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Quantum field theory on a cosmological, quantum space-time

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In loop quantum cosmology, Friedmann-LeMaître-Robertson-Walker space-times arise as well-defined approximations to specific *quantum* geometries. We initiate the development of a quantum theory of test scalar fields on these quantum geometries. Emphasis is on the new conceptual ingredients required in the transition from classical space-time backgrounds to quantum space-times. These include a ''relational time'' *à la* Leibniz, the emergence of the Hamiltonian operator of the test field from the quantum constraint equation, and ramifications of the quantum fluctuations of the background geometry on the resulting dynamics. The familiar quantum field theory on classical Friedmann-LeMaître-Robertson-Walker models arises as a well-defined reduction of this more fundamental theory.

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I. INTRODUCTION

Quantum field theory (QFT) on classical Friedmann-LeMaître-Robertson-Walker (FLRW) space-times is well developed and has had remarkable success in accounting for structure formation in inflationary cosmologies (see, e.g., [1]). In this analysis one assumes that the background space-time is adequately described by classical general relativity. During the inflationary era, this assumption is reasonable because, e.g., in the standard scenarios the matter density ρ, even at the onset of inflation, is less than $10^{-10}\rho_{\rm Pl}$, where $\rho_{\rm Pl}$ is the Planck density. However, even in an eternal inflation, the underlying classical space-time has a big-bang singularity [2]. The theory is thus incomplete. In particular, the presence of this singularity makes it awkward to introduce initial conditions, e.g., on the quantum state of matter.

To know what really happened in the Planck regime near the singularity, we need a quantum theory of gravity. While a fully satisfactory quantum gravity theory is still not available, over the past 2–3 years, loop quantum cosmology (LQC) has provided a number of concrete results on this Planck scale physics. (For recent reviews, see, e.g., [3,4].) LQC is a symmetry reduced version of loop quantum gravity (LQG) [5–7]), a nonperturbative, background-independent approach to the unification of general relativity and quantum physics. Here, space-time geometry is treated quantum mechanically from the start. In the symmetry reduced cosmological models these quantum geometry effects create a new repulsive force when space-time curvature enters the Planck regime. The force is so strong that the big bang is replaced by a specific type of quantum bounce [8–15]. The force rises very quickly once ρ exceeds $\sim 0.01\rho_{\rm Pl}$ to cause the bounce, but also dies very quickly after the bounce, once the density falls below this value. Therefore, the quantum space-time of LQC is very well approximated by the space-time continuum of general relativity once the curvature falls below the Planck scale. This scenario is robust in the sense that it is borne out for $k = 0$ models with or without a cosmological constant [16,17], $k = 1$ closed models, [18,19] $k = -1$ open models[1] [21], the Bianchi I model which incorporates anisotropies [22], and the $k = 0$ model with an inflationary potential with phenomenologically viable parameters [23].

In this paper, we will use the detailed quantum geometries that have been constructed in LQC for the $k = 0$, $\Lambda = 0$ FLRW models with a massless scalar field as a source. The full physical Hilbert space of LQC is infinite dimensional. Every physical state undergoes a quantum bounce in a precise sense [13]. However, for physical applications of interest here, we will consider only those states which are sharply peaked on a classical geometry at *some* late time and follow their evolution. Surprisingly, LQC predicts that dynamics of these states is well approximated by certain ''effective trajectories'' [24,25] in the gravitational phase space at *all* times, including the bounce point [12,13]. As one would expect, this effective trajectory departs sharply from the solution to Einstein's equation near the bounce. However, it does define a smooth space-time metric, but its coefficients now involve $\hbar$. These quantum corrections are extremely large in the Planck regime but, as indicated above, die off quickly, and the effective space-time is indistinguishable from the classical FLRW solution in the low curvature region.[2]

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[1]The current treatment of the $k = -1$ models is not entirely satisfactory because it regards the extrinsic curvature K_a^i as a connection and relies on holonomies constructed from it. However, a closer examination shows that this is not necessary [20].

[2]The availability of a singularity-free effective space-time can be extremely useful. For example, it has enabled one to show that, although Bousso's covariant entropy bound [26] is violated very near the singularity in classical general relativity, it is respected in the quantum space-time of LQC.

Thus, LQC provides specific, well-defined quantum geometries from which FLRW space-times emerge away from the Planck scale. At a fundamental level, one does not have a single classical metric but rather a probability amplitude for various metrics. So, the following question naturally arises: *How do quantum fields propagate on these quantum geometries?*

Availability of a satisfactory quantum theory of fields on a quantum geometry would provide new perspectives in a number of directions. First, it could provide a coherent theory of structure formation from first principles. For example, one may be able to specify the initial conditions either in the infinite past where quantum space-time is well approximated by a flat classical geometry, or at the bounce point which now replaces the big bang. Second, the theory is also of considerable importance from a more general conceptual perspective. Therefore, it should provide a bridge between quantum gravity and QFT in curved space-times. What precisely are the implications of the quantum fluctuations of geometry on the dynamics of other quantum fields? What, in particular, are the consequences of light cone fluctuations? Finally, this theory would lead to a rich variety of new avenues in mathematical physics. How is the relational time of quantum gravity related to the more familiar choices of time one makes in QFT in curved space-times? How do the standard anomalies of QFT on classical background geometries ''lift'' up to QFT on quantum geometries? What precisely are the approximations that enable one to pass from quantum QFT on quantum geometries to those on classical geometries?

The purpose of this paper is to provide the first steps to addressing these important issues. More precisely, we will present the basics of a framework to describe *test* quantum fields on the quantum FLRW geometries provided by LQC.

QFT in curved space-times has been developed in two directions. The first is the more pragmatic approach that cosmologists have developed to study structure formation, particle creation by given gravitational fields, and their backreaction on the geometry (see, e.g., [1]). Here, one uses the background geometry to make a mode decomposition and regards the quantum field as an assembly of oscillators. Typically, one focuses on one mode (or a finite number of modes) at a time and ignores the difficult functional analytical issues associated with the fact that the field in fact has an infinite number of degrees of freedom. The second direction is the more mathematical, algebraic approach that provides a conceptually complete home for the subject (see, e.g., [27,28]). Here the focus is on the structure of operator algebras, constructed ''covariantly'' using the background space-time geometry. States are treated as suitably regular positive linear functionals on the algebras. Not only is there no mode decomposition, but one does not tie oneself to any one Hilbert space. Our long-range goal is to generalize both sets of analyses to quantum space-times.

In this paper we will begin by following the more pragmatic approach: As in the literature on cosmology, we will use mode decomposition. However, in this analysis our emphasis will be on conceptual issues. First, in LQC one is led to a relational dynamics because there is no background space-time. More precisely, one ''deparametrizes'' the theory: The massless scalar field T—the matter source in the background space-time—is treated as the ''evolution parameter'' with respect to which the physical degrees of freedom—the density, volume, anisotropies, and other matter fields, if any—evolve. Therefore, in QFT on FLRW quantum geometries, it is natural to continue to use T as time. In QFT on classical FLRW space-times, on the other hand, one generally uses the conformal or proper time as the evolution parameter. We will resolve this conceptual tension. Second, in the quantum gravity perspective, dynamics is encoded in the quantum constraint equation. In QFT on a classical FLRW geometry, on the other hand, dynamics of the test quantum field is generated by a Hamiltonian. We will show how this Hamiltonian naturally emerges from the quantum constraint in a suitable approximation. The analysis is quite intricate because it involves different notions of time (or, equivalently, lapse fields) at different stages. Finally, we will be able to pinpoint the implications of the quantum fluctuations of geometry on the dynamics of the test quantum field. This discussion will, in turn, enable us to spell out the approximations that are essential to pass from the QFT on a quantum FLRW geometry to that on its classical counterpart.

The paper is organized as follows. In Sec. II we summarize key properties of quantum *space-time* geometries that emerge from LQC and recall the relevant features of QFT on a classical FLRW background. In Sec. III we introduce the Hamiltonian setup to describe test fields on classical and quantum background geometries, and in Sec. IV we show how the two are related. Section V contains a summary and presents the outlook.

