Generation of circular polarization of the cosmic microwave background

Stephon Alexander

Department of Physics and Astronomy, Haverford College, Haverford, Pennsylvania 19041, USA

Joseph Ochoa

Department of Physics, Institute for Gravitation and the Cosmos, The Pennsylvania State University,
104 Davey Lab, University Park, Pennsylvania 16802, USA

Arthur Kosowsky

Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, Pennsylvania 15208, USA

The standard cosmological model, which includes only Compton scattering photon interactions at energy scales near recombination, results in zero primordial circular polarization of the cosmic microwave background. In this paper we consider a particular renormalizable and gauge-invariant standard model extension coupling photons to an external vector field via a Chern-Simons term, which arises as a radiative correction if gravitational torsion couples to fermions. We compute the transport equations for polarized photons from a Boltzmann-like equation, showing that such a coupling will source circular polarization of the microwave background. For the particular coupling considered here, the circular polarization effect is always negligible compared to the rotation of the linear polarization orientation, also derived using the same formalism. We note the possibility that limits on microwave background circular polarization may probe other photon interactions and related fundamental effects such as violations of Lorentz invariance.

I. INTRODUCTION

One of the great successes of the standard cosmology is the prediction and measurement of the temperature anisotropies in the cosmic microwave background radiation. Most of these photons have freely propagated since the epoch of last scattering roughly 14×10^9 years ago and encode the initial conditions for structure formation. Measurements are now consistent to high precision with the simplest cosmological models with an initial power-law spectrum of adiabatic perturbations. Linear polarization of the microwave background fluctuations is also a generic result of these models; recent detections of the linear polarization power spectrum and of the cross-correlation between linear polarization and temperature are also consistent with the same cosmological models.

In general, radiation can have linear polarization, with 2 degrees of freedom (a polarization amplitude and orientation) as well as circular polarization, with a single degree of freedom. It is well-known that if an initially unpolarized photon field evolves solely via Compton scattering from free electrons plus free streaming, the resulting radiation field can have linear but not circular polarization. In the tight-coupling regime prior to last scattering when Compton scattering is rapid compared to the cosmological expansion time scale, the cosmic radiation field will be unpolarized. As the universe cools and the free electrons become bound into neutral hydrogen, a small linear polarization is generated from the balance of free-streaming and Compton scattering during this recombination process, but the resulting microwave background radiation today has

circular polarization which is identically zero. In this paper, we consider a generic class of interactions between photons and an external field which can produce circular polarization. The interactions have been considered in other contexts and are general enough to be expected in broad classes of theories beyond the standard model of particle physics. The same interaction can also arise if nonzero spacetime torsion impacts the microwave background radiation. The goal of this paper is two-fold. First, we provide an explicit calculation showing how circular polarization can be generically sourced in the microwave background, with the relevant evolution equations. Second, we demonstrate what the underlying microphysics might look like.

Consider the following extension of the photon sector of quantum electrodynamics:

$$\mathcal{L}' = \mathcal{L}_{\text{MAXWELL}} + \mathcal{L}_T$$
$$\equiv -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + g \epsilon^{\mu\nu\alpha\beta} A_\mu T_\nu F_{\alpha\beta}, \qquad (1)$$

where $\mathcal{L}_T$ is *CPT* odd and violates Lorentz invariance and g is the coupling constant of the interaction. Several authors have investigated such a Lorentz-invariance violating extension of QED for a constant 4-vector T^μ (see e.g. [1–3] and references therein). These so-called *Standard Model Extensions* have been shown to be renormalizable while maintaining gauge invariance [4]. We consider here only the flat-spacetime interaction term $\mathcal{L}_T$ for simplicity; this will be a good approximation in any cosmological context (see Refs. [5,6] and for the curved spacetime generaliza-

tion, which includes an extra factor of the square root of the metric determinant). If T_μ is fixed as a constant, the Lagrangian density in Eq. (1) is $U(1)$ gauge invariant apart from a boundary term; therefore in these cases the action is gauge invariant.

It is well-known that such an extension should result in optical activity in the propagation of electromagnetic radiation [7–10]; specifically, a modification to the dispersion relations of free electromagnetic radiation results in a rotation of the plane of linear polarization during propagation. We make no explicit assumption about whether T_μ is spacelike or timelike, although the timelike case appears pathological since it leads to a violation of causality and unitarity [11]. The magnitude of optical activity of electromagnetic radiation has been constrained by analysis of observational data from cosmological sources and from the microwave background radiation [7,12–20]. Such a term may arise as a radiative correction following the coupling of gravitational torsion with fermionic matter [21,22]. The same term has also been associated with the cancellation of gauge anomalies in QED when the background field T_μ is allowed to couple to the axial current (see e.g. [23]).

In this paper, we show that in addition to the well-known polarization rotation, such a term may also generate circular polarization, although for the specific case of Eq. (1) the circular polarization is always negligible compared to the polarization rotation. The generation of circular polarization following the optical activity produced by T_μ parallels the Faraday conversion and Faraday rotation effects for propagation in magnetized plasmas; for a discussion in the context of the microwave background, see [24]. The observation of circularly polarized microwave background radiation could be evidence of Lorentz-invariance violation and thus physics beyond the standard model; conversely, limits on circular polarization may constrain a certain class of standard model extensions. For a related analysis using an axionlike pseudoscalar coupling to the electromagnetic field, see Ref. [25], who also find a nonzero circular polarization and rotation of linear polarization.

In Sec. II, we review the usual description of polarized electromagnetic radiation in terms of Stokes parameters; linear polarization is described by the Q and U parameters, while circular polarization is described by a nonzero V parameter. Section III reviews the construction of the Boltzmann-type equation for the photon number density, starting from the quantum-mechanical evolution of the photon density matrix. In Secs. IV and V, we calculate the evolution of the Stokes parameters to first and second order, respectively, in the interaction term $\mathcal{L}_T$, deriving the evolution equation for the V polarization, which is generated from linear polarization due to the interaction term. We conclude in Sec. IV with estimates of the size of the V polarization in the microwave background for given interactions along with the magnitude of linear polarization

rotation. The mathematical details of evaluating the first and second-order interaction terms are relegated to Appendices A and B, while Appendix C addresses the issue of gauge invariance.

II. STOKES PARAMETERS

The polarization state of light is most easily described by making use of the Stokes parameters. For a complete review see, e.g., Refs. [26–28] or any optics text. Here we review the basic construction of the Stokes parameters in the classical and quantum mechanical contexts in order to motivate the quantum field theoretical construction. Consider a classical electromagnetic plane wave with electric field given by the components

$$E_1(t) = a_1 \sin(\omega t - \epsilon_1) \quad \text{and} \quad E_2(t) = a_2 \sin(\omega t - \epsilon_2), \tag{2}$$

where we assume, for simplicity, that the wave is nearly monochromatic with frequency ω, such that a_1, a_2, ϵ_1, and ϵ_2 only vary on time scales long compared to ω^{-1}. The Stokes parameters in the linear polarization basis are then defined as

$$I \equiv \langle (a_1)^2 + (a_2)^2 \rangle, \tag{3}$$

$$Q \equiv \langle (a_1)^2 - (a_2)^2 \rangle, \tag{4}$$

$$U \equiv \langle 2a_1 a_2 \cos\delta \rangle, \tag{5}$$

$$V \equiv \langle 2a_1 a_2 \sin\delta \rangle, \tag{6}$$

where $\delta \equiv \epsilon_2 - \epsilon_1$ and the brackets signify a time average over a time long compared to ω^{-1}. The I parameter measures the intensity of the radiation, while the parameters Q, U, and V each carry information about the polarization of the radiation. Unpolarized radiation is described by $Q = U = V = 0$. The linear polarization of the radiation is encoded in Q and U, while the parameter V is a measure of elliptical polarization with the special case of circular polarization ocurring when $a_1 = a_2$ and $\delta = \pm\pi/2$. From here on we will simply refer to V as the measure of circular polarization, which is technically correct if $Q = 0$. Note that while I and V are coordinate independent, Q and U depend on the orientation of the coordinate system used on the plane orthogonal to the direction of propagation. Under a rotation of the coordinate system by an angle ϕ, the parameters Q and U transform according to

$$Q' = Q \cos(2\phi) + U \sin(2\phi),$$

$$U' = -Q \sin(2\phi) + U \cos(2\phi),$$

while the angle defined by

$$\Phi = \frac{1}{2} \arctan\left(\frac{U}{Q}\right)$$

goes to $\Phi - \phi$ following a rotation by the angle ϕ. Therefore, Q and U only define an orientation and not a particular direction in the plane: after a rotation by π they are left unchanged. Physically, this is simply a manifestation of the oscillatory behavior of the electric field. These properties indicate that Q and U are part of a second-rank symmetric trace-free tensor P_{ab}, i.e. a spin-2 field in the plane orthogonal to the direction of propagation. Such a tensor can be represented as

$$P_{ij} = \begin{pmatrix} P & 0 \\ 0 & -P \end{pmatrix}, \qquad (7)$$

in an orthonormal eigenbasis, where $P = (Q^2 + U^2)^{1/2}$ is often called the magnitude of linear polarization.

In quantum mechanics we can express the state of a photon $\mathcal{A}$ as

$$|\mathcal{A}\rangle = \sum_i a_i |\epsilon_i\rangle, \qquad (8)$$

where $|\epsilon_i\rangle$ $(i = 1, 2)$ span the polarization state space and a_i are in general complex. The projection operators

$$\hat{I} \equiv |\epsilon_1\rangle\langle\epsilon_1| + |\epsilon_2\rangle\langle\epsilon_2|, \qquad (9)$$

$$\hat{Q} \equiv |\epsilon_1\rangle\langle\epsilon_1| - |\epsilon_2\rangle\langle\epsilon_2|, \qquad (10)$$

$$\hat{U} \equiv |\epsilon_1\rangle\langle\epsilon_2| + |\epsilon_2\rangle\langle\epsilon_1|, \qquad (11)$$

$$\hat{V} \equiv i|\epsilon_2\rangle\langle\epsilon_1| - i|\epsilon_1\rangle\langle\epsilon_2| \qquad (12)$$

have expectation values in single photon states which give the classical Stokes parameters, Eqs. (3)–(6). In a general mixed state, the density matrix ρ on the polarization state space encodes the intensity and polarization of the photon ensemble. For example,

$$\langle\hat{Q}\rangle = \frac{\text{tr}(\rho\hat{Q})}{\text{tr}(\rho)} = \frac{1}{\text{tr}(\rho)} \text{tr}\left[\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right]$$

$$= \frac{\rho_{11} - \rho_{22}}{\text{tr}(\rho)}.$$

Similar relations hold for the other "Stokes operators" such that the density matrix can be represented as

$$\rho = \frac{\text{tr}(\rho)}{2} \begin{pmatrix} 1 + Q & U - iV \\ U + iV & 1 - Q \end{pmatrix}, \qquad (13)$$

where $Q = \langle\hat{Q}\rangle$, $U = \langle\hat{U}\rangle$, and $V = \langle\hat{V}\rangle$.

