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Kaluza-Klein towers for spinors in warped spaces

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All the boundary conditions compatible with the reduction of a five-dimensional spinor field of bulk mass M in a compactified warped space to a four-dimensional brane are derived from the Hermiticity conditions of the relevant operator. The possible presence of metric singularities is taken into account. Examples of resulting Kaluza-Klein spinor towers are given for a representative set of values for the basic parameters of the model and of the parameters describing the allowed boundary conditions, within the hypothesis that there exists one-mass-scale-only, the Planck mass. In many cases, the lowest mass in the tower is small and very sensitive to the parameters while the other masses are much higher and become more regularly spaced. In these cases, if a basic fermion of the standard model (lepton or quark) happens to be the lowest mass of a Kaluza-Klein tower, the other masses would be much larger and weakly dependent on the fermion which defines the tower.

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I. INTRODUCTION

The Kaluza-Klein towers originate from the consideration of fields in compactified higher dimensions [1]. A large number of original and review articles have been written on the subject for scalar fields and for fields of higher spin [2]. In particular, some attention has been paid on the boundary conditions which are especially important for the structure and classification of the Kaluza-Klein towers. For fermions we are interested in here, see for example [3].

In recent articles, we have reanalyzed in great generality the generation of Kaluza-Klein mass states in five-dimensional theories with a fifth dimension compactified either to a strip or on a circle. This study was carried out considering a scalar field propagating freely in the bulk, either in a flat space or in a warped space [4], without or with metric singularities [5]. The approach relies on a careful study of the Hermiticity properties of the operators which arise in the Kaluza-Klein reduction equations and which are of second order in the derivatives. This lead us, considering different five-dimensional metric configurations (flat and warped), to focus our attention on the classification of all the sets of allowed boundary conditions and, from them, to study thoroughly the corresponding mass equations leading to the construction of the so-called Kaluza-Klein mass towers. Remember that the consideration of warp spaces offers the possibility to solve in a elegant way, with only one extra dimension, the hierarchy problem, in the sense that starting from the Planck mass as the only fundamental mass of the model, the observable low lying Kaluza-Klein masses can be made of the order of TeV without fine-tuning.

Having in mind that the future high energy colliders are expected to look for the possible appearance of Kaluza-Klein mass towers which could be of non zero spin as evidence for the existence of fields with spin propagating in higher dimensions, we were led to extend our work to spinor fields. In a previous article [5], we restricted ourselves, as a first step in a more general approach, to a five-dimensional flat space. Requesting the Dirac operator be a symmetric operator and taking into account the underlying symmetries of the Dirac equation in five dimensions, in particular, covariance and parity invariance in the brane, the whole set of allowed boundary conditions has been established leading to the mass equations from which the Kaluza-Klein mass towers are built.

In our preceding papers, illustrative numerical examples of Kaluza-Klein mass towers were given for the different configurations we considered.

In this article, we extend our study of Dirac fields by the consideration of five-dimensional compactified warp spaces, first without, and then with metric singularities. The article is organized as follows. In Sec. II, we consider the case of a warp space with no metric singularity. We establish the specific form of the Dirac equation and proceed with the Kaluza-Klein reduction. The whole set of allowed boundary conditions are obtained from the Hermiticity of the Dirac operator or equivalently from the least action principle (see also [3]). Within their classification, we have explored and exploited especially those, less studied in the literature, which relate together the values of the fields at different boundaries. The solutions for a free field with an arbitrary mass M propagating in the bulk are given explicitly. In Sec. III, we extend the same considerations to the case of warp spaces with an arbitrary number of metric singularities, focusing again on the determination of all the allowed boundary conditions.

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Section IV is devoted to various physical considerations concerning the determination and the interpretation of the Kaluza-Klein mass eigenstates, in particular, considerations about the possible choices of boundary conditions, the closure of the extra dimension strip to a circle, the mass scales of the model, the relation between Kaluza-Klein eigenmasses and observable masses and finally the mass state probability densities. In Sec. V, the general procedure adopted for the determination of the Kaluza-Klein mass towers is elaborated from the boundary conditions and the analytical expressions of the field. In Sec. VI, some illustrative numerical examples of Kaluza-Klein mass towers are given for specific boundary conditions, in the cases without and with metric singularities. Several types of towers result and some never appeared in print as of today.

In Appendix A, we show that the boundary relations derived from the application of the least action principle are identical to those we deducted from the symmetry of the Dirac operator. In Appendix. B, we developed some examples of boundary conditions in the general case of an arbitrary number of metric singularities.

II. THE DIRAC EQUATION IN A WARPED SPACE. NO METRIC SINGULARITIES

A. The dirac equation. invariances

We consider the Dirac equation (see Appendix A) with a bulk mass M

$$(i\gamma^A D_A - M)\Psi = 0 \tag{1}$$

and with the invariant scalar product between the spinors Φ and Ψ

$$(\Phi, \Psi) = \int \bar{\Phi}(x)\Psi(x)\sqrt{g}d^4x \tag{2}$$

in a five-dimensional warped space. With the following notation for the indices

$$\begin{aligned}&\text{warp space: } \{A\} \equiv \{\Sigma, S\} \equiv \{0, I, S\}, \qquad I = 1, 2, 3, \qquad S = 5\\ &\text{local space: } \{a\} \equiv \{\sigma, s\} \equiv \{0, i, s\}, \qquad i = 1, 2, 3, \qquad s = 5\end{aligned} \tag{3}$$

the warped metric is

$$\begin{aligned} dS^2 &= g_{AB}dx^A dx^B = g_{\Sigma\Theta}dx^\Sigma dx^\Theta - ds^2\\ &= \lambda^2 e^{-2\epsilon ks}\eta_{\sigma\theta}dx^\Sigma dx^\Theta - ds^2. \end{aligned} \tag{4}$$

In this equation, $\eta_{\sigma\theta}$, η_{ss} are the components of the five-dimensional flat space metric with signature $(+,-,-,-,-)$, λ is an arbitrary positive constant, introduced for later convenience, while, with k defined to be positive, the warp factor ϵk, $\epsilon = \pm 1$ can be chosen to be positive or negative. As in the four-dimensional space, the Dirac spinor is four-dimensional Ψ_α, $\alpha = 1, \ldots, 4$ in a five-dimensional space.

The non zero elements of the vielbein $e_A{}^c$ defined as usual by

$$g_{AB} = e_A{}^c \eta_{cd} e_B{}^d \tag{5}$$

are chosen as

$$e_\Sigma{}^\sigma = e_0{}^0 = e_1{}^1 = e_2{}^2 = e_3{}^3 = \lambda e^{-\epsilon ks}, \qquad e_S{}^s = 1 \tag{6}$$

$$e_\sigma{}^\Sigma = e_0{}^0 = e_1{}^1 = e_2{}^2 = e_3{}^3 = \lambda^{-1} e^{\epsilon ks}, \qquad e_s{}^S = 1. \tag{7}$$

The warped γ^A are given by

$$\gamma^A = e_a{}^A \gamma^a \tag{8}$$

with the local γ^a of the flat five-dimensional space built from those of the flat four-dimensional space ($[\gamma_a, \gamma_b]_+ = \eta_{ab}$, in particular $\gamma_s = \gamma_0\gamma_1\gamma_2\gamma_3$).

The Dirac equation is covariant under the diffeomorphisms and independently under local SO(4,1) transformations $\hat{e}_A{}^a = T^a{}_b e_A{}^b$, $\hat{\Psi}_\alpha = S_{\alpha\beta}\Psi_\beta$ with $S^{-1}\gamma^a S = T^a{}_b\gamma^b$ and $T^t \eta T = \eta$.

The covariant derivative of the four-component spinor field Ψ_α is given by

$$(D_A^{[s]}\Psi)_\alpha = \partial_A \Psi_\alpha + (G_A^{[s]})_{\alpha\beta}\Psi_\beta \tag{9}$$

where the spinor connection $G_A^{[s]}$, reduces to

$$G_\Sigma^{[s]} = -\frac{1}{2}\epsilon\lambda k e^{-\epsilon ks}\gamma_\sigma\gamma^5, \qquad G_S^{[s]} = 0. \tag{10}$$

Finally, one finds from (1) the specific Dirac equation of the warped space (4)

$$\left(\frac{e^{\epsilon ks}}{\lambda}(i\gamma^\sigma\partial_\Sigma) + (i\gamma^5)(\partial_5 - 2k\epsilon)\right)\Psi = M\Psi. \tag{11}$$

The five-dimensional mass M, the mass in the bulk, is an arbitrary parameter of the model.

B. Symmetry of the dirac operator

In order to have real M, the Dirac operator $\mathcal{D}$ in (1)

$$\mathcal{D} = i\gamma^A D_A \tag{12}$$

should be symmetric for the Hermitian scalar product (2), namely

$$(\Phi, \mathcal{D}\Psi) = (\mathcal{D}\Phi, \Psi). \tag{13}$$

The Eq. (13) reduces (up to a factor i) to the integral of a divergence

$$\int \partial_A(\bar{\Phi}\gamma^A\Psi\sqrt{g})d^dx = 0 \tag{14}$$

meaning that this operator is formally symmetric. This equation determines the boundary conditions which must be satisfied by Φ and Ψ in order for the Dirac operator to be fully symmetric. In this section, the discussion is carried on in a warped space without metric singularities and in Sec. III with an arbitrary number N of metric singularities.

C. Kaluza-Klein reduction. No metric singularity

We adopt the following Kaluza-Klein separation of variables

$$\Psi(x^\mu, s) = \sum_n (F^{[n]}(s) + iG^{[n]}(s)\gamma^5)\psi^{[n]}(x^\mu) \tag{15}$$

assuming that $F^{[n]}(s)$ and $G^{[n]}(s)$ are complex functions depending on s only while $\psi^{[n]}(x^\mu)$ is a x^μ dependent spinor.

In this form we have made the most general choice compatible with an SO(3,1) spinor invariance in the sense that Ψ and $\psi^{[n]}$ are supposed to transform in the same way under this subgroup of the spinor SO(4,1) transformations.

D. Boundary relation and conditions for the spinor fields. No metric singularity

For each variable x^A with range $[-\infty, \infty]$, the integration in (14) is identically zero for Ψ and Φ in the spinor Hilbert space (sufficient decrease of the fields at $\pm\infty$). For the variables s which has a finite range $[0, 2\pi R]$, the boundary relation is

$$\int d^4x[\bar{\Phi}\gamma^5\Psi\sqrt{g}]_{s=2\pi R} = \int d^4x[\bar{\Phi}\gamma^5\Psi\sqrt{g}]_{s=0}, \tag{16}$$

where the integration is carried on all the variables x^μ. In turn, the relation (16) implies conditions between the fields evaluated at $s = 2\pi R$ and $s = 0$. These boundary conditions will be written explicitly below for the case of a five-dimensional warped space without metric singularity or in Sec. III for the case with metric singularities.

We do not consider here the variable s with a semi-infinite range $[0, \infty]$ (up to a transformation $s' = \pm s + \beta$) which requires a special treatment.

In order to obtain the boundary conditions which must be satisfied by the components $F^{[n,\Psi]}$ and $G^{[n,\Psi]}$ of the field (with identical relations for $F^{[n,\Phi]}$ and $G^{[n,\Phi]}$), one introduces the reduction ansatz (15) in the boundary relation (16).

As in the flat case for spinors [5], there are two sets of possible boundary conditions. The first set

$$\text{Set BC1:} \left\{ \begin{pmatrix} F^{[n]}(2\pi R) \\ G^{[n]}(2\pi R) \end{pmatrix} = B \begin{pmatrix} F^{[n]}(0) \\ G^{[n]}(0) \end{pmatrix}, \right. \tag{17}$$

where B is a complex 2×2 matrix. After some computation one finds that B must be of the form

$$B = e^{4\epsilon\pi kR} e^{i\rho} \begin{pmatrix} \cosh(\omega) & \sinh(\omega) \\ \sinh(\omega) & \cosh(\omega) \end{pmatrix}, \tag{18}$$

where ρ is a real parameter with range $0 \le \rho < 2\pi$ and ω is an arbitrary real parameter. Compared to the flat case there is simply an extra $e^{4\epsilon\pi kR}$ factor.