*Remark.—*Much of the detailed, recent work in LQC assumes that the matter source is a massless scalar field T which, as we saw, plays the role of a global, relational time variable. The overall strategy is flexible enough to allow *additional* matter fields. The new issues that arise are technical, such as whether the relevant operators continue to be essentially self-adjoint. However, if, as in the simplest inflationary scenario, there is only a massive scalar field—and no massless ones—one faces new conceptual issues. In this case the scalar field serves as a good time variable only ''locally.'' That is, one has to divide evolution into ''epochs'' or ''patches,'' in each of which the scalar field is monotonic along dynamical trajectories. The discussion of the quantum bounce is not much more complicated because the bounce occurs in a single patch [23]. The problem of joining together these patches, on the other hand, is more complicated. Although it can be managed in principle (see, e.g., [29]), at present it seems difficult to handle in practice.

II. BACKGROUND QUANTUM GEOMETRY

LQC provides us with a nonperturbative quantum theory of FLRW cosmologies. Because it is based on a Hamiltonian treatment, the relation to the classical FLRW models was spelled out through dynamical trajectories in the classical phase space [12]. In particular, the emphasis has been on the relational Dirac observables, such as the matter density, anisotropies, and curvature at a given value of the scalar field. On the other hand, quantum field theory on classical FLRW backgrounds is developed on given classical space-times, rather than on dynamical trajectories in the phase space of general relativity. Therefore, as a first step we need to reformulate one of these descriptions using the paradigm used in the other. In this section, we will recast the LQC description, emphasizing space-times over phase space trajectories. Relation to the cosmology literature will then become more transparent.

We will focus on the $k = 0$, $\Lambda = 0$ FLRW models with a massless scalar field as a source. To avoid a discussion of boundary conditions on test fields in Sec. III, we will assume that the spatial three-manifold is $\mathbb{T}^3$, a torus with coordinates $x^i \in (0, \ell)$. It will be clear from our discussion that with appropriate changes the analysis can be extended to include a cosmological constant, or anisotropies, or closed $k = 1$ universes.

A. Space-time geometries and phase space trajectories

In this subsection we will clarify the relation between various notions of time that feature in LQC and set up a dictionary between the phase space and space-time descriptions.

Spatial homogeneity and isotropy imply that the space-time metric has the form

$$g_{ab}dx^a dx^b = -dt^2 + q_{ij}dx^i dx^j \equiv -dt^2 + a^2 d\vec{x}^2 \quad (2.1)$$

where q_{ij} is the physical spatial metric and a is the scale factor. Here the coordinate t is the proper time along the world lines of observers moving orthogonal to the homogeneous slices.

As explained in Sec. I, in LQC one uses a relational time defined by a massless scalar field which serves as a matter source. Because of this and because we will also have a test scalar field φ in Sec. III, we will denote the massless scalar source by T. Since T satisfies the wave equation with respect to g_{ab}, in LQC it is most natural to consider the *harmonic time coordinate* τ satisfying $\Box\tau = 0$. Then the space-time metric assumes the form

$$g_{ab}dx^a dx^b = -a^6 d\tau^2 + q_{ij}dx^i dx^j \equiv -a^6 d\tau^2 + a^2 d\vec{x}^2. \quad (2.2)$$

Let us now spell out the relation of this space-time metric to the phase space trajectories. In LQC, the gravitational part of the phase space is conveniently coordinatized by a canonically conjugate pair (ν, b), where ν is essentially the volume of the universe and b the Hubble parameter $\dot{a}/a$ (where, as usual, the "dot" refers to a derivative with respect to proper time t) [13,14]. More precisely, the volume is given by

$$V \equiv \ell^3 a^3 = 2\pi\gamma\ell_{\rm Pl}^2|\nu| \quad (2.3)$$

and the Hubble parameter by $\dot{a}/a = b/\gamma$, where γ is the so-called Barbero-Immirzi parameter of LQG.[3] (Its value, $\gamma \approx 0.24$, is fixed by black hole entropy calculations.) Throughout this paper, we will pass freely between V, ν, and the scale factor a.

The canonically conjugate pair for the scalar field is $(T, P_{(T)})$. Dynamics is generated by the Hamiltonian constraint, NC, where N is the lapse function and C the constraint function:

$$C = \frac{P_{(T)}^2}{2V} - \frac{3}{8\pi G}\frac{b^2}{\gamma^2}V \approx 0 \quad (2.4)$$

where, as usual, the weak equality holds on the constraint hypersurface. If one uses the time coordinate t, then it follows from (2.1) that the lapse is $N_t = 1$, while if one uses τ, (2.2) implies that the lapse is $N_\tau = a^3$. In the second case, the Hamiltonian constraint is

$$C_\tau := N_\tau C \equiv \frac{P_{(T)}^2}{2\ell^3} - \frac{3}{8\pi G}\frac{b^2}{\gamma^2}\frac{V^2}{\ell^3}, \quad (2.5)$$

whence the time evolution of the scalar field is given by

$$T = \frac{P_{(T)}}{\ell^3}\tau. \quad (2.6)$$

(Here we have set the integration constant to zero for simplicity). $P_{(T)}$ is a constant of motion which, for definiteness, will be assumed to be positive. Then, as one would expect, in any solution to the field equations the scalar field T grows linearly in the harmonic time τ. Thus, although T does not have the physical dimensions of time, it is a good evolution parameter. Therefore, following the LQC literature, we will refer to it as the *relational time* parameter. On any given solution, we can freely pass from τ to T and write the space-time metric as

[3]Following LQG, in LQC one uses orthonormal frames rather than metrics. Since these frames can be regarded as "square roots" of metrics, the configuration space is doubled. $\nu, b \in \mathbb{R}^2$ are constructed from the orthonormal frame and its time derivative, and the sign of ν depends on the orientation of the frame. The canonical commutation relations are $[\hat{b}, \hat{\nu}] = 2i$.

$$g_{ab}dx^a dx^b = -\frac{a^6\ell^6}{P_{(T)}^2}dT^2 + q_{ij}dx^i dx^j \equiv \frac{a^6\ell^6}{P_{(T)}^2}dT^2 + a^2 d\vec{x}^2. \tag{2.7}$$

The only difference from (2.2) is that the lapse is modified: $N_T = (\ell^3/P_{(T)})N_\tau$, whence, in any *given* space-time, two lapse functions are related just by a constant. However, in the *phase space*, $P_{(T)}$ varies from one dynamical trajectory to another, whence the relation is much more subtle. If we regard T as a parameter, τ evolves nontrivially on the full phase space, and vice versa. In quantum gravity, we do not have a fixed space-time, but a probability amplitude for various geometries. Therefore, the situation in the phase space is a better reflection of what happens in the quantum theory. Indeed, as we will see in Sec. III, the difference between τ and T plays a big role there.

Since the relation between the phase space and space-time notions is important for our subsequent discussion, we will conclude with a useful dictionary:

(i) A point in the phase space.— A homogeneous slice in space-time (i.e., $\mathbb{T}^3 \times \mathbb{R}$) equipped with the initial data for the gravitational and scalar fields;

(ii) A curve in the phase space along which T is monotonic.—A metric g_{ab} and a scalar field T on space-time;

(iii) A curve in the phase space along which T is monotonic *and* $P_{(T)}$ is constant.—A metric g_{ab} and a scalar field T satisfying $\Box T = 0$ on space-time; and, finally,

(iv) A dynamical trajectory in the phase space.—A solution (g_{ab}, T) to the Einstein-Klein-Gordon equation on space-time.

