\pi)^4$$

(123)

cannot be coupled to gravity in a Weyl invariant fashion. Given that, as we will see shortly, the Weyl invariant couplings we are constructing are not viable, we do not investigate this issue further. Also notice that when using a generic metric $g_{\mu\nu}$ rather than the conformally flat one of the last section, we have new independent higher-order curvature invariants, involving for instance the Riemann tensor, which for conformally flat metrics is instead determined by the Ricci tensor.

level. As a consequence, the de Sitter solution we found for π in general will be spoiled when the gravitational back-reaction is taken into account. Also notice that the problem of having higher-derivative terms for the metric with huge coefficients like in Eq. (126), will be shared by a large class of Lagrangians where π has important nonminimal couplings to $\hat{g}_{\mu\nu}$. The safest possibility from this viewpoint seems to be minimal coupling—it involves the least number of derivatives of $\hat{g}_{\mu\nu}$ compatible with diff-invariance. Let us therefore consider the case of minimal coupling. This amounts to covariantize the π action just by contracting Lorentz indices with $\hat{g}^{\mu\nu}$, replacing ordinary derivatives with covariant ones, and integrating in $d^4x\sqrt{-\hat{g}}$. Also, we include an Einstein term $M_{\mathrm{Pl}}^2\hat{R}$, to reproduce Newton's law. Scaling the various terms in the Galilean limit like in Eq. (77) and using the results of Sec. IV A for their relation to the other terms, the nth order Lagrangian term is schematically

$$\mathcal{L} \sim \frac{f^2}{H_0^{2n-2}} e^{(4-2n)\pi}(\partial\pi\partial\pi + \partial\partial\pi)^n. \quad (127)$$

Now, as we saw, a de Sitter configuration for π corresponds to Eq. (100). For cosmology, it is more convenient to use coordinates where the solution only depends on (conformal) time,

$$e^{\pi_{\mathrm{dS}}} = -\frac{1}{H_0\eta}. \quad (128)$$

The relation between this and Eq. (100) is just a special conformal transformation on π combined with a translation. However, as we discussed, when gravity is taken into account conformal transformations are no longer a symmetry. Therefore the two expressions for the de Sitter configuration are physically inequivalent—i.e., they produce different gravitational fields. If we are looking for FRW solutions for the metric, we have to stick to Eq. (128). The stress-energy tensor scales like the Lagrangian (127), which for our de Sitter configuration gives

$$T_{\mu\nu} \sim \frac{1}{\eta^4}\frac{f^2}{H_0^2}. \quad (129)$$

The curvature perceived by matter is a combination of that due to its conformal coupling to π, and that of $\hat{g}_{\mu\nu}$, which in turn is sourced by the π stress-energy tensor:

$$R_{\mu\nu} \sim \partial\pi\partial\pi + \partial\partial\pi + \hat{R}_{\mu\nu} \quad (130)$$

$$\sim \frac{1}{\eta^2}\left[1 + \frac{f^2}{M_{\mathrm{Pl}}^2}\frac{1}{H_0^2\eta^2}\right]. \quad (131)$$

The latter contribution becomes sizable and spoils the de Sitter geometry at conformal times

$$|\eta| \lesssim H_0^{-1}\cdot\frac{f}{M_{\mathrm{Pl}}}. \quad (132)$$

If we could make f smaller and smaller, we could postpone this moment at will. However we have already shown that tests of GR at subcosmological scales constrain f to be not significantly smaller than M_P. So we cannot play with f to make the solutions more stable. Anyway, since proper time t depends only logarithmically on η, this means that the geometry will look approximately de Sitter only for times of order Hubble. That is, gravitational interactions necessarily spoil the de Sitter solution at Hubble scales.