The second set

$$\text{Set BC1:} \begin{cases} G^{[n]}(0) = \epsilon_0 F^{[n]}(0), & \epsilon_0^2 = 1 \\ G^{[n]}(2\pi R) = \epsilon_R F^{[n]}(2\pi R), & \epsilon_R^2 = 1 \end{cases}, \tag{19}$$

where ϵ_0 and ϵ_R are two arbitrary signs is identical to the corresponding set in the flat case.

One supposes that the fields satisfy the S0(3,1) invariant boundary conditions.

E. Solutions. No metric singularity

Introducing the reduction ansatz (15) in the five-dimensional Dirac Eq. (1) and postulating that $\psi^{[n]}$ satisfies the four-dimensional parity invariant Dirac equation

$$(i\gamma^\mu\partial_\mu - m_n)\psi^{[n]} = 0 \tag{20}$$

one finds from (11) the two coupled equations for the components of the field

$$\begin{aligned} \partial_s G^{[n]} &= \left(M - e^{\epsilon ks}\frac{m_n}{\lambda}\right)F^{[n]} + 2\epsilon k G^{[n]} \\ \partial_s F^{[n]} &= 2\epsilon k F^{[n]} + \left(M + e^{\epsilon ks}\frac{m_n}{\lambda}\right)G^{[n]}. \end{aligned} \tag{21}$$

The solution can be written

$$\begin{aligned} F^{[n]}(s) &= \frac{1}{2}(F_+^{[n]}(s) + F_-^{[n]}(s)) \\ G^{[n]}(s) &= \frac{1}{2}(F_+^{[n]}(s) - F_-^{[n]}(s)) \end{aligned} \tag{22}$$

with $F_+^{[n]}(s)$ the following linear superposition of Bessel functions.

$$\begin{aligned} F_+^{[n]}(s) = \left(\frac{m_n e^{\epsilon ks}}{\lambda k}\right)^{5/2} \Bigg(\sigma_n J_{(\epsilon M/k)-(1/2)}\left(\frac{m_n e^{\epsilon ks}}{\lambda k}\right) \\ + \tau_n Y_{(\epsilon M/k)-(1/2)}\left(\frac{m_n e^{\epsilon ks}}{\lambda k}\right)\Bigg) \end{aligned} \tag{23}$$

with two arbitrary constants, and

$$F_-^{[n]}(s) = \frac{\lambda e^{-\epsilon ks}}{m_n}(-\partial_s F_+^{[n]}(s) + (M + 2\epsilon k)F_+^{[n]}(s)). \tag{24}$$

The constants σ_n and τ_n of (23) are determined by the boundary conditions (17) or (19).

III. APPLICATION TO A FIVE-DIMENSIONAL WARPED SPACE WITH METRIC SINGULARITIES

A. The five-dimensional warped space with N metric singularities

The extension of the preceding arguments to a warped space with an arbitrary number N of metric singularities situated at the points s_i, $i = 1, N$ with $s_0 = 0 < s_1 < s_2, \ldots, s_N < s_{N+1} = 2\pi R$ on the strip is straightforward.

By definition, the metric is of the general form (4) with $\epsilon = 1$ for some (possibly nonconnected) region of s and $\epsilon = -1$ for the complementary region. A singular point is a point which joins two regions of opposite values of ϵ. We moreover postulate, for physical reasons, that all the components of the metric are continuous at the singular points.

There are $N + 1$ intervals I_i, $i = 0, \ldots, N$

$$I_0 = [0, s_1], \qquad I_1 = [s_1, s_2], \ldots, I_{N-1} = [s_{N-1}, s_N], \quad I_N = [s_N, 2\pi R] \tag{25}$$

of respective length

$$l_0 = s_1, \qquad l_1 = s_2 - s_1, \quad l_2 = s_3 - s_2, \ldots, l_N = 2\pi R - s_N. \tag{26}$$

Defining

$$r_i = -2(-1)^{i+1}\left(\sum_{j=0}^{i-1}(-1)^j s_{i-j}\right) \tag{27}$$

(note $r_0 = 0$) equivalent to

$$r_{2i} = 2\sum_{j=1}^{i} l_{2j-1} \qquad r_{2i+1} = -2\sum_{j=0}^{i} l_{2j}, \tag{28}$$

the metric takes the explicit form

$$\text{for } s \in I_i\text{: } dS^2 = e^{-2k\epsilon((-1)^i s - r_i)} dx_\mu dx^\mu - ds^2 \quad (i = 0, \ldots, N) \tag{29}$$

chosen to be normalized to one at $s = 0$. The sign of the coefficient of s in the exponent alternates between ϵ and $-\epsilon$ for the intervals I_i with even and odd i. The end points of each interval are thus singular points, except $s = 0$ and $s = 2\pi R$ (see however the special case of a closure to a circle in Sec. IV B).

In each subspace, the Dirac equation derived from (1) assumes the form

$$\text{for } s \in I_i\text{: } (e^{\epsilon k((-1)^i s - r_i)}(i\gamma^\sigma \partial_\Sigma) + (i\gamma^5)(\partial_5 - 2(-1)^i k\epsilon))\Psi = M\Psi. \tag{30}$$

B. Kaluza-Klein reduction with metric singularities

We adopt the following Kaluza-Klein separation of variables analogous to the no singularity case (15), in each interval

$$\Psi^{[i]}(x^\mu, s) = \sum_n (F^{[n,i]}(s) + iG^{[n,i]}(s)\gamma^5)\psi^{[n]}(x^\mu) \tag{31}$$

assuming $\psi^{[n]}(x^\mu)$ to be a spinor depending on x^μ only, and independent of the interval I_i to which s belongs. The complex functions $F^{[n,i]}(s)$ and $G^{[n,i]}(s)$ are functions depending on s only. They are supposed to be smooth within the intervals I_i, where they may take different analytical forms.

Introducing the reduction (31) into the Dirac Eq. (30) in each subspace I_i and postulating that the $\psi^{[n]}(x_\mu)$ satisfies the four-dimensional Dirac Eq. (20), we find the two coupled equations

$$\text{for } s \in I_i \begin{cases} \partial_s G^{[n,i]} = (M - e^{\epsilon k((-1)^i s - r_i)} m_n) F^{[n,i]} + 2(-1)^i \epsilon k G^{[n,i]} \\ \partial_s F^{[n,i]} = 2(-1)^i \epsilon k F^{[n,i]} + (M + e^{\epsilon k((-1)^i s - r_i)} m_n) G^{[n,i]}. \end{cases} \tag{32}$$

C. Solutions with metric singularities

Following the same procedure as in the case without singularity II E we find that in the interval I_i, the solution is

$$F^{[n,i]}(s) = \frac{1}{2}(F_+^{[n,i]}(s) + F_-^{[n,i]}(s)) \quad G^{[n,i]}(s) = \frac{1}{2}(F_+^{[n,i]}(s) - F_-^{[n,i]}(s)). \tag{33}$$

The function $F_+^{[n,i]}(s)$ is the following linear superposition of Bessel functions.

$$F_+^{[n,i]}(s) = \left(\frac{m_n e^{\epsilon k((-1)^i s - r_i)}}{k}\right)^{5/2} \times \left(\sigma_{n,i} J_{(\epsilon(-1)^i M)/k-(1/2)}\left(\frac{m_n e^{\epsilon k((-1)^i s - r_i)}}{k}\right) + \tau_{n,i} Y_{(\epsilon(-1)^i M)/k-(1/2)}\left(\frac{m_n e^{\epsilon k((-1)^i s - r_i)}}{k}\right)\right) \tag{34}$$

depending on two arbitrary constants $\sigma_{n,i}$, $\tau_{n,i}$ and with

$$F_-^{[n,i]}(s) = \frac{e^{-\epsilon k((-1)^i s - r_i)}}{m_n}(-\partial_s F_+^{[n,i]}(s) + (M + 2\epsilon k(-1)^i) \times F_+^{[n,i]}(s)). \tag{35}$$

The constants $\sigma_{n,i}$ and $\tau_{n,i}$ (altogether $2(N + 1)$ parameters) must satisfy $2(N + 1)$ homogeneous linear boundary relations expressing the boundary conditions (see (52)). For given boundary conditions, in order to obtain a non trivial solution for the $\sigma_{n,i}$ and $\tau_{n,i}$, the related $2(N + 1) \times 2(N + 1)$ determinant must vanish, leading to a mass eigenvalue equation for the m_n.

D. Boundary relation and conditions for the spinor fields with metric singularities

In the boundary relation (14), the total derivative terms in Σ vanish since the fields are supposed to decrease sufficiently fast at infinity in the Σ directions. The fields are in general discontinuous at the metric singularity points. We define

$$\Psi^l(s_i) = \lim_{\eta \to 0^+} \Psi(x^\mu, s_i - \eta)$$
$$\Psi^r(s_i) = \lim_{\eta \to 0^+} \Psi(x^\mu, s_i + \eta). \tag{36}$$

The boundary relation (14), after integration over s, becomes

$$\int d^4x \left(\sum_{i=1}^{N+1} \bar{\Phi}^l(s_i)\gamma^5 \Psi^l(s_i)\sqrt{g(s_i)} - \sum_{i=0}^{N} \bar{\Phi}^r(s_i)\gamma^5\Psi^r(s_i)\sqrt{g(s_i)} \right) = 0. \tag{37}$$

Expanding Ψ and Φ according to the Kaluza-Klein reduction (31), leading to

$$\bar{\Phi}(x^\mu, s) = \sum_n \bar{\phi}^{[n]}(x^\mu)(C^{[n]*}(s) - iD^{[n]*}(s)\gamma^5), \tag{38}$$

one finds after some algebra

$$\sum_{i=0}^{N}(D^{[m]*r}(s_i)F^{[n]r}(s_i) - C^{[m]*r}(s_i)G^{[n]r}(s_i))\sqrt{g(s_i)} - \sum_{i=1}^{N+1}(D^{[m]*l}(s_i)F^{[n]l}(s_i) - C^{[m]*l}(s_i)G^{[n]l}(s_i))\sqrt{g(s_i)} = 0 \tag{39}$$

$$\sum_{i=0}^{N}(C^{[m]*r}(s_i)F^{[n]r}(s_i) - D^{[m]*r}(s_i)G^{[n]r}(s_i))\sqrt{g(s_i)} - \sum_{i=1}^{N+1}(C^{[m]*l}(s_i)F^{[n]l}(s_i) - D^{[m]*l}(s_i)G^{[n]l}(s_i))\sqrt{g(s_i)} = 0. \tag{40}$$

In terms of the left and right boundary values, we define the $4(N+1)$ dimensional vectors

$$\Phi = \begin{pmatrix} \sqrt[4]{g(s_0)}C^{[n]r}(s_0) \\ \sqrt[4]{g(s_0)}D^{[n]r}(s_0) \\ \sqrt[4]{g(s_1)}C^{[n]r}(s_1) \\ \sqrt[4]{g(s_1)}D^{[n]r}(s_1) \\ \vdots \\ \sqrt[4]{g(s_N)}C^{[n]r}(s_N) \\ \sqrt[4]{g(s_N)}D^{[n]r}(s_N) \\ \sqrt[4]{g(s_1)}C^{[n]l}(s_1) \\ \sqrt[4]{g(s_1)}D^{[n]l}(s_1) \\ \vdots \\ \sqrt[4]{g(s_N)}C^{[n]l}(s_N) \\ \sqrt[4]{g(s_N)}D^{[n]l}(s_N) \\ \sqrt[4]{g(s_{N+1})}C^{[n]l}(s_{N+1}) \\ \sqrt[4]{g(s_{N+1})}D^{[n]l}(s_{N+1}) \end{pmatrix}, \quad \Psi = \begin{pmatrix} \sqrt[4]{g(s_0)}F^{[n]r}(s_0) \\ \sqrt[4]{g(s_0)}G^{[n]r}(s_0) \\ \sqrt[4]{g(s_1)}F^{[n]r}(s_1) \\ \sqrt[4]{g(s_1)}G^{[n]r}(s_1) \\ \vdots \\ \sqrt[4]{g(s_N)}F^{[n]r}(s_N) \\ \sqrt[4]{g(s_N)}G^{[n]r}(s_N) \\ \sqrt[4]{g(s_1)}F^{[n]l}(s_1) \\ \sqrt[4]{g(s_1)}G^{[n]l}(s_1) \\ \vdots \\ \sqrt[4]{g(s_N)}F^{[n]l}(s_N) \\ \sqrt[4]{g(s_N)}G^{[n]l}(s_N) \\ \sqrt[4]{g(s_{N+1})}F^{[n]l}(s_{N+1}) \\ \sqrt[4]{g(s_{N+1})}G^{[n]l}(s_{N+1}) \end{pmatrix}. \tag{41}$$