B. Quantum FLRW space-times

In LQC one first constructs the quantum kinematics for the symmetry reduced models by faithfully mimicking the unique kinematics of LQG, selected by the requirement of background independence [30,31]. One then writes the quantum counterpart of the Hamiltonian constraint (2.5) as a self-adjoint operator on the kinematical Hilbert space:

$$\hat{C}_\tau \Psi_o(\nu, T) = -\frac{\hbar^2}{2\ell^3}(\partial_T^2 + \Theta)\Psi_o(\nu, T), \tag{2.8}$$

where Θ turns out to be a difference operator in ν given by

$$\Theta\Psi_o(\nu, T) = \frac{3\pi G}{\lambda^2}\nu[(\nu + 2\lambda)\Psi_o(\nu + 4\lambda) - 4\nu\Psi_o(\nu) + (\nu - 2\lambda)\Psi_o(\nu - 4\nu)]. \tag{2.9}$$

Here, $\lambda^2 = 4\sqrt{3}\pi\gamma\ell_{\rm Pl}^2$ is the smallest nonzero eigenvalue of the LQG area operator (on states relevant to homogeneity and isotropy) [3,22,32], and we use the subscript (or superscript) o to emphasizes that structures developed in this section refer to what will serve as the *background* quantum geometry. Physical states must satisfy[4]

$$\hat{C}_\tau \Psi_o(\nu, T) = 0. \tag{2.10}$$

A standard "group averaging procedure," which is applicable to a wide class of constrained systems, then provides the scalar product, enabling us to construct the physical Hilbert space $\mathcal{H}_{\rm phy}^o$. Since the form of the constraint (2.8) resembles the Klein-Gordon equation on a (fictitious) static space-time coordinatized by ν, T, as one might expect, $\mathcal{H}_{\rm phy}^o$ is built out of "positive-frequency solutions" to (2.8). More precisely, $\mathcal{H}_{\rm phy}^o$ consists of solutions to

$$-i\hbar\partial_T\Psi_o(\nu, T) = \hat{H}_o\Psi_o(\nu, T) \quad \text{where } \hat{H}_o = \hbar\sqrt{\Theta}, \tag{2.11}$$

with a finite norm with respect to the scalar product

$$\langle\Psi_o, \Psi_o\rangle = \frac{\lambda}{\pi}\sum_{\nu=4n\lambda}\frac{1}{|\nu|}\bar{\Psi}_o(\nu, T_0)\Psi_o'(\nu, T_0), \tag{2.12}$$

where the right side can be evaluated at any internal time T_0. Note that in their ν dependence, physical states have support on the lattice $\nu = 4n\lambda$, where n ranges over all integers (except zero). We will generally work in the Schrödinger representation. Then, the states can be regarded as functions $\Psi_o(\nu)$ of ν which have a finite norm (2.12) and which evolve via (2.11). The Hilbert space spanned by $\Psi(\nu)$ will be denoted by $\mathcal{H}_{\rm geo}$. For later use we note that the classical expression (2.3) of volume implies that the volume operator $\hat{V}$ acts on $\mathcal{H}_{\rm geo}$ simply by multiplication:

$$\hat{V}\Psi_o(\nu) = 2\pi\gamma\ell_{\rm Pl}^2|\nu|\Psi_o(\nu). \tag{2.13}$$

Every element Ψ_o of $\mathcal{H}_{\rm phy}^o$ represents a four-dimensional quantum geometry. However, to make contact with QFT on classical FLRW space-times, we are interested only in a subset of these states which can be described as follows. Choose a classical, expanding FLRW space-time in which $P_{(T)} \gg \hbar$ (in the classical units $G = c = 1$) and a homogeneous slice at a late time T_o, when the matter density and curvature are negligibly small compared to the Planck scale. This defines a point p in the classical phase space. Then, one can introduce coherent states $\Psi_o(\nu, T_o)$ in $\mathcal{H}_{\rm geo}$ which are sharply peaked at p [11–13]. Let us "evolve" them in the internal time T using (2.11). One can show [12,13] that these states remain sharply peaked on the classical trajectory passing through p for all $T > T_o$. In the backward time evolution, they do so till the density reaches approximately 1% of the Planck

[4]Recall from footnote 2 that $\nu \to -\nu$ just corresponds to changes in the orientation of the orthonormal frame, which does not change the metric. Since the theory does not involve any spinor fields, physics is insensitive to this orientation. Therefore, states must also satisfy $\Psi(\nu, T) = \Psi(-\nu, T)$.

density. As explained in Sec. I, even in the deep Planck regime the wave function remains sharply peaked but the peak now follows an effective trajectory which undergoes a quantum bounce. At the bounce point the matter density attains a maximum, $\rho_{\rm max} \approx 0.41\rho_{\rm Pl}$.[5] After the bounce the density and the space-time curvature start decreasing, and once the density falls below about 1% of the Planck density, the effective trajectory becomes essentially indistinguishable from a classical FLRW trajectory. Although the effective trajectory cannot be approximated by any classical solution in a neighborhood of the bounce point, $P_{(T)}$ is constant along the entire effective trajectory. The dictionary given at the end of Sec. II A then implies that the effective space-time has a contracting FLRW branch in the past and an expanding FLRW branch in the future. The scalar field T satisfies $\Box T = 0$ everywhere, but Einstein's equations break down completely in an intermediate region. Thanks to the quantum evolution equation (2.10), the two branches are joined in a deterministic fashion in this region. *By a quantum background geometry, we will mean a physical state* $\Psi_o(\nu, T)$ *with these properties*. There is a large class of such states, and our considerations will apply to all of them.

Of particular interest to us are the volume operators $\hat{V}_{T_0}$ on $\mathcal{H}^o_{\rm phy}$ representing the volume of the universe at any fixed instant T_0 of internal time:

$$[\hat{V}_{T_0}\Psi_o](\nu, T) = e^{(i/\hbar)\hat{H}_o(T-T_0)}(2\pi\gamma\ell_{\rm Pl}^2|\nu|) \times e^{-(i/\hbar)\hat{H}_o(T-T_0)}\Psi_o(\nu, T). \tag{2.14}$$

Thus, the action of $\hat{V}_{T_0}$ on any physical state $\Psi_o(\nu, T)$ is obtained by evolving that state to $T = T_0$, acting on it by the volume operator, and then evolving the resulting function of ν using (2.11). Each $\hat{V}_{T_0}$ is a positive-definite self-adjoint operator. Hence one can define any (measurable) function of $\hat{V}_{T_0}$—such as the scale factor $\hat{a}_{T_0}$—via its spectral decomposition. Finally, the matter density operator $\hat{\rho}_{T_0}$ at time $T = T_0$ is given by

$$\hat{\rho}_{T_0} = \frac{1}{2}\hat{V}_{T_0}^{-1}\hat{P}_{(T)}^2\hat{V}_{T_0}^{-1} \equiv \frac{\hbar^2}{2}\hat{V}_{T_0}^{-1}\Theta\hat{V}_{T_0}^{-1}. \tag{2.15}$$

As explained above, in background quantum geometries $\Psi_o(\nu, T)$ considered in this paper, the expectation values of $\hat{\rho}_T$ attain their maximum value $\rho_{\rm max} \approx 0.41\rho_{\rm Pl}$ at the bounce point.

In the kinematical setting, $\hat{\nu}$, $\hat{T}$, $\hat{P}_{(T)}$, Θ are independent self-adjoint operators. However, in the passage to the physical Hilbert space $\mathcal{H}^o_{\rm phy}$, a "deparametrization" occurs as in the quantum theory of a parametrized particle (see, e.g., [34]). On the physical sector we no longer have an operator $\hat{T}$ but just a parameter T, and the operator $\hat{P}^2_{(T)}$ gets identified with $\hbar^2\Theta$. Consequently, the space-time metric (2.7) can be represented as a self-adjoint operator on $\mathcal{H}^o_{\rm phy}$ as follows [33]:

$$\hat{g}_{ab}dx^a dx^b = -{:}\hat{V}_T^2\hat{H}_o^{-2}{:}dT^2 + \hat{q}_{ij}dx^i dx^j \equiv {:}\hat{V}_T^2\hat{H}_o^{-2}{:}dT^2 + \hat{V}_T^{2/3}d\vec{x}^2. \tag{2.16}$$

Thus, the geometry is quantum because the metric coefficients $\hat{g}_{TT}$ and $\hat{q}_{ij}$ are now quantum operators. In (2.16), a suitable factor ordering—denoted by: :—has to be chosen because the volume operator $\hat{V}_T$ does not commute with the Hamiltonian $\hat{H}_o$ of the background quantum theory. The simplest choice would be to use an anticommutator, but it would be more desirable if the ordering was determined by some general principles. (Note that $\hat{H}_o^{-2}$ is well defined because $\hat{H}_o$ is a positive self-adjoint operator.)