III. THE PHOTON BOLTZMANN EQUATION

We now review the construction of the evolution equation for the photon number operator under the influence of some perturbation to a free theory. The following formalism was developed to study neutrino mixing and damping [29,30]. It has also been applied to describe the generation of linear polarization in the microwave background due to

Compton scattering during recombination [31], generalizing an earlier kinetic equation treatment of microwave background temperature fluctuations [32].

Consider an ensemble of free photons. We will assume that the interaction $\mathcal{L}_T$ is slowly "turned on" and that the interactions of the photons with the external field T_μ are localized such that the photons can be considered free (with respect to the interaction $\mathcal{L}_T$) both before and after each point interaction. That is, we make the usual assumptions of scattering theory. We will not consider any possible interference effects which might occur between $\mathcal{L}_T$ and any other interaction.

The free photon field in the Coulomb (radiation) gauge can be expressed as

$$\hat{A}_\mu(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k^0} [\hat{a}_s(k)\epsilon_{s\mu}(k)e^{-ik\cdot x} + \hat{a}_s^\dagger(k)\epsilon_{s\mu}^*(k)e^{ik\cdot x}], \qquad (14)$$

where $\epsilon_{s\mu}(k)$ are the photon polarization 4-vectors and s, which takes the values 1 and 2, indexes the orthogonal transverse polarizations. The free creation and annihilation operators satisfy the canonical commutation relation

$$[\hat{a}_i(k), \hat{a}_j^\dagger(k')] = (2\pi)^3 2k^0 \delta_{ij}\delta^{(3)}(\mathbf{k} - \mathbf{k}'), \qquad (15)$$

where $k^0 = |\mathbf{k}|$.

We will be interested in the evolution of the polarization state of a photon ensemble, which is completely characterized by the density matrix ρ_{ij} defined via

$$\langle\hat{a}_i^\dagger(k)\hat{a}_j(k')\rangle = (2\pi)^3 2k^0\delta^{(3)}(\mathbf{k} - \mathbf{k}')\rho_{ij}(\mathbf{k}). \qquad (16)$$

The number operator $\hat{\mathcal{D}}_{ij}(\mathbf{k}) = \hat{a}_i^\dagger(\mathbf{k})\hat{a}_j(\mathbf{k})$, according to (16), is related to $\rho_{ij}(\mathbf{k})$ as

$$\langle\hat{\mathcal{D}}_{ij}(\mathbf{k})\rangle = (2\pi)^3 2k^0\delta^{(3)}(0)\rho_{ij}(\mathbf{k}). \qquad (17)$$

The infinite factor $\delta^{(3)}(0)$ is a remnant of the infinite quantization volume. As we show explicitly below, it cancels from all final expressions. Motivated by the construction in the quantum-mechanical system above, we can project out quantities analogous to the classical Stokes parameters:

$$\text{tr}(\sigma_0\rho(\mathbf{k})) \rightarrow I \propto \langle\hat{\mathcal{H}}_0(\mathbf{k})\rangle, \qquad (18)$$

$$\text{tr}(\sigma_z\rho(\mathbf{k})) \rightarrow Q \propto \langle\hat{\mathcal{D}}_{11}(\mathbf{k})\rangle - \langle\hat{\mathcal{D}}_{22}(\mathbf{k})\rangle, \qquad (19)$$

$$\text{tr}(\sigma_x\rho(\mathbf{k})) \rightarrow U \propto \langle\hat{\mathcal{D}}_{12}(\mathbf{k})\rangle + \langle\hat{\mathcal{D}}_{21}(\mathbf{k})\rangle, \qquad (20)$$

$$\text{tr}(\sigma_y\rho(\mathbf{k})) \rightarrow iV \propto \langle\hat{\mathcal{D}}_{12}(\mathbf{k})\rangle - \langle\hat{\mathcal{D}}_{21}(\mathbf{k})\rangle, \qquad (21)$$

where σ_0 is the 2×2 identity matrix, σ_i are the Pauli matrices and the trace is over polarization indices. $\hat{\mathcal{H}}_0(\mathbf{k})$

is the free energy density operator of the mode with wave number **k**.

We ignore any correlations such as $\langle \hat{a}_i(k)\hat{a}_j(k')\rangle$ and $\langle \hat{a}_i^\dagger(k)\hat{a}_j^\dagger(k')\rangle$ which might be generated via the interaction $\mathcal{L}_T$. In essence we are assuming that the background field T_μ varies slowly enough in time so that physical two-photon states are neither created nor destroyed by the interaction $\mathcal{L}_T$. If we define ω_T as a characteristic energy scale of the background field T_μ and ω as the frequency of a particular mode associated with free oscillations of the creation and annihilation operators, then we are requiring that $\omega_T \ll \delta\omega/\omega \ll \omega$ [33]. Note that $\delta\omega/\omega$ is the order at which mixing occurs between $\hat{a}_i$ and $\hat{a}_i^\dagger$, and such a mixing will result in a variation $\delta\langle \hat{a}_i(k)\hat{a}_j(k')\rangle \simeq (\delta\omega/\omega)\langle \hat{a}_i^\dagger(k)\hat{a}_j(k')\rangle$, which again we neglect. Naively, we then expect the following formalism to fail for low-frequency photon modes, although at precisely what scale the approximation breaks down depends on the characteristic scale of the background field T_μ.

The evolution of the number operator $\hat{\mathcal{D}}_{ij}(\mathbf{k})$ can be computed using a perturbative expansion in the interaction strength. The following reviews the construction detailed in [30]. Recall that the time evolution of any Heisenberg picture operator is given by

$$\frac{d}{dt}\hat{\mathcal{D}}_{ij} = i[\hat{H}, \hat{\mathcal{D}}_{ij}]. \qquad (22)$$

If the full Hamiltonian can be split into a free and interacting part, $\hat{H} = \hat{H}_{\text{free}} + \hat{H}_{\text{int}}$, then (22) becomes.

$$\frac{d}{dt}\hat{\mathcal{D}}_{ij} = i[\hat{H}_{\text{free}}, \hat{\mathcal{D}}_{ij}] + i[\hat{H}_{\text{int}}, \hat{\mathcal{D}}_{ij}]. \qquad (23)$$

A first-order perturbative approximation for the evolution of the number operator is given by replacing all operators on the right hand side of Eq. (23) by their free-theory counterparts, e.g.

$$\left\langle \frac{d}{dt}\hat{\mathcal{D}}_{ij}\right\rangle \simeq i\langle[\hat{H}_{\text{int}}^0, \hat{\mathcal{D}}^0]\rangle, \qquad (24)$$

where $\hat{\mathcal{O}}^0$ corresponds to the operator $\hat{\mathcal{O}}$ evaluated in terms of the operators of the free theory. The above assumes that $[\hat{H}_{\text{free}}^0, \hat{\mathcal{D}}^0] = 0$. We will refer to the term on the right hand side of (24) as the refractive term, or the forward scattering term. To determine a second-order perturbative approximation, we will use the fact that we can expand any operator to first order in interactions as

$$\hat{\xi}(t) \simeq \hat{\xi}^0(t) + i\int_0^t dt'[\hat{H}_{\text{int}}^0(t-t'), \hat{\xi}^0(t)], \qquad (25)$$

with the initial conditions $\hat{\xi}(0) = \hat{\xi}^0(0)$. The expansion Eq. (25) can be verified by explicitly taking the time derivative of both sides and seeing that one recovers the Heisenberg equation to first order in interactions. We now expand $[\hat{H}_{\text{int}}, \hat{\mathcal{D}}]$ as in (25), insert the result into (23), and

upon the evaluation of all operators in terms of the free-theory operators arrive at

$$\left\langle \frac{d}{dt}\hat{\mathcal{D}}_{ij}\right\rangle(t) \simeq i\langle[\hat{H}_{\text{int}}^0(t), \hat{\mathcal{D}}^0]\rangle$$
$$- \int_0^t dt'\langle[\hat{H}_{\text{int}}^0(t-t'), [\hat{H}_{\text{int}}^0(t), \hat{\mathcal{D}}^0]]\rangle. \qquad (26)$$

The second term in (26) will be referred to as the damping term or the nonforward scattering term. In terms of ρ, the evolution equation reads

$$(2\pi)^3\delta^3(0)2q^0\frac{d}{dt}\rho_{ij}(0, \mathbf{q})$$
$$= i\langle[\hat{H}_{\text{int}}^0(0), \hat{\mathcal{D}}_{ij}^0(\mathbf{q})]\rangle - \frac{1}{2}\int_{-\infty}^\infty dt'\langle[\hat{H}_{\text{int}}^0(t'),$$
$$[\hat{H}_{\text{int}}^0(0), \hat{\mathcal{D}}_{ij}^0(\mathbf{q})]]\rangle, \qquad (27)$$

where, as mentioned above, all factors of $\delta^3(0)$ will cancel from the final expressions. In going from (26) to (27), we have assumed the time step t in Eq. (26) is both small relative to the characteristic time scale of the evolution of ρ and large relative to the duration of a single interaction. This allows us to take $t \to \infty$ and set $\rho(t) = \rho(0)$ [30]. We have then replaced the integral $(\int_0^\infty dt)$ with $(\frac{1}{2}\int_{-\infty}^\infty dt)$, the difference being a principle part integral which is a second order correction to the refractive term. Equation (27) can be viewed as a *Generalized Boltzmann Equation* for the phase space function ρ. In this approximation, we have a set of differential equations for the components ρ_{ij} at $t = 0$; if the interactions are "forgotten" between intermediate collisions (an assumption known as molecular chaos in the derivation of the standard Boltzmann equation), then the differential equations will be valid for all times over which the interaction is relevant. The Liouville terms on the left side will incorporate any effects which result from a departure of the spacetime metric from a flat metric, including any weak inhomogeneities due to the presence of gravitational perturbations about a homogeneous cosmology. The case $H_{\text{int}}(t) = \int d\mathbf{x}\,\bar{\psi}\gamma^\mu A_\mu\psi$ as in full QED, where ψ is a spinor field associated with the electron, recovers the radiative transfer equations of Chandrasekhar [34] in the appropriate limits (see [31] for more details).

