Structure and evolution of self-gravitating objects and the orthogonal splitting of the Riemann tensor

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The full set of equations governing the structure and the evolution of self-gravitating spherically symmetric dissipative fluids with anisotropic stresses is written down in terms of five scalar quantities obtained from the orthogonal splitting of the Riemann tensor, in the context of general relativity. It is shown that these scalars are directly related to fundamental properties of the fluid distribution, such as energy density, energy density inhomogeneity, local anisotropy of pressure, dissipative flux, and the active gravitational mass. It is also shown that in the static case, all possible solutions to Einstein equations may be expressed explicitly through these scalars. Some solutions are exhibited to illustrate this point.

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I. INTRODUCTION

The gravitational collapse of massive stars and its resulting remanent (neutron star or black hole) represent one of the few observable scenarios where general relativity is expected to play a relevant role. Therefore, a detailed description of gravitational collapse and the modelling of the structure of compact objects under a variety of conditions remain among the most interesting problems that general relativity has to deal with. This fact explains the great attraction that these problems exert on the community of the relativists. Starting with the seminal papers by Oppenheimer and Snyder [1] and Oppeheimer and Volkoff [2], a long list of works has been presented trying to provide models of evolving gravitating spheres and compact objects (just as a sample, and without the pretension of being exhaustive, see [3–26] and references therein).

Motivated by the above arguments, we present in this work a study on self-gravitating relativistic fluids in terms of a set of scalars obtained from the orthogonal splitting of the Riemann tensor. As we shall see these scalars have a distinct physical meaning and appear to be particularly well suited for the description of self-gravitating fluids.

We shall assume our system to be spherically symmetric. This is a common assumption in the study of self-gravitating compact objects, since deviations from spherical symmetry are likely to be incidental rather than basic features of the process involved (see, however, the discussion in [27–29]).

For sake of generality we shall further assume the fluid to be locally anisotropic (principal stresses unequal) and dissipative.

The assumption of local anisotropy of pressure, which seems to be very sensible to describe matter distribution under a variety of circumstances, has been proved to be very useful in the study of relativistic compact objects and related problems (see [30] for a comprehensive review until 1997, and [31–72] and references therein, for more recent developments).

On the other hand, it is already an established fact that gravitational collapse is a highly dissipative process (see [73–75] and references therein). This dissipation is required to account for the very large (negative) binding energy of the resulting compact object (of the order of -10^{53} erg).

In fact, it seems that the only plausible mechanism to carry away the bulk of the binding energy of the collapsing star, leading to a neutron star or black hole, is neutrino emission [76].

As usual, we shall describe dissipation in two limiting cases. The first case (diffusion) applies whenever the mean free path of particles responsible for the propagation of energy is very small as compared with the typical length of the object. For example, for a main sequence star as the sun, the mean free path of photons at the center is of the order of 2 cm. Also, the mean free path of trapped neutrinos in compact cores of densities about 10^{12} g.cm^{-3} becomes smaller than the size of the stellar core [77,78]. Furthermore, the observational data collected from supernova 1987A indicates that the regime of radiation transport prevailing during the emission process is closer to the diffusion approximation than to the free-streaming limit [79]. In this case, it is assumed that the energy flux of radiation, as that of thermal conduction, is proportional to the gradient of temperature.

The second case (free streaming), applies when the mean free path of particles transporting energy is larger than (or equal to) the typical length of the object. Since this condition may hold for a large number of stellar scenarios, it is advisable to include simultaneously both limiting cases of radiative transport, diffusion, and free

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streaming, allowing for describing a wide range of situations.

In the next section we shall introduce the relevant physical variables and deploy the equations for describing a dissipative self-gravitating locally anisotropic fluid.

As mentioned before, a fundamental role in our study will be played by a set of scalars derived from the orthogonal splitting of the Riemann tensor. Such a splitting and the ensuing variables are presented in Sec. III. In Sec. IV the physical meaning of the above-mentioned scalars is discussed, and in Sec. V a general method to obtain all static anisotropic solutions in terms of those scalars is presented.

Finally, the results are discussed in the last section.

II. THE GENERAL FORMALISM

In this section we shall present the physical variables and the relevant equations for describing a dissipative self-gravitating locally anisotropic fluid. Here we shall closely follow the program outlined in [75], thus we refer the reader to that article for more details.

A. Einstein equations

We consider spherically symmetric distributions of collapsing fluid, which for the sake of completeness we assume to be locally anisotropic, undergoing dissipation in the form of heat flow (diffusion limit) and/or free-streaming radiation (free-streaming limit), bounded by a spherical surface Σ.

The line element is given in Schwarzschild-like (noncomoving) coordinates by

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where $\nu(t, r)$ and $\lambda(t, r)$ are functions of their arguments. We number the coordinates: $x^0 = t$; $x^1 = r$; $x^2 = \theta$; $x^3 = \phi$.

The metric (1) has to satisfy Einstein field equations

$$G^{\nu}_{\mu} = \kappa T^{\nu}_{\mu}, \quad (2)$$

with $\kappa = 8\pi$ and which in our case read [3]:

$$-\kappa T^0_0 = -\frac{1}{r^2} + e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right), \quad (3)$$

$$-\kappa T^1_1 = -\frac{1}{r^2} + e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right), \quad (4)$$

$$\begin{aligned} -\kappa T^2_2 &= -\kappa T^3_3 \\ &= -\frac{e^{-\nu}}{4}(2\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu})) \\ &\quad + \frac{e^{-\lambda}}{4}\left(2\nu'' + \nu'^2 - \lambda'\nu' + 2\frac{\nu' - \lambda'}{r}\right), \end{aligned} \quad (5)$$

$$-\kappa T_{10} = -\frac{\dot{\lambda}}{r}, \quad (6)$$

where dots and primes stand for partial differentiation with respect to t and r, respectively. In order to give physical significance to the T^{μ}_{ν} components we apply the Bondi approach [3].

Thus, following Bondi, let us introduce purely locally Minkowski coordinates (τ, x, y, z)

$$d\tau = e^{\nu/2}dt; \qquad dx = e^{\lambda/2}dr;$$
$$dy = rd\theta; \qquad dz = r\sin\theta d\phi.$$

Then, denoting the Minkowski components of the energy tensor by a bar, we have

$$\bar{T}^0_0 = T^0_0; \qquad \bar{T}^1_1 = T^1_1; \qquad \bar{T}^2_2 = T^2_2;$$
$$\bar{T}^3_3 = T^3_3; \qquad \bar{T}_{01} = e^{-(\nu+\lambda)/2}T_{01}.$$

Next, we suppose that when viewed by an observer moving relative to these coordinates with proper velocity ω in the radial direction, the physical content of space consists of an anisotropic fluid of energy density ρ, radial pressure P_r, tangential pressure $P_\perp$, radial heat flux q, and unpolarized radiation of energy density ϵ traveling in the radial direction. Thus, when viewed by this (comoving with the fluid) observer the covariant energy-momentum tensor in Minkowski coordinates is

$$\begin{pmatrix} \rho + \epsilon & -q - \epsilon & 0 & 0 \\ -q - \epsilon & P_r + \epsilon & 0 & 0 \\ 0 & 0 & P_\perp & 0 \\ 0 & 0 & 0 & P_\perp \end{pmatrix}.$$