The two boundary relations (39) and (40) can be written in matrix form

$$\Phi^+ S_j^{[4(N+1)]}\Psi = 0, \qquad j = 1, 2, \tag{42}$$

where S_j are square matrices with upper index $[4(N+1)]$ referring to their size. For (39), the antisymmetric matrix $S_1^{[4(N+1)]}$ has the following form

$$S_1^{[4(N+1)]} = \begin{pmatrix} S_1^{[2(N+1)]} & 0^{[2(N+1)]} \\ 0^{[2(N+1)]} & -S_1^{[2(N+1)]} \end{pmatrix} \tag{43}$$

with the zero matrix $0^{[2(N+1)]}$ and the antisymmetric block diagonal matrix $S_1^{[2(N+1)]}$

$$S_1^{[2(N+1)]} = \begin{pmatrix} -i\sigma_2 & 0^{[2]} & \dots \\ 0^{[2]} & -i\sigma_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}. \tag{44}$$

For (40), the matrix $S_2^{[4(N+1)]}$ is block diagonal

$$S_2^{[4(N+1)]} = \begin{pmatrix} S_2^{[2(N+1)]} & 0^{[2(N+1)]} \\ 0^{[2(N+1)]} & -S_2^{[2(N+1)]} \end{pmatrix} \tag{45}$$

with the diagonal matrix $S_2^{[2(N+1)]}$

$$S_2^{[2(N+1)]} = \begin{pmatrix} \sigma_3 & 0^{[2]} & \cdots \\ 0^{[2]} & \sigma_3 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \tag{46}$$

The allowed sets of boundary conditions can be obtained from the boundary relations (42), by the following general procedure. The boundary conditions are expressible in terms of $2(N+2)$ independent homogeneous linear relations among the components of the matrix Ψ (41) chosen in such a way as to guarantee the two boundary relations (39) and (40). The components of Φ have of course to satisfy the same linear relations. The boundary conditions are written

$$M_{\rm BC}\Psi = 0 \tag{47}$$

where $M_{\rm BC}$ is a $2(N+1)\times 4(N+1)$ matrix of rank $2(N+1)$. For any such $M_{\rm BC}$, there exists a $4(N+1)\times 4(N+1)$ permutation matrix P such that, defining

$$\Psi_P \equiv P\Psi, \tag{48}$$

the $2(N+1)$ boundary conditions are equivalent to

$$\Psi_P = V_P^{[4(N+1)]}\Psi_P \tag{49}$$

with the matrix $V_P^{[4(N+1)]}$ written in terms of a matrix $V_P^{[2(N+1)]}$ (depending on P) and the unit matrix $1^{[2(N+1)]}$ as

$$V_P^{[4(N+1)]} = \begin{pmatrix} 1^{[2(N+1)]} & 0^{[2(N+1)]} \\ V_P^{[2(N+1)]} & 0^{[2(N+1)]} \end{pmatrix}. \tag{50}$$

Writing Ψ_P in terms of its $2(N+1)$ upper elements Ψ_P^u and its down elements Ψ_P^d

$$\Psi_P = \begin{pmatrix} \Psi_P^u \\ \Psi_P^d \end{pmatrix} \tag{51}$$

one finds that the $2(N+1)$ first equations are trivial while the last $2(N+1)$ equations express the boundary conditions equivalent to (47)

$$\Psi_P^d = V_P^{[2(N+1)]}\Psi_P^u. \tag{52}$$

This is in agreement with the observation that, from (47), there exists always a permutation P of the component of Ψ such that the $2(N+1)$ components (Ψ_P^d) are linear functions of the $2(N+1)$ other components (Ψ_P^u).

Writing $S_{Pj}^{[4(N+1)]}$ ($j = 1, 2$) the transformed of $S_j^{[4(N+1)]}$ under P

$$S_{Pj}^{[4(N+1)]} = PS_j^{[4(N+1)]}P^{-1}, \tag{53}$$

the matrix $V_P^{[4(N+1)]}$ expressing the allowed boundary conditions (49), must satisfy the two matrix equations

$$(V_P^{[4(N+1)]})^+ S_{Pj}^{[4(N+1)]} V_P^{[4(N+1)]} = 0. \tag{54}$$

This follows from the fact that the boundary relations (42) then depend on Ψ_P^u and Φ_P^{u+} only which are all independent and arbitrary.

With the four $2(N+1)\times 2(N+1)$ matrices $S_{Pj}^{[2(N+1)],r}$, $r = 1, \ldots, 4$, $j = 1, 2$ defined for each $S_{Pj}^{[4(N+1)]}$ as

$$S_{Pj}^{[4(N+1)]} = \begin{pmatrix} S_{Pj}^{[2(N+1)],1} & S_{Pj}^{[2(N+1)],2} \\ S_{Pj}^{[2(N+1)],3} & S_{Pj}^{[2(N+1)],4} \end{pmatrix}, \tag{55}$$

the boundary relations (42) lead explicitly to two equations for the matrix $V_P^{[2(N+1)]}$

for $j = 1, 2$

$$S_{Pj}^{[2(N+1)],1} + (V_P^{[2(N+1)]}))^+ S_{Pj}^{[2(N+1)],3} + S_{Pj}^{[2(N+1)],2} V_P^{[2(N+1)]} + (V_P^{[2(N+1)]})^+ S_{Pj}^{[2(N+1)],4} V_P^{[2(N+1)]} = 0. \tag{56}$$

It should be stressed that different choices of P may lead to equivalent, differently expressed, boundary conditions, in particular, by multiplying P by further permutations within the elements of Ψ^u or within the elements of Ψ^d. A few examples of boundary conditions are given in Appendix A.

IV. PHYSICAL CONSIDERATIONS

In our previous article [5], we have given a detailed discussion of the physical relevance of the main aspects underlying the Kaluza-Klein construction for scalars. We summarize here the points which apply to the spinor case.

A. Physical discussion of the generalized boundary conditions

It happens that the boundary conditions (52) may connect the values of the components F and G of the field (not their derivatives as in the scalar case) at different points of the s-domain i.e. at the N metric singular points and at the two edges. In this case, the field explores, in fact, its full domain at once. This is tantamount to action at a distance or non locality, which we argued previously not to be in contradiction with quantum mechanics (see Sec. 5.1 of the third article in [5]).

In our numerical applications (6) however, we restrict ourselves either to fully local boundary conditions (locality at the metric singular points as well as at the edges (Appendix B 2) or to partially local boundary conditions (excluding locality at the edges (Appendix B 1).

B. Closure to a circle

The strip could be closed in a circle by identifying the points $s = 0$ and $s = 2\pi R$ under the following requirements.

There must be an even number $2p(p > 0)$ of singularities. By rotation around the circle, the first singularity can always be placed at the closure point $s = 0$. Then $N \equiv 2p - 1$. The total range where the sign of s in the exponential in the metric is positive must be equal to the total range where it is negative (25) and (26)

$$\sum_{j=0}^{j=p-1} l_{2j} = \sum_{j=0}^{j=p-1} l_{2j+1} = \pi R. \tag{57}$$

C. Physical discussion of the singularities

The physical meaning of the metric singularities is, we feel, a delicate question which is probably not yet fully understood. If it happens that the fifth dimension is a strip, there is no need for singularities although they may exist. However, as we just saw, if the strip is closed to a circle, at least two singularities must be present, and the question of their interpretation cannot be avoided. It has been argued, in the Z_2 orbifold case, that the invoked visible and hidden branes would contain localized fields (or energy) [4] related to the corresponding metric singularities. With this interpretation, why would not then one or more singularities appear on an interval or an arbitrary even number of singularities on a closed circle, meaning one or more positions with localized fields on the extra dimension axis? We have chosen not to be prejudiced and decided to envisage an arbitrary number of singularities. Indeed, if brane localized fields are allowed, there is no good argument to exclude their presence in one or more locations.

D. The ''one-mass-scale-only'' hypothesis

By assumption, there is only one high mass scale in the theory which is chosen to be the Planck mass

$$M_{\rm Pl} \approx 1.22\ 10^{16}\ \text{TeV}. \tag{58}$$

The dimensionfull parameters k, R and M can be written in terms of reduced parameters $\bar{k}$, $\bar{R}$ and $\bar{M}$

$$k = \bar{k} M_{\rm Pl} \qquad R = \bar{R}(M_{\rm Pl})^{-1} \qquad M = \bar{M} M_{\rm Pl}. \tag{59}$$

We call the assumption that the reduced parameters are neither large nor small numbers (except 0) the ‘‘one-mass-scale-only’’ hypothesis. The parameter $\bar{k}\bar{R} = kR$ governs the reduction from the high mass scale to the TeV scale of the low lying masses in the Kaluza-Klein towers.

Finally, let us note that by rescaling the parameter $\bar{k}$ can always be chosen to be equal to one

$$\bar{k} = 1. \tag{60}$$

Since the mass eigenvalue equation are covariant under a rescaling of all the reduced parameters $\bar{p}$ according to their energy dimension d_p

$$\bar{p} \longrightarrow \lambda^{d_p} \bar{p}, \tag{61}$$

one finds that the mass eigenvalues for a given $\bar{k}$ can be obtained from eigenvalues corresponding to our choice $\bar{k} = 1$ (using $\lambda = 1/\bar{k}$) by

$$m_n(\{\bar{k}, \bar{R}, \bar{M}\}) = \bar{k} m_n\left(\left\{1, \bar{k}\bar{R}, \frac{\bar{M}}{\bar{k}}\right\}\right). \tag{62}$$

E. The physical masses

For a four-dimensional observer supposed to be sitting at $s = s_{\rm obs}$ in a given I_i interval (25), the metric (29)

$$dS^2 = e^{-2\epsilon k((-1)^i s_{\rm obs} - r_i)} dx_\mu dx^\mu - ds^2 \tag{63}$$

can be transformed in canonical form

$$dS^2 = d\tilde{x}_\mu d\tilde{x}^\mu - ds^2 \tag{64}$$

by the following rescaling

$$\tilde{x}_\mu = e^{-\epsilon k((-1)^i s_{\rm obs} - r_i)} x_\mu. \tag{65}$$

According to (20), we have

$$\begin{aligned} i\gamma^\mu \tilde{\partial}_\mu \psi^{[n]} &= e^{\epsilon k((-1)^i s_{\rm obs} - r_i)} (i\gamma^\mu \partial_\mu \psi^{[n]}) \\ &= e^{\epsilon k((-1)^i s_{\rm obs} - r_i)} m_n \psi^{[n]}. \end{aligned} \tag{66}$$

The mass as seen in by the observer in the brane at $s = s_{\rm obs} \in I_i$ is thus related to the mass eigenvalue m_n by

$$m_n^{\rm obs} = e^{\epsilon k((-1)^i s_{\rm obs} - r_i)} m_n. \tag{67}$$

For $s_{\rm obs} = 0$, the physical mass is just equal to the mass eigenvalue.

