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Stellar oscillations in tensor-vector-scalar theory

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An alternative theory of gravity has recently been proposed by Bekenstein, named tensor-vector-scalar (TeVeS) theory, which can explain many galactic and cosmological observations without the need for dark matter. While this theory passes basic Solar System tests, and has been scrutinized with considerable detail in other weak-field regimes, comparatively little has been done in the strong-field limit of the theory. In this article, with Cowling approximation, we examine the oscillation spectra of neutron stars in TeVeS. As a result, we find that the frequencies of fundamental modes in TeVeS could become lager than those expected in general relativity, while the dependence of frequency of higher overtone on gravitational theory is stronger than that of lower modes. These imprints of TeVeS make it possible to distinguish the gravitational theory in strong-field regime via the observations of gravitational waves, which can provide unique confirmation of the existence of scalar field.

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I. INTRODUCTION

Tests of gravitational theories in the strong-field regime are extremely important because, unlike the weak-field, they are still largely unconstrained by observations. However, with developing technology, it is becoming possible to observe compact objects with high accuracy. For example, observations of emitted X-rays and γ-rays from compact objects could be used to directly test the strong-field regime of a gravity theory [1]. Furthermore, future observational developments, for example, with gravitational waves, will allow us to obtain different physical properties for compact objects, which will further allow for testing in the strong-field regime. From a theoretical point of view, there are attempts to test theories of gravity in the strong-field regime by using surface atomic line redshifts [2] or gravitational waves from the neutron stars [3]. In these investigations, the possibility of distinguishing scalar-tensor theory, proposed by Damour and Esposito-Farèse [4], from general relativity (GR) was discussed (see Psaltis [1] for a review). While the existence of scalar fields has not been experimentally verified, several experiments in the weak-field limit of GR set severe limits on the existence and strength of such fields [5].

Recently, tensor-vector-scalar (TeVeS) theory has attracted considerable attention as an alternative gravitational theory. TeVeS was proposed by Bekenstein [6] as a relativistic theory for modified Newtonian dynamics [7]. As such, it explains galaxy rotation curves and the Tully-Fisher law without the existence of dark matter. TeVeS has also successfully explained strong gravitational lensing [8] as well as key features of the cosmic microwave background [9] and galaxy distributions through an evolving Universe [10] without cold dark matter. While in the strong-field regime of TeVeS, Giannios found the Schwarzschild solution [11], and Sagi and Bekenstein generalized this to the Reissner-Nordström solution [12]. Furthermore, Contaldi, Wiseman, and Withers have found vacuum solutions for a constant scalar field [13]. More recently, the Tolman-Oppenheimer-Volkoff (TOV) equations in TeVeS were derived by Lasky, Sotani, and Giannios [14], with which one can produce static, spherically symmetric neutron stars, and they showed the possibility of distinguishing TeVeS from GR by way of redshift observations. In this article, we examine whether observations of gravitational waves associated with the neutron star oscillations can provide an alternative way of probing the gravitational theory in the strong-field regime.

The attempt to estimate the stellar parameters, such as mass, radius, and equation of state (EOS), via their oscillation properties is not a new idea. Helioseismology is established in astronomy and one could know the information about the interior of our sun. In the late 1990s, it was suggested possible to reveal the compact star properties through the oscillation spectra [15], which is called gravitational wave asteroseismology. The stellar mass, radius, and EOS can be deduced by an analysis of the oscillation spectrum of fundamental, pressure, and spacetime modes, i.e., f, p, and w modes (e.g., [16,17]). The rotation period of a compact star can be revealed by the examination of r mode oscillations (e.g., [18,19]), where such frequencies are proportional to the rotation rate. Furthermore, the detailed analysis of the gravitational waves makes it possible to determine the radius of the accretion disk around supermassive black hole [20] or to know the magnetic effect during the stellar collapse [21].

In general, the oscillations of a neutron star in TeVeS could produce not only gravitational but also scalar and vector waves, which is similar to the case in the scalar-tensor theory [22], and the direct detection of scalar and/or vector waves would be a unique probe for the gravitational theory. Still, we will show that it is not necessary to detect

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these waves, because the obvious imprints due to the existence of scalar and vector fields will be apparent in the spectrum of gravitational waves associated with the stellar oscillations. Although we adopt Cowling approximation in this article, the more complicated analysis including the metric, vector, and scalar perturbations will be seen in the near future.

This article is organized as follows. In the next section, we describe the fundamental parts of TeVeS and TOV equations in TeVeS, where we also show the neutron star models. In Sec. III we derive the perturbation equations with Cowling approximation. Then the oscillation spectra of neutron stars in TeVeS are shown in Sec. IV; finally we discuss the results related to gravitational wave asteroseismology in Sec. V. In this article, we adopt the unit of $c = G = 1$, where c and G denote the speed of light and the gravitational constant, respectively, and the metric signature is $(-, +, +, +)$.