Then a Lorentz transformation gives

$$T^0_0 = \bar{T}^0_0 = \frac{\rho + P_r\omega^2}{1 - \omega^2} + \frac{2\omega q}{1 - \omega^2} + \frac{\epsilon(1 + \omega)}{1 - \omega}, \quad (7)$$

$$T^1_1 = \bar{T}^1_1 = -\frac{P_r + \rho\omega^2}{1 - \omega^2} - \frac{2\omega q}{1 - \omega^2} - \frac{\epsilon(1 + \omega)}{1 - \omega}, \quad (8)$$

$$T^2_2 = T^3_3 = \bar{T}^2_2 = \bar{T}^3_3 = -P_\perp, \quad (9)$$

$$\begin{aligned} T_{01} &= e^{(\nu+\lambda)/2}\bar{T}_{01} \\ &= -\frac{(\rho + P_r)\omega e^{(\nu+\lambda)/2}}{1 - \omega^2} - \frac{q e^{(\lambda+\nu)/2}}{1 - \omega^2}(1 + \omega^2) \\ &\quad - \frac{e^{(\lambda+\nu)/2}\epsilon(1 + \omega)}{1 - \omega}. \end{aligned} \quad (10)$$

Note that the coordinate velocity in the (t, r, θ, ϕ) system, dr/dt, is related to ω by

$$\omega = \frac{dr}{dt}e^{(\lambda-\nu)/2}. \quad (11)$$

Feeding back (7)–(10) into (3)–(6), we get the field equations in the form

$$\frac{\rho + P_r\omega^2}{1-\omega^2} + \frac{2\omega q}{1-\omega^2} + \frac{\epsilon(1+\omega)}{1-\omega} = -\frac{1}{\kappa}\left\{-\frac{1}{r^2} + e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right)\right\}, \tag{12}$$

$$\frac{P_r + \rho\omega^2}{1-\omega^2} + \frac{2\omega q}{1-\omega^2} + \frac{\epsilon(1+\omega)}{1-\omega} = -\frac{1}{\kappa}\left\{\frac{1}{r^2} - e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right)\right\}, \tag{13}$$

$$P_\perp = -\frac{1}{\kappa}\left\{\frac{e^{-\nu}}{4}(2\ddot\lambda + \dot\lambda(\dot\lambda - \dot\nu)) - \frac{e^{-\lambda}}{4}\left(2\nu'' + \nu'^2 - \lambda'\nu' + 2\frac{\nu' - \lambda'}{r}\right)\right\}, \tag{14}$$

$$\frac{(\rho + P_r)\omega e^{(\lambda+\nu)/2}}{1-\omega^2} + \frac{q e^{(\lambda+\nu)/2}}{1-\omega^2}(1+\omega^2) + \frac{e^{(\lambda+\nu)/2}\epsilon(1+\omega)}{1-\omega} = -\frac{\dot\lambda}{\kappa r}. \tag{15}$$

Next, the four-velocity vector is defined as

$$u^\alpha = \left(\frac{e^{-(\nu/2)}}{(1-\omega^2)^{1/2}}, \frac{\omega e^{-(\lambda/2)}}{(1-\omega^2)^{1/2}}, 0, 0\right), \tag{16}$$

from which we can calculate the four acceleration, $a^\alpha = u^\alpha_{;\beta}u^\beta$

$$\omega a_1 = -a_0 e^{(\lambda-\nu)/2} = -\frac{\omega}{1-\omega^2}\left[\left(\frac{\omega\omega'}{1-\omega^2} + \frac{\nu'}{2}\right) + e^{(\lambda-\nu)/2}\left(\frac{\omega\dot\lambda}{2} + \frac{\dot\omega}{1-\omega^2}\right)\right], \tag{17}$$

the shear tensor $\sigma_{\mu\nu}$, and the expansion Θ

$$\sigma_{\mu\nu} = u_{\mu;\nu} + u_{\nu;\mu} - u_\mu a_\nu - u_\nu a_\mu - \frac{2}{3}\Theta h_{\mu\nu}, \tag{18}$$

with

$$h_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu \qquad \Theta = u^\mu_{;\mu}, \tag{19}$$

$$\Theta = \frac{e^{-\nu/2}}{2(1-\omega^2)^{1/2}}\left(\dot\lambda + \frac{2\omega\dot\omega}{1-\omega^2}\right) + \frac{e^{-\lambda/2}}{2(1-\omega^2)^{1/2}}\left(\omega\nu' + \frac{2\omega'}{1-\omega^2} + \frac{4\omega}{r}\right), \tag{20}$$

$$\sigma_{11} = -\frac{2}{3(1-\omega^2)^{3/2}}\left[e^{\lambda}e^{-\nu/2}\left(\dot\lambda + \frac{2\omega\dot\omega}{1-\omega^2}\right) + e^{\lambda/2}\left(\omega\nu' + \frac{2\omega'}{1-\omega^2} - \frac{2\omega}{r}\right)\right], \tag{21}$$

$$\sigma_{22} = -\frac{e^{-\lambda}r^2(1-\omega^2)}{2}\sigma_{11}, \tag{22}$$

$$\sigma_{33} = -\frac{e^{-\lambda}r^2(1-\omega^2)}{2}\sin^2\theta\sigma_{11}, \tag{23}$$

$$\sigma_{00} = \omega^2 e^{-\lambda}e^{\nu}\sigma_{11}, \tag{24}$$

$$\sigma_{01} = -\omega e^{(\nu-\lambda)/2}\sigma_{11}. \tag{25}$$

We may write the shear tensor also as

$$\sigma_{\alpha\beta} = \tfrac{1}{2}\sigma(s_\alpha s_\beta + \tfrac{1}{3}h_{\alpha_\beta}), \tag{26}$$

with

$$\sigma = -\frac{1}{(1-\omega^2)^{1/2}}\left[e^{-(\nu/2)}\left(\dot\lambda + \frac{2\omega\dot\omega}{1-\omega^2}\right) + e^{-(\lambda/2)}\left(\omega\nu' + \frac{2\omega'}{1-\omega^2} - \frac{2\omega}{r}\right)\right], \tag{27}$$

and s^μ being defined by

$$s^\mu = \left(\frac{\omega e^{-(\nu/2)}}{(1-\omega^2)^{1/2}}, \frac{e^{-(\lambda/2)}}{(1-\omega^2)^{1/2}}, 0, 0\right), \tag{28}$$

with the properties $s^\mu u_\mu = 0$, $s^\mu s_\mu = -1$.

It will be convenient to write the energy-momentum tensor (7)–(10) as

$$T^\mu_\nu = \tilde\rho u^\mu u_\nu - \hat P h^\mu_\nu + \Pi^\mu_\nu + \tilde q(s^\mu u_\nu + s_\nu u^\mu), \tag{29}$$

with

$$\Pi^\mu_\nu = \Pi\left(s^\mu s_\nu + \frac{1}{3}h^\mu_\nu\right), \qquad \tilde q^\mu = \tilde q s^\mu,$$

$$\hat P = \frac{\tilde P_r + 2P_\perp}{3}, \qquad \tilde\rho = \rho + \epsilon, \qquad \tilde P_r = P_r + \epsilon,$$

$$\tilde q = q + \epsilon, \qquad \Pi = \tilde P_r - P_\perp.$$

For the exterior of the fluid distribution, the spacetime is that of Vaidya, given by

$$ds^2 = \left(1 - \frac{2M(u)}{\mathcal{R}}\right)du^2 + 2dud\mathcal{R} - \mathcal{R}^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{30}$$

where u is a coordinate related to the retarded time, such that $u =$ constant is (asymptotically) a null cone open to the future and $\mathcal{R}$ is a null coordinate ($g_{\mathcal{R}\mathcal{R}} = 0$).