F. Probability density

Once all the parameters defining a specific model are chosen and the mass eigenvalue tower is determined, there exists a unique field $\psi^{[n]}(x^\mu, s)$ (see (31)) for each mass eigenvalue leading to a naive probability density field distribution $D^{[n]}(x^\mu, s)$ which depends both on x^μ and s

$$D^{[n]}(x^\mu, s) = \sqrt{g}(\bar{\psi}^{[n]}(x^\mu, s)\psi^{[n]}(x^\mu, s)). \tag{68}$$

Note that the shape of this density distribution depends in general on the interval I_i to which s belongs. As observed and discussed in [5], these probability densities are fast varying functions of s. The total normalized probability density for a Kaluza-Klein particle to be present in a brane situated at $s = s_{\rm obs}$ is

$$D^{[n]}(s_{\rm obs}) = \frac{\int d^4x D^{[n]}(x^\mu, s_{\rm obs})}{\int d^4x ds D^{[n]}(x^\mu, s)}. \tag{69}$$

Remember that the physical mass as seen by the observer is also a function of the $s_{\rm obs}$ position (67).

More generally, this raises the question of the underlying dynamics of the production of the tower states and of their interactions. In this work, we have just focused on the construction of the towers and have neglected the much more difficult and model dependent problems related to the dynamics.

V. TOWERS

In the absence of metric singularities, the two arbitrary parameters σ and τ which appear in the solution (22)–(24) of the Dirac Eq. (1) in the five-dimensional space after the Kaluza-Klein (KK) reduction (15) have to satisfy two homogeneous linear equations expressing an allowed set of boundary conditions, belonging either to the set BC1 (17) or to the set BC2 (19). The condition for the existence of a non trivial σ, τ solution is the vanishing of the related determinant. This leads in each case to a mass equations from which the KK mass towers can be derived. In Sec. VI A, numerical examples of KK mass towers are given for each of the two sets of boundary conditions, for different values of the basic parameters of the model, i.e. the warp factors ϵ, k, kR, the bulk mass M, as well as for different values of the parameters ρ, ω or ϵ_0, ϵ_R defining the boundary conditions considered.

In the general case, when there are N metric singularities, there are $N+1$ parameters $\sigma_{n,i}$ and $N+1$ parameters $\tau_{n,i}$ appearing in the solution (22), (34), and (35) of the Dirac equation after the Kaluza-Klein reduction (31). These parameters have to satisfy the $2(N+1)$ homogeneous linear Eqs. (52) resulting from the imposition of the $2(N+1)$ boundary conditions on the $2(N+1)$ values of the fields at the edges of the $N+1$ intervals I_i in the s-range (36). Indeed, for a given singularity configuration, there exists a set of $2(N+1)$ boundary conditions resulting from the two boundary relations (42) expressing the condition of Hermiticity of the Dirac operator. As in the preceding case, the requested vanishing of the determinant of the coefficients of the $2(N+1)$ boundary conditions with respect to the $2(N+1)$ parameters $(\sigma_{n,i}, \tau_{n,i})$ leads to the corresponding KK mass equation. In Sec. VI B, a few examples of towers are given when there is one singularity.

For completeness, let us list all the parameters. They are the basic parameters of the warp model k, ϵ, kR, the bulk mass M, the positions s_i of the N metric singularities and the boundary parameters defining the matrix $V_P^{[2(N+1)]}$ subject to the two conditions (56). Once all these parameters are chosen, the vanishing of the above determinant is generally a transcendental function of the eigenvalues m_n.

VI. EXAMPLES OF TOWERS

For an illustration of the types of spinor towers which appear in warped spaces, we construct examples of the eight lowest mass eigenvalues for simple specific boundary conditions. We first discuss the case when there is no metric singularity, then when there is one metric singularity. We would like to stress that, in order to perform the numerical computations, high precision is mandatory.

A. Examples of towers. No metric singularity

In this subsection, a few illustrative numerical examples of Kaluza-Klein spinor towers in warped spaces are presented for each of the two sets BC1 (17) and BC2 (19) of boundary conditions and for some chosen values of the bulk mass M and of the parameters fixing the boundary conditions. In general, the Kaluza-Klein mass eigenvalues are irregularly spaced. With the adopted values of the basic parameters of the warp model, i.e. k arbitrarily normalized to the Planck mass ($\bar{k}=1$, see (59)) and $kR \approx 6.3$, all the low lying Kaluza-Klein masses are of the order TeV. In the tables, kr is fixed to

$$kR = 6.3 \tag{70}$$

and the Kaluza-Klein tower masses denoted with $\tilde{m}_i$ are given in TeV. As a general rule, the values of $\tilde{m}_i$ increase (exponentially) when kR decreases, hence fixing the overall scale of the masses in the tower (for example for $kR = 6$ the masses are multiplied roughly by a factor of $e^{2\pi(6.3-6)} \approx 6.5$). It should be noted that choosing the value of the bulk mass M to zero or to values of the order of the Planck mass, within the one-mass-scale-only IV D, does not lead to substantially different Kaluza-Klein towers.

In Table I, the eight low lying mass eigenvalues ($\tilde{m}_i$, $i = 1, \ldots, 8$) of Kaluza-Klein towers are given in the case of boundary conditions BC1 (17) for zero bulk mass M, for different values of the parameter ρ, and for each of them, for different values of the parameter ω (18). One observe that the first mass of the towers, $\tilde{m}_1$, is relatively sensitive to the value of ω, particularly for small values of the parameter ρ. For $\omega = \rho = 0$, the Kaluza-Klein tower exhibits some characteristic features: it is the only tower to possess a zero mass state while the higher masses are doubly degenerate. Indeed, one sees that, for $\rho = 0$, when ω decreases toward zero, pairs of adjacent masses in the towers are getting closer and closer and take the same value when ω reaches the value zero.

In Tables II, III, IV, and V, the Kaluza-Klein mass towers are similarly presented for a representative choice of bulk mass values, respectively $\bar{M} = 0.01$, $\bar{M} = 0.1$ and $\bar{M} = 1$ (59). It appears, as a general rule, that the lowest lying mass of the towers $\tilde{m}_1$ vanishes when $\rho = 0$ and when the parameter ω takes exactly the value $\omega_{\bar{M}}$

$$\{\rho = 0 \quad \text{and} \quad \omega = \omega_{\bar{M}} = 2\pi RM\} \leftrightarrow m_1 = 0. \tag{71}$$

This agrees with the analogous result for $M = 0$ as seen in Table I. Moreover, for $\rho = 0$ and for any $\bar{M}$ of the order 1, the value of $\tilde{m}_1$ depends almost exactly linearly on the

TABLE I. Mass towers for $M = 0$ and for the boundary conditions BC1 (no metric singularity) (17) and (18). The towers are symmetric under $\omega \leftrightarrow -\omega$. The mass eigenvalues $\tilde{m}_i$ are in TeV.

BC1 (no singularity). Case $M = 0$ and $kR = 6.3$									
ρ	ω	$\tilde{m}_1$	$\tilde{m}_2$	$\tilde{m}_3$	$\tilde{m}_4$	$\tilde{m}_5$	$\tilde{m}_6$	$\tilde{m}_7$	$\tilde{m}_8$
0	0	0	0.49362	0.49362	0.98725	0.98725	1.48087	1.48087	1.9740
	0.01	0.00078561	0.49283	0.49441	0.98646	0.98803	1.4801	1.4817	1.9737
	0.1	0.00784	0.48578	0.50147	0.97941	0.99509	1.473	1.4887	1.9667
	1	0.068017	0.42561	0.56164	0.91923	1.0553	1.4129	1.5489	1.9065
	5	0.12235	0.37128	0.61597	0.8649	1.1096	1.3585	1.6032	1.8522
	100	0.12341	0.37022	0.61703	0.86384	1.1107	1.3575	1.6043	1.8511
$\pi/10$	0	0.024681	0.46894	0.5183	0.9625	1.01193	1.4562	1.5055	1.9498
	0.01	0.024693	0.46893	0.51832	0.96256	1.0119	1.4562	1.5056	2.0004
	0.1	0.025858	0.46777	0.51948	0.96139	1.0131	1.455	1.5067	1.9486
	1	0.071234	0.42239	0.56486	0.91602	1.0585	1.4096	1.5521	1.9033
	5	0.1224	0.37123	0.61603	0.86485	1.1097	1.3585	1.6033	1.8521
	100	0.12341	0.37022	0.61703	0.86384	1.1107	1.3575	1.6043	1.8511
$\pi/4$	0	0.061703	0.43192	0.55533	0.92555	1.049	1.4192	1.5426	1.9128
	0.01	0.061707	0.43192	0.55534	0.92554	1.049	1.4192	1.5426	1.9128
	0.1	0.062094	0.43153	0.55572	0.92516	1.0493	1.4188	1.543	1.9124
	1	0.08601	0.40762	0.57963	0.90124	1.0733	1.3949	1.5669	1.8885
	5	0.12266	0.37097	0.61628	0.86459	1.1099	1.3582	1.6035	1.8518
	100	0.12341	0.37022	0.61703	0.86384	1.1107	1.3575	1.6043	1.8511
$\pi/2$	[0, 100]	0.12341	0.37022	0.61703	0.86384	1.1107	1.3575	1.6043	1.8511
$3\pi/4$	0	0.18511	0.30852	0.67874	0.80214	1.1724	1.2958	1.666	1.7894
	0.1	0.18472	0.30891	0.67834	0.80253	1.172	1.2962	1.6656	1.7898
	1	0.1608	0.33282	0.65443	0.82645	1.1481	1.3201	1.6417	1.8137
	5	0.12415	0.36947	0.61778	0.8631	1.1114	1.3567	1.605	1.8503
	100	0.12341	0.37022	0.61703	0.86384	1.1107	1.3575	1.6043	1.851
$9\pi/10$	0	0.22213	0.27149	0.71576	0.76512	1.2094	1.2587	1.703	1.7524
	0.1	0.22095	0.27267	0.71458	0.7663	1.2082	1.2599	1.7018	1.7535
	1	0.17558	0.31805	0.6692	0.81167	1.1628	1.3053	1.6565	1.7989
	5	0.12441	0.36921	0.61804	0.86284	1.1117	1.3565	1.6053	1.8501
	100	0.12341	0.37022	0.61703	0.86384	1.1107	1.3575	1.6043	1.8511
π	0	0.24681	0.24682	0.74044	0.74044	1.2341	1.2341	1.7277	1.7277
	0.1	0.23897	0.25466	0.73259	0.74829	1.2262	1.2419	1.7198	1.7355
	1	0.1788	0.31483	0.67242	0.80846	1.166	1.3021	1.6597	1.7957
	5	0.12447	0.36916	0.61809	0.86279	1.1117	1.3564	1.6053	1.85
	100	0.12341	0.37022	0.61703	0.86384	1.1107	1.3575	1.6043	1.8511

value of the parameter ω from about $\omega = \omega_{\bar{M}} - 1$ up to values very close to $\omega_{\bar{M}}$ and on the other side from ω very close to $\omega_{\bar{M}}$ up to about $\omega_{\bar{M}} + 1$.

In general, except for the first mass of the towers in the case $\rho = 0$, all the other masses in the towers do not show a strong dependence on the value of the ω parameter. The fact that the first mass of the tower can take values between 0 and about 0.1 TeV, and hence can be small when ρ is not large, allows one, by a suitable choice of the parameters kR, M, ω to associate a tower to a particular fermion of the standard model be it a lepton or a quark and assuming it to be the lowest state of a Kaluza-Klein tower in a five-dimensional warped space. From the second mass on, the intervals between successive masses are generally much larger and more regular.