II. STELLAR MODELS IN TEVES

A. TeVeS

Since details of TeVeS can be found in [6], we only mention here the fundamental parts of the theory that are necessary for the present calculations. TeVeS is based on three dynamical gravitational fields: an Einstein metric $g_{\mu\nu}$, a timelike 4-vector field $\mathcal{U}^\mu$, and a scalar field φ. There is also a nondynamical scalar field σ. The vector field fulfills the normalization condition $g_{\mu\nu}\mathcal{U}^\mu\mathcal{U}^\nu = -1$ and the physical metric is given by

$$\tilde{g}_{\mu\nu} = e^{-2\varphi}g_{\mu\nu} - 2\mathcal{U}_\mu\mathcal{U}_\nu \sinh(2\varphi), \tag{2.1}$$

$$\tilde{g}^{\mu\nu} = e^{2\varphi}g^{\mu\nu} + 2\mathcal{U}^\mu\mathcal{U}^\nu \sinh(2\varphi). \tag{2.2}$$

All quantities in the physical frame are denoted with a tilde, and any quantity without a tilde is in the Einstein frame. The total action of TeVeS S contains contributions from the three dynamical fields and a matter contribution (see [6] for details). These include two positive dimensionless parameters k and K which are the coupling parameters for the scalar and vector fields, respectively. There also exists a dimensionless free function F, a constant length scale ℓ, and a spacetime dependent Lagrange multiplier λ.

By varying the total action S with respect to $g^{\mu\nu}$, one can obtain the field equations for the tensor field

$$G_{\mu\nu} = 8\pi G[\tilde{T}_{\mu\nu} + (1 - e^{-4\varphi})\mathcal{U}^\alpha\tilde{T}_{\alpha(\mu}\mathcal{U}_{\nu)} + \tau_{\mu\nu}] + \Theta_\nu, \tag{2.3}$$

where $\tilde{T}_{\mu\nu}$ is the energy-momentum tensor in the physical frame, $\tilde{T}_{\alpha(\mu}\mathcal{U}_{\nu)} \equiv \tilde{T}_{\alpha\mu}\mathcal{U}_\nu + \tilde{T}_{\alpha\nu}\mathcal{U}_\mu$, and $G_{\mu\nu}$ is the Einstein tensor in the Einstein frame. Conservation of energy-momentum is therefore given in the physical frame as $\tilde{\nabla}_\mu\tilde{T}^{\mu\nu} = 0$. The other sources in Eq. (2.3) are given by

$$\begin{aligned}\tau_{\mu\nu} = \sigma^2\Big[&\varphi_{,\mu}\varphi_{,\nu} - \frac{1}{2}g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta}g_{\mu\nu} \\ &- \frac{G\sigma^2}{4\ell^2}F(kG\sigma^2)g_{\mu\nu} \\ &- \mathcal{U}^\alpha\varphi_{,\alpha}\Big(\mathcal{U}_{(\mu}\varphi_{,\nu)} - \frac{1}{2}\mathcal{U}^\beta\varphi_{,\beta}g_{\mu\nu}\Big)\Big],\end{aligned} \tag{2.4}$$

$$\begin{aligned}\Theta_{\mu\nu} = K\Big(&g^{\alpha\beta}\mathcal{U}_{[\alpha,\mu]}\mathcal{U}_{[\beta,\nu]} \\ &- \frac{1}{4}g^{\gamma\delta}g^{\alpha\beta}\mathcal{U}_{[\gamma,\alpha]}\mathcal{U}_{[\delta,\beta]}g_{\mu\nu}\Big) - \lambda\mathcal{U}_\mu\mathcal{U}_\nu,\end{aligned} \tag{2.5}$$

where $\mathcal{U}_{[\alpha,\beta]} \equiv \mathcal{U}_{\alpha,\beta} - \mathcal{U}_{\beta,\alpha}$. Similarly, by varying S with respect to $\mathcal{U}_\mu$ and φ, one obtains the field equations for the vector and scalar fields,

$$\begin{aligned}&K\mathcal{U}^{[\alpha;\beta]}{}_{;\beta} + \lambda\mathcal{U}^\alpha + 8\pi G\sigma^2\mathcal{U}^\beta\varphi_{,\beta}g^{\alpha\gamma}\varphi_{,\gamma} \\ &\quad = 8\pi G(1 - e^{-4\varphi})g^{\alpha\mu}\mathcal{U}^\beta\tilde{T}_{\mu\beta},\end{aligned} \tag{2.6}$$

$$\begin{aligned}[\mu(k\ell^2h^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu})h^{\alpha\beta}\varphi_{,\alpha}]_{;\beta} = kG[&g^{\alpha\beta} + (1 + e^{-4\varphi}) \\ &\times \mathcal{U}^\alpha\mathcal{U}^\beta]\tilde{T}_{\alpha\beta},\end{aligned} \tag{2.7}$$

where $h^{\alpha\beta} = g^{\alpha\beta} - \mathcal{U}^\alpha\mathcal{U}^\beta$ and $\mu(x)$ is a function defined by $2\mu F(\mu) + \mu^2 dF(\mu)/d\mu = -2x$. With this function μ, the nondynamical scalar field σ is determined by

$$kG\sigma^2 = \mu(k\ell^2h^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta}). \tag{2.8}$$

Therefore, the field equations of TeVeS are Eqs. (2.3), (2.6), and (2.8). It has been shown in the strong-field limit that $\mu = 1$ is an excellent approximation [11,12]. On cosmological scales this is not a good choice [6], however in this article we only consider regions not too far from neutron stars, and we therefore set $\mu = 1$. This implies from Eq. (2.6) that $\sigma^2 = 1/(kG)$. Moreover, while the functional form of F is not predicted by the theory, one can show that when $\mu = 1$, the contribution of F to the field equations vanishes [6,11,12]. Therefore, our results are independent of this function and we drop it from the remaining discussion.