V. DISCUSSION AND OUTLOOK

We conclude with a few remarks on our results and on possible future developments. First, let us summarize our findings. We have considered theories that modify the infrared behavior of gravity where, at least in a local patch smaller than the cosmological horizon:

(i) The modification of gravitational interactions is due to an effective relativistic scalar degree of freedom π kinetically mixed with the metric, or equivalently, universally coupled to matter via the interaction $\pi T^\mu{}_\mu$.[8]

(ii) This scalar degree of freedom generically decouples from matter at shorter scales thanks to important derivative self-interactions (Vainshtein effect).[9]

In this class of theories, we have argued that the dynamics of π must enjoy an internal Galilean symmetry. In 4D there are only five possible Lagrangian terms with such a symmetry. This makes a thorough analysis of very symmetric solutions possible. First, we demanded the existence and stability of a de Sitter solution in the absence of a cosmological constant—a self-accelerating solution *à la* DGP, but with no ghost instabilities. Then, we imposed that with these de Sitter asymptotics there exist spherically symmetric Vainshtein-like solutions, describing the π configuration about compact sources, and that these configurations are also stable against small perturbations. We found some tension, if no contradiction, in satisfying these requirements. It is plausible—although we have no proof of this—that demanding the existence and stability of more complicated solutions, like, e.g., with several compact sources in generic positions, may lead to contradictions. However, we also found that small excitations are *forced* to be superluminal at *large* distances from a compact source,

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[8]This is the case for instance in Fierz-Pauli massive gravity and in the DGP model, whereas massive gravity theories that crucially rely on spontaneous breaking of Lorentz invariance are left out.

[9]Some form of nonlinearities are necessary for screening a universally coupled scalar at short scales, thus recovering agreement with solar-system tests of GR. The only robust alternative to the Vainshtein effect we are aware of is the Chameleon mechanism [26]. There, the nonlinearities are in the coupling of the scalar to matter, rather than in its self-interactions.

where π is in the linear regime, even if we do not require the existence of self-accelerating solutions.

We ran into trouble also as a result of the opposite problem: the very slow speed of angular fluctuations when approaching the source due to the presence of the Galilean terms $\mathcal{L}_{4,5}$. This makes a static computation of the background not trustable because of huge retardation effects, which are also at the origin of the very low-energy cutoff of the theory. To avoid these difficulties we considered the subclass of Lagrangians with $c_4 = c_5 = 0$. This requirement is still compatible with the absence of ghosts around the de Sitter background and with the existence a stable Vainshtein-like solution around compact sources. The kinetic term in the angular directions is now enhanced, as a result quantum fluctuations are suppressed and the effective cutoff increases like in DGP above the naive estimate $\Lambda \sim (1000 \text{ km})^{-1}$.

Finally, we considered whether our local Galileon π can be promoted to a global scalar degree of freedom, coupled someway to four-dimensional GR. First, neglecting gravity, we promoted the Poincaré group and our local Galilean invariance to the conformal group, with π as the dilaton. The de Sitter group is one of the maximal subgroups of the conformal group, and the existence of de Sitter-like solutions for π may be generically expected by symmetry considerations. We showed that our local Galilean-invariant terms are in fact the short-distance limit of globally defined, conformally-invariant Lagrangian terms for π. The problem however is in coupling this system to gravity. This necessarily breaks conformal invariance, and as a consequence spoils the de Sitter geometry at times of order of the Hubble time, and longer. Although this is not a disaster—in fact this model may well yield a realistic cosmology, for we have no idea of whether our Universe's future resembles de Sitter space—the symmetry motivation for the "conformal completion" of the Galilean symmetry is somewhat lost.

Let us stress once again that our local analysis of the Galilean model is quite generic—it applies to all theories obeying conditions (i) and (ii) above. Therefore, the partial failure of our attempt to "IR-complete" the Galilean model should not be seen as impairing the local analysis. Our local analysis applies for instance to the DGP model—as we showed in Sec. II C. DGP at short distances corresponds to a Galilean model with nonvanishing c_2 and c_3 only. There however, it is manifest that the IR completion is not a four-dimensional local theory—it is in fact a five-dimensional one, and our π degree of freedom stops making sense at scales of order of the cosmological horizon, where its interactions and mixings with the whole continuum of Kaluza-Klein modes cannot be neglected anymore. We do not know whether this is going to be the case for a generic Galilean model, or whether instead there exist purely four-dimensional local IR completions that realize the de Sitter symmetry group as a subgroup of a larger

internal symmetry group, other than the conformal one. Perhaps, as we suggest in Sec. IV A, there exists one such theory where the internal symmetry group is the 5D Poincaré group. This in a sense would yield a purely four-dimensional version of DGP, without the extra complications of 5D gravity.