In Table VI, Kaluza-Klein towers are presented for the set of boundary conditions BC2 (set (19)) for the two possible choices of the product $\epsilon_0\epsilon_R$ of the boundary condition parameters, and for each of them, for some values of the reduced bulk mass $\bar{M}$. In general, the tower masses have a fairly mild dependence with respect to the bulk mass, with the exception of the first mass in the towers when $\epsilon_0\epsilon_R$ is equal to -1, in which case, starting from a value of about 0.1 TeV for $\bar{M} = 0$, it falls to less than 10^{-10} TeV when $\bar{M}$ is equal to one or higher. This feature is again of importance in view of practical applications of the warp model to fermions, either to the leptons or to the quarks of the standard model. Indeed, by an adequate choice kR and of the bulk mass M, the first mass of a Kaluza-Klein tower could be made equal to the mass of a

TABLE II. Mass towers for $\bar{M} = 0.01$ (59) and for the boundary conditions BC1 (no metric singularity) (17) and (18). Here $\omega_{0.01} = 0.3958406\ldots$ (71). The mass eigenvalues $\tilde{m}_i$ are in TeV.

BC1 (no singularity). Case $\bar{M} = 0.01$ and $kR = 6.3$										
$\bar{M}$	ρ	ω	$\tilde{m}_1$	$\tilde{m}_2$	$\tilde{m}_3$	$\tilde{m}_4$	$\tilde{m}_5$	$\tilde{m}_6$	$\tilde{m}_7$	$\tilde{m}_8$
0.01	0	-100	0.12486	0.37153	0.61832	0.86511	1.1119	1.3587	1.6055	1.8523
		-5	0.12413	0.37227	0.61758	0.86585	1.1112	1.3595	1.6048	1.8531
		-1	0.08607	0.41059	0.57881	0.90447	1.0722	1.3983	1.5657	1.892
		0	0.03042	0.46598	0.5221	0.96011	1.0152	1.454	1.5085	1.9479
		0.01	0.029692	0.4667	0.52135	0.96084	1.0145	1.4548	1.5078	1.9486
		0.05	0.026728	0.46963	0.51834	0.96377	1.0115	1.4577	1.5048	1.9515
		0.3	0.0075237	0.48849	0.49891	0.98262	0.99202	1.4765	1.4853	1.9704
		$\omega_{0.01} - 10^{-3}$	$7.8549\ 10^{-5}$	0.49095	0.49623	0.98406	0.99035	1.4774	1.4843	1.9708
		$\omega_{0.01}$	0	0.49087	0.49630	0.98398	0.99043	1.47730	1.48435	1.97071
		$\omega_{0.01} + 10^{-3}$	$7.8546\ 10^{-5}$	0.4908	0.49637	0.98391	0.9905	1.4772	1.4844	1.9706
		1	0.04456	0.44608	0.53983	0.93924	1.0339	1.4326	1.5278	1.926
		2	0.09130	0.3982	0.56746	0.88909	1.0634	1.3812	1.5583	1.8738
		5	0.1204	0.37043	0.61423	0.86409	1.1079	1.3577	1.6015	1.8513
		10	0.12194	0.36891	0.61574	0.86258	1.1094	1.3562	1.603	1.8499
		100	0.12195	0.3689	0.61575	0.86257	1.1094	1.3562	1.603	1.8498
	$\pi/10$	-100	0.12486	0.37153	0.61832	0.86511	1.1119	1.3587	1.6055	1.8523
		-5	0.12417	0.37223	0.61761	0.86582	1.1112	1.3594	1.6048	1.8531
		-1	0.088106	0.4085	0.5809	0.90237	1.0743	1.3961	1.5678	1.8899
		$5\ 10^{-3}$	0.038561	0.45728	0.53078	0.95128	1.024	1.4451	1.5175	1.9389
		0.3	0.025784	0.46849	0.51891	0.96221	1.0124	1.4559	1.506	1.9495
		$\omega_{0.01} - 10^{-3}$	0.024677	0.46877	0.51841	0.96233	1.0121	1.4559	1.5057	1.9495
		$\omega_{0.01}$	0.024677	0.46876	0.51841	0.96232	1.0121	1.4559	1.5057	1.9495
		$\omega_{0.01} + 10^{-3}$	0.024676	0.46876	0.51842	0.96231	1.0121	1.4559	1.5058	1.9495
		0.5	0.025923	0.46665	0.52028	0.96005	1.0141	1.4535	1.5079	1.9471
		1	0.050207	0.44069	0.54522	0.93391	1.0392	1.4273	1.5331	1.9208
		5	0.12048	0.37036	0.6143	0.86402	1.108	1.3576	1.6016	1.8513
		100	0.12195	0.3689	0.61575	0.86257	1.1094	1.3562	1.603	1.8498
	$\pi/2$	-100	0.12486	0.37153	0.61832	0.86511	1.1119	1.3587	1.6055	1.8523
		0	0.12393	0.37068	0.61748	0.86428	1.1111	1.3579	1.6047	1.8515
		$\omega_{0.01}$	0.12338	0.37018	0.61699	0.8638	1.1106	1.3574	1.6042	1.851
		5	0.12195	0.3689	0.61575	0.86257	1.1094	1.3562	1.603	1.8498
		100	0.12195	0.3689	0.61575	0.86257	1.1094	1.3562	1.603	1.8498
	π	-100	0.12486	0.37153	0.61832	0.86511	1.1119	1.3587	1.6055	1.8523
		-5	0.12558	0.3708	0.61905	0.86437	1.1127	1.358	1.6063	1.8516
		-1	0.16353	0.33224	0.65755	0.82548	1.1514	1.3189	1.6451	1.8124
		0	0.21866	0.27577	0.71308	0.76862	1.2071	1.2619	1.701	1.7552
		$\omega_{0.01}$	0.24457	0.24900	0.73739	0.74341	1.23063	1.23740	1.724	1.7313
		2	0.15266	0.33865	0.646	0.83258	1.1395	1.3264	1.633	1.8201
		5	0.12349	0.36737	0.61727	0.86106	1.1109	1.3547	1.6045	1.8483
		100	0.12195	0.3689	0.61575	0.86257	1.1094	1.3562	1.603	1.8498
	$3\pi/2$	0	0.12393	0.37068	0.61748	0.86428	1.1111	1.3579	1.6047	1.8515
		$\omega_{0.01}$	0.12338	0.37018	0.61699	0.8638	1.1106	1.3574	1.6042	1.851
		5	0.12195	0.3689	0.61575	0.86257	1.1094	1.3562	1.603	1.8498

given lepton, for example, the muon, leading to identify the tower as associated to that lepton. Considering for example $kR = 6.3$ and the case of the muon, with mass equal to $1.057\ 10^{-4}$ TeV as the lowest mass in a tower, the associated Kaluza-Klein tower would result from adopting a value around 0.65 for the reduced bulk mass of the five-dimensional fermion $\bar{M}_\mu$ associated to the muon. A not very different reduced bulk mass $\bar{M}_e = 0.8$ would produce the electron of mass equal to $5.11\ 10^{-7}$ TeV as its first mass. It is interesting to remark that bulk fermions with rather close reduced bulk masses (0.65, 0.8) would lead to the observed fermions with masses in the large ratio $m_\mu/m_e = 206.8$. It should be noted that the Kaluza-Klein tower masses associated to either of these two lep-

TABLE III. Mass towers for $\bar{M} = 0.1$ (59) and for the boundary conditions BC1 (no metric singularity) (17) and (18). Here $\omega_{0.1} = 3.958406\ldots$ (71). The mass eigenvalues $\tilde{m}_i$ are in TeV (18).

BC1 (no singularity). Case $\bar{M} = 0.1$ and $kR = 6.3$										
$\bar{M}$	ρ	ω	$\tilde{m}_1$	$\tilde{m}_2$	$\tilde{m}_3$	$\tilde{m}_4$	$\tilde{m}_5$	$\tilde{m}_6$	$\tilde{m}_7$	$\tilde{m}_8$
0.1	0	-100	0.13756	0.38327	0.62981	0.87651	1.1232	1.37	1.6168	1.8636
		0	0.13405	0.3982	0.56746	0.88909	1.0634	1.3812	1.5583	1.8738
		0.1	0.13368	0.38757	0.62526	0.88117	1.1184	1.3749	1.6118	1.8686
		2	0.11163	0.41108	0.59908	0.90652	1.0906	1.4013	1.5829	1.8958
		3.5	0.035687	0.47886	0.51167	0.97459	1.0017	1.4687	1.4939	1.9622
		$\omega_{0.1} - 10^{-3}$	$7.6983\ 10^{-5}$	0.4667	0.52135	0.96084	1.0145	1.4548	1.5078	1.9486
		$\omega_{0.1}$	0	0.46285	0.51711	0.95083	1.0148	1.4412	1.5107	1.9326
		$\omega_{0.1} + 10^{-3}$	$7.6968\ 10^{-5}$	0.48849	0.49891	0.98262	0.99202	1.4765	1.4853	1.9704
		5	0.09129	0.3982	0.56746	0.88909	1.0634	1.3812	1.5583	1.8738
		100	0.10836	0.35695	0.60413	0.85111	1.098	1.3449	1.5917	1.8386
	$\pi/10$	-100	0.13756	0.38327	0.62981	0.8765	1.1232	1.37	1.6168	1.8636
		-5	0.13754	0.3833	0.62978	0.87653	1.1232	1.37	1.6168	1.8636
		0	0.13422	0.38697	0.6259	0.88051	1.1191	1.3742	1.6125	1.8679
		2.5	0.09733	0.42499	0.58206	0.92123	1.0726	1.4164	1.5644	1.9112
		3.5	0.04326	0.46584	0.52471	0.96012	1.0162	1.4537	1.5089	1.9471
		$\omega_{0.1} - 10^{-3}$	0.024178	0.4536	0.52631	0.94278	1.0228	1.4338	1.5181	1.9255
		$\omega_{0.1}$	0.02418	0.45366	0.52627	0.94284	1.0227	1.4338	1.5181	1.9256
		$\omega_{0.1} + 10^{-3}$	0.024178	0.4536	0.52631	0.94278	1.0228	1.4338	1.5181	1.9255
		5.5	0.082178	0.3808	0.58249	0.873	1.0776	1.3658	1.572	1.8588
		9	0.10756	0.35766	0.60346	0.85176	1.0974	1.3455	1.5911	1.8392
		15	0.10836	0.35695	0.60413	0.85111	1.098	1.3449	1.5917	1.8386
		100	0.10836	0.35695	0.60413	0.85111	1.098	1.3449	1.5917	1.8386
	$\pi/2$	$[-100, 0]$	0.13755	0.38325	0.62979	0.87648	1.1232	1.37	1.6168	1.8636
		$\omega_{0.1}$	0.12055	0.36645	0.61279	0.85929	1.1059	1.3525	1.5991	1.8458
		[8, 100]	0.10837	0.35695	0.60414	0.85111	1.098	1.3449	1.5917	1.8386
	π	-100	0.13756	0.38327	0.62981	0.8765	1.1232	1.37	1.6168	1.8636
		-5	0.13759	0.38324	0.62984	0.87647	1.1233	1.37	1.6168	1.8636
		0	0.14106	0.37936	0.6339	0.87225	1.1276	1.3656	1.6213	1.859
		2	0.16286	0.35411	0.65892	0.84469	1.154	1.3367	1.6486	1.8292
		3.5	0.22906	0.26911	0.72706	0.75623	1.2218	1.2476	1.7155	1.7403
		$\omega_{0.1}$	0.22176	0.26629	0.70634	0.76627	1.1958	1.2629	1.6868	1.7584
		4.5	0.18156	0.29665	0.6679	0.79554	1.1584	1.2915	1.6501	1.7867
		5	0.15425	0.31828	0.64339	0.81563	1.1351	1.3109	1.6275	1.8056
		100	0.10836	0.35695	0.60413	0.85111	1.098	1.3449	1.5917	1.8386
	$3\pi/2$	0	0.13755	0.38325	0.62979	0.87648	1.1232	1.37	1.6168	1.8636
		$\omega_{0.1}$	0.12055	0.36645	0.61279	0.85929	1.1059	1.3525	1.5991	1.8458
		8	0.10837	0.35695	0.60414	0.85111	1.098	1.3449	1.5917	1.8386

tons would be hardly distinguishable beyond the first mass. One should also be aware that the Kaluza-Klein towers associated to a given fermion would be of a different structure depending on the set (BC1 or BC2) of boundary conditions considered.