As we will see below, this construction allows linearizing the right side of Eq. (27) in ρ. We can expand the photon density matrix about a uniform unpolarized distribution (ignoring any small inhomogeneities) as

$$\rho_{ij}(t, k, \hat{k}) = \rho_{ij}^{(0)}(t, k) + \rho_{ij}^{(1)}(t, k, \hat{k}), \qquad (28)$$

where $\rho_{11}^{(0)} = \rho_{22}^{(0)}$ and $\rho_{12}^{(0)} = \rho_{21}^{(0)} = 0$. As a consistency check, the right side of Eq. (27) should vanish when evaluated in terms of $\rho^{(0)}$ so that in, for example, an FRW background of zero spatial curvature with scale factor $a(t)$ we have [31]

$$\frac{d}{dt}\rho_{11}^{(0)} = \frac{\partial \rho_{11}^{(0)}}{\partial t} - \frac{\dot{a}}{a}k\frac{\partial \rho_{11}^{(0)}}{\partial k} = 0, \tag{29}$$

the solution of which is $\rho_{11}^{(0)}(t, k) = \rho_{11}^{(0)}(ka)$, recovering the uniform redshift due to cosmological expansion.

In order to make contact with the measurable Stokes parameters, we define the normalized brightness perturbations [31]

$$\Delta_I \equiv \left[\frac{q}{4}\frac{\partial \rho_{11}^{(0)}(q)}{\partial q}\right]^{-1}(\rho_{11}^{(1)} + \rho_{22}^{(1)}), \tag{30}$$

$$\Delta_Q \equiv \left[\frac{q}{4}\frac{\partial \rho_{11}^{(0)}(q)}{\partial q}\right]^{-1}(\rho_{11}^{(1)} - \rho_{22}^{(1)}), \tag{31}$$

$$\Delta_U \equiv \left[\frac{q}{4}\frac{\partial \rho_{11}^{(0)}(q)}{\partial q}\right]^{-1}(\rho_{12}^{(1)} + \rho_{21}^{(1)}), \tag{32}$$

$$\Delta_V \equiv -i\left[\frac{q}{4}\frac{\partial \rho_{11}^{(0)}(q)}{\partial q}\right]^{-1}(\rho_{12}^{(1)} - \rho_{21}^{(1)}), \tag{33}$$

where $q = ka$ is the comoving photon momentum and we have expanded the density matrix ρ in a linear polarization basis.

IV. THE FIRST-ORDER INTERACTION TERM

We now evaluate the right side of the Generalized Boltzmann Eq. (27) for an interaction Hamiltonian which is linear in the Hamiltonian density

$$\hat{\mathcal{H}}_T = -g\epsilon^{\mu\nu\alpha\beta}:\hat{A}_\mu T_\nu \hat{F}_{\alpha\beta}:, \tag{34}$$

where $\hat{A}_\mu$ for the free theory is given by Eq. (14) and $\hat{F}_{\mu\nu} = 2\partial_{[\mu}\hat{A}_{\nu]}$ is the free electromagnetic field strength operator. We will treat T_μ as a classical background field, the dynamics of which are not influenced by the electromagnetic interaction Eq. (34) and are assumed to be externally prescribed. The symbol $: \cdots :$ denotes normal ordering of the enclosed operator products.

Following a local $U(1)$ gauge transformation $\delta_G A_\mu = \partial_\mu \lambda$ of A_μ, the resulting change in the Hamiltonian density Eq. (34) is given by

$$\delta_G \mathcal{H}_T = -g\epsilon^{\mu\nu\alpha\beta}(\partial_\mu \lambda)T_\nu F_{\alpha\beta}$$
$$= g\epsilon^{\mu\nu\alpha\beta}\lambda(\partial_{[\mu}T_{\nu]})F_{\alpha\beta} - g\epsilon^{\mu\nu\alpha\beta}\partial_\mu(\lambda T_\nu F_{\alpha\beta}), \tag{35}$$

where the square brackets denote antisymmetrization and we have used $\partial_{[\mu}F_{\alpha\beta]} = 0$. If $\partial_{[\mu}T_{\nu]} = 0$ then the interaction Hamiltonian is gauge invariant apart from a boundary term. We assume that the external field $T_\mu(x)$ does indeed satisfy this condition allowing us to maintain gauge invariance, which is detailed in the second-order calculation in Appendix C. Furthermore, we assume that $T_\mu(x)$ is

a pseudovector: under a parity transformation the external field transforms as

$$T_\mu(t, -\mathbf{x}) = \begin{cases} -T_\mu(t, \mathbf{x}), & \mu = 0 \\ +T_\mu(t, \mathbf{x}), & \mu = 1, 2, 3 \end{cases} \tag{36}$$

and under time reversal the external field transforms as

$$T_\mu(-t, \mathbf{x}) = \begin{cases} +T_\mu(t, \mathbf{x}), & \mu = 0 \\ -T_\mu(t, \mathbf{x}), & \mu = 1, 2, 3. \end{cases} \tag{37}$$

We will find it useful to consider the Fourier transform of T_μ,

$$\tilde{T}_\mu(p) = \int d^4x T_\mu(x)e^{-ip\cdot x}. \tag{38}$$

In momentum space the gauge invariance restrictions can be expressed by the condition

$$p_{[\mu}\tilde{T}_{\nu]}(p) = 0. \tag{39}$$

In order to determine whether circular polarization can be sourced at some order of (35), we will not need to impose any further restrictions on $T_\mu(x)$ aside from those listed above. Specifically, our calculations do not assume that $T_\mu(x)$ is either timelike or spacelike. As indicated by the investigation in [11], violations of causality and unstable solutions may arise for the case of timelike $T_\mu(x)$, and the results here must be interpreted with care in this case.

In Appendix A below we detail the calculation of the first-order interaction Hamiltonian which is linear in the Hamiltonian density Eq. (34), as well as the refractive and damping terms of Eq. (27) due to first-order processes. Quoting the results of Appendix A, the interaction Hamiltonian for first-order processes is given by

$$^1\hat{H}_{\text{int}}(t) = -2ig\int dpd\mathbf{k}\frac{\tilde{T}_\nu(p)e^{i(k_0+p_0-|\mathbf{k}+\mathbf{p}|)t}}{2|\mathbf{k}+\mathbf{p}|}\hat{a}_s^\dagger(k)\hat{a}_r(\tilde{k})$$
$$\times \epsilon_{s\mu}^*(\mathbf{k})(\epsilon^{\mu\nu0\beta}(|\mathbf{k}| + |\mathbf{k}+\mathbf{p}|)$$
$$+ \epsilon^{\mu\nu j\beta}(2k_j + p_j))\epsilon_{r\beta}(\mathbf{k}+\mathbf{p}). \tag{40}$$

The refractive term of Eq. (27) for this first-order interaction Hamiltonian is given by

$$i\langle[^1\hat{H}_{\text{int}}(0), \hat{\mathcal{D}}_{uv}(\mathbf{q})]\rangle = 4gq^0(2\pi)^3\delta^3(0)\epsilon_{s\mu}(\mathbf{q})(\delta_{ur}\rho_{sv}(\mathbf{q})$$
$$- \delta_{vs}\rho_{ur}(\mathbf{q}))\mathcal{A}^{\mu\beta}(q)\epsilon_{r\beta}(\mathbf{q}), \tag{41}$$

where $\mathcal{A}^{\mu\beta}$ is defined via

$$(2\pi)^3\delta^3(0)\mathcal{A}^{\mu\beta}(q) = (2\pi)^3\delta^3(0)\mathcal{A}^{[\mu\beta]}(q)$$
$$= \int \frac{dp}{2|\mathbf{q}+\mathbf{p}|}(2\pi)^3\delta^3(-\mathbf{p})\tilde{T}_\nu(p)$$
$$\times \epsilon^{\mu\nu\alpha\beta}(2q_\alpha + \tilde{p}_\alpha), \tag{42}$$

and we have for convenience defined $\tilde{p} \equiv (\Delta q, \mathbf{p})$, $\Delta q \equiv |\mathbf{q} + \mathbf{p}| - |\mathbf{q}|$.

As detailed in Appendix A the damping term of Eq. (27) due to first-order processes is given by

$$
\begin{aligned}
D_{uv} &\equiv \int_{-\infty}^{\infty} dt' \langle [{}^1\hat{H}_{\text{int}}(t'), [{}^1\hat{H}_{\text{int}}(0), \hat{\mathcal{D}}_{uv}(\mathbf{q})]]\rangle \\
&= -(2ig)^2 \epsilon_{m\kappa} \epsilon_{s\mu} \int \frac{dp}{2|\mathbf{q}+\mathbf{p}|} \tilde{T}_\lambda(\tilde{p}) q_\rho \epsilon^{\sigma\lambda\rho\kappa} \tilde{T}_\nu(p) \\
&\times (2q+\tilde{p})_\alpha \epsilon^{\mu\nu\alpha\beta} \tilde{\epsilon}_{n\sigma} \tilde{\epsilon}_{r\beta} (\delta_{nr}\delta_{us}\rho_{mv}(\mathbf{q}) \\
&+ \delta_{rn}\delta_{vs}\rho_{um}(\mathbf{q}) - \delta_{us}\delta_{mv}\rho_{rn}(\mathbf{q}+\mathbf{p}) \\
&- \delta_{um}\delta_{sv}\rho_{nr}(\mathbf{q}+\mathbf{p})),
\end{aligned} \tag{43}
$$

where we have defined $\epsilon_{r\mu} \equiv \epsilon_{r\mu}(\mathbf{q})$ and $\tilde{\epsilon}_{r\mu} \equiv \epsilon_{r\mu}(\mathbf{q}+\mathbf{p})$.