The two coordinate systems (t, r, θ, ϕ) and $(u, \mathcal{R}, \theta, \phi)$ are related at the boundary surface by

$$u = t - r - 2M\ln\left(\frac{r}{2M} - 1\right), \tag{31}$$

$$\mathcal{R} = r. \tag{32}$$

In order to match smoothly the two metrics above on the boundary surface $r = r_\Sigma(t)$, we require the continuity of the first and the second fundamental forms across that surface, yielding (see [15] for details)

$$e^{\nu_\Sigma} = 1 - \frac{2M}{\mathcal{R}_\Sigma}, \tag{33}$$

$$e^{-\lambda_\Sigma} = 1 - \frac{2M}{\mathcal{R}_\Sigma}, \tag{34}$$

$$[P_r]_\Sigma = [q]_\Sigma, \tag{35}$$

where, from now on, subscript Σ indicates that the quantity is evaluated on the boundary surface Σ, and (35) expresses the discontinuity of the radial pressure in the presence of heat flow, which is a well-known result [80].

Equations (33)–(35) are the necessary and sufficient conditions for a smooth matching of the two metrics (1) and (30) on Σ.

B. The Riemann and the Weyl tensor

We know that the Riemann tensor may be expressed through the Weyl tensor $C^\rho_{\alpha\beta\mu}$, the Ricci tensor $R_{\alpha\beta}$, and the scalar curvature R as

$$R^\rho_{\alpha\beta\mu} = C^\rho_{\alpha\beta\mu} + \tfrac{1}{2}R^\rho_\beta g_{\alpha\mu} - \tfrac{1}{2}R_{\alpha\beta}\delta^\rho_\mu + \tfrac{1}{2}R_{\alpha\mu}\delta^\rho_\beta - \tfrac{1}{2}R^\rho_\mu g_{\alpha\beta} - \tfrac{1}{6}R(\delta^\rho_\beta g_{\alpha\mu} - g_{\alpha\beta}\delta^\rho_\mu). \tag{36}$$

In the spherically symmetric case, the magnetic part of the Weyl tensor vanishes and we can express the Weyl tensor in terms of its electric part ($E_{\alpha\beta} = C_{\alpha\gamma\beta\delta}u^\gamma u^\delta$) as

$$C_{\mu\nu\kappa\lambda} = (g_{\mu\nu\alpha\beta}g_{\kappa\lambda\gamma\delta} - \eta_{\mu\nu\alpha\beta}\eta_{\kappa\lambda\gamma\delta})u^\alpha u^\gamma E^{\beta\delta}, \tag{37}$$

with $g_{\mu\nu\alpha\beta} = g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}$, and $\eta_{\mu\nu\alpha\beta}$ denoting the Levi-Civita tensor. Observe that $E_{\alpha\beta}$ may also be written as

$$E_{\alpha\beta} = E\left(s_\alpha s_\beta + \frac{1}{3}h_{\alpha\beta}\right), \tag{38}$$

with

$$E = \frac{e^{-\nu}}{4}\left[\ddot{\lambda} + \frac{\dot{\lambda}(\dot{\lambda} - \dot{\nu})}{2}\right] - \frac{e^{-\lambda}}{4}\left[\nu'' + \frac{\nu'^2 - \lambda'\nu'}{2} - \frac{\nu' - \lambda'}{r} + \frac{2(1 - e^\lambda)}{r^2}\right], \tag{39}$$

satisfying the following properties:

$$E^\alpha{}_\alpha = 0, \qquad E_{\alpha\gamma} = E_{(\alpha\gamma)}, \qquad E_{\alpha\gamma}u^\gamma = 0. \tag{40}$$

C. The mass function and the Tolman mass

Here we shall introduce the two most commonly used definitions for the mass of a sphere interior to the surface Σ, as well as some interesting relationships between them and the Weyl tensor. These will be used later to provide physical meaning to the five scalars quantities which will be derived from the orthogonal splitting of the Riemann tensor.

1. The mass function

For the line element (1) the mass function m is defined by

$$R^3_{232} = 1 - e^{-\lambda} = \frac{2m}{r}. \tag{41}$$

Then, using (2), (36), and (38) we may write

$$\frac{3m}{r^3} = \frac{\kappa}{2}\tilde{\rho} + \frac{\kappa}{2}(P_\perp - \tilde{P}_r) + E. \tag{42}$$

Observe that from (3) and (41) the mass function may also be written as

$$m = \frac{\kappa}{2}\int_0^r r^2 T^0_0 dr. \tag{43}$$

Another interesting relationship for the mass function may be obtained as follows. From field equations (12)–(14), (36), (38), and (43) we get

$$m = \frac{\kappa}{6}r^3(T^0_0 + T^1_1 - T^2_2) + \frac{r^3 E}{3}. \tag{44}$$

Next, differentiating (44) with respect to r and using (43), it follows

$$\left(\frac{r^3 E}{3}\right)' = -\frac{\kappa}{6}r^3(T^0_0)' + \frac{\kappa}{6}[r^3(T^2_2 - T^1_1)]', \tag{45}$$

and integrating

$$E = -\frac{\kappa}{2r^3}\int_0^r r^3(T^0_0)' dr + \frac{\kappa}{2}(T^2_2 - T^1_1). \tag{46}$$

Finally, inserting (46) into (44) we obtain

$$m(r, t) = \frac{\kappa}{6}r^3 T^0_0 - \frac{\kappa}{6}\int_0^r r^3(T^0_0)' dr. \tag{47}$$

Now, there are three specific situations when $T^0_0 = \tilde{\rho}$ and $T^2_2 - T^1_1 = \Pi$, namely:

(i) In the static regime, i.e. when ω as well as all time derivatives vanish.

(ii) In the quasistatic regime, where (see [15])

$$\omega^2 \approx \dot{\omega} \approx \ddot{\nu} \approx \ddot{\lambda} \approx \dot{\nu}\,\dot{\lambda} \approx \dot{\lambda}^2 \approx 0. \tag{48}$$

(iii) Immediately after the system departs from equilibrium, i.e. on a time scale of the order of (or smaller than) the hydrostatic time, in which case $\omega \approx \dot{\lambda} \approx \dot{\nu} \approx 0$; $\dot{\omega} \neq 0$.

Thus in the three cases above, (46) and (47) become

$$E = -\frac{\kappa}{2r^3}\int_0^r r^3(\tilde{\rho})' dr + \frac{\kappa}{2}\Pi, \tag{49}$$

$$m(r,t) = \frac{\kappa}{6} r^3 \tilde{\rho} - \frac{\kappa}{6} \int_0^r r^3 (\tilde{\rho})' dr. \tag{50}$$

The first of these equations relates the Weyl tensor to two fundamental physical properties of the fluid distribution, namely, density inhomogeneity and local anisotropy of pressure. The second one expresses the mass function in terms of its value in the case of a homogeneous energy density distribution, plus the change induced by density inhomogeneity.