B. Examples of towers. One metric singularity. Semilocal boundary conditions

When there is one metric singularity, the number of arbitrary parameters increases. Besides kR and M, the position of the singularity on the strip $[0, 2\pi R]$ appears as a new parameter

$$s_1 = (2\pi R)\bar{s}_1, \qquad 0 \le \bar{s}_1 \le 1. \tag{72}$$

which is complemented by the boundary condition parameters.

In order to keep the mass eigenvalues roughly of the order of TeV, we are led to adapt the value of kR to the value chosen for $\bar{s}_1$. Satisfactory choices are

$$\begin{aligned} \bar{s}_1 = 1 \leftrightarrow kR = 6.3 \qquad \bar{s}_1 = 0.9 \leftrightarrow kR = 6.9 \\ \bar{s}_1 = 0.75 \leftrightarrow kR = 8.3 \qquad \bar{s}_1 = 0.5 \leftrightarrow kR = 12.5. \end{aligned} \tag{73}$$

There are many possible sets of allowed boundary conditions as seen in the discussion of Appendix B. To build

TABLE IV. Mass towers for $\bar{M} = 1$ (59) as a function of ω for $\rho = 0$ and for the boundary conditions BC1 (no metric singularity) (17) and (18). Here $\omega_1 = 39.58406\ldots$ (71)). The masses m_2 to m_8 are essentially independent of ρ. The mass m_1 is also independent of ρ except for ω in a range close to ω_1, approximatively in the range $[\omega_1 - 10,\ \omega_1 + 10]$. In this range, the variation of m_1 as a function of ρ is given in Table V. The mass tower is symmetric under $(\rho) \leftrightarrow (2\pi - \rho)$. The mass eigenvalues $\tilde{m}_i$ are in TeV.

BC1 (no singularity). Case $\bar{M} = 1$ and $kR = 6.3$										
$\bar{M}$	ρ	ω	$\tilde{m}_1$	$\tilde{m}_2$	$\tilde{m}_3$	$\tilde{m}_4$	$\tilde{m}_5$	$\tilde{m}_6$	$\tilde{m}_7$	$\tilde{m}_8$
1	0	$[-100, 5]$	0.24681	0.49363	0.74044	0.98725	1.2341	1.4809	1.7277	1.9745
		15	0.2468	0.49359	0.74039	0.98718	1.234	1.4808	1.7276	1.9744
		19	0.20097	0.41953	0.65138	0.88974	1.1313	1.3746	1.6189	1.8638
		20	0.10356	0.36447	0.61361	0.86141	1.1088	1.3559	1.603	1.85
		25	$7.4472\ 10^{-4}$	0.35302	0.60692	0.85666	1.1051	1.3529	1.6004	1.8478
		30	$5.0175\ 10^{-6}$	0.35302	0.60692	0.85666	1.1051	1.3529	1.6004	1.8478
		39	$3.8500\ 10^{-8}$	0.35302	0.60692	0.85666	1.1051	1.3529	1.6004	1.8478
		$\omega_1 - 10^{-3}$	$3.4548\ 10^{-13}$	0.35302	0.60692	0.85666	1.1051	1.3529	1.6004	1.8478
		ω_1	0	0.35302	0.60692	0.85666	1.1051	1.3529	1.6004	1.8478
		$\omega_1 + 10^{-3}$	$3.4514\ 10^{-13}$	0.35302	0.60692	0.85666	1.1051	1.3529	1.6004	1.8478
		41	$2.6150\ 10^{-10}$	0.35302	0.60692	0.85666	1.1051	1.3529	1.6004	1.8478
		50	$3.453\ 10^{-10}$	0.35302	0.60692	0.85666	1.1051	1.3529	1.6004	1.8478
		[55–200]	$3.4531\ 10^{-10}$	0.35302	0.60692	0.85666	1.1051	1.3529	1.6004	1.8478

our examples, we have limited ourselves to what we call semilocal boundary conditions: the fields on the left and on the right of the singularity are related and, separately, the fields at $s = 0$ are related to the fields at $s = 2\pi R$. Both boundary conditions are taken of the form BC1 (17) and (18) and hence are defined by four parameters

$$\begin{aligned}&\text{BC1 with parameters } \omega_b, \rho_b \text{ at the edges } 0 \text{ and } 2\pi R\\ &\text{BC1 with parameters } \omega_s, \rho_s \text{ on both sides of the sigularity } s_1.\end{aligned} \tag{74}$$

The conditions for $m_1 = 0$ are analogous to the conditions in the case of no metric singularity (71)

$$\{\rho_b - \rho_s = 0 \quad \text{and} \quad \omega_b - \omega_s = \omega_{\bar{M}} = 2\pi R M\} \leftrightarrow m_1 = 0. \tag{75}$$

In Table VII, for $\bar{M} = 1$, numerical examples of towers are given for some arbitrarily chosen positions s_1 of the metric singularity and some values of the boundary parameters (74). Similar results, respectively, for $\bar{M} = 0.1$ and $\bar{M} = 0$, are presented in Tables VIII and IX.

Again in view of applications to leptons and quarks, it should be noted that, when the parameters almost satisfy the mass zero conditions (75), the tower consists of a low mass $\tilde{m}_1$ accompanied, as a signature, by almost regularly separated doublets of higher masses with $\tilde{m}_2$ much larger than $\tilde{m}_1$.

TABLE V. The lowest mass eigenvalue $\tilde{m}_1$ in the towers for $\bar{M} = 1$ (59), as a function of ρ and of ω in the range $[\omega_1 - 10,\ \omega_1 + 10]$, and for the boundary conditions BC1 (no metric singularity) (17) and (18). Here $\omega_1 = 39.58406\ldots$ (71)). The mass $\tilde{m}_1$ (in TeV) is symmetric under $(\rho) \leftrightarrow (2\pi - \rho)$.

BC1 (no singularity). The first mass eigenvalue $\tilde{m}_1$ for $\bar{M} = 1$ and $kR = 6.3$												
							ω					
		30	36	38	39	$\omega_1 - 10^{-3}$	ω_1	$\omega_1 + 10^{-3}$	41	42	44	50
ρ	0	$5.02\ 10^{-6}$	$1.21\ 10^{-8}$	$1.34\ 10^{-9}$	$2.74\ 10^{-10}$	$3.46\ 10^{-13}$	0	$3.46\ 10^{-13}$	$2.62\ 10^{-10}$	$3.15\ 10^{-10}$	$3.41\ 10^{-10}$	$3.45\ 10^{-10}$
	$10^{-10}\pi$	$5.02\ 10^{-6}$	$1.21\ 10^{-8}$	$1.34\ 10^{-9}$	$2.74\ 10^{-10}$	$3.45\ 10^{-13}$	$1.08\ 10^{-19}$	$3.45\ 10^{-13}$	$2.62\ 10^{-10}$	$3.14\ 10^{-10}$	$3.41\ 10^{-10}$	$3.45\ 10^{-10}$
	$10^{-5}\pi$	$5.02\ 10^{-6}$	$1.21\ 10^{-8}$	$1.34\ 10^{-9}$	$2.74\ 10^{-10}$	$3.45\ 10^{-13}$	$1.08\ 10^{-14}$	$3.45\ 10^{-13}$	$2.62\ 10^{-10}$	$3.14\ 10^{-10}$	$3.41\ 10^{-10}$	$3.45\ 10^{-10}$
	0.1π	$5.02\ 10^{-6}$	$1.21\ 10^{-8}$	$1.36\ 10^{-9}$	$3.10\ 10^{-10}$	$1.08\ 10^{-10}$	$1.08\ 10^{-10}$	$1.08\ 10^{-10}$	$2.67\ 10^{-10}$	$3.16\ 10^{-10}$	$3.41\ 10^{-10}$	$3.45\ 10^{-10}$
	0.2π	$5.02\ 10^{-6}$	$1.22\ 10^{-8}$	$1.42\ 10^{-9}$	$3.96\ 10^{-10}$	$2.14\ 10^{-10}$	$2.13\ 10^{-10}$	$2.13\ 10^{-10}$	$2.82\ 10^{-10}$	$3.21\ 10^{-10}$	$3.42\ 10^{-10}$	$3.45\ 10^{-10}$
	0.3π	$5.02\ 10^{-6}$	$1.22\ 10^{-8}$	$1.51\ 10^{-9}$	$5.01\ 10^{-10}$	$3.14\ 10^{-10}$	$3.14\ 10^{-10}$	$3.13\ 10^{-10}$	$3.04\ 10^{-10}$	$3.28\ 10^{-10}$	$3.43\ 10^{-10}$	$3.45\ 10^{-10}$
	0.4π	$5.02\ 10^{-6}$	$1.23\ 10^{-8}$	$1.61\ 10^{-9}$	$6.09\ 10^{-10}$	$4.06\ 10^{-10}$	$4.06\ 10^{-10}$	$4,06\ 10^{-10}$	$3.29\ 10^{-10}$	$3.37\ 10^{-10}$	$3.44\ 10^{-10}$	$3.45\ 10^{-10}$
	0.5π	$5.02\ 10^{-6}$	$1.24\ 10^{-8}$	$1.72\ 10^{-9}$	$7.09\ 10^{-10}$	$4.89\ 10^{-10}$	$4.88\ 10^{-10}$	$4,88\ 10^{-10}$	$3.55\ 10^{-10}$	$3.47\ 10^{-10}$	$3.45\ 10^{-10}$	$3.45\ 10^{-10}$
	0.6π	$5.02\ 10^{-6}$	$1.25\ 10^{-8}$	$1.82\ 10^{-9}$	$7.97\ 10^{-10}$	$5.59\ 10^{-10}$	$5.59\ 10^{-10}$	$5.58\ 10^{-10}$	$3.80\ 10^{-10}$	$3.56\ 10^{-10}$	$3.47\ 10^{-10}$	$3.45\ 10^{-10}$
	0.7π	$5.02\ 10^{-6}$	$1.26\ 10^{-8}$	$1.91\ 10^{-9}$	$8.68\ 10^{-10}$	$6.16\ 10^{-10}$	$6.15\ 10^{-10}$	$6.15\ 10^{-10}$	$4.00\ 10^{-10}$	$3.64\ 10^{-10}$	$3.48\ 10^{-10}$	$3.45\ 10^{-10}$
	0.8π	$5.02\ 10^{-6}$	$1.27\ 10^{-8}$	$1.97\ 10^{-9}$	$9.21\ 10^{-10}$	$6.57\ 10^{-10}$	$6.57\ 10^{-10}$	$6.56\ 10^{-10}$	$4.16\ 10^{-10}$	$3.71\ 10^{-10}$	$3.49\ 10^{-10}$	$3.45\ 10^{-10}$
	0.9π	$5.02\ 10^{-6}$	$1.28\ 10^{-8}$	$2.01\ 10^{-9}$	$9.54\ 10^{-10}$	$6.82\ 10^{-10}$	$6.82\ 10^{-10}$	$6.82\ 10^{-10}$	$4.26\ 10^{-10}$	$3.75\ 10^{-10}$	$3.49\ 10^{-10}$	$3.45\ 10^{-10}$
	π	$5.02\ 10^{-6}$	$1.28\ 10^{-8}$	$2.03\ 10^{-9}$	$9.65\ 10^{-10}$	$6.91\ 10^{-10}$	$6.91\ 10^{-10}$	$6.90\ 10^{-10}$	$4.29\ 10^{-10}$	$3.76\ 10^{-10}$	$3.49\ 10^{-10}$	$3.45\ 10^{-10}$

TABLE VI. Mass towers for the boundary conditions BC2 (19) (no metric singularity). The mass eigenvalues $\tilde{m}_i$ are in TeV.