III. THE TEST QUANTUM FIELD

This section is divided into two parts. In the first we summarize the essential features of QFT on classical FLRW backgrounds in a language that is well suited for our generalization to quantum backgrounds, and in the second we carry out the generalization.

A. QFT on classical FLRW backgrounds

As in Sec. II, let us fix a four-manifold $M = \mathbb{T}^3 \times \mathbb{R}$, equipped with coordinates $x^j \in (0, \ell)$ and $x_0 \in \mathbb{R}$. Consider on it a FLRW four-metric g_{ab} given by

$$g_{ab}dx^a dx^b = -N_{x_0}^2(x_0)dx_0^2 + a^2(x_0)d\vec{x}^2, \tag{3.1}$$

where, as usual, the lapse function N_{x_0} depends on the choice of time coordinate x_0. Consider a real, massive, test Klein-Gordon field φ satisfying $(\Box - m^2)\,\varphi = 0$ on this classical space-time (M, g_{ab}). Note that φ is *not* required to be homogeneous. Quantum theory of this field can be described with various degrees of rigor and generality. As explained in Sec. I, in this paper, we will consider the simplest version in terms of mode decomposition.

The canonically conjugate pair for the test scalar field consists of fields $(\varphi, \pi_{(\varphi)})$ on a $x_0 = $ const slice. Let us perform Fourier transforms:

$$\varphi(x_j, x_0) = \frac{1}{(2\pi)^{3/2}}\sum_{\vec{k}\in\mathcal{L}}\varphi_{\vec{k}}(x_0)e^{ik_jx^j} \quad \text{and}$$
$$\pi_{(\varphi)}(x_j, x_0) = \frac{1}{(2\pi)^{3/2}}\sum_{\vec{k}\in\mathcal{L}}\pi_{\vec{k}}(x_0)e^{ik_jx^j}, \tag{3.2}$$

where $\mathcal{L}$ is the three-dimensional lattice spanned by $(k_1, k_2, k_3) \in ((2\pi/\ell)\mathbb{Z})^3$, $\mathbb{Z}$ being the set of integers. The Fourier coefficients are canonically conjugate, $\{\varphi_{\vec{k}}, \pi_{\vec{k}'}\} = \delta_{\vec{k},-\vec{k}'}$, and since $\varphi(\vec{x}, x_0)$ is real, they satisfy

[5] The existence of this maximum value does *not* follow simply from the fact that $|\nu|$ is bounded below by 4λ. Its origin is more subtle [13,33]: $\hat{\rho} = (1/2)\hat{V}^{-1}\hat{P}^2_{(T)}\hat{V}^{-1}$ and the maximum value, $0.41\rho_{\rm Pl}$, of $\langle\hat{\rho}\rangle$ is the same no matter how large $P_{(T)} = \langle\hat{P}_{(T)}\rangle$ is.

the conditions $\varphi_{\vec{k}} = \bar{\varphi}_{-\vec{k}}$ and $\pi_{\vec{k}} = \bar{\pi}_{-\vec{k}}$. The time-dependent Hamiltonian (generating evolution in x_0) is given by

$$H_{\varphi}(x_0) = \frac{1}{2}\int \frac{N_{x_0}(x_0)}{a^3(x_0)}[\pi^2_{(\varphi)} + a^4(x_0)(\partial_i\varphi)^2 + m^2a^6(x_0)\varphi^2]d^3x = \frac{N_{x_0}(x_0)}{2a^3(x_0)}\sum_{\vec{k}\in\mathcal{L}} \bar{\pi}_{\vec{k}}\pi_{\vec{k}} + (\vec{k}^2a^4(x_0) + a^6(x_0)m^2)\bar{\varphi}_{\vec{k}}\varphi_{\vec{k}}. \quad (3.3)$$

In the literature, the test scalar field φ is often regarded as an assembly of harmonic oscillators, one for each mode. To pass to this description, first note that, because of the reality conditions, the Fourier modes are interrelated. One can find an independent set by, e.g., considering the sublattices $\mathcal{L}^{\pm}$ of $\mathcal{L}$ as follows:

$$\mathcal{L}^+ = \{\vec{k}\colon k_3 > 0\} \cup \{\vec{k}\colon k_3 = 0, k_2 > 0\} \cup \{\vec{k}\colon k_3 = 0, k_2 = 0, k_1 > 0\} \quad \text{and} \quad \mathcal{L}^- = \{\vec{k}\colon -\vec{k} \in \mathcal{L}^+\}. \quad (3.4)$$

Then, for each $\vec{k} \in \mathcal{L}^+$, we can introduce real variables $q_{\pm\vec{k}}$, $p_{\pm\vec{k}}$,

$$\varphi_{\vec{k}} = \frac{1}{\sqrt{2}}(q_{\vec{k}} + iq_{-\vec{k}}) \quad \text{and} \quad \pi_{\vec{k}} = \frac{1}{\sqrt{2}}(p_{\vec{k}} + ip_{-\vec{k}}). \quad (3.5)$$

The pair $(q_{\pm\vec{k}}, p_{\pm\vec{k}})$ is canonically conjugate for each $\vec{k} \in \mathcal{L}^+$. In terms of these variables, the Hamiltonian becomes

$$H_{\varphi}(x_0) = \frac{N_{x_0}(x_0)}{2a^3(x_0)}\sum_{\vec{k}\in\mathcal{L}} p^2_{\vec{k}} + (\vec{k}^2a^4(x_0) + m^2a^6(x_0))q^2_{\vec{k}} \quad (3.6)$$

where we have set $q_0 := \varphi_{\vec{k}=0}$ and $\pi_0 := \pi_{\vec{k}=0}$. Thus, the Hamiltonian for the test field is the same as that for an assembly of harmonic oscillators, one for each $\vec{k} \in \mathcal{L}$.

To pass to the quantum theory, let us focus on just one mode $\vec{k}$. Then we have a single harmonic oscillator. So the Hilbert space is given by $H_{\vec{k}} = L^2(\mathbb{R})$, the operator $\hat{q}_{\vec{k}}$ acts by multiplication, $\hat{q}_{\vec{k}}\psi(q_{\vec{k}}) = q_{\vec{k}}\psi(q_{\vec{k}})$, and $\hat{p}_{\vec{k}}$ acts by differentiation, $\hat{p}_{\vec{k}}\psi(q_{\vec{k}}) = -i\hbar d\psi/dq_{\vec{k}}$. The time evolution is dictated by the time-dependent Hamiltonian operator $\hat{H}_{\vec{k}}(x_0)$:

$$i\hbar\partial_{x_0}\psi(q_{\vec{k}}, x_0) = \hat{H}_{\vec{k}}(x_0)\psi(q_{\vec{k}}, x_0) \equiv \frac{N_{x_0}(x_0)}{2a^3(x_0)}[\hat{p}^2_{\vec{k}} + (\vec{k}^2a^4(x_0) + m^2a^6(x_0))\hat{q}^2_{\vec{k}}]\psi(q_{\vec{k}}, x_0). \quad (3.7)$$

In this theory, there is considerable freedom in choosing the time coordinate x_0 (and hence the lapse function N_{x_0}). One generally chooses x_0 to be either the conformal time η or the proper time t. However, as we saw in Sec. II B, in quantum geometry only the relational time T is a parameter; η, t, and even the harmonic time τ become operators [33]. Therefore, in QFT on a quantum geometry, while it is relatively straightforward to analyze evolution with respect to T, conceptually and technically it is more subtle to describe evolution with respect to conformal, proper, or harmonic time (as it requires the introduction of conditional probabilities). In the standard QFT on classical FLRW space-times, on the other hand, T plays no role; indeed, the source of the background geometry never enters the discussion. This tension is conceptually significant and needs to be resolved to relate QFT on classical and quantum FLRW geometries.