On the other hand, since DGP locally reduces to a particular Galilean model, perhaps a generic one can be IR completed by a generalization of DGP. Possible generalizations include higher-than-five dimensional setups [27], or the standard five-dimensional setup with nonminimal bulk or brane operators. In the latter case, we might try adding higher-curvature invariants in the bulk, but in order to keep second-order EOM for π—a crucial feature of our Galilean-invariant Lagrangians—we are only allowed to use Lovelock combinations. But in 5D we just have one such combination, whereas to reproduce a generic Galilean model we need several independent terms. The other possibility is to add localized terms on the brane involving powers of the extrinsic curvature. Indeed for small gravitational fields, the extrinsic curvature is $K_{\mu\nu} \propto \partial_\mu \partial_\nu \pi$, so such terms will directly yield derivative self-interactions for π. The problem with this is that they involve too many derivatives—two per π field. Another possibility would be to add new five-dimensional degrees of freedom besides gravity, or, as we said, to consider higher-dimensional spacetimes. We leave this for future investigations.

We now turn our attention to the issue of the *ultraviolet* completion. From this viewpoint, our model is just a generalization of the 4D effective action of DGP that gets rid of the ghost on the self-accelerated branch, but that nevertheless shares with DGP the other pathologies identified in Ref. [28]. Namely, (a) the superluminalilty of scalar excitations about some physically relevant solutions, and (b) the excessive softness of certain scattering amplitudes at low energies. These features are in fact related to each other, and although they do not necessarily signal any inconsistency of the low-energy effective theory—unless closed timelike curves may form—they do indicate that our model cannot arise as the low-energy description of a Lorentz-invariant microscopic theory obeying the standard axioms of the S-matrix theory, like, e.g., any renormalizable relativistic quantum field theory [28]. This is to be contrasted with what happens for GR. The Weinberg-Witten theorem makes the quest for a UV completion of GR within local Lorentz-invariant quantum field theories particularly hard—perhaps hopeless [29]. Nevertheless, looking at GR as a perturbative Lorentz-invariant effective theory of gravitons living in Minkowski space, one finds that gravitational interactions cannot lead to superluminal propagation with respect to the underlying Minkowski light cone as long as sources obey the null energy condition [28,30]. Therefore, although probably not local, the UV completion of GR may well be Lorentz invariant (like, e.g., string theory).

From a more theoretical perspective, we have constructed a class of models with peculiar derivative interactions. On the one hand, π's self-interactions involve many derivatives, more than one derivative per field. On the other hand, the particular combinations we isolated yield second-order equations of motion. At the classical level, this means that no new ghostlike degree of freedom shows up in the nonlinear regime. This is a necessary condition for the quantum effective theory not to break down for sizable classical nonlinearities. In fact with a consistent, if not generic choice of counterterms, our effective theories can be extrapolated deep into the nonlinear regime [21]. In a forthcoming publication, we show that these peculiarities allow our models to violate the null energy condition with no manifest pathologies in the low-energy theory. In particular, our models subtly evade the theorems of refs. [7,31].

ACKNOWLEDGMENTS

APPENDIX A: EXISTENCE AND UNIQUENESS OF GALILEAN-INVARIANT TERMS AT EACH ORDER IN π

As we discussed in Sec. II D, the generic structure of a Galilean-invariant Lagrangian term with $n + 1$ π factors is $(\partial^2\pi)^{n-1}\partial\pi\partial\pi$ and the corresponding equation of motion has exactly two derivatives acting on every π, $(\partial\partial\pi)^n = 0$. However, because of the shift symmetry of the π Lagrangian, the EOM for π can also be written as the divergence of the conserved current associated to this symmetry, schematically: $\partial^\mu(\partial^{2n-1}\pi^n)_\mu = (\partial\partial\pi)^n = 0$. This equivalence can be read as the statement that the combination $(\partial\partial\pi)^n$ that appears in the EOM derived from a Galilean-invariant term is a total derivative.

In the following we show that there exists one and only one total derivative $(\partial\partial\pi)^n$ for every n; for this reason there can be at most one Galilean-invariant term at each order and eventually we conclude that such a term exists for every n by constructing it explicitly, starting from the corresponding total derivative.