B. TOV in TeVeS

First, the Tolman-Oppenheimer-Volkoff (TOV) equations in TeVeS are derived by Lasky, Sotani, and Giannios [14]. Here we make an brief description of TOV equations. A static, spherically symmetric metric can be expressed as

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -e^{\nu(r)}dt^2 + e^{\zeta(r)}dr^2 + r^2d\Omega^2, \tag{2.9}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and $e^{-\zeta} = 1 - 2m(r)/r$. In general, the vector field for a static, spherically symmetric spacetime can be described as $\mathcal{U}^\mu = (\mathcal{U}^t, \mathcal{U}^r, 0, 0)$, where $\mathcal{U}^t$ and $\mathcal{U}^r$ are functions of r.

Giannios [11] showed that in vacuum, the parametrized post-Newtonian (PPN) coefficients for a spherically symmetric, static spacetime with a nonzero $\mathcal{U}^r$ can violate observational restrictions. In this article, we therefore only consider the case where $\mathcal{U}^r = 0$. In this case, the vector field can be fully determined from the normalization condition, and is found to be $\mathcal{U}^\mu = (e^{-\nu/2}, 0, 0, 0)$. Moreover, one can show that the vector field equation (2.6) is now trivially satisfied. With this vector field, the physical metric is

$$d\tilde{s}^2 = \tilde{g}_{\alpha\beta}dx^\alpha dx^\beta = -e^{\nu+2\varphi}dt^2 + e^{\zeta-2\varphi}dr^2 + e^{-2\varphi}r^2d\Omega^2, \quad (2.10)$$

and the fluid four-velocity is $\tilde{u}_\mu = e^\varphi \mathcal{U}_\mu$. We further assume the stellar matter content to be a perfect fluid, i.e., $\tilde{T}_{\mu\nu} = (\tilde{\rho} + \tilde{P})\tilde{u}_\mu\tilde{u}_\nu + \tilde{P}\tilde{g}_{\mu\nu}$, from which one can show that the full system of equations with $k \neq 0$ and $K \neq 0$ reduces to

$$\left(1 - \frac{K}{2}\right)m' = \frac{Km}{2r} + 4\pi G r^2 e^{-2\varphi}(\tilde{\rho} + 2K\tilde{P}) + \left[\frac{2\pi r2}{k}\psi^2 - \frac{Kr\nu'}{4}\left(1 + \frac{r\nu'}{4}\right)\right]e^{-\zeta}, \quad (2.11)$$

$$\frac{Kr}{4}\nu' = -1 + \left[1 + K\left(\frac{4\pi G r^3\tilde{P}e^{-2\varphi} + m}{r - 2m} + \frac{2\pi r^2}{k}\psi^2\right)\right]^{1/2}, \quad (2.12)$$

$$\tilde{P}' = -\frac{\tilde{P} + \tilde{\rho}}{2}(2\psi + \nu'), \quad (2.13)$$

$$\varphi' = \psi, \quad (2.14)$$

$$\psi' = \left[\frac{m'r - m}{r(r - 2m)} - \frac{\nu'}{2} - \frac{2}{r}\right]\psi + kGe^{-2\varphi+\zeta}(\tilde{\rho} + 3\tilde{P}), \quad (2.15)$$

where a prime denotes a derivative with respect to r. (See [14] for the derivation of these equations and for a discussion with $k = 0$ and/or with $K = 0$.) This system of equations is closed with the addition of an equation of state (EOS). The stellar radius in physical frame R is determined by $R \equiv e^{-\varphi(r_s)}r_s$, where r_s is the position of the stellar surface defined as the point where $\tilde{P} = 0$. Note that on exterior region the scalar field still exists although there is no fluid.

We integrate the above equations from the center $r = 0$ to the stellar surface $r = r_s$. Moreover, the interior boundary conditions are given by $\tilde{P}(0) = \tilde{P}_0$, $\tilde{\rho}(0) = \tilde{\rho}_0$, $\nu(0) = \nu_0$, $\varphi(0) = \varphi_0$, $\psi(0) = 0$, and $m(0) = 0$, which are determined by Taylor series expansions of the above equations near $r = 0$ (see [14] for details). The exact values for ν_0 and φ_0 are determined by matching the functions $\nu(r)$ and $\varphi(r)$ to their asymptotic behavior, which is found by performing a coordinate transformation on (2.10) to bring it into an asymptotically flat form. We define new coordinates $\hat{t} \equiv te^{\varphi_c}$ and $\hat{r} \equiv re^{-\varphi_c}$, where φ_c denotes the cosmological value of the scalar field. Then performing an asymptotic expansion of all the equations and dropping the hats on the new coordinates for simplicity in the expressions implies

$$\tilde{g}_{tt} = -1 + \frac{2M_{\mathrm{ADM}}}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), \quad (2.16)$$

$$\tilde{g}_{rr} = 1 + \frac{2M_{\mathrm{ADM}}}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), \quad (2.17)$$

$$\varphi = \varphi_c - \frac{kGM_\varphi}{4\pi e^{\varphi_c}r} + \mathcal{O}\left(\frac{1}{r^2}\right). \quad (2.18)$$

Here, M_{ADM} is the total Arnowitt-Deser-Misner (ADM) mass given by

$$M_{\mathrm{ADM}} = \left(m_\infty + \frac{kGM_\varphi}{4\pi}\right)e^{-\varphi_c}, \quad (2.19)$$

where m_∞ is the mass function evaluated at radial infinity. Also, M_φ is the scalar mass [6], which is constant outside the star and is defined everywhere as

$$M_\varphi = 4\pi\int_0^r r^2(\tilde{\rho} + 3\tilde{P})e^{(\nu+\zeta)/2-2\varphi}dr. \quad (2.20)$$

We adopt the same EOS as in [3], which are polytropic ones derived by fitting functions to tabulated data of realistic EOS known as EOS A and EOS II. The maximum masses of neutron stars in GR are $M = 1.65M_\odot$ with $R = 8.9$ km for EOS A and $M = 1.95M_\odot$ with $R = 10.9$ km for EOS II. That is, EOS A is considered soft, while EOS II is an intermediate EOS.