B. QFT on quantum FLRW backgrounds

Recall first that in full general relativity dynamics is generated by constraints. Our system of interest is general relativity coupled to a massless scalar field T and a massive scalar field φ, where T is spatially homogeneous and φ is, in general, inhomogeneous but regarded as a *test* field propagating on the homogeneous, isotropic geometry created by T. Therefore, we can start with the constraint functions on the *full* phase space of the gravitational field, T and φ, but impose isotropy and homogeneity on the gravitational field and T and retain terms which are at most quadratic in φ and $\pi_{(\varphi)}$. The fact that we are ignoring the backreaction of φ on the gravitational field implies that, among the infinitely many constraints of this theory, only the zero mode of the scalar constraint is relevant for us. That is, we need to smear the scalar constraint *only* with homogeneous lapse functions (and can ignore the Gauss and the vector constraints). For concreteness, as in Sec. II A, we will choose the harmonic time coordinate τ and the corresponding lapse function $N_\tau = a^3$. Then, in the truncated theory now under consideration, the scalar constraint (Sec. II A) is replaced by

$$C_\tau \equiv N_\tau C = \frac{P^2_{(T)}}{2\ell^3} - \frac{3}{8\pi G}\frac{b^2}{\gamma^2}\frac{V}{\ell^3} + \frac{1}{2}\int[\pi^2_{(\varphi)} + a^4(\partial_i\varphi)^2 + m^2a^6\varphi^2]d^3x \approx 0. \quad (3.8)$$

(Recall that the volume and the scale factor are related by $V = \ell^3a^3$.) If we focus just on the $\vec{k}$th mode, the constraint simplifies further:

$$C_{\tau,\vec{k}} = \frac{P^2_{(T)}}{2\ell^3} - \frac{3}{8\pi G}\frac{b^2}{\gamma^2}\frac{V}{\ell^3} + H_{\tau,\vec{k}}, \quad (3.9)$$

where

$$H_{\tau,\vec{k}} = \tfrac{1}{2}[p_{\vec{k}}^2 + (\vec{k}^2 a^4 + m^2 a^6) q_{\vec{k}}^2]. \tag{3.10}$$

In quantum theory, then, physical states $\Psi(\nu, q_{\vec{k}}, T)$ must be annihilated by this constraint, i.e., must satisfy

$$-\hbar^2 \partial_T^2 \Psi(\nu, q_{\vec{k}}, T) = [\hat{H}_o^2 - 2\ell^3 \hat{H}_{\tau,\vec{k}}]\Psi(\nu, q_{\vec{k}}, T), \tag{3.11}$$

where, as in Sec. II B, $\hat{H}_o^2 = \hbar^2\Theta$ is the difference operator defined in (2.9). (Although $\hat{a}$ is an operator, it commutes with $\hat{q}_{\vec{k}}$ and $\hat{p}_{\vec{k}}$ on the kinematical Hilbert space. So, there are no factor ordering subtleties in the definition of $\hat{H}_{\tau,\vec{k}}$.) As in Sec. II B, the construction of the physical inner product requires us to take the "positive-frequency" square root of this equation. More precisely, on the tensor product $\mathcal{H}_{\rm geo} \otimes L^2(\mathbb{R})$ of the quantum geometry Hilbert space $\mathcal{H}_{\rm geo}$ and the $\vec{k}$-mode Hilbert space $L^2(\mathbb{R})$, the operator $[\hat{H}_o^2 - 2\ell^3 \hat{H}_{\tau,\vec{k}}]$ on the right-hand side of (3.11) is symmetric, and we assume that it can be made self-adjoint on a suitable domain. On the physical Hilbert space, this operator gets identified with $\hat{P}_{(T)}^2$. Since classically $P_{(T)}^2$ is a positive Dirac observable, we are led to restrict ourselves to the positive part of the spectrum of $[\hat{H}_o^2 - 2\ell^3 \hat{H}_{\tau,\vec{k}}]$ and then solve the evolution equation

$$\begin{aligned} -i\hbar\partial_T \Psi(\nu, q_{\vec{k}}, T) &= [\hat{H}_o^2 - 2\ell^3 \hat{H}_{\tau,\vec{k}}]^{1/2}\Psi(\nu, q_{\vec{k}}, T) \\ &=: \hat{H}\Psi(\nu, q_{\vec{k}}, T). \end{aligned} \tag{3.12}$$

The solutions are in the physical Hilbert space $\mathcal{H}_{\rm phy}$ of the truncated theory, provided they have a finite norm with respect to the inner product:

$$\langle\Psi_1|\Psi_2\rangle = \frac{\lambda}{\pi}\sum_{\nu=4n\lambda}\frac{1}{|\nu|} \times \int_{-\infty}^{\infty} dq_{\vec{k}} \bar{\Psi}_1(\nu, q_{\vec{k}}, T_0)\Psi_2(\nu, q_{\vec{k}}, T_0) \tag{3.13}$$

where the right side is evaluated at *any* fixed instant of internal time T_0. As one might expect, the physical observables of this theory are the Dirac observables of the background geometry—such as the time-dependent density and volume operators $\hat{\rho}(T)$ and $\hat{V}(T)$—and observables associated with the test field, such as the mode operators $\hat{q}_{\vec{k}}$ and $\hat{p}_{\vec{k}}$.

Formally, this completes the specification of the quantum theory of the test field $\hat{\phi}$ on a quantum FLRW background geometry. We have presented this theory (as well as the QFT on a classical background in Sec. III A) using the Schrödinger picture because this is the description one is naturally led to when, following Dirac, one imposes quantum constraints to select physical states. However, at the end of the process it is straightforward to reexpress the theory in the Heisenberg picture.

*Remark,—*In this section we began with the constraint (3.8) on the classical phase space spanned by $(\nu, b; T, P_{(T)}; \varphi, \pi_{(\varphi)})$. Solutions to this theory do include a backreaction of the field φ but just on the homogeneous mode of the classical geometry. In the final quantum theory, the Hamiltonian of the field φ features on the right side of (3.12) whence, in the Heisenberg picture, it affects the evolution of geometric operators. As in the classical theory, this evolution incorporates backreaction of the field $\hat{\varphi}$ but just on the homogeneous mode of the quantum geometry. Mathematically, we have a closed system involving $\hat{\nu}$, $\hat{\varphi}$, T, whence this inclusion of the backreaction is consistent. However, physically it is not as meaningful because we have ignored the backreaction at the same order that would add inhomogeneities to the quantum geometry. So, from a physical viewpoint, *all* corrections to quantum geometry which are quadratic in $\hat{\varphi}$ should be consistently ignored. We will explicitly impose this restriction in Sec. IV C. However, the classical theory determined by (3.8) and the quantum theory constructed in this section can be directly useful in some applications where it is meaningful to ignore inhomogeneous metric perturbations and study the homogeneous mode, including the backreaction corrections.

IV. COMPARISON

In this section we will compare QFT on a classical background discussed in Sec. III A and QFT on quantum FLRW geometries discussed in Sec. III B. The discussion is divided into three subsections which provide the successively stronger simplifications of the dynamical equation (3.12) that are needed to arrive at the dynamical equation (3.7) on a classical FLRW space-time.