First of all, notice that it is easy to write down a total derivative at the nth order:

$$\sum_p (-1)^p \eta^{\mu_1 p(\nu_1)} \dots \eta^{\mu_n p(\nu_n)} \partial_{\mu_1}\partial_{\nu_1}\pi \dots \partial_{\mu_n}\partial_{\nu_n}\pi$$

$$\equiv T^{\mu_1\nu_1\dots\mu_n\nu_n} \partial_{\mu_1}\partial_{\nu_1}\pi \dots \partial_{\mu_n}\partial_{\nu_n}\pi, \qquad (A1)$$

here the sum runs over all the permutations p of the indices

ν_i and $(-1)^p$ is the parity of the permutation. We now prove that this is the only one for a given n. In what follows we use the same notation of Sec. II D: Π is the matrix of second derivatives of π, $\Pi^\mu_\nu = \partial^\mu\partial_\nu\pi$ and the brackets $[\mathcal{M}]$ denote the trace of the matrix $\mathcal{M}$, hence $[\Pi^n]$ is the cyclic contraction of order n.

Suppose that there are two total derivatives with the same n. Take the linear combination $\mathcal{L}_n$ of the two such that the terms with the highest number $m \leq n - 1$ of $\Box\pi$ are $c_m^i (\Box\pi)^m [\Pi^{i_1}]\dots[\Pi^{i_k}]$ with $i_1 + \dots + i_k = n - m$ and $i_j \geq 2$ for $j = 1, \dots, k$: actually we can always cancel in $\mathcal{L}_n$ at least the term $(\Box\pi)^n$. The multi-index i counts the different contractions that can appear in $(\partial\partial\pi)^{n-m}$.

Now if we consider the combination $\mathcal{L}_n$ as a Lagrangian density, by definition of total derivative the equation of motion for π that we get from $\mathcal{L}_n$ must vanish. Let us see what are the consequences of this condition on the coefficients c_m^i. To find the EOM we can integrate by parts twice and isolate a π shifting the two derivatives acting on it to all the other Π in turn, in particular, when we choose a π in the jth cyclic contraction and we let the two derivatives act on the π close to the one we are starting with in the same cycle, we end up with the structure $[\Box\Pi\dots\Pi^{i_j-2}]$. Therefore among many terms, $\mathcal{L}_n$ will give a contribution to the EOM proportional to

$$c_m^i (\Box\pi)^m [\Pi^{i_1}]\dots[\Pi^{i_{j-1}}][\Box\Pi\dots\Pi^{i_j-2}][\Pi^{i_{j+1}}]\dots[\Pi^{i_k}]. \qquad (A2)$$

In order to have a vanishing equation of motion, this must be cancelled by a contribution coming from another term in $\mathcal{L}_n$. The only possible contraction that gives something proportional to (A2) in the equation of motion would be given by

$$\mathcal{L}_n \supset (\Box\pi)^{m+1} [\Pi^{i_1}]\dots[\Pi^{i_{j-1}}][\Pi^{i_j-1}][\Pi^{i_{j+1}}]\dots[\Pi^{i_k}]. \qquad (A3)$$

In fact if we integrate by parts and we shift the $\Box$ of the $(m + 1)$th $\Box\pi$ to one of the Π in the jth cycle we get exactly $\pi\dots$ (A2): this is the only other possibility to have in the EOM a $\Box$ in the structure $[\Box\Pi\dots\Pi^{i_j-2}]$, maintaining at the same time $(\Box\pi)^m$ and the cycles $i_1, \dots, i_{j-1}$, $i_{j+1}, \dots, i_k$.

However, terms with $(\Box\pi)^{m+1}$ like (A3) are absent from $\mathcal{L}_n$ by construction, and as a consequence $\mathcal{L}_n$ can be a total derivative only if $c_m^i = 0$. By induction on m we conclude that the whole $\mathcal{L}_n$ must be zero: the two total derivatives we started with are the same apart from an irrelevant overall constant factor. In other words, the structure $(\partial\partial\pi)^n$ can be a total derivative only if it contains $\Box\pi$ to all powers from 0 to n, and this can happen only once for a given n.