2. Tolman mass

An alternative definition to describe the energy content of a fluid sphere was proposed by Tolman many years ago. The Tolman mass for a spherically symmetric distribution of matter is given by (Eq. 24 in [81])

$$m_T = \frac{\kappa}{2} \int_0^{r_\Sigma} r^2 e^{(\nu+\lambda)/2} (T_0^0 - T_1^1 - 2T_2^2) dr + \frac{1}{2} \times \int_0^{r_\Sigma} r^2 e^{(\nu+\lambda)/2} \frac{\partial}{\partial t} \left(\frac{\partial L}{\partial[\partial(g^{\alpha\beta}\sqrt{-g})/\partial t]} \right) g^{\alpha\beta} dr, \tag{51}$$

where L denotes the usual gravitational Lagrangian density (Eq. (10) in [81]). Although Tolman's formula was introduced as a measure of the total energy of the system, with no commitment to its localization, we shall define the mass within a sphere of radius r, completely inside Σ, as

$$m_T = \frac{\kappa}{2} \int_0^{r} r^2 e^{(\nu+\lambda)/2} (T_0^0 - T_1^1 - 2T_2^2) dr + \frac{1}{2} \times \int_0^{r} r^2 e^{(\nu+\lambda)/2} \frac{\partial}{\partial t} \left(\frac{\partial L}{\partial[\partial(g^{\alpha\beta}\sqrt{-g})/\partial t]} \right) g^{\alpha\beta} dr. \tag{52}$$

This extension of the global concept of energy to a local level [82] is suggested by the conspicuous role played by m_T as the "active gravitational mass," which will be exhibited below.

Now, it can be shown after some lengthy calculations (see [83] for details) that

$$m_T = e^{(\nu+\lambda)/2} \left[m(r,t) - \frac{\kappa}{2} r^3 T_1^1 \right]. \tag{53}$$

Replacing T_1^1 by (4) and m by (41), one also finds

$$m_T = e^{(\nu-\lambda)/2} \nu' \frac{r^2}{2}. \tag{54}$$

This last equation brings out the physical meaning of m_T as the active gravitational mass. Indeed, as follows from (17), the gravitational acceleration ($a = -s^\nu a_\nu$) of a test particle, instantaneously at rest in a static gravitational field, is given by (see also [84])

$$a = \frac{e^{-\lambda/2}\nu'}{2} = \frac{e^{-\nu/2} m_T}{r^2}. \tag{55}$$

Another expression for m_T, which appears to be more suitable for the discussion in Sec. IV, may be obtained as follows. Taking the r derivative of (54) (see [83] for details, but notice some minor misprints in Eqs. (31) and (38) in that reference as well as slight changes in notation) and using (44) and (53) and field equations, we obtain

$$rm_T' - 3m_T = e^{(\nu+\lambda)/2} r^3 \left[\frac{\kappa}{2}(T_1^1 - T_2^2) - E \right] + \frac{e^{(\lambda-\nu)/2} r^3}{2} \left(\ddot{\lambda} + \frac{\dot{\lambda}^2}{2} - \frac{\dot{\lambda}\dot{\nu}}{2} \right), \tag{56}$$

which can be formally integrated to give

$$m_T = (m_T)_\Sigma \left(\frac{r}{r_\Sigma} \right)^3 - r^3 \int_r^{r_\Sigma} \frac{e^{(\nu+\lambda)/2}}{r} \left[\frac{\kappa}{2}(T_1^1 - T_2^2) - E \right] dr - r^3 \int_r^{r_\Sigma} \frac{e^{(\lambda-\nu)/2}}{2r} \left(\ddot{\lambda} + \frac{\dot{\lambda}^2}{2} - \frac{\dot{\lambda}\dot{\nu}}{2} \right) dr, \tag{57}$$

or, using (46)

$$m_T = (m_T)_\Sigma \left(\frac{r}{r_\Sigma} \right)^3 - r^3 \int_r^{r_\Sigma} e^{(\nu+\lambda)/2} \left[\frac{\kappa}{r}(T_1^1 - T_2^2) + \frac{1}{r^4} \int_0^r \frac{\kappa}{2} \tilde{r}^3 (T_0^0)' d\tilde{r} \right] dr - r^3 \int_r^{r_\Sigma} \frac{e^{(\lambda-\nu)/2}}{2r} \times \left(\ddot{\lambda} + \frac{\dot{\lambda}^2}{2} - \frac{\dot{\lambda}\dot{\nu}}{2} \right) dr. \tag{58}$$

For the three cases considered above (i.e. the static regime, the quasistatic regime, and immediately after the system departs from equilibrium) we have $T_0^0 = \tilde{\rho}$ and $T_2^2 - T_1^1 = \Pi$. Thus, in any of these cases, (58) expresses the Tolman mass of a sphere of radius r interior to Σ, in terms of its value in the case of a homogeneous energy density and locally isotropic fluid in equilibrium (first term), plus the change induced by density inhomogeneity and local anisotropy (second term), plus changes derived from the fact that the system is not in equilibrium (last term). For a discussion on this last term, see [83]. We shall come back to this expression in Sec. IV.

The important point to stress here is that the second integral in (57) (or (58)) describes the contribution of density inhomogeneity and local anisotropy of pressure to the Tolman mass. It is also worth noticing that when the system is evaluated immediately after its departure from equilibrium, the value of ω remains unchanged. Therefore the physical meaning of m_T, as the active gravitational mass obtained for the static (and quasistatic) case, may be safely extrapolated to the nonstatic case within that (hydrostatic) time scale.

D. Structure and evolution equations

As shown in [75] the following set of equations may be derived to describe the self-gravitating fluid:

$$\tilde{\rho}^{*} + (\tilde{\rho} + \tilde{P}_r)\theta = \frac{2}{3}\left(\theta + \frac{\sigma}{2}\right)\Pi - \tilde{q}^{\dagger} - 2\tilde{q}\left(a + \frac{s^1}{r}\right), \tag{59}$$

$$\tilde{P}_r^{\dagger} + (\tilde{\rho} + \tilde{P}_r)a + \frac{2s^1}{r}\Pi = \frac{\tilde{q}}{3}(\sigma - 4\theta) - \tilde{q}^{*}, \tag{60}$$

$$\theta^{*} + \frac{\theta^2}{3} + \frac{\sigma^2}{6} - a^{\dagger} - a^2 - \frac{2as^1}{r} = -\frac{\kappa}{2}(\tilde{\rho} + 3\tilde{P}_r) + \kappa\Pi, \tag{61}$$

$$\left(\frac{\sigma}{2} + \theta\right)^{\dagger} = -\frac{3\sigma s^1}{2r} + \frac{3\kappa}{2}\tilde{q}, \tag{62}$$

$$E + \frac{\kappa}{2}\Pi = -a^{\dagger} - a^2 - \frac{\sigma^{*}}{2} - \frac{\theta\sigma}{3} + \frac{as^1}{r} + \frac{\sigma^2}{12}, \tag{63}$$