A. Simplification of the evolution equation

Let us begin by using the test field approximation. Since the backreaction of the scalar field φ is neglected, the theory constructed in Sec. III B can be physically trusted only on the sector on which $\hat{H}_o^2$ dominates over $2\ell^3\hat{H}_{\tau,\vec{k}}$. On this sector, one can expand out the square root on the right side of (3.12) in a useful fashion. For this, we will use an operator identity: Given self-adjoint operators A, B (which need not commute) such that A is positive definite, we have the following expansion:

$$\begin{aligned} (A+B)^{1/2} &= A^{1/4}(1 + A^{-(1/2)}BA^{-(1/2)})^{1/2}A^{1/4} \\ &= A^{1/4}(1 + \tfrac{1}{2}A^{-(1/2)}BA^{-(1/2)} + \ldots)A^{1/4}. \end{aligned} \tag{4.1}$$

If we set $A = \hat{H}_o^2$, $B = -2\ell^3\hat{H}_{\tau,\vec{k}}$ and ignore the higher order ... terms, the right side of (3.12) simplifies:

$$-i\hbar\partial_T\Psi(\nu, q_{\vec{k}}, T) = (\hat{H}_o - (\ell^{-3}\hat{H}_o)^{-(1/2)} \times \hat{H}_{\tau,\vec{k}}(\ell^{-3}\hat{H}_o)^{-(1/2)})\Psi(\nu, q_{\vec{k}}, T). \quad (4.2)$$

This evolution equation has several noteworthy features. First, there was no factor ordering freedom; a specific ordering naturally emerged from the general expansion (4.1). Next, we will now show that the second term on the right side of (4.2) has a direct interpretation. In the classical theory, $H_{\tau,\vec{k}}$ is the Hamiltonian generating evolution in harmonic time τ. Since the corresponding lapse function N_τ is related to the lapse function N_T corresponding to the relational time T via $N_T = (P_T\ell^3)^{-1}N_\tau$, the Hamiltonian generating evolution in T is given by $H_{T,\vec{k}} = (\ell^{-3}P_T)^{-1}H_{\tau,\vec{k}} \approx (\ell^{-3}H_o)^{-1}H_{\tau,\vec{k}}$, where in the last step we have again used the test field approximation. The second term on the right side of (4.2) is *precisely* a specific quantization of $H_{T,\vec{k}}$. This is just as one would physically expect because the left side of (4.2) is the derivative of the quantum state with respect to T. Thus, we can rewrite (4.2) as

$$-i\hbar\partial_T\Psi(\nu, q_{\vec{k}}, T) = (\hat{H}_o - \hat{H}_{T,\vec{k}})\Psi(\nu, q_{\vec{k}}, T). \quad (4.3)$$

The nontriviality lies in the fact that this evolution equation arose from a systematic quantization of the (ν, φ, T) system, where geometry is also quantum. As in LQC we began with the quantum constraint operator associated with the harmonic time, and then used the group averaging procedure to find the physical Hilbert space. This naturally led us to take a square root of the quantum constraint. Then an expansion, which is valid in the test field approximation, automatically provided the extra factor to rescale the lapse operator just in the right manner to pass from the harmonic to the relational time. Thus, there is coherence between the constrained dynamics, various notions of time involved, deparametrization of the full theory, and the test field approximation.

B. Interaction picture

The simplified evolution equation (4.3) is rather analogous to the Schrödinger equation (3.7) in QFT on a classical FLRW background. However, there are two key differences. First, in (4.3) the background geometry appears through *operators* $\hat{V}$ and $\hat{H}_o$, while in (3.7) it appears through the *classical* scale factor $a(x_0)$ and (if we set $x_0 = T$) the constant $\ell^3/P_{(T)} = N_T/a^3$ determined by the momentum of the background scalar field. The fact that there are operators on the Hilbert space $\mathcal{H}_{\rm geo}$ of quantum geometry in the first case and classical fields on space-time M in the second is not surprising. But there is also a more subtle, second difference. The operators $\hat{H}_o$ and $\hat{V}$ which features on the right side of (4.3) do *not* depend on time[6]: $\hat{V}\Psi(\nu, q_{\vec{k}}, T) = 2\pi\gamma\ell_{\rm Pl}^2|\nu|\Psi(\nu, q_{\vec{k}}, T)$ and $\hat{H}_o\Psi(\nu, q_{\vec{k}}, T) = \hbar\sqrt{\Theta}\Psi(\nu, q_{\vec{k}}, T)$. The scale factor $a(x_0)$ that appears in (3.7), on the other hand, is explicitly time dependent. This is because while (4.2) provides a quantum evolution equation for the state $\Psi(\nu, q_{\vec{k}}, T)$ that depends on (the $\vec{k}$th mode of) the test field φ *and* the quantum geometry (encoded in ν), (3.7) evolves the state $\psi(q_{\vec{k}}, T)$ just of the test scalar field on the given time-dependent background geometry [encoded in $a(x_0)$].

To make the two evolutions comparable, therefore, we need to recast (4.3) in such a way that the test field evolves on a background, *time-dependent* quantum geometry. This can be readily achieved by working in the "interaction picture." More precisely, it is natural to regard $\hat{H}_o$ in (4.3) as the Hamiltonian of the heavy degree of freedom and $\hat{H}_{T,\vec{k}}$ as a perturbation governing the light degree of freedom, and, as in the interaction picture, set

$$\Psi_{\rm int}(\nu, q_{\vec{k}}, T) := e^{-(i/\hbar)\hat{H}_o(T-T_0)}\Psi(\nu, q_{\vec{k}}, T), \quad (4.4)$$

where T_0 is any fixed instant of relational time. Then, (4.3) yields the following evolution equation for $\Psi_{\rm int}$:

$$i\hbar\partial_T\Psi_{\rm int}(\nu, q_{\vec{k}}, T) = \tfrac{1}{2}(\ell^3\hat{H}_o)^{-(1/2)}[p_{\vec{k}}^2 + (\vec{k}^2\hat{a}^4(T) + m^2\hat{a}^6(T))q_{\vec{k}}^2] \times (\ell^3\hat{H}_o)^{-(1/2)}\Psi_{\rm int}(\nu, q_{\vec{k}}, T) =: \hat{H}_{T,\vec{k}}^{\rm int}\Psi_{\rm int}(\nu, q_{\vec{k}}, T). \quad (4.5)$$

Here the operators $\hat{a}(T)$ (and their powers) are defined on the Hilbert space $\mathcal{H}_{\rm geo}$ of quantum geometry (now tied to the internal time T_o):

$$\hat{a}(T) = e^{-(i/\hbar)\hat{H}_o(T-T_o)}\hat{a}e^{(i/\hbar)\hat{H}_o(T-T_o)} \quad \text{with} \quad \hat{a} = \frac{1}{\ell}|\hat{V}|^{1/3}. \quad (4.6)$$

Thus, in this interaction picture, quantum geometry is in effect described in the Heisenberg picture—states of quantum geometry are "frozen" at time $T = T_o$ but the scale factor operators evolve—while the test field is described using the Schrödinger picture. Therefore, the quantum evolution equation (4.5) is now even more similar to the Schrödinger equation (3.7) for the test field on a classical background. However, the lapse $\hat{N}_T$ and powers of the scale factor $\hat{a}$ are still operators on $\mathcal{H}_{\rm geo}$. In the next

[6]This also occurs in the classical theory. There, in place of the Hamiltonian, we have the constraint function $C = P_{(T)}^2/2V - (3/8\pi G)(b^2V/\gamma^2)$ on the phase space. b, V which appear in the expression are determined just by the point at which C is evaluated; there is no time parameter on which they could depend. This is actually the origin of the fact that the $\hat{V}$ and $\hat{H}_o$ in (4.3) do not depend on time.

subsection we will specify the approximations necessary to reduce (4.5) to (3.7).