When it comes to the study of the structure of neutron stars, TeVeS introduces three new parameters, k, K, and φ_c, with respect to GR. Since the value of k is tightly constrained by both cosmological models and also planetary motions in the outer Solar System [6], we accordingly set $k = 0.03$ for the remainder of the article. With respect to the value of φ_c, Lasky, Sotani, and Giannios showed that φ_c can have a minimum value of around 0.001, based on causality issues inside the neutron star [14]. Therefore, for this article we use $\varphi_c = 0.003$. Details of neutron star models where these parameters are allowed to vary are given in Lasky, Sotani, and Giannios [14], in which they showed that the dependences on k and on φ_c are minimal for neutron star models. On the other hand, restrictions on K are less severe, and have not been discussed in great detail in the literature. In this article we consider the range $0 < K < 2$, because for $K > 2$ one can show that the pressure diverges from the stellar center outward, and therefore stellar models are not possible [14] (Sagi and Bekenstein [12] also showed that physical black hole solutions are only valid for $K < 2$).

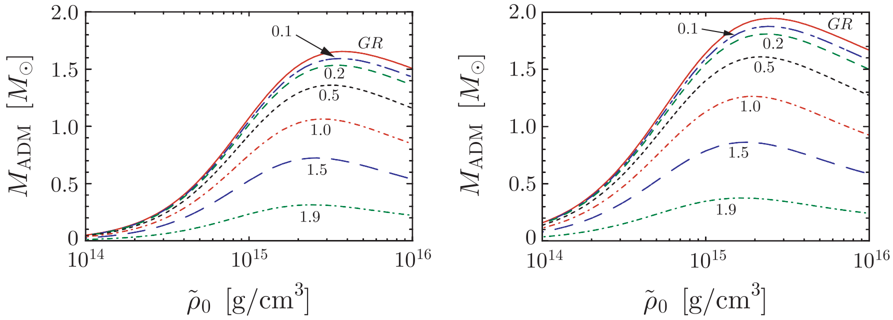

FIG. 1 (color online). Relation between the mass and central density of neutron stars in GR and in TeVeS with $k = 0.03$, $\varphi_c = 0.003$, where the left and right panels are corresponding to the stellar properties given by EOS A and EOS II, respectively. In the figure the solid line denotes the case of GR while the other lines are corresponding to the stellar models with different values of K in TeVeS.

C. Neutron star models in TeVeS

Figure 1 shows the ADM mass as a function of the central density of neutron stars. Different lines correspond to different values of K, whose values are indicated. Additionally, we plot the stellar model for GR with the solid line. Note that for spherically symmetric neutron stars, stellar models for the region $\partial M_{\rm ADM}/\partial\tilde\rho_0 < 0$ could be unstable. From this figure we can see that, although the central density giving the maximum mass is almost independent of the value of K, the corresponding maximum mass depends strongly on the parameter K, i.e., for larger values of K the maximum mass becomes smaller. For example, with EOS A for $K = 0.5$ the maximum mass is 18% smaller than that of GR.

Figure 2 shows the relation between the ADM mass and stellar radius with different values of K and also for the GR case. In general, one requires a softer EOS near the stellar surface, which implies the stellar radius becomes larger, but with high central density for typical neutron star, the stellar models are almost independent from the consideration of softer EOS near the stellar surface. This figure further implies that there exists a minimum radius, which usually corresponds to the maximum mass [23]. Considering this minimum radius, it can be seen in Fig. 2 that neutron stars in TeVeS are smaller than in the GR case. For example, we can see that the minimum radius for a star with EOS A in TeVeS with $K = 0.5$ is 7.7 km, whereas for GR it is 8.9 km, which is a 13.5% difference.

III. PERTURBATION EQUATIONS IN THE COWLING APPROXIMATION

In this section we derive the perturbation equations for nonradial oscillations of spherically symmetric neutron stars in TeVeS. For simplicity, we adopt the Cowling approximation, in which the fluid is perturbed on a fixed background. That is, the perturbations of the spacetime, vector field, and scalar field are frozen: $\delta\tilde g_{\mu\nu} = 0$, $\delta\mathcal{U}^\mu = 0$, and $\delta\varphi = 0$. Note that with the Cowling approximation we can study only the oscillation modes related to the fluid perturbations, such as f, p, and g modes, while it is impossible to study the other emissions of scalar waves, vector waves, and gravitational waves connected to the oscillation of spacetime. Additionally, we should notice that the Cowling approximation in GR is typically very good for the axial type of oscillations while for the polar type of oscillations the error for typical relativistic stellar models could become less than 20% for f modes and around 10% for p modes [24].