Once the density matrix is expanded as given in Eq. (28), it is straightforward to see that the refractive term, Eq. (41), vanishes when evaluated in terms of $\rho^{(0)}(|\mathbf{q}|)$. For the damping term, Eq. (43), we must perform an expansion of $\rho^{(0)}(|\mathbf{q}+\mathbf{p}|) = \rho^{(0)}(|\mathbf{k}|)$, in the p integral of Eq. (43), about $|\mathbf{q}|$:

$$
\rho^{(0)}(|\mathbf{k}|) = \rho^{(0)}(|\mathbf{q}|) + \mathcal{O}\left(\frac{d\rho^{(0)}}{d|\mathbf{k}|}(|\mathbf{q}|)\right), \tag{44}
$$

which is a suitable approximation as long as $\tilde{T}(p^0, \mathbf{p})$ has support solely over $|\mathbf{p}| \ll |\mathbf{q}|$, where $|\mathbf{q}|$ is the energy of the scattering photons. Then to lowest order in $\rho^{(0)}$ the damping term vanishes and we have

$$
\frac{d}{dt}\rho^{(0)} = 0 + \mathcal{O}\left(g^2 \frac{d\rho^{(0)}(|\mathbf{q}|)}{d|\mathbf{q}|}\right). \tag{45}
$$

In terms of the Stokes brightness perturbations defined in Eqs. (30)–(33), to first order in g the evolution of the polarization of the photon ensemble becomes

$$
\frac{d}{dt}\Delta_I = 0, \tag{46}
$$

$$
\frac{d}{dt}\Delta_Q = -g\alpha(q)\Delta_U, \tag{47}
$$

$$
\frac{d}{dt}\Delta_U = g\alpha(q)\Delta_Q, \tag{48}
$$

$$
\frac{d}{dt}\Delta_V = 0, \tag{49}
$$

where we have defined the quantity

$$
\alpha(q) \equiv 4\epsilon_{1\mu}(q)\mathcal{A}^{\mu\beta}(q)\epsilon_{2\beta}(q). \tag{50}
$$

For processes which are first order in the Hamiltonian density Eq. (34), according to Eq. (49) no circular polarization is generated to $\mathcal{O}(g)$ in our approximation. In fact, it is easy to see that to $\mathcal{O}(g)$ (the refractive term), Eqs. (46)–(49) reproduce the well-known effect of optical activity of the electromagnetic radiation, rotating the plane of linear polarization during propagation [7–10]. This is a useful check of the calculations.

The relevant linear combinations of the damping term Eq. (43) which source the polarization of the photon ensemble to $\mathcal{O}(g^2)$ and due to first-order processes are

$$
\begin{aligned}
D_{11} + D_{22} &= -(2ig)^2 \int \frac{dp}{2|\mathbf{q}+\mathbf{p}|} \tilde{T}_\lambda(\tilde{p}) q_\rho \epsilon^{\sigma\lambda\rho\kappa} \tilde{T}_\nu(p)(2q+\tilde{p})_\alpha \{(\tilde{\epsilon}_{1\sigma}\tilde{\epsilon}_{1\beta} + \tilde{\epsilon}_{2\sigma}\tilde{\epsilon}_{2\beta})[2\epsilon_{1\kappa}\epsilon_{1\mu}\rho_{11}^{(1)}(\mathbf{q}) + 2\epsilon_{2\kappa}\epsilon_{2\mu}\rho_{22}^{(1)}(\mathbf{q}) \\
&+ (\epsilon_{2\kappa}\epsilon_{1\mu} + \epsilon_{1\kappa}\epsilon_{2\mu})[\rho_{12}^{(1)}(\mathbf{q}) + \rho_{21}^{(1)}(\mathbf{q})]] - (\epsilon_{1\kappa}\epsilon_{1\mu} + \epsilon_{2\kappa}\epsilon_{2\mu})[2\tilde{\epsilon}_{1\beta}\tilde{\epsilon}_{1\sigma}\rho_{11}^{(1)}(\mathbf{q}+\mathbf{p}) + 2\tilde{\epsilon}_{2\beta}\tilde{\epsilon}_{2\sigma}\rho_{22}^{(1)}(\mathbf{q}+\mathbf{p}) \\
&+ (\tilde{\epsilon}_{1\beta}\tilde{\epsilon}_{2\sigma} + \tilde{\epsilon}_{2\beta}\tilde{\epsilon}_{1\sigma})[\rho_{12}^{(1)}(\mathbf{q}+\mathbf{p}) + \rho_{21}^{(1)}(\mathbf{q}+\mathbf{p})]]\} + \mathcal{O}\left(g^2 \frac{d\rho^{(0)}(|\mathbf{q}|)}{d|\mathbf{q}|}\right),
\end{aligned}
$$

$$
\begin{aligned}
D_{11} - D_{22} &= -(2ig)^2 \int \frac{dp}{2|\mathbf{q}+\mathbf{p}|} \tilde{T}_\lambda(\tilde{p}) q_\rho \epsilon^{\sigma\lambda\rho\kappa} \tilde{T}_\nu(p)(2q+\tilde{p})_\alpha \{(\tilde{\epsilon}_{1\sigma}\tilde{\epsilon}_{1\beta} + \tilde{\epsilon}_{2\sigma}\tilde{\epsilon}_{2\beta})[2\epsilon_{1\kappa}\epsilon_{1\mu}\rho_{11}^{(1)}(\mathbf{q}) - 2\epsilon_{2\kappa}\epsilon_{2\mu}\rho_{22}^{(1)}(\mathbf{q}) \\
&+ (\epsilon_{2\kappa}\epsilon_{1\mu} - \epsilon_{1\kappa}\epsilon_{2\mu})[\rho_{12}^{(1)}(\mathbf{q}) + \rho_{21}^{(1)}(\mathbf{q})]] - (\epsilon_{1\kappa}\epsilon_{1\mu} - \epsilon_{2\kappa}\epsilon_{2\mu})[2\tilde{\epsilon}_{1\beta}\tilde{\epsilon}_{1\sigma}\rho_{11}^{(1)}(\mathbf{q}+\mathbf{p}) + 2\tilde{\epsilon}_{2\beta}\tilde{\epsilon}_{2\sigma}\rho_{22}^{(1)}(\mathbf{q}+\mathbf{p}) \\
&+ (\tilde{\epsilon}_{1\beta}\tilde{\epsilon}_{2\sigma} + \tilde{\epsilon}_{2\beta}\tilde{\epsilon}_{1\sigma})[\rho_{12}^{(1)}(\mathbf{q}+\mathbf{p}) + \rho_{21}^{(1)}(\mathbf{q}+\mathbf{p})]]\} + \mathcal{O}\left(g^2 \frac{d\rho^{(0)}(|\mathbf{q}|)}{d|\mathbf{q}|}\right),
\end{aligned}
$$

$$
\begin{aligned}
D_{12} + D_{21} &= -(2ig)^2 \int \frac{dp}{2|\mathbf{q}+\mathbf{p}|} \tilde{T}_\lambda(\tilde{p}) q_\rho \epsilon^{\sigma\lambda\rho\kappa} \tilde{T}_\nu(p)(2q+\tilde{p})_\alpha \{(\tilde{\epsilon}_{1\sigma}\tilde{\epsilon}_{1\beta} + \tilde{\epsilon}_{2\sigma}\tilde{\epsilon}_{2\beta})[(\epsilon_{1\kappa}\epsilon_{1\mu} + \epsilon_{2\kappa}\epsilon_{2\mu})[\rho_{12}^{(1)}(\mathbf{q}) \\
&+ \rho_{21}^{(1)}(\mathbf{q})] + 2\epsilon_{1\kappa}\epsilon_{2\mu}\rho_{11}^{(1)}(\mathbf{q}) + 2\epsilon_{2\kappa}\epsilon_{1\mu}\rho_{22}^{(1)}(\mathbf{q})] - (\epsilon_{2\kappa}\epsilon_{1\mu} + \epsilon_{1\kappa}\epsilon_{2\mu})[2\tilde{\epsilon}_{1\beta}\tilde{\epsilon}_{1\sigma}\rho_{11}^{(1)}(\mathbf{q}+\mathbf{p}) \\
&+ 2\tilde{\epsilon}_{2\beta}\tilde{\epsilon}_{2\sigma}\rho_{22}^{(1)}(\mathbf{q}+\mathbf{p}) + (\tilde{\epsilon}_{1\beta}\tilde{\epsilon}_{2\sigma} + \tilde{\epsilon}_{1\sigma}\tilde{\epsilon}_{2\beta})[\rho_{12}^{(1)}(\mathbf{q}+\mathbf{p}) + \rho_{21}^{(1)}(\mathbf{q}+\mathbf{p})]]\} + \mathcal{O}\left(g^2 \frac{d\rho^{(0)}(|\mathbf{q}|)}{d|\mathbf{q}|}\right),
\end{aligned}
$$

$$D_{12} - D_{21} = -(2ig)^2 \int \frac{dp}{2|\mathbf{q} + \mathbf{p}|} \tilde{T}_\lambda(\tilde{p}) q_\rho \epsilon^{\sigma\lambda\rho\kappa} \tilde{T}_\nu(p)$$

$$\times (2q + \tilde{p})_\alpha \{(\tilde{\epsilon}_{1\sigma} \tilde{\epsilon}_{1\beta} + \tilde{\epsilon}_{2\sigma} \tilde{\epsilon}_{2\beta})$$

$$\times (\epsilon_{1\kappa} \epsilon_{1\mu} + \epsilon_{2\kappa} \epsilon_{2\mu})[\rho^{(1)}_{12}(\mathbf{q}) - \rho^{(1)}_{21}(\mathbf{q})]$$

$$- (\epsilon_{2\kappa} \epsilon_{1\mu} - \epsilon_{1\kappa} \epsilon_{2\mu})(\tilde{\epsilon}_{1\beta} \tilde{\epsilon}_{2\sigma} - \tilde{\epsilon}_{1\sigma} \tilde{\epsilon}_{2\beta})$$

$$\times [\rho^{(1)}_{12}(\mathbf{q} + \mathbf{p}) - \rho^{(1)}_{21}(\mathbf{q} + \mathbf{p})]\}.$$

Given the above expressions it is easy to see that no mixing occurs between Δ_V and the set $\{\Delta_I, \Delta_Q, \Delta_U\}$ as a result of the damping term Eq. (43). Therefore, in our approxima-

tion, no circular polarization is generated by first-order processes up to $\mathcal{O}(g^2)$.