C. Replacing geometric operators by their mean values

Let us now assume that the state $\Psi_{\rm int}(\nu, q_{\vec{k}}, T)$ factorizes as $\Psi_{\rm int}(\nu, q_{\vec{k}}, T) = \Psi_o(\nu, T_0) \otimes \psi(q_{\vec{k}}, T)$, where $\Psi_o(\nu, T_0)$ is a quantum geometry state introduced in Sec. II B, peaked at an effective LQC geometry of the (ν, φ) system. This assumption is justified because φ is a test field; i.e., its backreaction is ignored. Then, (4.5) further simplifies as follows:

$$\begin{aligned}\Psi_o(\nu, T_0) \otimes [i\hbar\partial_T \psi(q_{\vec{k}}, T)] = {} & \tfrac{1}{2}[(\ell^{-3}\hat{H}_o)^{-1}\Psi_o(\nu, T_0)] \\ & \otimes [\hat{p}^2_{\vec{k}}\psi(q_{\vec{k}}, T)] \\ & + \tfrac{1}{2}[(\ell^{-3}\hat{H}_o)^{-(1/2)}(\vec{k}^2\hat{a}^4(T) \\ & + m^2\hat{a}^6(T))(\ell^{-3}\hat{H}_o)^{-(1/2)} \\ & \times \Psi_o(\nu, T_0)] \otimes [\hat{q}^2_{\vec{k}}\psi(q_{\vec{k}}, T)].\end{aligned} \tag{4.7}$$

Let us now suppose that $\Psi_o(\nu, T_o)$ is normalized and take the scalar product of (4.7) with $\Psi_o(\nu, T_0)$. Then, we obtain

$$\begin{aligned}i\hbar\partial_T \psi(q_{\vec{k}}, T) = {} & \tfrac{1}{2}\langle(\ell^{-3}\hat{H}_o)^{-1}\rangle\hat{p}^2_{\vec{k}}\psi(q_{\vec{k}}, T) \\ & + \tfrac{1}{2}[\vec{k}^2\langle(\ell^{-3}\hat{H}_o)^{-(1/2)}\hat{a}^4(T)(\ell^{-3}\hat{H}_o)^{-(1/2)}\rangle \\ & + m^2\langle(\ell^{-3}\hat{H}_o)^{-(1/2)}\hat{a}^6(T) \\ & \times (\ell^{-3}\hat{H}_o)^{-(1/2)}\rangle]\hat{q}^2_{\vec{k}}\psi(q_{\vec{k}}, T),\end{aligned} \tag{4.8}$$

where $\langle\hat{A}\rangle$ denotes the expectation value of the operator $\hat{A}$ in the quantum geometry state Ψ_0. Thus, in this equation all geometrical quantities are c numbers. Nonetheless, (4.8) is, in general, different from (3.7) because expectation values of products of operators do not equal products of expectation values of operators. We discuss the differences and analogies below.

Equation (4.8) tells us how the quantum state of the mode $q_{\vec{k}}$ "evolves," but the background geometry is neither classical nor quantum in the sense of Sec. II B. The mode knows about the background geometry only through the three expectation values that feature on the right side of (4.8). Therefore, one is led to ask if there is an *effective* classical FLRW space-time such that the Schrödinger equation (3.7) on it is equivalent to (4.8).

To address this question, let us begin with the plausible assumption that the quantum geometry state Ψ_0 is sharply peaked at the expectation values $\bar{P}_{(T)}$ and $\bar{a}$ of $\hat{H}$ and $\hat{a}$, respectively, and, *furthermore*, work in the approximation in which quantum fluctuations of geometry can be ignored. *A priori* this is a very strong simplification but, for cosmological applications, this approximation can be justified because the quantum geometries $\Psi_o(\nu, T)$ have incredibly small dispersions along the entire effective trajectory [35].

Then, (4.8) reduces to

$$\begin{aligned}i\hbar\partial_T \psi(q_{\vec{k}}, T) = {} & \frac{\bar{N}_T}{2\bar{a}^3}[\hat{p}^2_{\vec{k}} + (\vec{k}^2\bar{a}^4(x_0) \\ & + m^2\bar{a}^6(x_0))\hat{q}^2_{\vec{k}}]\psi(q_{\vec{k}}, x_0).\end{aligned} \tag{4.9}$$

This is exactly the Schrödinger equation (3.7) governing the dynamics of the test quantum field on a classical space-time with scale factor $\bar{a}$ containing a massless scalar field T with momentum $\bar{P}_{(T)} = \bar{a}^2\ell^3/\bar{N}_T$. This is the precise sense in which the dynamics of a test quantum field on a classical background emerges from a more complete QFT on quantum FLRW backgrounds. Note however that, even with this strong simplification, the classical space-time is *not* a FLRW solution of the Einstein-Klein-Gordon equation. Rather, it is the effective space-time $(M, \bar{g}_{ab})$ *à la* LQC on which the quantum geometry $\Psi_o(\nu, T)$ is sharply peaked. But as discussed in Secs. I and II A, away from the Planck regime, $(M, \bar{g}_{ab})$ is extremely well approximated by a classical FLRW space-time (M, g^o_{ab}). Thus, starting from quantum geometry and making a series of well-motivated approximations, we have arrived at a QFT of a test field φ which is a nontrivial extension of the QFT on a standard (M, g^o_{ab}). It has the same structure as the standard theory but is defined on a much larger space-time in which the big bang is replaced by a quantum bounce and there is an infinite pre-big branch. Therefore, although the theory developed in this section describes a test quantum field $\hat{\varphi}$ on classical backgrounds and approximates the standard QFT on classical FLRW geometries at late times, it also contains a lot of new physics, particularly in the Planck regime around the bounce.

Next, it is interesting to return to Eq. (4.8) and *not* make additional simplifications. One can still ask if there is a classical metric tensor

$$g'_{ab}dx^a dx^b = -N'^2(T)dT^2 + a'^2(T)d\vec{x}^2 \tag{4.10}$$

such that (4.8) agrees with the Schrödinger equation (3.7) on (M, g'_{ab}). For this agreement to hold, the scale factor $a'(T)$ and the lapse function $N'(T)$ should satisfy the following system of equations:

$$N'(T) = \ell^3 a'^3(T)\langle\hat{H}_o^{-1}\rangle, \tag{4.11}$$

$$N'(T)a'(T) = \ell^3\langle\hat{H}_o^{-(1/2)}\hat{a}^4(T)\hat{H}_o^{-(1/2)}\rangle, \tag{4.12}$$

$$m^2 N'(T)a'^3(T) = m^2\ell^3\langle\hat{H}_o^{-(1/2)}\hat{a}^6(T)\ell^{-3}\hat{H}_o^{-(1/2)}\rangle. \tag{4.13}$$

In the case when the test field is massless, the third equation disappears and there is clearly a solution $(N'(T), a'(T))$. But note that the interpretation of (4.8) as the evolution equation for $\psi(q_{\vec{k}}, T)$ on the classical space-time (M, g'_{ab}) is not entirely satisfactory because, if the quantum geometry state is sharply peaked at $\langle\hat{a}\rangle = \bar{a}$ and $\langle\hat{P}_{(T)}\rangle = \bar{P}_{(T)}$, then $a'(T) \neq \bar{a}(T)$ and $N'(T) \neq \ell^3\bar{a}^3/\bar{P}_{(T)}$.

Thus, deductions about the quantum geometry made from the dynamics of the test scalar field would be different from those made by observing the geometry directly, e.g., from the measurement of the Hubble parameter or of the volume at the bounce point. Finally, in the case when the test scalar field φ has mass, on the other hand, if the quantum geometry fluctuations are not negligible, dynamics of the test field given by (4.8) cannot be interpreted as dynamics of the test field on *any* classical FLRW background.

V. DISCUSSION

Consider QFT of a massive, test, scalar field $\hat{\varphi}$ on a classical FLRW space-time (M, g^o_{ab}) with a massless scalar field T as its matter source. Our main goal was to derive this theory from that of the scalar field $\hat{\varphi}$ on a quantum geometry $\Psi_o(\nu, T)$ that replaces (M, g^o_{ab}) in LQC. Conceptually, the two theories are quite distinct:

(i) They use very different notions of time. In particular, the conformal time η and the proper time t used in the first theory are nontrivial operators in the second [33].