Since we proved that the only total derivative for every n is given by (A1), the last step of our argument is to show that this specific contraction can be always derived as the equation of motion of a Galilean-invariant term. The

Galilean term can be constructed by replacing in turn each derivative ∂_{μ_i} with $\partial_{\mu_i}\pi$:

$$\mathcal{L}_{\text{Galilean}} = T^{\mu_1 \nu_1 \dots \mu_n \nu_n}(\partial_{\mu_1}\pi \partial_{\nu_1}\pi \partial_{\mu_2}\partial_{\nu_2}\pi \dots \partial_{\mu_n}\partial_{\nu_n}\pi$$
$$+ \dots + \partial_{\mu_1}\partial_{\nu_1}\pi \dots \partial_{\mu_{n-1}}\partial_{\nu_{n-1}}\pi \partial_{\mu_n}\pi \partial_{\nu_n}\pi). \tag{A4}$$

It is easy to check that the equation of motion we find starting from (A4) is precisely $-2nT^{\mu_1 \nu_1 \dots \mu_n \nu_n} \partial_{\mu_1}\partial_{\nu_1}\pi \dots \partial_{\mu_n}\partial_{\nu_n}\pi = 0$; in fact terms with 3 or 4 derivatives acting on the same π must have 2 derivatives with a ν index like $\partial_{\nu_i}\partial_{\nu_j}\partial_{\mu_k}\pi$ or $\partial_{\nu_i}\partial_{\nu_j}\partial_{\mu_k}\partial_{\mu_l}\pi$ and because of the antisymmetry of $T^{\mu_1 \nu_1 \dots \mu_n \nu_n}$ in the exchange of $\nu_i \leftrightarrow \nu_j$ due to the factor $(-1)^p$, they are all zero.

APPENDIX B: $c_i \leftrightarrow d_i$

We want to expand the equation of motion about the de Sitter solution,

$$\pi \to -\tfrac{1}{4}H^2 x_\mu x^\mu + \pi,$$
$$\partial_\mu \partial_\nu \pi \to -\tfrac{1}{2}H^2 \eta_{\mu\nu} + \partial_\mu \partial_\nu \pi. \tag{B1}$$

Each Galilean term (39)–(43) will then yield a linear combination of itself and the lower-order ones. The contributions to the tadpole $\mathcal{E}_1$ coming from all the terms will cancel—since we are assuming that the de Sitter background is a solution to the EOM We will therefore neglect the tadpole and concentrate on the other terms. Upon the replacement (B1) we get

$$\mathcal{E}_2 \to \mathcal{E}_2 \tag{B2}$$

$$\mathcal{E}_3 \to \mathcal{E}_3 - 3H^2 \mathcal{E}_2 \tag{B3}$$

$$\mathcal{E}_4 \to \mathcal{E}_4 - 3H^2 \mathcal{E}_3 + \tfrac{9}{2}H^4 \mathcal{E}_2 \tag{B4}$$

$$\mathcal{E}_5 \to \mathcal{E}_5 - 2H^2 \mathcal{E}_4 + 3H^4 \mathcal{E}_3 - 3H^6 \mathcal{E}_2 \tag{B5}$$

Therefore, comparing Eq. (45) with Eq. (49) we have

$$\begin{bmatrix} d_2 \\ d_3 \\ d_4 \\ d_5 \end{bmatrix} = \begin{bmatrix} 1 & -3H^2 & \tfrac{9}{2}H^4 & -3H^6 \\ 0 & 1 & -3H^2 & 3H^4 \\ 0 & 0 & 1 & -2H^2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix}. \tag{B6}$$

The inverse relation is

$$\begin{bmatrix} c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} 1 & 3H^2 & \tfrac{9}{2}H^4 & 3H^6 \\ 0 & 1 & 3H^2 & 3H^4 \\ 0 & 0 & 1 & 2H^2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} d_2 \\ d_3 \\ d_4 \\ d_5 \end{bmatrix}. \tag{B7}$$

Notice that the two matrices that describe the mappings $\{c\} \to \{d\}$ and $\{d\} \to \{c\}$ are exactly the same, upon replacing H^2 with $-H^2$. This had to be expected, for we can think of the trivial configuration as an excitation of the de Sitter one with $\pi = +\tfrac{1}{4}H^2 x^2$. Then, like in (B1), to get the equation of motion for π fluctuations about the trivial configuration we have to perform the replacement

$$\pi \to +\tfrac{1}{4}H^2 x_\mu x^\mu + \pi \tag{B8}$$

everywhere in the π EOM about the de Sitter solution. Of course this way we will not recover the tadpole term, which we can however reconstruct by imposing the existence of a de Sitter solution with given H^2.