$$\left(\frac{\kappa}{2}\tilde{P}_r + \frac{3m}{r^3}\right)\left(\theta + \frac{\sigma}{2}\right) + \left(E - \frac{\kappa}{2}\Pi + \frac{\kappa}{2}\tilde{\rho}\right)^{*} = -\frac{3\kappa s^1}{2r}\tilde{q}, \tag{64}$$

$$\left(E + \frac{\kappa}{2}\tilde{\rho} - \frac{\kappa}{2}\Pi\right)^{\dagger} = \frac{3s^1}{r}\left(\frac{\kappa}{2}\Pi - E\right) + \frac{\kappa}{2}\tilde{q}\left(\frac{\sigma}{2} + \theta\right), \tag{65}$$

$$\frac{3m}{r^3} = \frac{\kappa}{2}\tilde{\rho} + \frac{\kappa}{2}(P_{\perp} - \tilde{P}_r) + E, \tag{66}$$

with $f^{\dagger} = f_{,\alpha}s^{\alpha}$, $f^{*} = f_{,\alpha}u^{\alpha}$, $a^{\alpha} = as^{\alpha}$, and

$$\frac{\sigma}{2} + \theta = \frac{3\omega s^1}{r}. \tag{67}$$

These equations are not independent and, of course, provide no more information than the one contained in Einstein equations, however, we present them all, since depending on the problem under consideration, it may be more advantageous using one set instead of the other.

The first two equations come from the "conservation" equations $T^{\mu}_{\nu;\mu} = 0$. Equations (61) (Raychaudhuri equation) and (62) are derived from the Ricci identities, whereas Eq. (63) is a consequence of (2) and (36) and Ricci identities. The next two equations follow from the Bianchi identities written in terms of the Weyl tensor (see [75] for details). Finally, (66) is just (42).

We shall next present the orthogonal splitting of the Riemann tensor and express it in terms of the variables considered so far.

III. THE ORTHOGONAL SPLITTING OF THE RIEMANN TENSOR

The orthogonal splitting of the Riemann tensor was first considered by Bel [85], here we shall follow closely (with some changes) the notation in [86].

Thus following Bel, let us introduce the following tensors:

$$Y_{\alpha\beta} = R_{\alpha\gamma\beta\delta}u^{\gamma}u^{\delta}, \tag{68}$$

$$Z_{\alpha\beta} = {}^{*}R_{\alpha\gamma\beta\delta}u^{\gamma}u^{\delta} = \tfrac{1}{2}\eta_{\alpha\gamma\epsilon\rho}R^{\epsilon\rho}{}_{\beta\delta}u^{\gamma}u^{\delta}, \tag{69}$$

$$X_{\alpha\beta} = {}^{*}R^{*}_{\alpha\gamma\beta\delta}u^{\gamma}u^{\delta} = \tfrac{1}{2}\eta_{\alpha\gamma}{}^{\epsilon\rho}R^{*}_{\epsilon\rho\beta\delta}u^{\gamma}u^{\delta}, \tag{70}$$

with $R^{*}_{\alpha\beta\gamma\delta} = \frac{1}{2}\eta_{\epsilon\rho\gamma\delta}R_{\alpha\beta}{}^{\epsilon\rho}$.

It can be shown that the Riemann tensor can be expressed through these tensors in what is called the orthogonal splitting of the Riemann tensor (see [86] for details). Now, instead of using the explicit form of the splitting of the Riemann tensor (Eq. (4.6) in [86]), we shall proceed as follows.

Using the Einstein equations we may write (36) as

$$R^{\alpha\gamma}{}_{\beta\delta} = C^{\alpha\gamma}{}_{\beta\delta} + 2\kappa T^{[\alpha}_{[\beta}\delta^{\gamma]}{}_{\delta]} + \kappa T\left(\frac{1}{3}\delta^{\alpha}{}_{[\beta}\delta^{\gamma}{}_{\delta]} - \delta^{[\alpha}{}_{[\beta}\delta^{\gamma]}{}_{\delta]}\right), \tag{71}$$

then feeding back (29) into (71) we split the Riemann tensor as

$$R^{\alpha\gamma}{}_{\beta\delta} = R^{\alpha\gamma}_{(\mathrm{I})\beta\delta} + R^{\alpha\gamma}_{(\mathrm{II})\beta\delta} + R^{\alpha\gamma}_{(\mathrm{III})\beta\delta}, \tag{72}$$

where

$$R^{\alpha\gamma}_{(\mathrm{I})\beta\delta} = 2\kappa\tilde{\rho}u^{[\alpha}u_{[\beta}\delta^{\gamma]}{}_{\delta]} - 2\kappa\hat{P}h^{[\alpha}{}_{[\beta}\delta^{\gamma]}{}_{\delta]} + \kappa(\tilde{\rho} - 3\hat{P})\left(\frac{1}{3}\delta^{\alpha}{}_{[\beta}\delta^{\gamma}{}_{\delta]} - \delta^{[\alpha}{}_{[\beta}\delta^{\gamma]}{}_{\delta]}\right)$$

$$R^{\alpha\gamma}_{(\mathrm{II})\beta\delta} = 2\kappa(\Pi^{[\alpha}{}_{[\beta}\delta^{\gamma]}{}_{\delta]} + \tilde{q}s^{[\alpha}u_{[\beta}\delta^{\gamma]}{}_{\delta]} + \tilde{q}u^{[\alpha}s_{[\beta}\delta^{\gamma]}{}_{\delta]})$$

$$R^{\alpha\gamma}_{(\mathrm{III})\beta\delta} = 4u^{[\alpha}u_{[\beta}E^{\gamma]}{}_{\delta]} - \epsilon^{\alpha\gamma}{}_{\mu}\epsilon_{\beta\delta\nu}E^{\mu\nu} \tag{73}$$

with

$$\epsilon_{\alpha\gamma\beta} = u^{\mu}\eta_{\mu\alpha\gamma\beta}, \qquad \epsilon_{\alpha\gamma\beta}u^{\beta} = 0, \tag{74}$$

and where the vanishing, due to the spherical symmetry, of the magnetic part of the Weyl tensor ($H_{\alpha\beta} = {}^{*}C_{\alpha\gamma\beta\delta}u^{\gamma}u^{\delta}$) has been used.