With Cowling approximation, the perturbed energy-momentum tensor, $\delta\tilde T^{\mu\nu}$, is given as

$$\delta\tilde T^{\mu\nu} = (\delta\tilde\rho + \delta\tilde P)\tilde u^\mu\tilde u^\nu + (\tilde\rho + \tilde P)(\delta\tilde u^\mu\tilde u^\nu + \tilde u^\mu\delta\tilde u^\nu) + \delta\tilde P\tilde g^{\mu\nu}. \tag{3.1}$$

Introducing the Lagrangian displacement vector, the perturbed variables in $\delta\tilde T^{\mu\nu}$ such as $\delta\tilde u^\mu$, $\delta\tilde\rho$, and $\delta\tilde P$, can be described explicitly. The Lagrangian displacement vector

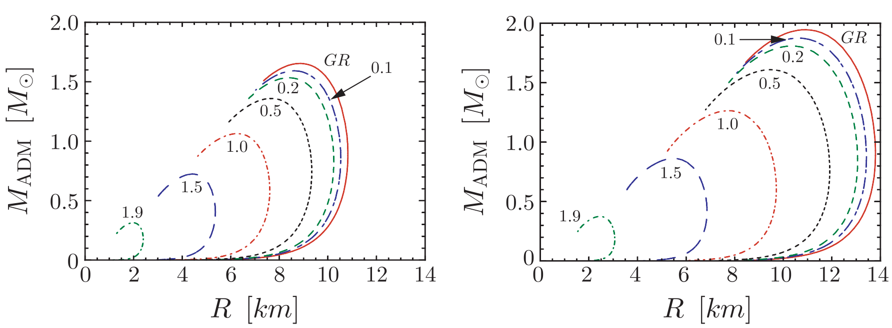

FIG. 2 (color online). Mass-radius relation for neutron stars in GR and in TeVeS with $k = 0.03$, $\varphi_c = 0.003$, where the left and right panels are corresponding to the stellar properties given by EOS A and EOS II, respectively.

for the fluid perturbations is

$$\tilde{\xi}^{i} = (\tilde{\xi}^{r}, \tilde{\xi}^{\theta}, \tilde{\xi}^{\phi}) = (W, -V\partial_{\theta}, -V\sin^{-2}\theta\partial_{\phi})\frac{1}{r^2}Y_{\ell m}, \tag{3.2}$$

where W and V are functions of t and r. Then the perturbations of four-velocity $\delta\tilde{u}^{\mu}$ can be written as

$$\delta\tilde{u}^{\mu} = (0, \dot{W}, -\dot{V}\partial_{\theta}, -\dot{V}\sin^{-2}\theta\partial_{\phi})e^{-\varphi-\nu/2}\frac{1}{r^2}Y_{\ell m}, \tag{3.3}$$

where dots on the variables denote the partial derivative with respect to t. On the other hand, using the first law of thermodynamics, we can get the following relation between the adiabatic changes of the density and the baryon number density:

$$\Delta\tilde{\rho} = \frac{\tilde{\rho}+\tilde{P}}{\tilde{n}}\Delta\tilde{n}, \tag{3.4}$$

where $\tilde{n}$ denotes the baryon number density. So if we use the relationship between the Lagrangian perturbation $\Delta\tilde{\rho}$ and Eulerian perturbation $\delta\tilde{\rho}$ such as

$$\Delta\tilde{\rho} \simeq \delta\tilde{\rho} + \tilde{\xi}^{r}\partial_{r}\tilde{\rho}, \tag{3.5}$$

we can express the Eulerian density variation as

$$\delta\tilde{\rho} = (\tilde{\rho}+\tilde{P})\frac{\Delta\tilde{n}}{\tilde{n}} - \frac{\tilde{\rho}'W}{r^2}Y_{\ell m}. \tag{3.6}$$

Additionally, with the definition of the adiabatic constant

$$\gamma \equiv \left(\frac{\partial\ln\tilde{P}}{\partial\ln\tilde{n}}\right)_{s} = \frac{\tilde{n}\Delta\tilde{P}}{\tilde{P}\Delta\tilde{n}}, \tag{3.7}$$

we can derive the Eulerian variation of the pressure

$$\delta\tilde{P} = \gamma\tilde{P}\frac{\Delta\tilde{n}}{\tilde{n}} - \frac{\tilde{P}'W}{r^2}Y_{\ell m}. \tag{3.8}$$

Note that with Eqs. (3.4) and (3.7) we can get the useful expression for γ as

$$\gamma = \frac{\tilde{\rho}+\tilde{P}}{\tilde{P}}\left(\frac{\partial\tilde{P}}{\partial\tilde{\rho}}\right)_{s}. \tag{3.9}$$

Finally, the Lagrangian variation of the baryon number density, which comes on the expressions of $\delta\tilde{\rho}$ and $\delta\tilde{P}$, is determined by the relation as

$$\frac{\Delta\tilde{n}}{\tilde{n}} = -\tilde{\nabla}^{(3)}_{k}\tilde{\xi}^{k} - \frac{\delta\tilde{g}}{2\tilde{g}}, \tag{3.10}$$

where $\tilde{\nabla}^{(3)}_{k}$ and $\tilde{g}$ denote the covariant derivative in a three-dimension with metric $\tilde{g}_{\mu\nu}$ and the determinant of $\tilde{g}_{\mu\nu}$, respectively. In this article, since we assume the Cowling approximation, the second term is neglected. Then the Lagrangian variation of the baryon number density can be written as

$$\frac{\Delta\tilde{n}}{\tilde{n}} = -\left[W' + \frac{1}{2}(\zeta' - 6\varphi')W + \ell(\ell+1)V\right]\frac{1}{r^2}Y_{\ell m}. \tag{3.11}$$