BC2 (no singularity). Case $kR = 6.3$									
$\epsilon_0\epsilon_R$	$\bar{M}$	$\tilde{m}_1$	$\tilde{m}_2$	$\tilde{m}_3$	$\tilde{m}_4$	$\tilde{m}_5$	$\tilde{m}_6$	$\tilde{m}_7$	$\tilde{m}_8$
1	0	0.24681	0.49363	0.74044	0.98725	1.2341	1.4809	1.7277	1.9745
	0.1	0.25789	0.5053	0.75233	0.99925	1.2461	1.493	1.7398	1.9867
	0.3	0.27967	0.52839	0.77592	1.0231	1.2701	1.5171	1.764	2.0109
	0.6	0.31158	0.56244	0.81084	1.0585	1.3058	1.553	1.8001	2.0471
	0.7	0.32204	0.57366	0.82237	1.0702	1.3177	1.5649	1.812	2.0591
	1	0.35302	0.60692	0.85666	1.1051	1.3529	1.6004	1.8478	2.095
	1.5	0.40347	0.66128	0.91289	1.1624	1.411	1.659	1.9067	2.1542
	2	0.45279	0.71453	0.96813	1.2189	1.4683	1.7169	1.9651	2.2129
	5	0.73502	1.0187	1.2849	1.544	1.7995	2.0527	2.3045	2.5552
−1	0	0.12341	0.37022	0.61703	0.86384	1.1107	1.3575	1.6043	1.8511
	0.1	0.10836	0.35695	0.60413	0.85111	1.098	1.3449	1.5917	1.8386
	0.3	$7.3598\ 10^{-2}$	0.32963	0.57794	0.82537	1.0725	1.3195	1.5665	1.8134
	0.50627	$1.5682\ 10^{-2}$	0.30426	0.55445	0.80258	1.0501	1.2974	1.5445	1.7915
	0.6	$9.9522\ 10^{-4}$	0.31159	0.56246	0.81086	1.0585	1.3059	1.553	1.8001
	0.7	$2.8062\ 10^{-5}$	0.32204	0.57366	0.82237	1.0702	1.3177	1.5649	1.812
	0.8	$6.8302\ 10^{-7}$	0.33244	0.5848	0.83385	1.0819	1.3295	1.5768	1.824
	0.9	$1.5627\ 10^{-8}$	0.34276	0.59589	0.84528	1.0935	1.3412	1.5886	1.8359
	1	$3.4531\ 10^{-10}$	0.35302	0.60692	0.85666	1.1051	1.3529	1.6004	1.8478
	1.1	$7.4593\ 10^{-12}$	0.36321	0.61789	0.86799	1.1166	1.3646	1.6122	1.8596
	1.2	$1.5857\ 10^{-13}$	0.37335	0.62881	0.87928	1.1281	1.3762	1.624	1.8714

TABLE VII. Mass towers for $\bar{M} = 1$ (59) and for the semilocal boundary conditions (one metric singularity) (74). Here $\omega_1 = 2\pi(kR)$ (71). The mass eigenvalues $\tilde{m}_i$ are in TeV.

Semilocal BC (one singularity at $s_1 = 2\pi R\bar{s}_1$). Case $\bar{M} = 1$														
$\bar{M}$	$\bar{s}_1$	kR	ω_b	ω_s	ρ_b	ρ_s	$\tilde{m}_1$	$\tilde{m}_2$	$\tilde{m}_3$	$\tilde{m}_4$	$\tilde{m}_5$	$\tilde{m}_6$	$\tilde{m}_7$	$\tilde{m}_8$
1	0.9	6.9	2	0	0	0	0.28251	0.52712	0.74625	0.96836	1.1863	1.4056	1.6232	1.8416
			15	0	0	0	0.28251	0.52712	0.74625	0.96836	1.1863	1.40558	1.6232	1.8416
			20.6	0	0	0	0.28044	0.40672	0.53557	0.75335	0.97614	1.1951	1.4157	1.6346
			20.9	0	0	0	0.27248	0.31334	0.53237	0.75223	0.97546	1.1947	1.4153	1.6342
			21.1	0	0	0	0.24109	0.29079	0.53166	0.75187	0.97522	1.1945	1.4151	1.6341
			21.5	0	0	0	0.1648	0.28597	0.53108	0.75152	0.97497	1.1943	1.4150	1.6339
			22	0	0	0	0.10037	0.28481	0.53068	0.75127	0.97478	1.1941	1.4148	1.6338
			25	0	0	0	$5.007\ 10^{-3}$	0.28481	0.53068	0.75127	0.97478	1.1941	1.4148	1.6338
			28	0	0	0	$2.493\ 10^{-4}$	0.28481	0.53068	0.75127	0.97478	1.1941	1.4148	1.6338
			35	0	0	0	$2.272\ 10^{-7}$	0.28481	0.53068	0.75127	0.97478	1.1941	1.4148	1.6338
			42	0	0	0	$1.538\ 10^{-10}$	0.28481	0.53068	0.75127	0.97478	1.1941	1.4148	1.6338
			$\omega_1 - 10^{-3}$	0	0	0	$5.355\ 10^{-14}$	0.28481	0.53068	0.75127	0.97478	1.1941	1.4148	1.6338
			ω_1	0	0	0	0	0.28481	0.53068	0.75127	0.97478	1.1941	1.4148	1.6338
			$\omega_1 + 10^{-3}$	0	0	0	$5.349\ 10^{-14}$	0.28481	0.53068	0.75127	0.97478	1.1941	1.4148	1.6338
			80	0	0	0	$5.3520\ 10^{-11}$	0.28481	0.53068	0.75127	0.97478	1.1941	1.4148	1.6338
			$\omega_1 + 2$	−2	0	0	$3.492\ 10^{-10}$	0.06222	0.60707	0.64552	1.0558	1.0959	1.4971	1.5395
			ω_1	0	$\pi/3$	$-\pi/3$	$9.2699\ 10^{-11}$	0.28481	0.53068	0.75127	0.97478	1.1941	1.4148	1.6338
			0	2	0	0	$4.5\ 10^{-5}$	0.415	0.45234	0.84995	0.8872	1.2846	1.3218	1.7192
			2	0	$\pi/3$	0	0.28481	0.53068	0.75127	0.97478	1.1941	1.4148	1.6338	1.8536
1	0.75	8.3	2	0	0	0	0.2571	0.47971	0.67912	0.88125	1.0795	1.2791	1.4772	1.6758
			$\omega_1 - 10^{-3}$	0	0	0	$5.9671\ 10^{-16}$	0.25713	0.47971	0.67912	0.88125	1.0795	1.2791	1.4771
			ω_1	0	0	0	0	0.25713	0.47971	0.67912	0.88125	1.0795	1.2791	1.4771
1	0.5	12.5	2	0	0	0	0.21972	0.40998	0.5804	0.75315	0.92261	1.0932	1.2624	1.4322
			ω_1	0	0	0	0	0.21973	0.40997	0.5804	0.75315	0.92261	1.0932	1.2624
			$\omega_1 + 2$	-2	$\pi/3$	$-\pi/3$	$7.0721\ 10^{-18}$	0.043199	0.46927	0.49822	0.81672	0.84566	1.1586	1.1875

TABLE VIII. Mass towers for $\bar{M} = 0.1$ (59) and for the semilocal boundary conditions (one metric singularity) (74). Here $\omega_{0.1} = 0.2\pi(kR)$ (71). The mass eigenvalues $\tilde{m}_i$ are in TeV.

Semilocal BC (one singularity at $s_1 = 2\pi R\bar{s}_1$). Case $\bar{M} = 0.1$															
$\bar{M}$	$\bar{s}_1$	kR	ω_b	ω_s	ρ_b	ρ_s	$\tilde{m}_1$	$\tilde{m}_2$	$\tilde{m}_3$	$\tilde{m}_4$	$\tilde{m}_5$	$\tilde{m}_6$	$\tilde{m}_7$	$\tilde{m}_8$
1	0.9	6.9	2	0	0	0	0.1108	0.3680	0.5441	1.246	1.413	1.684	1.849	2.121
			$\omega_{0.1} - 10^{-3}$	0	0	0	$6.711\ 10^{-5}$	0.3962	0.4625	0.8241	0.906	1.257	1.347	1.691
			$\omega_{0.1} = 4.34\ldots$	0	0	0	0	0.3963	0.4625	0.8242	0.9060	1.257	1.347	1.691
			$\omega_{0.1} + 10^{-3}$	0	0	0	$6.708\ 10^{-5}$	0.3962	0.4625	0.8241	0.906	1.257	1.347	1.691
			0	$-\omega_{0.1} - 10^{-3}$	0	0	$2.181\ 10^{-6}$	0.4537	0.4575	0.8892	0.8970	1.324	1.336	1.759
			0	$-\omega_{0.1}$	0	0	0	0.4537	0.4575	0.8892	0.8970	1.324	1.336	1.759
			0	2	0	0	0.2216	0.2640	0.6553	0.6991	1.091	1.136	1.527	1.573
			0	$\omega_{0.1}$	0	0	0.2385	0.2473	0.6717	0.6828	1.106	1.120	1.541	1.559
			2	0	$\pi/3$	0	0.1189	0.3579	0.5549	0.7954	0.9903	1.233	1.426	1.670
			2	0	0	$\pi/3$	0.1189	0.3579	0.5549	0.7954	0.9903	1.233	1.426	1.670
			2	0	$\pi/2$	0	0.1271	0.3481	0.5656	0.7840	1.439	1.657	1.875	2.094
			$\omega_{0.1}$	0	$\pi/3$	0	0.07014	0.3521	0.5060	0.7855	0.9442	1.221	1.382	1.657

TABLE IX. Mass towers for $M = 0$ and for the semilocal boundary conditions (one metric singularity) (74). Here $\omega_0 = 0$ (71). The mass eigenvalues $\tilde{m}_i$ are in TeV.

Semilocal BC (one singularity at $s_1 = 2\pi R\bar{s}_1$). Case $M = 0$															
$\bar{M}$	$\bar{s}_1$	kR	ω_b	ω_s	ρ_b	ρ_s	$\tilde{m}_1$	$\tilde{m}_2$	$\tilde{m}_3$	$\tilde{m}_4$	$\tilde{m}_5$	$\tilde{m}_6$	$\tilde{m}_7$	$\tilde{m}_8$
0	0.9	6.9	-2	0	0	0	0.09061	0.3467	0.5279	0.7841	0.9653	1.221	1.403	1.659
			-10^{-3}	0	0	0	$6.966\ 10^{-5}$	0.4373	0.4374	0.8746	0.8747	1.312	1.312	1.749
			0	0	0	0	0	0.4373	0.4373	0.8746	0.8746	1.312	1.312	1.749
			10^{-3}	0	0	0	$6.961\ 10^{-5}$	0.4373	0.4374	0.8746	0.8747	1.3119	1.3121	1.7492
			2	0	0	0	0.09061	0.3467	0.5279	0.7841	0.9653	1.221	1.403	1.659
			$2 + 10^{-3}$	2	$\pi/3$	$\pi/3$	$1.849\ 10^{-5}$	0.4346	0.4401	0.8691	0.8803	1.304	1.320	1.738

VII. CONCLUSIONS

In this article, we have extended our previous study of the generation of Kaluza-Klein mass towers for spinor fields propagating in a five-dimensional flat space with the fifth dimension compactified either on a strip or on a circle. We have studied spinor fields propagating in five-dimensional compactified warp spaces. Should Kaluza-Klein towers exist and be discovered at high energy colliders, most likely as higher recurrences of quarks and leptons, they would be the fingerprint of higher dimensions, essentially independently of the underlying dynamics.

We first considered the case of a warp space without metric singularity. We established the specific Dirac equation in the relevant five-dimensional warp space for a spinor field with an arbitrary bulk mass M and proceeded with the Kaluza-Klein reduction considering the most general choice of separation of variables compatible with a SO(3,1) spinor covariance. The reducted components of the Dirac fields are found to satisfy a system of two coupled equations for which the most general solutions for a four-dimensional mass m are given in terms of Bessel functions.

From the requirement of Hermiticity of the Dirac operator, we have established all the allowed sets of boundary conditions which have to be imposed on the fields. We found that these boundary conditions belong to two essentially different sets BC1 (17) and BC2 (19), leading to the mass equations from which the Kaluza-Klein mass towers can be built. The same considerations have been extended to the case of warp spaces with an arbitrary number of metric singularities.