From (74) it follows that $\epsilon^{\mu\gamma\nu}\epsilon_{\nu\alpha\beta} = u^{\sigma}u^{\rho}\eta^{\mu\gamma\nu}_{\rho}\eta_{\sigma\nu\alpha\beta}$, producing

$$\epsilon^{\mu\gamma\nu}\epsilon_{\nu\alpha\beta} = \delta^{\gamma}_{\alpha}h^{\mu}_{\beta} - \delta^{\mu}_{\alpha}h^{\gamma}_{\beta} + u_{\alpha}(u^{\mu}\delta^{\gamma}_{\beta} - \delta^{\mu}_{\beta}u^{\gamma}), \tag{75}$$

or, contracting α with μ in (75)

$$\epsilon^{\mu\gamma\nu}\epsilon_{\nu\mu\beta} = -2h^{\gamma}_{\beta}. \tag{76}$$

Using the results above, we can now find the explicit expressions for the three tensors $Y_{\alpha\beta}$, $Z_{\alpha\beta}$, and $X_{\alpha\beta}$ in terms of the physical variables, and we obtain

$$Y_{\alpha\beta} = \frac{\kappa}{6}(\tilde{\rho} + 3\hat{P})h_{\alpha\beta} + \frac{\kappa}{2}\Pi_{\alpha\beta} + E_{\alpha\beta}, \tag{77}$$

$$Z_{\alpha\beta} = \frac{\kappa}{2}\tilde{q}s^{\mu}\epsilon_{\alpha\mu\beta}, \tag{78}$$

and

$$X_{\alpha\beta} = \frac{\kappa}{3}\tilde{\rho}h_{\alpha\beta} + \frac{\kappa}{2}\Pi_{\alpha\beta} - E_{\alpha\beta}. \tag{79}$$

From the above, we can obtain expressions for two quantities endowed with a profound physical meaning (see [86–88] and references therein). They appear in the orthogonal splitting of the Bel tensor [89]. One of them is the Bel superenergy, defined by

$$\bar{W} = \tfrac{1}{2}(X^{\alpha\beta}X_{\alpha\beta} + Y^{\alpha\beta}Y_{\alpha\beta}) + Z^{\alpha\beta}Z_{\alpha\beta}, \tag{80}$$

the other is the super-Poynting vector, defined as

$$\bar{P}_{\alpha} = \epsilon_{\alpha\beta\gamma}(Y^{\gamma}_{\delta}Z^{\beta\delta} - X^{\gamma}_{\delta}Z^{\delta\beta}). \tag{81}$$

Similar quantities may also be defined from the orthogonal splitting of the Bel-Robinson tensor [90]. However, it should be noticed that due to the vanishing of the magnetic part of the Weyl tensor in the spherically symmetric case, the super-Poynting vector associated to the Bel-Robinson tensor vanishes. Also, in the conformally flat case the superenergy associated to the Bel-Robinson tensor vanishes. In other words, these two definitions cover a much wider range of situations, when defined from the Bel tensor. Of course, in vacuum both sets of definitions coincide. Then, from (80) and (81), using (77)–(79) we find

$$\bar{W} = \frac{\kappa^2}{24}(5\tilde{\rho}^2 + 6\tilde{\rho}\,\hat{P} + 9\hat{P}^2 + 4\Pi^2) + \frac{2}{3}E^2 + \frac{\kappa^2}{2}\tilde{q}^2, \tag{82}$$

$$\bar{P}_{\alpha} = \frac{\kappa^2}{2}\tilde{q}(\tilde{\rho} + \tilde{P}_r)s_{\alpha}. \tag{83}$$

Observe that the superenergy associated to the Bel-Robinson tensor (W), which is defined by

$$W = E^{\alpha\beta}E_{\alpha\beta}, \tag{84}$$

(assuming the magnetic part of the Weyl tensor vanishes, as it happens in our case), takes the form

$$W = \tfrac{2}{3}E^2. \tag{85}$$

Combining (85) with (82) we found

$$\bar{W} - W = \frac{\kappa^2}{24}(5\tilde{\rho}^2 + 6\tilde{\rho}\,\hat{P} + 9\hat{P}^2 + 4\Pi^2) + \frac{\kappa^2}{2}\tilde{q}^2. \tag{86}$$

It is also worth noticing that the super-Poynting vector vanishes if and only if there is not dissipative flux. This fact clearly illustrates the physical meaning of this vector and fully justifies its name.

A. Five relevant scalars

We shall now derive five scalar quantities (hereafter referred to as structure scalars), in terms of which we shall write our Eqs. (59)–(66).

Let us first observe that tensors $X_{\alpha\beta}$ and $Y_{\alpha\beta}$ can be splitted in terms of their traces and the corresponding trace-free tensor, i.e.

$$X_{\alpha\beta} = \tfrac{1}{3}\mathrm{Tr}\,Xh_{\alpha\beta} + X_{\langle\alpha\beta\rangle}, \tag{87}$$

with $\mathrm{Tr}\,X = X^{\alpha}_{\alpha}$ and

$$X_{\langle\alpha\beta\rangle} = h^{\mu}_{\alpha}h^{\nu}_{\beta}(X_{\mu\nu} - \tfrac{1}{3}\mathrm{Tr}\,Xh_{\mu\nu}). \tag{88}$$

From (79) we have

$$\mathrm{Tr}\;X \equiv X_T = \kappa\tilde{\rho}, \tag{89}$$

and

$$X_{\langle\alpha\beta\rangle} = X_{TF}\left(s_{\alpha}s_{\beta} + \frac{h_{\alpha\beta}}{3}\right), \tag{90}$$

where

$$X_{TF} \equiv \left(\frac{\kappa\Pi}{2} - E\right). \tag{91}$$

In a similar way we obtain

$$\mathrm{Tr}\;Y \equiv Y_T = \frac{\kappa}{2}(\tilde{\rho} + 3\tilde{P}_r - 2\Pi) \tag{92}$$

and

$$Y_{\langle\alpha\beta\rangle} = Y_{TF}\left(s_{\alpha}s_{\beta} + \frac{h_{\alpha\beta}}{3}\right), \tag{93}$$

with

$$Y_{TF} \equiv \left(\frac{\kappa\Pi}{2} + E\right). \tag{94}$$

Finally a fifth scalar may be defined from (78) as

$$Z = \sqrt{Z_{\alpha\beta}Z^{\alpha\beta}} = \frac{\kappa}{\sqrt{2}}\tilde{q}. \tag{95}$$

From the above it follows that local anisotropy of pressure is determined by X_{TF} and Y_{TF} by

$$\kappa\Pi = X_{TF} + Y_{TF}. \tag{96}$$

We can now rewrite Eqs. (59)–(66) in terms of our five structure scalars (X_T, X_{TF}, Y_T, Y_{TF}, Z):

$$\frac{\kappa}{2}\tilde{\rho}^* + \frac{1}{3}(X_T + Y_T + X_{TF} + Y_{TF})\theta = \frac{1}{3}\left(\theta + \frac{\sigma}{2}\right)(X_{TF} + Y_{TF}) - \sqrt{2}\left(\frac{Z^{\dagger}}{2} + aZ + \frac{s^1}{r}Z\right), \tag{97}$$

$$\frac{\kappa}{2}\tilde{P}_r^{\dagger} + \frac{1}{3}(X_T + Y_T + X_{TF} + Y_{TF})a + \frac{s^1}{r}(X_{TF} + Y_{TF}) = \frac{\sqrt{2}}{3}\left(\frac{\sigma}{2} - 2\theta\right)Z - \sqrt{2}\frac{Z^*}{2}, \tag{98}$$

$$\theta^* + \frac{1}{3}\theta^2 + \frac{\sigma^2}{6} - a^{\dagger} - a^2 - \frac{2}{r}as^1 = -Y_T, \tag{99}$$

$$\left(\frac{\sigma}{2} + \theta\right)^{\dagger} = -\frac{3\sigma s^1}{2r} + \frac{3\sqrt{2}}{2}Z \tag{100}$$

$$a^{\dagger} + a^2 + \frac{\sigma^*}{2} + \frac{1}{3}\theta\sigma - \frac{a}{r}s^1 - \frac{1}{12}\sigma^2 = -Y_{TF}, \tag{101}$$