V. THE SECOND-ORDER INTERACTION TERM

We now move on to calculate the contribution to the evolution of the photon density matrix from scattering processes which are second order in the interaction Hamiltonian density operator Eq. (34). Details of the calculation of the second-order interaction Hamiltonian and the corresponding refractive term are presented in Appendix B. The second-order interaction Hamiltonian is given by

$$^2\hat{H}_{\text{int}}(t) = -\frac{i(2\pi)^3}{2}(2g)^2 \int \frac{d^3\mathbf{p}_1}{(2\pi)^3 2p_1^0} dl_1 dl_2 \frac{d^3\mathbf{p}_2}{(2\pi)^3 2p_2^0} \epsilon^{\mu\nu\alpha\beta} \epsilon^{\rho\sigma\lambda\kappa} \tilde{T}_\nu(l_1) \tilde{T}_\sigma(l_2)(-ig_{\mu\rho}) \Bigg[((p_1 + p_2 + l_2)_\alpha (2p_2 + l_2)_\lambda)$$

$$\times e^{-i((p_1)^0 - (l_1)^0 - (p_2)^0 - (l_2)^0)t} \delta^3(\mathbf{p}_1 - \mathbf{l} - \mathbf{p}_2 - \mathbf{l}) \frac{\epsilon^*_{s\kappa}(p_2) \epsilon_{s\beta}(p_1) \hat{a}^\dagger_s(p_2) \hat{a}_r(p_1)}{(p_2 + l_2)^2 + i\epsilon}$$

$$+ ((p_1 + p_2 - l_2)_\alpha (2p_2 - l_2)_\lambda) e^{i((p_1)^0 + l_1^0 - (p_2)^0 + l_2^0)t} \delta^3(\mathbf{p}_1 + \mathbf{l} - \mathbf{p}_2 + \mathbf{l}) \frac{\epsilon^*_{s\beta}(p_1) \epsilon_{r\kappa}(p_2) \hat{a}^\dagger_s(p_1) \hat{a}_r(p_2)}{(p_2 - l_2)^2 + i\epsilon} \Bigg]. \quad (51)$$

The refractive term of Eq. (27) due to this interaction Hamiltonian is

$$i\langle[^2\hat{H}_{\text{int}}(0), \hat{\mathcal{D}}_{uv}(\mathbf{q})]\rangle = i(2\pi)^3 \delta^3(0)(2g)^2(\delta_{ur}\rho_{sv}(\mathbf{q})$$

$$- \delta_{vs}\rho_{ur}(\mathbf{q}))\epsilon^\mu_s(q)\epsilon^\nu_r(q)\mathcal{T}_{\mu\nu}(q),$$

where we have defined $\mathcal{T}_{ij}(q)$ via

$$(2\pi)^3 \delta^3(0)\mathcal{T}_{ij}(q) = -\int dl_1 dl_2 (2\pi)^3 \delta^3(\mathbf{l} + \mathbf{l}) \tilde{T}_j(l_2^0, \mathbf{l})$$

$$\times \tilde{T}_i(l_1^0, \mathbf{l})[q \cdot (l_1 + l_2)]$$

$$\times \Bigg[\frac{1}{(l_2)^2 + 2l_2 \cdot q + i\epsilon}$$

$$- \frac{1}{(l_2)^2 - 2l_2 \cdot q + i\epsilon} \Bigg]. \quad (52)$$

Note that the interaction Hamiltonian Eq. (51) and the refractive term Eq. (52) were computed using the photon propagator in the Feynman gauge Eq. (B3), but a demonstration of gauge invariance of these results is detailed in Appendix C. For an unpolarized photon ensemble, it is easy to see that the right side of Eq. (52) vanishes when evaluated in terms of $\rho^{(0)}$ defined in Eq. (28) above. The contribution to the perturbed density matrix, $\rho^{(1)}$, to second order in the interaction coupling g is

$$\frac{d}{dt}\rho^{(1)}_{uv}(\mathbf{q}) = \frac{2ig^2}{q^0}(\delta_{ur}\rho^{(1)}_{sv}(\mathbf{q})$$

$$- \delta_{vs}\rho^{(1)}_{ur}(\mathbf{q}))\epsilon^\mu_s(q)\epsilon^\nu_r(q)\mathcal{T}_{\mu\nu}(q). \quad (53)$$

Here we have used Eq. (27) and have ignored the damping term, which is of $\mathcal{O}(g^4)$. Expressing Eq. (53) explicitly in

terms of photon density matrix components gives the evolution equations

$$\frac{d}{dt}\rho^{(1)}_{11}(\mathbf{q}) = \frac{2ig^2}{q^0}(-\epsilon^\mu_1\epsilon^\nu_2\rho^{(1)}_{12}(\mathbf{q}) + \epsilon^\mu_2\epsilon^\nu_1\rho^{(1)}_{21}(\mathbf{q}))\mathcal{T}_{\mu\nu}(q)$$

$$(54)$$

$$\frac{d}{dt}\rho^{(1)}_{22}(\mathbf{q}) = \frac{2ig^2}{q^0}(\epsilon^\mu_1\epsilon^\nu_2\rho^{(1)}_{12}(\mathbf{q}) - \epsilon^\mu_2\epsilon^\nu_1\rho^{(1)}_{21}(\mathbf{q}))\mathcal{T}_{\mu\nu}(q)$$

$$(55)$$

$$\frac{d}{dt}\rho^{(1)}_{12}(\mathbf{q}) = \frac{2ig^2}{q^0}([\epsilon^\mu_1\epsilon^\nu_1 - \epsilon^\mu_2\epsilon^\nu_2]\rho^{(1)}_{12}(\mathbf{q})$$

$$- \epsilon^\mu_2\epsilon^\nu_1[\rho^{(1)}_{11}(\mathbf{q}) - \rho^{(1)}_{22}(\mathbf{q})])\mathcal{T}_{\mu\nu}(q) \quad (56)$$

$$\frac{d}{dt}\rho^{(1)}_{21}(\mathbf{q}) = \frac{2ig^2}{q^0}(-[\epsilon^\mu_1\epsilon^\nu_1 - \epsilon^\mu_2\epsilon^\nu_2]\rho^{(1)}_{21}(\mathbf{q})$$

$$+ \epsilon^\mu_1\epsilon^\nu_2[\rho^{(1)}_{11}(\mathbf{q}) - \rho^{(1)}_{22}(\mathbf{q})])\mathcal{T}_{\mu\nu}(q). \quad (57)$$

All polarization vectors ϵ^μ_r in the above expression depend on $\mathbf{q}$, the same photon momentum as in the argument of the photon density matrix. Using the Stokes brightness perturbations in Eqs. (30)–(33), we can now express the evolution of the polarization of the photon ensemble due to processes mediated by $^2\hat{H}_{\text{int}}(t)$ as

$$\frac{d}{dt}\Delta_I(\mathbf{q}) = 0, \quad (58)$$

$$\frac{d}{dt}\Delta_Q(\mathbf{q}) = -\frac{g^2}{q^0}(\zeta(\hat{\mathbf{q}})\Delta_V(\mathbf{q}) + i\psi(\hat{\mathbf{q}})\Delta_U(\mathbf{q})), \quad (59)$$

$$\frac{d}{dt}\Delta_U(\mathbf{q}) = -\frac{g^2}{q^0}(\chi(\hat{\mathbf{q}})\Delta_V(\mathbf{q}) - i\psi(\hat{\mathbf{q}})\Delta_Q(\mathbf{q})), \quad (60)$$

$$\frac{d}{dt}\Delta_V(\mathbf{q}) = \frac{g^2}{q^0}(\chi(\hat{\mathbf{q}})\Delta_U(\mathbf{q}) + \zeta(\hat{\mathbf{q}})\Delta_Q(\mathbf{q})), \quad (61)$$

where we have defined the contractions

$$\zeta(\hat{\mathbf{q}}) \equiv -2\mathcal{T}_{\mu\nu}(\epsilon_1^\mu \epsilon_2^\nu + \epsilon_2^\mu \epsilon_1^\nu), \quad (62)$$

$$\chi(\hat{\mathbf{q}}) \equiv 2\mathcal{T}_{\mu\nu}(\epsilon_1^\mu \epsilon_1^\nu - \epsilon_2^\mu \epsilon_2^\nu), \quad (63)$$

$$\psi(\hat{\mathbf{q}}) \equiv 2\mathcal{T}_{\mu\nu}(\epsilon_1^\mu \epsilon_2^\nu - \epsilon_2^\mu \epsilon_1^\nu) \quad (64)$$

and the quantity $\mathcal{T}_{\mu\nu}$ is defined via the integral expression Eq. (52).

The circular polarization brightness Δ_V is sourced by terms which are proportional to either Δ_Q or Δ_U: a linearly polarized photon ensemble in the presence of the interaction Eq. (34) will acquire circular polarization due to processes which are of second order in the interaction. As long as the interaction acts, both Stokes brightnesses Δ_Q and Δ_U are rotated with Δ_V. Note that the above equations do not depend on the time component of $T_\mu(x)$ since the polarization vectors are purely spatial.

VI. DISCUSSION

The evolution Eqs. (58) to (61), along with Eqs. (46) to (49), are the central result of this paper. Other source terms associated with the usual Compton scattering effects will also appear on the right sides. While the polarization brightnesses will be zero prior to recombination, during recombination Δ_Q and Δ_U become nonzero, with an amplitude a factor of 20 smaller than the intensity fluctuations Δ_I. It is easy to see by inspection of the evolution equations that at that point, Δ_Q and Δ_V will rotate into each other with a characteristic angular frequency $\omega_{QV} = g^2\zeta/k^0$, Δ_U and Δ_V will rotate into each other with a characteristic frequency $\omega_{UV} = g^2\chi/k^0$, and Δ_Q and Δ_U will rotate into each other with a characteristic frequency $\omega_{QU} = g\alpha$, along with an exponential decay or growth of Δ_Q and Δ_U associated with the first-order damping effects. All of these source terms are active whenever the interaction Eq. (34) is nonzero, in contrast to the conventional Compton scattering terms, which are only significant when the photons propagate through ionized regions of the universe.