Finally we can get the equations describing the fluid perturbations by taking a variation of the energy-momentum conservation law $\tilde{\nabla}_{\nu}\tilde{T}^{\mu\nu} = 0$. With Cowling approximation, this equation becomes $\tilde{\nabla}_{\nu}\delta\tilde{T}^{\mu\nu} = 0$. The explicit forms with $\mu = r, \theta$ are

$$\begin{aligned}\partial_{r}\bigg[\frac{\gamma\tilde{P}}{r^2}\Big\{W' + \frac{1}{2}(\zeta' - 6\varphi')W + \ell(\ell+1)V\Big\} + \frac{\tilde{P}'W}{r^2}\bigg] \\ - (\tilde{\rho}+\tilde{P})e^{-4\varphi-\nu+\zeta}\frac{\ddot{W}}{r^2} - \frac{\tilde{\rho}'+\tilde{P}'}{\tilde{\rho}+\tilde{P}}\bigg[\frac{\gamma\tilde{P}}{r^2} \\ \times\Big\{W' + \frac{1}{2}(\zeta' - 6\varphi')W + \ell(\ell+1)V\Big\} + \frac{\tilde{P}'W}{r^2}\bigg] = 0,\end{aligned} \tag{3.12}$$

$$\begin{aligned}(\tilde{\rho}+\tilde{P})e^{-4\varphi-\nu}\ddot{V} + \frac{\gamma\tilde{P}}{r^2}\Big\{W' + \frac{1}{2}(\zeta' - 6\varphi')W \\ + \ell(\ell+1)V\Big\} + \frac{\tilde{P}'W}{r^2} = 0,\end{aligned} \tag{3.13}$$

where we use the relation that $\delta\tilde{\rho}/\delta\tilde{P} = \tilde{\rho}'/\tilde{P}'$ and Eq. (2.13). By assuming a harmonic dependence on time, the perturbative variables will be written as $W(t,r) = W(r)e^{i\omega t}$ and $V(t,r) = V(r)e^{i\omega t}$. To make the above equations simpler, by calculating the combination of the form $d(\text{Eq.}\,(3.13))/dr - (\text{Eq.}\,(3.12))$ and substituting Eq. (3.13) again, we can get

$$V' = (4\varphi' + \nu')V - e^{\zeta}\frac{W}{r^2}. \tag{3.14}$$

Thus, from Eqs. (3.13) and (3.14), we can obtain the following simple equation system for the perturbations of fluid:

$$\begin{aligned}W' = \frac{d\tilde{\rho}}{d\tilde{P}}\bigg[\omega^2 r^2 e^{-4\varphi-\nu}V + \frac{1}{2}(2\varphi' + \nu')W\bigg] \\ + \frac{1}{2}(6\varphi' - \zeta')W - \ell(\ell+1)V,\end{aligned} \tag{3.15}$$

$$V' = (4\varphi' + \nu')V - e^{\zeta}\frac{W}{r^2}. \tag{3.16}$$

With the appropriate boundary conditions at the center and stellar surface, the above equation system constitutes an eigenvalue problem for the parameter ω. One can find the behavior of W and V near the stellar center as $W(r) = Br^{\ell+1} + \mathcal{O}(r^{\ell+3})$ and $V(r) = -Br^{\ell}/\ell + \mathcal{O}(r^{\ell+2})$, where B is an arbitrary constant, while the boundary condition at the stellar surface is the vanishing the Lagrangian perturbation of the pressure, i.e., $\Delta\tilde{P} = 0$. Since the Lagrangian perturbation of the pressure is described by $\Delta\tilde{P} = \gamma\tilde{P}\Delta\tilde{n}/\tilde{n}$, with the help of Eq. (3.15) we can get the boundary condition at the stellar surface as

$$2\omega^2 r^2 e^{-4\varphi-\nu} V + (2\varphi' + \nu')W = 0. \tag{3.17}$$

IV. OSCILLATION SPECTRA

With respect to the neutron star models shown in Sec. II, in this section we examine the stellar oscillations. Especially, we focus on the stellar models whose central density is in the range from $\tilde{\rho}_0 = 10^{14}$ g/cm^3 up to the value given for the maximum ADM mass. The stellar parameters with maximum ADM mass are summarized in Tables I and II, where φ_0 and z are central values of φ and surface redshift, respectively. In general, the oscillation spectrum is directly related to the stellar parameter, such as mass, radius, and EOS, but the frequencies of fundamental oscillation modes, i.e., f modes, can be connected to the stellar average density $(M_{\rm ADM}/R^3)^{1/2}$. The reason is physically explained by considering the relation between the sound speed and the time that fluid perturbation needs to propagate across the star. Actually, for the stellar models in GR, Andersson and Kokkotas found the empirical formula for the frequency of f mode as a function of stellar average density [25]. The f mode frequencies for the stellar models in GR constructed with almost all EOS are subject to this empirical formula. While, the frequencies of f mode for the stellar models in TeVeS with the above two EOS are shown in Fig. 3. The deviation from GR is clearly cognized for typical neutron stars and depending on the value of parameter K, the frequencies become around 20% larger than those expected in a general relativistic neutron star. This can be an observable effect and one might distinguish the gravitational theory in strong gravitational field by using the observations of gravitational waves.

TABLE I. Stellar parameters for models with EOS A and with maximum ADM mass, where we choose that $k = 0.03$ and $\varphi_c = 0.003$.