$$\frac{1}{3}[(Y_T + Y_{TF}) - 2X_{TF} + X_T]\left(\theta + \frac{1}{2}\sigma\right) + \left(\frac{X_T}{2} - X_{TF}\right)^* = -\frac{3}{2r}s^1\sqrt{2}Z, \tag{102}$$

$$\left(\frac{\kappa}{2}\tilde{\rho} - X_{TF}\right)^{\dagger} = \frac{3s^1}{r}X_{TF} + \frac{\sqrt{2}}{2}Z\left(\frac{\sigma}{2} + \theta\right), \tag{103}$$

$$\frac{3m}{r^3} = \frac{X_T}{2} - X_{TF}. \tag{104}$$

Whereas for the Bel superenergy and the super-Poynting vector we find

$$\bar{W} = \tfrac{1}{6}(X_T^2 + Y_T^2) + \tfrac{1}{3}(X_{TF}^2 + Y_{TF}^2) + Z^2 \tag{105}$$

and

$$\bar{P}_{\alpha} = \tfrac{\sqrt{2}}{3}Z(X_T + Y_T + X_{TF} + Y_{TF})s_{\alpha}. \tag{106}$$

B. On the physical meaning of the structure scalars

Let us now focus on the physical meaning of the scalars introduced in the previous subsection.

The physical meaning of X_T and Z is evident and does not require further clarification.

Let us now consider X_{TF}. From (103) it follows that in the absence of dissipation ($Z = 0$), (using the regular center condition),

$$\tilde{\rho}^{\dagger} = 0 \Leftrightarrow X_{TF} = 0. \tag{107}$$

In other words, in the absence of dissipation, X_{TF} controls inhomogeneities in the energy density.

The role of density inhomogeneities in the collapse of dust [91] and, in particular, in the formation of naked singularities [92–99], has been extensively discussed in the literature.

In the nondissipative locally isotropic case, we obtain from (103), $\tilde{\rho}^{\dagger} = 0 \Leftrightarrow E = 0$. This link between the Weyl tensor and energy density inhomogeneity and the fact that tidal forces tend to make the gravitating fluid more inhomogeneous as the evolution proceeds, led Penrose to propose a gravitational arrow of time in terms of the Weyl tensor [100] (see also [101] and references therein).

However, the fact that such a relationship is no longer valid in the presence of local anisotropy of the pressure and/or dissipative processes, already discussed in [75], explains its failure in scenarios where the above-mentioned factors are present [102].

Here we see that the single scalar X_{TF} (in the absence of dissipation) controls density inhomogeneities and therefore should be the fundamental ingredient in the definition of a gravitational arrow of time. If dissipative processes are present, the scalar Z should be incorporated into that definition.

To establish the physical meaning of Y_T and Y_{TF} let us get back to Eqs. (57), using (94) (for the three cases considered in II C 1) we get

$$m_T = (m_T)_{\Sigma}\left(\frac{r}{r_{\Sigma}}\right)^3 + r^3\int_r^{r_{\Sigma}} \frac{e^{(\nu+\lambda)/2}}{r}Y_{TF}dr - r^3\int_r^{r_{\Sigma}} \frac{e^{(\lambda-\nu)/2}}{2r}\ddot{\lambda}dr. \tag{108}$$

Comparing the above expression with (58) we see that Y_{TF} describes the influence of the local anisotropy of pressure and density inhomogeneity on the Tolman mass. It is also worth recalling that Y_{TF}, together with X_{TF}, determines the local anisotropy of the fluid distribution.

Finally, we observe that for a system in equilibrium or quasiequilibrium, the Tolman mass (52) becomes

$$m_T = \frac{\kappa}{2}\int_0^r r^2 e^{(\nu+\lambda)/2}(T_0^0 - T_1^1 - 2T_2^2)dr, \tag{109}$$

which for those two regimes may be written as

$$m_T = \int_0^r r^2 e^{(\nu+\lambda)/2} Y_T dr. \tag{110}$$

Thus Y_T appears to be proportional to the Tolman mass "density" for systems in equilibriun or quasiequilibrium.

IV. ALL STATIC ANISOTROPIC SPHERES

In this section we shall restrict ourselves to static systems. We shall see how the metric corresponding to any static anisotropic sphere can be expressed in terms of the structure scalars. We shall explore three possible alternatives.

A. First alternative

From (41) and (104) it follows at once that

$$e^{-\lambda} = 1 - \tfrac{2}{3}r^2(\tfrac{1}{2}X_T - X_{TF}). \tag{111}$$

Next, using (99) and (101) in the static case, we may write

$$a = \frac{r}{3s^1}(Y_{TF} + Y_T). \tag{112}$$

On the other hand we have in the static case

$$a = e^{-(\lambda/2)}\frac{\nu'}{2}, \qquad s^1 = e^{-(\lambda/2)}, \tag{113}$$

where (17) and (28) have been used. Feeding back (113) into (112) and integrating we obtain

$$e^{\nu} = Ce^{\int (2r/3)(Y_{TF}+Y_T)/(1-(2r^2/3)((1/2)X_T - X_{TF}))dr}, \tag{114}$$

where C is a constant of integration easily obtained from (33).

Then, using (114) and (118) the line element (1) in the static case may be written as

$$ds^2 = Ce^{\int (2r/3)(Y_{TF}+Y_T)/(1-(2r^2/3)((1/2)X_T - X_{TF}))dr}dt^2 - \frac{1}{1-\frac{2}{3}r^2(\frac{1}{2}X_T - X_{TF})}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{115}$$

Thus we see that all possible spacetimes generated by static anisotropic fluids are fully determined by two scalars, namely, $Y_{TF} + Y_T$ and $\frac{1}{2}X_T - X_{TF}$.

B. Second alternative

Alternatively we may proceed as follows. From (43), (89), and (104) we obtain

$$m(r) = \frac{r^3}{3}\left(\frac{m'}{r^2} - X_{TF}\right), \tag{116}$$

which after integration produces

$$m(r) = r^3\left(\int \frac{X_{TF}}{r}dr + c_1\right). \tag{117}$$

Or, using (41)

$$e^{-\lambda} = 1 - 2r^2\left(\int \frac{X_{TF}}{r}dr + c_1\right), \tag{118}$$

where the constant of integration c_1 may be easily calculated from (34).

Next, the field equation (4) in the static case may be written as

$$\frac{\kappa}{2}P_r = \frac{1}{2}\frac{e^{-\lambda}-1}{r^2} + e^{-\lambda}\frac{\nu'}{2r}, \tag{119}$$

or, using (41), (92), (94), (112), and (113),

$$\frac{\kappa}{2}P_r + \frac{m}{r^3} = e^{-\lambda}\frac{\nu'}{2r} = Y_h, \tag{120}$$

with

$$Y_h = \tfrac{1}{3}(Y_T + Y_{TF}). \tag{121}$$

We can now integrate (120) to obtain

$$e^{\nu} = c_2 e^{\int 2rY_h/(1-2r^2(\int (X_{TF}/r)dr + c_1))dr}, \tag{122}$$

where the constant of integration c_2 may be obtained from (33).