The rotation between Δ_Q and Δ_U can be constrained from current measurements in a straightforward way. Linear polarization on the sky is conveniently expressed in a different basis, corresponding to the "gradient" and "curl" pieces of the polarization tensor field [35,36], also known as the E/B decomposition [37]. This decomposition is useful because scalar density perturbations in the universe, which evolve into the structures we see today via gravitational instability, generate only E-mode polariza-

tion. Subsequent rotation of the polarization plane as the wave propagates rotates E-mode into B-mode. Current limits on the amplitude of B-mode polarization (see Refs. [38–40] for some recent linear polarization measurements) can be translated into limits on the total rotation of linear polarization between the time of last scattering and today; see Refs. [41,42] for corresponding limits on magnetic fields due to Faraday rotation. Precise limits on the interaction studied here from the first-order rotation effect Eqs. (47) and (48) can be obtained similarly, and will be computed elsewhere. But we know that the total amount of rotation must be small at frequencies between 50 GHz and 150 GHz where good measurements of the primordial linear polarization have been made. Given this observational constraint on linear polarization rotation, can some realistic cosmological model generate detectable circular polarization via Eqs. (59)–(61)? First, note that ω_{QU} has dimensions of $[gT]$, and that for this rotation to be below current limits,

$$\omega_{QU} \simeq gT \ll H_0; \quad (65)$$

otherwise, as the microwave background photons propagate a Hubble distance from last scattering until today, we would have substantial rotation of E-mode into B-mode polarization. We are aware of no other laboratory or theoretical constraints on this class of interactions.

Now note that any first order damping effects will contribute an extra factor of length compared to $\alpha(q)$, arising from an additional time integral over the field $T_\mu(x)$; if the field is active for all times, this leads to roughly a factor of H_0^{-1}. Then the time scale for exponential growth or decay of linear polarization is, by dimensional estimate, $\omega_{QU}(\omega_{QU}/H_0)$, which is small compared to ω_{QU}: we can always neglect the exponential growth or decay of linear polarization compared to its rotation.

For generation of V polarization, ω_{QV} and ω_{UV} both have dimensions $[g^2T^2/k^0]$, differing only by a geometric factor related to the propagation directions of the photons and the direction of the field $T^\mu(x)$. So a dimensional estimate for both is $\omega_{QV} \simeq \omega_{UV} \simeq \omega_{QU}(\omega_{QU}/k^0)$. A typical microwave background photon today will have a frequency of $k^0 \simeq 100$ GHz or 10^{11} s^{-1}, while the characteristic size of ω_{QU} at the observational limit is H_0, a huge mismatch in scales. So in the case considered here, the generation of circular polarization is always vastly subdominant to the rotation of the linear polarization, and can be neglected.

The calculation presented here demonstrates generally that, given additional interactions beyond Compton scattering, circular polarization is not necessarily zero, and elaborates the framework for calculating it for a given microphysical interaction. Other interactions may well induce circular polarization without optical activity from the linear interaction term, and for these cases circular polarization could be the most constraining probe. We

have also only considered a constant field $\mathcal{T}^\mu$ for simplicity; calculations for a nonconstant field are messier but straightforward, involving convolutions over the field and photon distributions. Spatial or temporal variations in the field could change the relative importance of the optical activity and circular polarization generation effects. In particular, the torsion field necessarily couples to fermions via the interaction $\mathcal{L}_{TF} = g_1 \mathcal{T}_\mu \bar{\psi} \gamma^\mu \psi$ as well as a torsion-induced four-fermion interaction $\mathcal{L}_{FF} = g_2 \bar{\psi} \gamma^5 \gamma^\mu \psi \bar{\psi} \gamma_5 \gamma_\mu \psi$ [43–45], where g_1 and g_1 are renormalized couplings. It would be interesting to study the coupled system of torsion and fermions subject to our formalism for evaluating V, since backreaction effects from the torsion-fermion interaction could enhance the amplitude and modify the scale of V; we leave this to future work.

We encourage experimenters to make measurements testing the standard lore that circular polarization of the cosmic microwave background radiation should be identically zero, and theorists to consider the effects of any nonstandard photon couplings on microwave background polarization as photons propagate over cosmological distances.

ACKNOWLEDGMENTS

APPENDIX A: FIRST-ORDER CALCULATION

In this appendix we determine the contribution to the time evolution of the photon density matrix due to processes which are first order in the interaction Hamiltonian density Eq. (34). We first detail the calculation of the refractive term and then the damping term of Eq. (27). The first-order interaction Hamiltonian is simply given by

$$\hat{H}_{\rm int}^{(1)}(t) = \int d^3\mathbf{x}\, \hat{\mathcal{H}}_T. \tag{A1}$$

In the approximation employed here, all operators in the collision terms of Eq. (27) are from the free theory. From this point on, all operators represent free-theory operators and we drop the "0" superscript used in Sec. III. Ignoring any processes in which two physical photons are either annihilated or created leads to the following expression for the interaction Hamiltonian:

$$
\begin{aligned}
{}^1\hat{H}_{\rm int}(t) = 2g \int d^3\mathbf{x}\, dp\, d\mathbf{k}\, d\mathbf{k}' \epsilon^{\mu\nu\alpha\beta} \tilde{T}_\nu(p)(-ik'_\alpha) \\
\times (\hat{a}_s^\dagger(k)\hat{a}_r(k')\epsilon_{s\mu}^*(k)\epsilon_{r\beta}(k')e^{i(p+k-k')\cdot x} \\
- \hat{a}_r^\dagger(k')\hat{a}_s(k)\epsilon_{r\beta}^*(k')\epsilon_{s\mu}(k)e^{i(p+k'-k)\cdot x}), \tag{A2}
\end{aligned}
$$

where we have used the expression for $\hat{A}_\mu$ given in Eq. (14) and we have used the shorthand notation

$$\int dp \equiv \int \frac{d^4p}{(2\pi)^4}, \qquad \int d\mathbf{k} \equiv \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k^0}.$$

Performing the spatial integral and relabeling dummy momentum variables and spacetime indices in the second term of (A2) gives

$$
\begin{aligned}
{}^1\hat{H}_{\rm int}(t) = -2ig \int dp\, d\mathbf{k}\, d\mathbf{k}'(2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{p} - \mathbf{k}') \\
\times e^{i(k_0 + p_0 - k_0')t} \epsilon^{\mu\nu\alpha\beta} \tilde{T}_\nu(p)\hat{a}_s^\dagger(k)\hat{a}_r(k') \\
\times \epsilon_{s\mu}^*(k)(k + k')_\alpha \epsilon_{r\beta}(k'). \tag{A3}
\end{aligned}
$$

Now perform the $\int d\mathbf{k}'$ integral, after which we have

$$
\begin{aligned}
{}^1\hat{H}_{\rm int}(t) &= -2ig \int dp\, d\mathbf{k}\, \frac{\tilde{T}_\nu(p)e^{i(k_0+p_0-|\mathbf{k}+\mathbf{p}|)t}}{2|\mathbf{k}+\mathbf{p}|}\hat{a}_s^\dagger(k)\hat{a}_r(\tilde{k}) \\
&\times \epsilon_{s\mu}^*(k)(\epsilon^{\mu\nu0\beta}(|\mathbf{k}| + |\mathbf{k}+\mathbf{p}|) \\
&+ \epsilon^{\mu\nu j\beta}(2k_j + p_j))\epsilon_{r\beta}(\tilde{k}), \\
&= -2ig \int dp\, d\mathbf{k}\, \frac{\epsilon^{\mu\nu\alpha\beta}\tilde{T}_\nu(p)e^{i(k_0+p_0-|\mathbf{k}+\mathbf{p}|)t}}{2|\mathbf{k}+\mathbf{p}|} \\
&\times \hat{a}_s^\dagger(k)\hat{a}_r(\tilde{k})\epsilon_{s\mu}^*(k)(k+\tilde{k})_\alpha \epsilon_{r\beta}(\tilde{k}), \tag{A4}
\end{aligned}
$$

where $(\tilde{k})_\alpha = (|\mathbf{k} + \mathbf{p}|, \mathbf{k} + \mathbf{p})$. Equation (A4) is our first-order interaction Hamiltonian. Now the commutator necessary for the refractive term of Eq. (27) is given by

$$
\begin{aligned}
[{}^1\hat{H}_{\rm int}(t), \hat{\mathcal{D}}_{uv}(\mathbf{q})] &= -2ig \int dp\, d\mathbf{k}\, \frac{\tilde{T}_\nu(p)e^{i(k_0+p_0-|\mathbf{k}+\mathbf{p}|)t}}{2|\mathbf{k}+\mathbf{p}|} \\
&\times \epsilon_{s\mu}^*(k)(\epsilon^{\mu\nu\alpha\beta}(k+\tilde{k})_\alpha)\epsilon_{r\beta}(\tilde{k}) \\
&\times (2\pi)^3 2q^0(\delta_{ur}\delta^3(\mathbf{q}-\mathbf{k}-\mathbf{p}) \\
&\times \hat{a}_s^\dagger(k)\hat{a}_v(q) - \delta_{vs}\delta^3(\mathbf{q}-\mathbf{k}) \\
&\times \hat{a}_u^\dagger(q)\hat{a}_r(\tilde{k})) \tag{A5}
\end{aligned}
$$

where we have used the canonical commutation relations between the free creation and annihilation operators Eq. (15). Taking the expectation value of (A5) and using the relationship between the number operator and density matrix given by Eq. ((17), we arrive at the following expression:

$$
\begin{aligned}
i\langle[{}^1\hat{H}_{\rm int}(t), \hat{\mathcal{D}}_{uv}(\mathbf{q})]\rangle &= 2g \int dp\, d\mathbf{k}\, \frac{\tilde{T}_\nu(p)e^{i(k_0+p_0-|\mathbf{k}+\mathbf{p}|)t}}{2|\mathbf{k}+\mathbf{p}|} \\
&\times \epsilon_{s\mu}^*(k)\epsilon^{\mu\nu\alpha\beta}(k+\tilde{k})_\alpha \epsilon_{r\beta}(\tilde{k})(2\pi)^6 \\
&\times (2q^0)^2 \delta^3(\mathbf{q}-\mathbf{k}-\mathbf{p})\delta^3(\mathbf{q}-\mathbf{k}) \\
&\times (\delta_{ur}\rho_{sv}(\mathbf{q}) - \delta_{vs}\rho_{ur}(\mathbf{q})). \tag{A6}
\end{aligned}
$$