[1] A. G. Riess *et al.* (Supernova Search Team Collaboration), Astron. J. **116**, 1009 (1998); S. Perlmutter *et al.* (Supernova Cosmology Project Collaboration), Astrophys. J. **517**, 565 (1999).
[2] S. Weinberg, Phys. Rev. Lett. **59**, 2607 (1987).
[3] N. Arkani-Hamed, H. C. Cheng, M. A. Luty, and S. Mukohyama, J. High Energy Phys. 05 (2004) 074.
[4] V. A. Rubakov, arXiv:hep-th/0407104.
[5] S. L. Dubovsky, J. High Energy Phys. 10 (2004) 076.
[6] G. R. Dvali, G. Gabadadze, and M. Porrati, Phys. Lett. B **485**, 208 (2000).
[7] S. Dubovsky, T. Gregoire, A. Nicolis, and R. Rattazzi, J. High Energy Phys. 03 (2006) 025.
[8] V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. **125B**, 139 (1983).
[9] G. Dvali, G. Gabadadze, and M. Shifman, Phys. Rev. D **67**, 044020 (2003).
[10] C. Deffayet, Phys. Lett. B **502**, 199 (2001); C. Deffayet, G. R. Dvali, and G. Gabadadze, Phys. Rev. D **65**, 044023 (2002).
[11] A. I. Vainshtein, Phys. Lett. **39B**, 393 (1972).
[12] H. van Dam and M. J. Veltman, Nucl. Phys. **B22**, 397 (1970); V. I. Zakharov, JETP Lett. **12**, 312 (1970).
[13] N. Arkani-Hamed, H. Georgi, and M. D. Schwartz, Ann. Phys. (N.Y.) **305**, 96 (2003).
[14] M. A. Luty, M. Porrati, and R. Rattazzi, J. High Energy Phys. 09 (2003) 029.
[15] P. Creminelli, A. Nicolis, M. Papucci, and E. Trincherini, J. High Energy Phys. 09 (2005) 003.
[16] C. Deffayet and J. W. Rombouts, Phys. Rev. D **72**, 044003 (2005).
[17] D. G. Boulware and S. Deser, Phys. Rev. D **6**, 3368 (1972).
[18] C. Deffayet, G. R. Dvali, G. Gabadadze, and A. I. Vainshtein, Phys. Rev. D **65**, 044026 (2002).
[19] A. Gruzinov, New Astron. Rev. **10**, 311 (2005).
[20] M. Porrati, Phys. Lett. B **534**, 209 (2002).

[21] A. Nicolis and R. Rattazzi, J. High Energy Phys. 06 (2004) 059.

[22] D. Gorbunov, K. Koyama, and S. Sibiryakov, Phys. Rev. D **73**, 044016 (2006).

[23] S. Weinberg, Phys. Rev. D **67**, 123504 (2003).

[24] S. Fubini, Nuovo Cimento Soc. Ital. Fis. **34A**, 521 (1976).

[25] J. Polchinski, Nucl. Phys. **B303**, 226 (1988).

[26] J. Khoury and A. Weltman, Phys. Rev. Lett. **93**, 171104 (2004).

[27] C. de Rham *et al.*, Phys. Rev. Lett. **100**, 251603 (2008).

[28] A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis, and R. Rattazzi, J. High Energy Phys. 10 (2006) 014.

[29] S. Weinberg and E. Witten, Phys. Lett. **96B**, 59 (1980).

[30] M. Visser, B. Bassett, and S. Liberati, Nucl. Phys. B, Proc. Suppl. **88**, 267 (2000).

[31] P. Creminelli, M. A. Luty, A. Nicolis, and L. Senatore, J. High Energy Phys. 12 (2006) 080.