(ii) In the first theory, dynamics is generated by a Hamiltonian, while in the second, it has to be teased out of a constraint.

(iii) In the first theory, there is a fixed classical metric g^o_{ab} in the background which is used repeatedly in the construction of the QFT, while in the second, there is only a probability distribution for various metrics encoded in $\Psi_o(\nu, T)$.

(iv) While in the first theory the scale factor a is a given function on M, in the second theory we are confronted with quantum fluctuations of (different powers of) the operator $\hat{a}$.

Our first task was to set up an appropriate framework to explore the relation between the two theories in detail.

To construct the second of these theories, in Sec. III we began with the constrained quantum system for the gravitational field coupled with the scalar fields T and φ but made simplifications to encode the idea that the space-time geometry and T are homogeneous and φ is (inhomogeneous but) a test field whose backreaction is ignored. This theory was deparametrized by singling out T as the relational time variable with respect to which the gravitational field and φ evolve. The states of the coupled system are then functions $\Psi(\nu, \varphi, T)$ of volume ν (or, equivalently, the scale factor) of the universe, the massive test field φ, and the massless scalar field T. We found that their inner product is given by (3.13) and their dynamics is governed by the Schrödinger equation (3.12). Thus, a quantum theory of the test field φ on quantum geometries could be constructed, although we do not have a fixed classical metric or a fixed causal structure in the background.

In Sec. IV we made successive approximations to simplify (3.12), all of which are well motivated by the setup of the problem:

(i) We regarded variables (ν, T), which provide the background geometry, as the heavy degree of freedom and the test field φ as the light degree to simplify the Hamiltonian operator in (3.12).

(ii) We assumed that the state $\Psi(\nu, \varphi, T)$ can be expanded as $\Psi(\nu, \varphi, T) = \Psi_o(\nu, T) \otimes \psi(\varphi, T)$, where $\Psi_o(\nu, T)$ is the quantum geometry that replaces the classical FLRW space-time in LQC, and took the scalar product of the evolution equation (3.12) with respect to the quantum geometry state $\Psi_o(\nu, T)$ to obtain an evolution equation for $\psi(\varphi)$.

(iii) To simplify this equation on $\psi(\varphi)$, we ignored the quantum fluctuations of geometry by replacing the expectation values of products of geometrical operators by products of their expectation values. The result was the standard Schrödinger equation (4.9) for a test field φ on a classical background.

However, Eq. (4.9) has two nonstandard features. First, the classical background is *not* a FLRW space-time (M, g^o_{ab}) but rather an effective space-time $(M, \bar{g}_{ab})$ on which the LQC state $\Psi_o(\nu, T)$ is sharply peaked. Second, the Schrödinger equation naturally arises with T as the time variable. This is unusual from the perspective of QFT on classical backgrounds because T is the massless scalar field that acts as the source of the gravitational field, while QFT on classical backgrounds, as normally formulated, has no knowledge of the source. Rather, the time variables that are normally used—the conformal time η or the proper time t—arise directly from the metric g^o_{ab}. However, from the perspective of quantum geometry, these are unnatural because while T is a parameter in that theory, as we noted above, η and t are not; they get promoted to operators. Of course, once we have arrived at the "lower" theory—i.e., QFT on the classical space-time $(M, \bar{g}_{ab})$—it is straightforward to reformulate dynamics in terms of either η or t. But at a more fundamental level, it is the relational time T that appears to be the natural time parameter. Finally, let us return to the first difference. The effective space-time $(M, \bar{g}_{ab})$ is a nontrivial extension of the FLRW solution (M, g^o_{ab}) in which the big bang is replaced by a bounce and there is an infinite pre-big-bang branch. However, FLRW solutions (M, g^o_{ab}) are excellent approximations to the effective space-times $(M, \bar{g}_{ab})$ in the expanding, post-big-bang branch *away from the Planck regime.* Furthermore, our QFT on effective space-times does reduce to the standard one on FLRW solutions when the space-time curvature is smaller than the Planck scale. Moreover, it provides a physically interesting extension near and to the past of the big bounce. Because $(M, \bar{g}_{ab})$ is nonsingular, this theory opens a new window to the Planck scale physics which was inaccessible to QFT on classical FLRW solutions.

Thus, in this paper we have laid down foundations for further work with applications to cosmology as well as mathematical physics. We will conclude by indicating directions that are currently being pursued. First, we need to include the backreaction of φ on geometry, treating it as a perturbation. As far as the homogeneous mode of the gravitational field is concerned, this is already achieved in the evolution equation (3.12) (see the remark at the end of Sec. III B). Inclusion of inhomogeneous gravitational perturbations remains an open issue. Second, we have to analyze the quantum dynamics of the gauge invariant combinations Φ of φ and the scalar perturbations of the metric. Here the important step is to construct the Mukhanov variable Φ starting from the full quantum constraint. Existing literature on cosmological perturbations in the LQG setting [36,37] is likely to be directly useful in this task. The mathematical theory of propagation of Φ on the quantum background geometry $\Psi_o(\nu, T)$ would be rather similar to that of φ analyzed in this paper. Third, we have to account for the origin of the massless scalar field T which plays the role of time for us. It seems most natural to have a single scalar field Φ, the homogeneous mode of which would provide the relational time parameter T, and the inhomogeneous modes of which would provide the physical perturbations that lead to structure formation. This seems feasible. However, it is likely that the resulting relational time will not be global. Thus, as remarked at the end of Sec. I, the analysis in quantum geometry may have to be divided into "epochs," in each of which the homogeneous part of Φ will serve as a relational time variable. If these three steps can be carried out to completion, we will have a coherent framework to analyze cosmological perturbations and structure formation which is free from the limitations of a big-bang singularity. In particular, one will then be able to evolve perturbations across the big bounce and study phenomenological implications. Immediately after the big bounce, there is a short epoch of superinflation in LQC (see [38] and especially [39]). The possibility that ramifications of this sudden and very rapid expansion may be observable has drawn considerable attention of cosmologists recently. A more complete QFT on quantum geometries will provide a systematic avenue to analyze these issues.

The second direction for further work is motivated by mathematical physics (although it too has some implications to cosmology). In this paper we focused on a single mode of the scalar field φ. Inclusion of a finite number of modes is completely straightforward. Inclusion of all modes, on the other hand, involves functional analytic subtleties. Recall, however, that in quantum geometry, the volume operator has a nonzero minimum value, $2\pi\gamma\ell_{\rm Pl}^2|\nu|_{\rm min} = 8\pi\gamma\lambda\ell_{\rm Pl}^2$. Therefore, in a certain sense there is a built-in ultraviolet cutoff. A careful examination may well reveal that this cutoff descends to the test scalar field φ, in which case φ would have only a finite number of modes and the treatment presented here will suffice. However, if this possibility is not realized, one would have to resolve the functional analytical difficulties. Our first task is to address these issues. Second, a number of ideas related to the algebraic approach are being explored. This approach can be applied directly to the effective space-times $(M, \bar{g}_{ab})$ that emerge from LQC. What can one say about the (regularized) stress tensor of φ and its backreaction on the geometry? Is there a sense in which the Schrödinger equation (3.12) already includes these effects? More importantly, can one extend the algebraic approach systematically to cosmological *quantum* geometries? At first the extension seems very difficult, if not impossible, because so many of the structures normally used in the algebraic approach to QFT on classical space-times use the fact that we have access to a *fixed* space-time metric. However, in the cosmological context, additional structures—such as a preferred foliation—naturally become available, and they enable one to construct the required $\star$ algebras of field operators in the canonical setting. Also, the background quantum geometries $\Psi_o(\nu, T)$ are rather well controlled, and one may be able to use the fact that they are extremely sharply peaked around effective space-times [35]. Can one exploit this setting to introduce the analogs of Hadamard states? We believe that such generalizations are now within reach.

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