Perform the $\int d\mathbf{k}$ integral gives

$$i\langle[{}^1\hat{H}_{\text{int}}(t), \hat{\mathcal{D}}_{uv}(\mathbf{q})]\rangle = 2g \int dp \frac{\tilde{T}_\nu(p)e^{i(q_0+p_0-|\mathbf{q}+\mathbf{p}|)t}}{4|\mathbf{q}||\mathbf{q}+\mathbf{p}|}$$
$$\times \epsilon^*_{s\mu}(q)\epsilon^{\mu\nu\alpha\beta}(q+\tilde{q})_\alpha\epsilon_{r\beta}(\tilde{q})(2\pi)^3$$
$$\times (2q^0)^2\delta^3(-\mathbf{p})(\delta_{ur}\rho_{sv}(\mathbf{q})$$
$$- \delta_{vs}\rho_{ur}(\mathbf{q})) \qquad (A7)$$

where as before we have defined $(\tilde{q})_\alpha = (|\mathbf{q}+\mathbf{p}|, \mathbf{q}+\mathbf{p})$ so that the above can be expressed as

$$i\langle[{}^1\hat{H}_{\text{int}}(t), \hat{\mathcal{D}}_{uv}(\mathbf{q})]\rangle = 2g\epsilon^*_{s\mu}(q)(2q^0)(\delta_{ur}\rho_{sv}(\mathbf{q})$$
$$- \delta_{vs}\rho_{ur}(\mathbf{q}))\int dp (2\pi)^3\delta^3(-\mathbf{p})$$
$$\times \frac{\tilde{T}_\nu(p)e^{i(q_0+p_0-|\mathbf{q}+\mathbf{p}|)t}}{2|\mathbf{q}+\mathbf{p}|}$$
$$\times (\epsilon^{\mu\nu0\beta}(|\mathbf{q}|+|\mathbf{q}+\mathbf{p}|) + \epsilon^{\mu\nu j\beta}$$
$$\times (2q_j+p_j))\epsilon_{r\beta}(\mathbf{q}+\mathbf{p}). \qquad (A8)$$

Define the quantity

$$(2\pi)^3\delta^3(0)\mathcal{A}^{\mu\beta}(q) = (2\pi)^3\delta^3(0)\mathcal{A}^{[\mu\beta]}(q),$$
$$= \int dp (2\pi)^3\delta^3(-\mathbf{p})\frac{\tilde{T}_\nu(p)}{2|\mathbf{q}+\mathbf{p}|}$$
$$\times (\epsilon^{\mu\nu0\beta}(|\mathbf{q}|+|\mathbf{q}+\mathbf{p}|)$$
$$+ \epsilon^{\mu\nu j\beta}(2q_j+p_j)), \qquad (A9)$$

where we have anticipated the fact that the delta-distribution factor will be present once an appropriate form for the external field $T_\mu(x)$ is chosen. The refractive term of Eq. (27) due to first-order processes can then be expressed as

$$i\langle[{}^1\hat{H}_{\text{int}}(0), \hat{\mathcal{D}}_{uv}(\mathbf{q})]\rangle = 4gq^0(2\pi)^3\delta^3(0)\epsilon^*_{s\mu}(\mathbf{q})(\delta_{ur}\rho_{sv}(\mathbf{q})$$
$$- \delta_{vs}\rho_{ur}(\mathbf{q}))\mathcal{A}^{\mu\beta}(q)\epsilon_{r\beta}(\mathbf{q}). \qquad (A10)$$

Next we consider the damping term of Eq. (26), the integrand of which involves the double commutator

$$[{}^1\hat{H}_{\text{int}}(t-t'), [{}^1\hat{H}_{\text{int}}(t), \hat{\mathcal{D}}_{uv}(\mathbf{q})]] = (2ig)^2 \int dl\,dp\,d\mathbf{k}_2\,d\mathbf{k}_1 \frac{\tilde{T}_\nu(p)e^{i((k_1)_0+p_0-|\mathbf{k}_1+\mathbf{p}|)t}}{2|\mathbf{k}_1+\mathbf{p}|} \frac{\tilde{T}_\lambda(l)e^{i((k_2)_0+l_0-|\mathbf{k}_2+l|)(t-t')}}{2|\mathbf{k}_2+l|}$$
$$\times \epsilon^*_{s\mu}(k_1)\epsilon^{\mu\nu\alpha\beta}(k_1+\tilde{k}_1)_\alpha\epsilon_{r\beta}(\tilde{k}_1)\epsilon^*_{n\sigma}(k_2)\epsilon^{\sigma\lambda\rho\kappa}(k_2+\tilde{k}_2)_\rho\epsilon_{m\kappa}(\tilde{k}_2)(2\pi)^6 2q^0$$
$$\times (\delta_{ur}\delta^3(\mathbf{q}-\mathbf{k}_1-\mathbf{p})(2(k_1)^0\delta_{ms}\delta^3(\mathbf{k}_2+l-\mathbf{k}_1)\hat{a}^\dagger_n(k_2)\hat{a}_v(q)$$
$$- 2q^0\delta_{nv}\delta^3(\mathbf{k}_2-\mathbf{q})\hat{a}^\dagger_s(k_1)\hat{a}_m(\tilde{k}_2)) - \delta_{vs}\delta^3(\mathbf{q}-\mathbf{k}_1)(2q^0\delta_{mu}\delta^3(\mathbf{k}_2+l-\mathbf{q})\hat{a}^\dagger_n(k_2)\hat{a}_r$$
$$\times (\tilde{k}_1) - 2(k_2)^0\delta_{nr}\delta^3(\mathbf{k}_2-\mathbf{k}_1-\mathbf{p})\hat{a}^\dagger_u(q)\hat{a}_m(\tilde{k}_2))), \qquad (A11)$$

where we have used the canonical commutation relations between the free creation and annihilation operators, Eq. (15), and have defined $(\tilde{k}_1)_\alpha = (|\mathbf{k}_1+\mathbf{p}|, \mathbf{k}_1+\mathbf{p})$ and $(\tilde{k}_2)_\alpha = (|\mathbf{k}_2+l|, \mathbf{k}_2+l)$. Now the $\int d\mathbf{k}_1$ and the $\int d\mathbf{k}_2$ integrals can be performed, giving

$$[{}^1\hat{H}_{\text{int}}(t-t'), [{}^1\hat{H}_{\text{int}}(t), \hat{\mathcal{D}}_{uv}(\mathbf{q})]] = (2ig)^2 2q^0 \int dl\,dp\,\tilde{T}_\lambda(l)\epsilon^{\sigma\lambda\rho\kappa}\tilde{T}_\nu(p)\epsilon^{\mu\nu\alpha\beta}\Bigg(\frac{e^{i(|\mathbf{q}-\mathbf{p}|+p_0-|\mathbf{q}|)t}\epsilon^*_{s\mu}(\tilde{q}_{-p})(\tilde{q}_{-p}+q)_\alpha\epsilon_{r\beta}(q)\delta_{ur}}{4|\mathbf{q}-\mathbf{p}||\mathbf{q}|}$$
$$\times \Bigg[2|\mathbf{q}-\mathbf{p}|\frac{e^{i(|\mathbf{q}-\mathbf{p}-l|+l_0-|\mathbf{q}-\mathbf{p}|)(t-t')}}{4|\mathbf{q}-\mathbf{p}-l||\mathbf{q}-\mathbf{p}|}\epsilon^*_{n\sigma}(\tilde{q}_{-p-l})(\tilde{q}_{-p-l}+\tilde{q}_{-p})_\rho\epsilon_{m\kappa}(\tilde{q}_{-p})\delta_{ms}\hat{a}^\dagger_n(\tilde{q}_{-p-l})$$
$$\times \hat{a}_v(q) - 2q^0\frac{e^{i(q_0+l_0-|\mathbf{q}+l|)(t-t')}}{4|\mathbf{q}||\mathbf{q}+l|}\epsilon^*_{n\sigma}(q)(q+\tilde{q}_{+l})_\rho\epsilon_{m\kappa}(\tilde{q}_{+l})\delta_{nv}\hat{a}^\dagger_s(\tilde{q}_{-p})\hat{a}_m(\tilde{q}_{+l})\Bigg]$$
$$- \frac{e^{i((q)_0+p_0-|\mathbf{q}+\mathbf{p}|)t}\epsilon^*_{s\mu}(q)(q+\tilde{q}_{+p})_\alpha\epsilon_{r\beta}(\tilde{q}_{+p})\delta_{vs}}{4|\mathbf{q}+\mathbf{p}||\mathbf{q}|}\Bigg[2q^0\frac{e^{i(|\mathbf{q}-l|+l_0-|\mathbf{q}|)(t-t')}}{4|\mathbf{q}-l||\mathbf{q}|}$$
$$\times \epsilon^*_{n\sigma}(\tilde{q}_{-l})(\tilde{q}_{-l}+q)_\rho\epsilon_{m\kappa}(q)\delta_{mu}\hat{a}^\dagger_n(\tilde{q}_{-l})\hat{a}_r(\tilde{q}_{+p}) - 2|\mathbf{q}+\mathbf{p}|\frac{e^{i(|\mathbf{q}+\mathbf{p}|+l_0-|\mathbf{q}+\mathbf{p}+l|)(t-t')}}{4|\mathbf{q}+\mathbf{p}||\mathbf{q}+\mathbf{p}+l|}$$
$$\times \epsilon^*_{n\sigma}(\tilde{q}_{+p})(\tilde{q}_{+p}+\tilde{q}_{+p+l})_\rho\epsilon_{m\kappa}(\tilde{q}_{+p+l})\delta_{nr}\hat{a}^\dagger_u(q)\hat{a}_m(\tilde{q}_{+p+l})\Bigg]\Bigg). \qquad (A12)$$