Thus using (118) and (122), the line element (1) in the static case may be written as

$$ds^2 = c_2 e^{\int 2rY_h/(1-2r^2(\int (X_{TF}/r)dr + c_1))dr}dt^2 - \frac{1}{1-2r^2(\int \frac{X_{TF}}{r}dr + c_1)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{123}$$

allowing for the representation of all possible metrics in terms of two scalar functions, Y_h and X_{TF}.

C. Third alternative

In the two previous alternatives we have seen that all line elements corresponding to an anisotropic fluid may be determined by two scalars. However, in both cases these two scalars are a combination of four (alternative I) or three (alternative II) of our structure scalars. Here we shall present a third alternative, which allows for describing any line element in terms of only two structure scalars, namely X_{TF} and Y_{TF}.

Now, in the static case we may write from (39),

$$E = -\frac{e^{-\lambda}}{2}\left[\frac{\nu''}{2} + \left(\frac{\nu'}{2}\right)^2 + \frac{\nu'}{2}\left(-\frac{\lambda'}{2} - \frac{1}{r}\right) + \frac{\lambda'}{2r} + \frac{1-e^{\lambda}}{r^2}\right], \tag{124}$$

then introducing new variables

$$y = e^{-\lambda}, \qquad \frac{\nu'}{2} = \frac{u'}{u}, \tag{125}$$

(124) becomes

$$y' + 2y\frac{u'' - \frac{u'}{r} + \frac{u}{r^2}}{u' - \frac{u}{r}} = \frac{2u(1-2r^2E)}{r^2(u' - \frac{u}{r})}, \tag{126}$$

which after integration yields

$$y = e^{-\int k(r)dr}\left(\int e^{\int k(r)dr}f(r)dr + C_1\right), \tag{127}$$

with

$$k(r) = 2\frac{d}{dr}\left[\ln(u' - \frac{u}{r})\right], \qquad f(r) = \frac{2u(1-2r^2E)}{r^2(u' - \frac{u}{r})},$$

and where C_1 is a constant of integration easily obtained from the junction conditions.

Then, getting back to original variables, (127) becomes

$$\frac{\nu'}{2} - \frac{1}{r} = \frac{e^{\lambda/2}}{r} \times \sqrt{(1-2r^2E) + e^{-\nu}r^2C_1 + e^{-\nu}r^2\int (2r^2E)'\frac{e^{\nu}}{r^2}dr}. \tag{128}$$

Next let us introduce the new variable z by

$$e^{\nu} = \frac{e^{2\int z dr}}{r^2}, \tag{129}$$

producing

$$z(r) = \frac{\nu'}{2} + \frac{1}{r}. \tag{130}$$

Using (129) and (130) in (128) we get a link between E and z

$$z(r) = \frac{2}{r} + \frac{e^{\lambda/2}}{r}\sqrt{(1 - 2r^2E) + r^4 e^{-\int 2z(r)dr} C_1 + r^4 e^{-\int 2z(r)dr}\int (2r^2E)' \frac{e^{\int 2z(r)dr}}{r^4} dr}. \tag{131}$$

Next, from field equations (4) and (5) it follows

$$\kappa\Pi = e^{-\lambda}\left[-\frac{\nu''}{2} - \left(\frac{\nu'}{2}\right)^2 + \frac{\nu'}{2r} + \frac{1}{r^2}\right] + e^{-\lambda}\frac{\lambda'}{2}\left(\frac{\nu'}{2} + \frac{1}{r}\right) - \frac{1}{r^2}, \tag{132}$$

which, in terms of the variables z and y introduced above, becomes

$$y' + y\left(\frac{2z'}{z} + 2z - \frac{6}{r} + \frac{4}{r^2 z}\right) = -\frac{2}{z}\left(\frac{1}{r^2} + \kappa\Pi\right). \tag{133}$$

Integrating (133) we obtain for λ:

$$e^{\lambda(r)} = \frac{z^2(r) e^{\int ((4/r^2 z(r)) + 2z(r))dr}}{r^6(-2\int \frac{z(r)(1+\kappa\Pi(r)r^2) e^{\int ((4/r^2 z(r)) + 2z(r))dr}}{r^8} dr + C)}, \tag{134}$$

where C is a constant of integration.

In terms of X_{TF} and Y_{TF}, Eqs. (131) and (134) may be written as

$$z(r) = \frac{2}{r} + \frac{e^{\lambda/2}}{r}\sqrt{[1 - r^2(Y_{TF} - X_{TF})] + r^4 e^{-\int 2z(r)dr}(C_1 + \int [r^2(Y_{TF} - X_{TF})]' \frac{e^{\int 2z(r)dr}}{r^4} dr}, \tag{135}$$

and

$$e^{\lambda(r)} = \frac{z^2(r) e^{\int ((4/r^2 z(r)) + 2z(r))dr}}{r^6(-2\int \frac{z(r)[1+(X_{TF}+Y_{TF})r^2] e^{\int ((4/r^2 z(r)) + 2z(r))dr}}{r^8} dr + C)}. \tag{136}$$

Thus, given X_{TF} and Y_{TF} and using (136) in (135) we obtain z, which by virtue of (129) allows to find ν. Then using z in (136) determines λ.

This approach is essentially equivalent to the method for obtaining static anisotropic solutions presented in [68] (see also [103]).

As a simple example let us consider all conformally flat anisotropic fluids. Conformal flatness implies $Y_{TF} = X_{TF}$. Feeding back this condition into (135) we obtain for z

$$z = \frac{2}{r} + \frac{e^{\lambda/2}}{r}\tanh\left(\int \frac{e^{\lambda/2}}{r} \mathrm{dr}\right). \tag{137}$$

In order to specify a single solution we have to provide another condition on our scalars. Thus, for example, if we assume further the energy density to be constant, then $X_{TF} = 0$ implying $Y_{TF} = 0$, leading to the well-known Schwarzschild interior solution.

V. CONCLUSIONS

We have presented a systematic study of spherically symmetric self-gravitating relativistic fluids, based on scalars functions derived from the orthogonal splitting of the Riemann tensor. In the most general case (dissipative and anisotropic fluid) we have five scalars, which reduce to two, in the case of dissipationless dust and static anisotropic fluids, and to one for static isotropic fluids.

The motivation to present such a study and to consider further these scalars in the study of self-gravitating objects stems from their distinct physical meaning. As we have seen, two of them (X_T and Z) define the energy density and the dissipative flux, respectively. In the absence of dissipation, X_{TF} controls the inhomogeneity of energy density and therefore is the relevant quantity in any definition of a gravitational arrow of time *à la* Penrose. Of course, in the dissipative case Z must also enter into that definition.

The two scalars Y_{TF} and Y_T are related in a conspicuous way to the Tolman mass. On the one hand Y_{TF} describes the influence of both energy density inhomogeneity and local anisotropy of pressure on the Tolman mass. On the other hand Y_T acts as a Tolman mass density. It is worth noticing that these two scalars are the only ones that appear in the "kinematical" Eqs. (99) and (101). Also observe that Z is the only structure scalar appearing in (100).

In the static case, Einstein equations reduce to three ordinary differential equations for five variables $(\rho, P_r, P_\perp, \nu, \lambda)$, implying that any specific solution is determined by two scalar functions. This was illustrated in the three subsections above. Furthermore, these two scalar functions may be X_{TF} and Y_{TF} as shown in the third alternative developed in subsection IV C. This brings out further the physical relevance of the structure scalars.

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