K	$M_{\rm ADM}/M_\odot$	$\tilde{\rho}_0$ [g/cm^3]	R [km]	φ_0	$M_{\rm ADM}/R$	z
0.2	1.56	3.54×10^{15}	8.36	8.10×10^{-4}	0.271	0.479
0.5	1.42	3.30×10^{15}	7.60	1.26×10^{-3}	0.265	0.459
1.0	1.15	3.00×10^{15}	6.16	1.91×10^{-3}	0.255	0.433
1.5	0.81	2.81×10^{15}	4.32	2.48×10^{-3}	0.248	0.414
1.9	0.36	2.70×10^{15}	1.92	2.90×10^{-3}	0.242	0.401

TABLE II. Stellar parameters for models with EOS II and with maximum ADM mass, where we choose that $k = 0.03$ and $\varphi_c = 0.003$.

K	$M_{\rm ADM}/M_\odot$	$\tilde{\rho}_0$ [g/cm^3]	R [km]	φ_0	$M_{\rm ADM}/R$	z
0.2	1.84	2.40×10^{15}	10.32	9.38×10^{-4}	0.259	0.441
0.5	1.67	2.25×10^{15}	9.39	1.35×10^{-3}	0.253	0.425
1.0	1.36	2.06×10^{15}	7.63	1.96×10^{-3}	0.245	0.403
1.5	0.96	1.92×10^{15}	5.36	2.51×10^{-3}	0.238	0.385
1.9	0.43	1.80×10^{15}	2.40	2.91×10^{-3}	0.231	0.370

The possibility to probe the gravitational theory by using observations of gravitational waves can be also seen in Fig. 4, where we plot the normalized frequencies of f and p_1 modes as functions of ADM mass. In the figures, the solid line denotes the frequency in GR while the other broken lines are corresponding to those in TeVeS with several values of K. One can easily observe that the frequencies expected in TeVeS are quite different from those in GR. Since this distinction results from the difference of gravitational theory, which is created by the presence of a scalar field, observing more than one mode of gravitational wave could tell us the existence of the scalar field. This statement might become more obvious by seeing the dependence of frequencies of gravitational waves on the parameter K. In Fig. 5, we plot the normalized frequencies of the first four modes, i.e., f, p_1, p_2, and p_3, as functions of parameter K, where the ADM masses are fixed to be $1.4M_\odot$. The allowed maximum values of K to produce the stellar models with $M_{\rm ADM} = 1.4M_\odot$ are $K = 0.54$ for EOS A and $K = 0.94$ for EOS II. From these figures, we can see that the qualitative dependences of frequency on

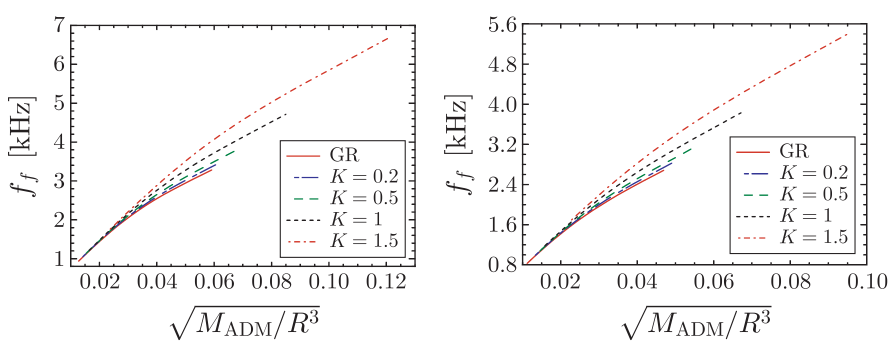

FIG. 3 (color online). The frequency of f mode as a function of the stellar average density $(M_{\rm ADM}/R^3)^{1/2}$, where f_f is defined as $f_f \equiv \omega_f/(2\pi)$. The left panel corresponds to EOS A and the right panel to EOS II. The solid line corresponds to the frequency in GR, while the other broken lines are corresponding to the frequencies in TeVeS with several values of K. Notice that the unit of average density is [1/km] in the geometrical unit, where $c = G = 1$.

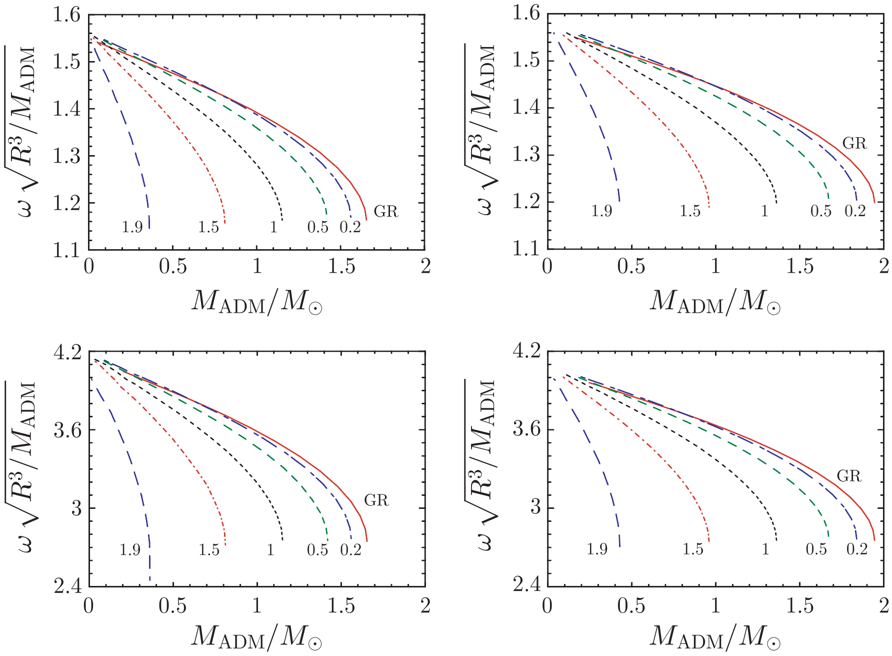

FIG. 4 (color online). The normalized frequencies of first two fluid modes are plotted as functions of the ADM mass, where upper panels correspond to f and the lower panels correspond to p_1 modes. The left panels correspond to EOS A and the right panels to EOS II. In these figures, the solid line corresponds to the frequency in GR, while the other broken lines are corresponding to the frequencies in TeVeS with several values of K.