In view of the interpretation of the Kaluza-Klein mass eigenstates, specific physical considerations have been made about the possible choices of boundary conditions, about the particular situation in which the extra dimension strip could be closed to a circle, about the mass scale of the model, about the relation between the Kaluza-Klein masses and the physical masses as observed in a brane and also about the mass state probability densities. In particular, all the parameters with energy dimension are scaled to the Plank mass within the only-one-mass-scale hypothesis.

Finally, illustrative numerical examples of Kaluza-Klein mass towers are given when there is no metric singularity for each of the two sets of boundary conditions, BC1 and BC2. When there is one metric singularity we have exemplified towers for some boundary conditions belonging to what we call the semilocal set. With $kR = 6.3$ or around, it happens that the low lying masses are of the order of TeV, thereby solving the hierarchy problem without fine-tuning.

In the different situations considered, the towers have been established for several choices of the basic parameters of the warp model, i.e. the mass reduction parameter $kR = 6.3$ (suitably readjusted to the value of s_1), the bulk mass M, the position s_1 of the singularity on the extra dimension if any, as well as of the parameters defining the boundary conditions. In general, the mass towers are irregularly spaced, and a zero mass state or a small mass state exists which depends on the boundary parameters and on the value given to the bulk mass M. This situation allows one, by a suitable choice of the parameters of the model, to associate a mass tower to any particular fermion of the standard model whose mass would be the smallest in the tower. In the assumption that the known leptons and quarks propagate in the bulk under consideration, one would expect to observe the next low lying masses at high energy colliders, in particular, at the LHC.

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APPENDIX A: LEAST ACTION PRINCIPLE

The most general invariant action linear in Ψ and $\bar{\Psi}$ and of first order in their derivatives is, using (9),

$$\mathcal{A} = i\alpha \int \bar{\Psi}\gamma^A(\vec{D}_A\Psi)\sqrt{g}d^5x + i\beta \int (\bar{\Psi}\overleftarrow{D}_A)\gamma^A\Psi\sqrt{g}d^5x - m\int \bar{\Psi}\Psi\sqrt{g}d^5x. \quad \text{(A1)}$$

The Lagrangian is Hermitian if

$$\beta = -\alpha^*, \qquad m = m^*. \quad \text{(A2)}$$

Requesting then the variation of the action (A1) to vanish for arbitrary variations of the fields Ψ and $\bar{\Psi}$, one finds (with $\alpha - \beta \neq 0$) the Dirac equations

$$i\gamma^A(\vec{D}_A\Psi) - m\Psi = 0 \qquad i(\bar{\Psi}\overleftarrow{D}_A)\gamma^A + m\bar{\Psi} = 0 \quad \text{(A3)}$$

independently of the boundary conditions and identical to those obtained from the usual least action principle, i.e. with vanishing variations at the boundaries (A3).

However, further attention has to be devoted to the variation of the action arising from the boundary terms. Suppose that there are N metric singularities located at the points s_i, $i = 1, N$ in the s space extending from $s_0 = 0$ to $s_{N+1} = 2\pi R$. Denote by $\Psi^l(s_i)$ and $\Psi^r(s_i)$ the values of the fields at the left and at the right of the metric singularities, and similarly for their variations. The boundary relations expressed from the boundary terms become

$$\sum_{i=1}^{N+1} \bar{\Psi}^l(s_i)\gamma^s\delta\Psi^l(s_i)\sqrt{g(s_i)} - \sum_{i=0}^{N} \bar{\Psi}^r(s_i)\gamma^s\delta\Psi^r(s_i)\sqrt{g(s_i)} = 0 \quad \text{(A4)}$$

$$\sum_{i=1}^{N+1} \bar{\delta}\,\Psi^l(s_i)\gamma^s\Psi^l(s_i)\sqrt{g(s_i)} - \sum_{i=0}^{N} \bar{\delta}\,\Psi^r(s_i)\gamma^s\Psi^r(s_i)\sqrt{g(s_i)} = 0. \quad \text{(A5)}$$

It is natural to suppose that the variations $\delta\Psi$, $\delta\bar{\Psi}$ and the fields Ψ and $\bar{\Psi}$ belong to the same Hilbert space i.e. satisfy the same boundary conditions. The relations (A4) and (A5) then imply boundary conditions which turn out to be identical to those obtained from (14) in the main part of the article and which resulted from the requirement of symmetry of the Dirac operator (12).

APPENDIX B: EXAMPLES OF BOUNDARY RELATIONS

There are many inequivalent sets of boundary conditions related to various choices of the permutation P in (48). Let us give a few.

1. P=1. Local boundary conditions at the metric singular points. Non-local conditions at the edges of the *s*-domain

With $P = 1$, one can obtain boundary conditions which satisfy the locality conditions (see Sec. IV A) at the singular points but not at the edges. The form of $V_P^{[2(N+1)]}$ compatible with this partial-locality is

$$V_P^{[2(N+1)]} = \begin{pmatrix} 0^{[2]} & V_1^{[2]} & 0^{[2]} & 0^{[2]} & \dots & 0^{[2]} \\ 0^{[2]} & 0^{[2]} & V_2^{[2]} & 0^{[2]} & \dots & 0^{[2]} \\ 0^{[2]} & 0^{[2]} & 0^{[2]} & V_3^{[2]} & \dots & 0^{[2]} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0^{[2]} & 0^{[2]} & 0^{[2]} & 0^{[2]} & \dots & V_N^{[2]} \\ V_{N+1}^{[2]} & 0^{[2]} & 0^{[2]} & 0^{[2]} & \dots & 0^{[2]} \end{pmatrix}. \quad \text{(B1)}$$

Introducing this form of the matrix in the Eqs. (56), one finds for all j ($j = 1, \dots, N+1$)

$$V_j^{[2]+}(i\sigma_2)V_j^{[2]} = i\sigma_2 \quad \text{(B2)}$$

$$V_j^{[2]+}(\sigma_3)V_j^{[2]} = \sigma_3. \quad \text{(B3)}$$

From (B2), all the $V_j^{[2]}$ are complex symplectic 2×2 matrices restricted by the further condition (B3). Their resulting general form is

$$V_j^{[2]} = e^{i\rho_j}\begin{pmatrix} \cosh(\omega_j) & \sinh(\omega_j) \\ \sinh(\omega_j) & \cosh(\omega_j) \end{pmatrix}, \qquad j = 1, \dots, N+1 \quad \text{(B4)}$$

and depends on $2(N+1)$ arbitrary parameters. Hence the explicit boundary conditions at the metric singularities $j = 1, \dots, N$ are

$$\begin{pmatrix} F^{[n]l}(s_j) \\ G^{[n]l}(s_j) \end{pmatrix} = e^{i\rho_j}\begin{pmatrix} \cosh(\omega_j) & \sinh(\omega_j) \\ \sinh(\omega_j) & \cosh(\omega_j) \end{pmatrix}\begin{pmatrix} F^{[n]r}(s_j) \\ G^{[n]r}(s_j) \end{pmatrix}. \tag{B5}$$

At the edges, the boundary conditions are non local

$$\begin{pmatrix} F^{[n]l}(2\pi R) \\ G^{[n]l}(2\pi R) \end{pmatrix} = \sqrt[4]{\frac{g(0)}{g(2\pi R)}}e^{i\rho_{N+1}}\begin{pmatrix} \cosh(\omega_{N+1}) & \sinh(\omega_{N+1}) \\ \sinh(\omega_{N+1}) & \cosh(\omega_{N+1}) \end{pmatrix}\begin{pmatrix} F^{[n]r}(0) \\ G^{[n]r}(0) \end{pmatrix}. \tag{B6}$$

since they connect the values of the fields at $s = 2\pi R$ to the values of the fields at $s = 0$ (a long distance effect). When the conditions (57) for the closure of the strip to a circle are met, these would also be local boundary conditions.

2. Local boundary conditions both at the metric singular points and at the edges of the *s*-domain

A way to obtain fully local boundary conditions is to perform the following permutation

$$P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}. \tag{B7}$$

and to take $V_P^{[2(N+1)]}$ of the form (B1) with $V_{N+1}^{[2]}$ diagonal. This leads, on one side to a relation between $G^{[N]l}(s_{N+1} = 2\pi R)$ and $F^{[N]l}(s_{N+1} = 2\pi R)$, on the other side to a relation between $G^{[N]r}(s_0 = 0)$ and $F^{[N]r}(s_0 = 0)$. The conditions for $V_j^{[2]}$ $(j = 1, \dots, N)$ are the same as in the preceding case (B2) and (B3), leading to the same conditions at each of the singular points (B4). For the diagonal $V_{N+1}^{[2]}$,

$$V_{N+1}^{[2]+}\sigma_3 = \sigma_3 V_{N+1}^{[2]} \tag{B8}$$

$$V_{N+1}^{[2]+}\sigma_3 V_{N+1}^{[2]} = \sigma_3. \tag{B9}$$

Introducing (B8) in (B9) on see that $(V_{N+1}^{[2]})^2 = 1^{[2]}$.

This leads to boundary conditions at the singularities (s_j, $j = 1, \dots, N$) as above (B5) and to

$$\begin{aligned} G^{[n]r}(0) &= \epsilon_0 F^{[n]r}(0), & \epsilon_0^2 = 1 \\ G^{[n]l}(2\pi R) &= \epsilon_R F^{[n]l}(2\pi R), & \epsilon_R^2 = 1 \end{aligned} \tag{B10}$$

at the edges.

3. General boundary conditions for $P = 1$

When $P = 1$, i.e. when the boundary conditions express the values at the left of the exceptional points (singularities and edges) in terms of the values at the right, the two equations ($j = 1, 2$) (56) take the simplified form with $V^{[2(N+1)]} \equiv V_{P=1}^{[2(N+1)]}$, $S_1^{[2(N+1)]}$ from (44) and $S_2^{[2(N+1)]}$ from (46)

$$(V^{[2(N+1)]})^+ S_j^{[2(N+1)]} V^{[2(N+1)]} = S_j^{[2(N+1)]}. \tag{B11}$$

The matrix $V^{[2(N+1)]}$ must be an element in the intersection of the complex sympletic group $Sp(2(N+1))$ (from the relation for $j = 1$) and of the pseudounitary group $U(N+1, N+1)$ (from the relation $j = 2$). The dimension of the parameter space can be obtained by writing $V^{[2(N+1)]}$ infinitesimally close to the identity $1^{[2(N+1]}$

$$V^{[2(N+1)]} = 1^{[2(N+1)]} + i\eta H^{[2(N+1)]}, \qquad \eta \to 0 \tag{B12}$$

in 2×2 blocks. One finds that there are, for $H^{[2(N+1)]}$, $N+1$ diagonal 2×2 blocks H_{jj} $(j = 1, \dots, N+1)$, each depending on two real parameters

$$H_{jj} = \begin{pmatrix} p_j & iq_j \\ iq_j & p_j \end{pmatrix}, \qquad p_j, q_j \text{ real} \tag{B13}$$

and $N(N+1)$ independent non diagonal 2×2 blocks H_{jk}, $j < k = 1, \dots, N+1$, each depending on two complex (four real) parameters p_{jk} and q_{jk}

$$H_{jk} = \begin{pmatrix} p_{jk} & q_{jk} \\ q_{jk} & p_{jk} \end{pmatrix}, \qquad H_{kj} = \begin{pmatrix} p_{jk}^* & -q_{jk}^* \\ -q_{jk}^* & p_{jk}^* \end{pmatrix}. \tag{B14}$$

Hence, the set of boundary conditions for $P = 1$ is indexed by $2(N+1)^2$ real parameters.

Let us finally remark that contrary to what happens for the scalar fields where the boundary conditions relate the fields and their derivatives, the boundary parameters have zero energy dimension (59) and, hence, there is no need to introduce reduced parameters in the spinor case.

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