the value of K are independent of EOS and the kinds of eigenmode. That is, the normalized frequencies of fluid modes are decreasing as the value of K becomes large. On the other hand, it is also found that the higher overtone is quantitatively more sensitive against the value of K than the lower modes. This point can be seen in Table III, where we summarize the ratio of difference between the frequencies for the stellar models with $K = 0.05$ and with the allowed maximum values of K. In other words, the frequencies of higher overtone is more helpful to distinguish TeVeS from GR via the gravitational wave observations. Anyway, through Figs. 4 and 5, we can find that with the

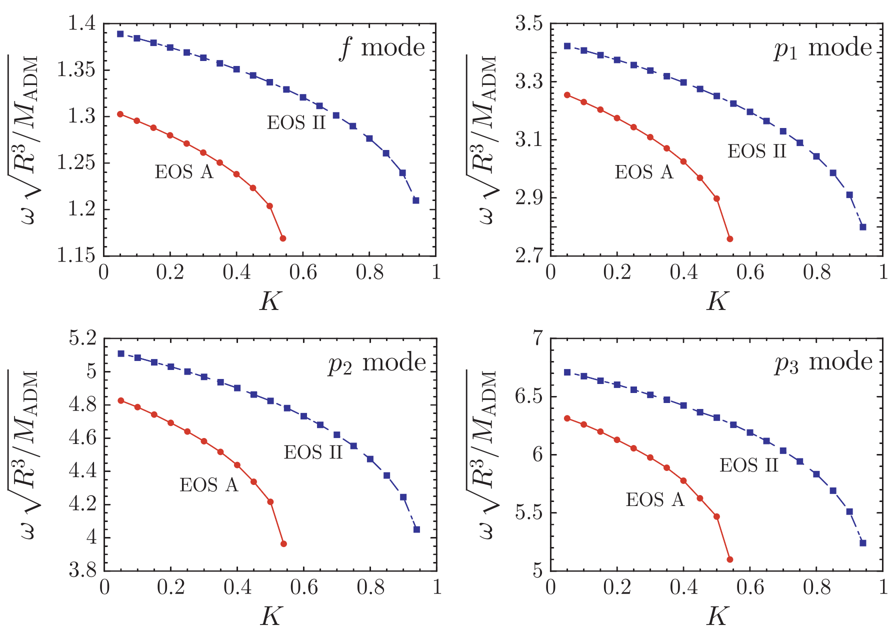

FIG. 5 (color online). For the stellar models with $M_{\rm ADM} = 1.4M_\odot$, the normalized eigenvalues ω of the first few modes (f, p_1, p_2, and p_3) are shown as functions of parameter K with EOS A (solid lines) and EOS II (broken line). The values of k and φ_c are fixed as $k = 0.03$ and $\varphi_c = 0.003$, respectively.

TABLE III. The relative frequency change of each eigenmode in Fig. 5 defined as $(\omega_{K=0.05} - \omega_{K=\max})/\omega_{K=0.05}$, where $\omega_{K=0.05}$ and $\omega_{K=\max}$ denote the frequencies for the stellar models with $K = 0.05$ and with the allowed maximum values of K, respectively.

Mode	EOS A	EOS II
f	10.28%	12.89%
p_1	15.21%	18.18%
p_2	17.88%	20.73%
p_3	19.23%	21.89%

help of observation of stellar mass, it is possible to probe the gravitational theory in the strong-field regime by using observations of gravitational waves.

V. CONCLUSION

In this article, to examine the effect of the tensor-vector-scalar (TeVeS) theory on the oscillation spectra of neutron stars, we have derived the perturbation equations of neutron stars in TeVeS and calculated their eigenfrequencies. Depending on the parameter of TeVeS, the frequencies of fundamental oscillation could be off the well-known empirical formula in GR and they become larger than those expected in GR. We can also see the deviation from GR in the frequencies of higher overtones and they have stronger dependence on the parameter K than the lower oscillation modes. Since these imprints of TeVeS come from the presence of the scalar field, by using the observations of gravitational waves associated with the stellar oscillations, it will be possible not only to distinguish the gravitational theory in the strong-field regime, but also to probe the existence of the scalar field.

For simplicity, we assumed the Cowling approximation in this study, which restricts our examination to only stellar oscillations. This means that we should do a more detailed study including the metric, vector, and scalar field perturbations. Via these oscillations, we could obtain the additional information in the gravitational spectrum and combining those with results shown in this article would provide more accurate constraints on the gravitational theory in the strong-field regime. Furthermore the introduction of the stellar magnetic effect also might be important. For example, recent observation of quasiperiodic oscillation in the giant flares are believed to be related to the oscillations of strong magnetized neutron stars [26–28]. Considering the magnetic effects, one might be able to get the further constraint in the theory.

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