After taking the expectation value, this becomes

$$\langle [^1\hat{H}_{\text{int}}(t - t'), [^1\hat{H}_{\text{int}}(t), \hat{\mathcal{D}}_{uv}(\mathbf{q})]]\rangle = (2ig)^2 \int dl dp \tilde{T}_\lambda(l)\epsilon^{\sigma\rho\kappa}\tilde{T}_\nu(p)\epsilon^{\mu\nu\alpha\beta}(2\pi)^3 \delta^3(l + \mathbf{p})e^{i(p_0+l_0)t}$$

$$\times \left(\frac{e^{-i(|\mathbf{q}|+l_0-|\mathbf{q}+l|)t'}\epsilon^*_{s\mu}(\tilde{q}_{+l})(\tilde{q}_{+l} + q)_\alpha\epsilon_{r\beta}(q)\delta_{ur}}{2|\mathbf{q} + l|}[\epsilon^*_{n\sigma}(q)(q + \tilde{q}_{+l})_\rho\epsilon_{m\kappa}(\tilde{q}_{+l})\delta_{ms}\rho_{nv}\right.$$

$$\times (\mathbf{q}) - \epsilon^*_{n\sigma}(q)(q + \tilde{q}_{+l})_\rho\epsilon_{m\kappa}(\tilde{q}_{+l})\delta_{nv}\rho_{sm}(\mathbf{q} + l)]$$

$$-\frac{e^{-i(|\mathbf{q}+\mathbf{p}|+l_0-|\mathbf{q}|)t'}\epsilon^*_{s\mu}(q)(q + \tilde{q}_{+p})_\alpha\epsilon_{r\beta}(\tilde{q}_{+p})\delta_{vs}}{2|\mathbf{q} + \mathbf{p}|}[\epsilon^*_{n\sigma}(\tilde{q}_{+p})(\tilde{q}_{+p} + q)_\rho\epsilon_{m\kappa}(q)$$

$$\left.\times \delta_{mu}\rho_{nr}(\mathbf{q} + \mathbf{p}) - \epsilon^*_{n\sigma}(\tilde{q}_{+p})(\tilde{q}_{+p} + q)_\rho\epsilon_{m\kappa}(q)\delta_{nr}\rho_{um}(\mathbf{q})]\right). \tag{A13}$$

Interchanging l and p in the first set of terms gives

$$\langle [^1\hat{H}_{\text{int}}(t-t'), [^1\hat{H}_{\text{int}}(t), \hat{\mathcal{D}}_{uv}(\mathbf{q})]]\rangle = (2ig)^2 \int dl dp(2\pi)^3 \delta^3(l+\mathbf{p})e^{i(p_0+l_0)t}$$

$$\times \left(\frac{e^{-i(|\mathbf{q}|+p_0-|\mathbf{q}+\mathbf{p}|)t'}p_\lambda\phi(p)\epsilon^{\sigma\rho\kappa}l_\nu\phi(l)\epsilon^{\mu\nu\alpha\beta}\epsilon^*_{s\mu}(\tilde{q}_{+p})(\tilde{q}_{+p}+q)_\alpha\epsilon_{r\beta}(q)\delta_{ur}}{2|\mathbf{q}+\mathbf{p}|}\right.$$

$$\times [\epsilon^*_{n\sigma}(q)(q+\tilde{q}_{+p})_\rho\epsilon_{m\kappa}(\tilde{q}_{+p})\delta_{ms}\rho_{nv}(\mathbf{q}) - \epsilon^*_{n\sigma}(q)(q+\tilde{q}_{+p})_\rho\epsilon_{m\kappa}(\tilde{q}_{+p})\delta_{nv}\rho_{sm}(\mathbf{q}+\mathbf{p})]$$

$$-\frac{e^{-i(|\mathbf{q}+\mathbf{p}|+l_0-|\mathbf{q}|)t'}\tilde{T}_\lambda(l)\epsilon^{\sigma\rho\kappa}\tilde{T}_\nu(p)\epsilon^{\mu\nu\alpha\beta}\epsilon^*_{s\mu}(q)(q+\tilde{q}_{+p})_\alpha\epsilon_{r\beta}(\tilde{q}_{+p})\delta_{vs}}{2|\mathbf{q}+\mathbf{p}|}$$

$$\left.\times [\epsilon^*_{n\sigma}(\tilde{q}_{+p})(\tilde{q}_{+p}+q)_\rho\epsilon_{m\kappa}(q)\delta_{mu}\rho_{nr}(\mathbf{q}+\mathbf{p}) - \epsilon^*_{n\sigma}(\tilde{q}_{+p})(\tilde{q}_{+p}+q)_\rho\epsilon_{m\kappa}(q)\delta_{nr}\rho_{um}(\mathbf{q})]\right). \tag{A14}$$

After relabeling some spacetime and polarization indices in first set of terms, the above becomes

$$\langle [^1\hat{H}_{\text{int}}(t - t'), [^1\hat{H}_{\text{int}}(t), \hat{\mathcal{D}}_{uv}(\mathbf{q})]]\rangle = (2ig)^2 \int dl dp(2\pi)^3 \delta^3(l + \mathbf{p})e^{i(p_0+l_0)t}$$

$$\times \frac{\tilde{T}_\nu(p)\epsilon^{\sigma\rho\kappa}\tilde{T}_\lambda(l)\epsilon^{\mu\nu\alpha\beta}\epsilon^*_{n\sigma}(\tilde{q}_{+p})(\tilde{q}_{+p} + q)_\rho\epsilon_{m\kappa}(q)\epsilon^*_{s\mu}(q)(q + \tilde{q}_{+p})_\alpha\epsilon_{r\beta}(\tilde{q}_{+p})}{2|\mathbf{q} + \mathbf{p}|}$$

$$\times \{e^{-i(|\mathbf{q}|+p_0-|\mathbf{q}+\mathbf{p}|)t'}\delta_{um}[\delta_{rn}\rho_{sv}(\mathbf{q}) - \delta_{sv}\rho_{nr}(\mathbf{q} + \mathbf{p})] - e^{-i(|\mathbf{q}+\mathbf{p}|+l_0-|\mathbf{q}|)t'}$$

$$\times \delta_{vs}[\delta_{mu}\rho_{nr}(\mathbf{q} + \mathbf{p}) - \delta_{nr}\rho_{um}(\mathbf{q})]\}. \tag{A15}$$

Now integrate over $\int_{-t}^t dt'$ ($t \to \infty$) and define $\Delta q = |\mathbf{q} + \mathbf{p}| - |\mathbf{q}|$ to arrive at

$$D_{uv} \equiv \int_{-\infty}^\infty dt'\langle [^1\hat{H}_{\text{int}}(t'), [^1\hat{H}_{\text{int}}(0), \hat{\mathcal{D}}_{uv}(\mathbf{q})]]\rangle$$

$$= (2ig)^2 \int dl dp \frac{(2\pi)^4 \delta^3(l + \mathbf{p})}{2|\mathbf{q} + \mathbf{p}|}\tilde{T}_\nu(p)\epsilon^{\sigma\rho\kappa}\tilde{T}_\lambda(l)\epsilon^{\mu\nu\alpha\beta}\tilde{\epsilon}_{n\sigma}(\tilde{q}_{+p} + q)_\rho(q + \tilde{q}_{+p})_\alpha\tilde{\epsilon}_{r\beta}\{\delta(p_0 - \Delta q)$$

$$\times [\delta_{rn}\delta_{um}\epsilon_{m\kappa}\epsilon_{s\mu}\rho_{sv}(\mathbf{q}) - \delta_{um}\delta_{sv}\epsilon_{m\kappa}\epsilon_{s\mu}\rho_{nr}(\mathbf{q} + \mathbf{p})] + \delta(l_0 + \Delta q)[\delta_{rn}\delta_{vs}\epsilon_{m\kappa}\epsilon_{s\mu}\rho_{um}(\mathbf{q})$$

$$- \delta_{um}\delta_{sv}\epsilon_{m\kappa}\epsilon_{s\mu}\rho_{nr}(\mathbf{q} + \mathbf{p})]\}, \tag{A16}$$

where for convenience we have defined D_{uv} above, as well as the abbreviations $\epsilon_{r\mu} = \epsilon_{r\mu}(\mathbf{q})$ and $\tilde{\epsilon}_{r\mu} = \epsilon_{r\mu}(\mathbf{q} + \mathbf{p})$. Note that since ρ is expressed in a linear polarization basis all polarization vectors above have been assumed to be real. Now in the $\int dl$ integral above, the relevant factor can be simplified as

$$\tilde{\epsilon}_{n\sigma}\epsilon_{m\kappa}\epsilon_{s\mu}\tilde{\epsilon}_{r\beta}\int dl\,\delta(l+\mathbf{p})\tilde{T}_\lambda(l)(q+\tilde{q}_{+p})_\rho\epsilon^{\sigma\lambda\rho\kappa}\tilde{T}_\nu(p)(q+\tilde{q}_{+p})_\alpha\epsilon^{\mu\nu\alpha\beta}$$

$$= \tilde{\epsilon}_{n\sigma}\epsilon_{m\kappa}\epsilon_{s\mu}\tilde{\epsilon}_{r\beta}\{\tilde{T}_i(l_0,-\mathbf{p})[|\mathbf{q}|+|\mathbf{q}+\mathbf{p}|]\epsilon^{\sigma i0\kappa} - \tilde{T}_0(l_0,-\mathbf{p})(2q_i+p_i)\epsilon^{\sigma i0\kappa}\}$$

$$\times\{\tilde{T}_j(p_0,\mathbf{p})[|\mathbf{q}|+|\mathbf{q}+\mathbf{p}|-p_0]\epsilon^{\mu j0\beta} - 2\tilde{T}_0(p_0,\mathbf{p})q_j\epsilon^{\mu j0\beta}\},$$

$$= \tilde{\epsilon}_{n\sigma}\epsilon_{m\kappa}\epsilon_{s\mu}\tilde{\epsilon}_{r\beta}\{\tilde{T}_i(l_0,\mathbf{p})[(|\mathbf{q}|+|\mathbf{q}+\mathbf{p}|)+l_0]\epsilon^{\sigma i0\kappa} + 2\tilde{T}_0(l_0,\mathbf{p})q_i\epsilon^{\sigma i0\kappa}\}$$

$$\times\{\tilde{T}_j(p_0,\mathbf{p})[|\mathbf{q}|+|\mathbf{q}+\mathbf{p}|-p_0]\epsilon^{\mu j0\beta} - 2\tilde{T}_0(p_0,\mathbf{p})q_j\epsilon^{\mu j0\beta}\},$$

where in getting to the final expression we have used $p_{[\mu}\tilde{T}_{\nu]}(p)=0$, the fact that $\epsilon_{n\mu}(k)$ can be fixed as purely spatial for any momentum k, and the assumed behavior of $T_\mu(x)$ under a parity transformation, Eq. (36). Now perform the appropriate energy integrals made trivial by the presence of delta-distributions in Eq. (A16). The relevant factor from the second term of Eq. (A16) becomes

$$\tilde{\epsilon}_{n\sigma}\epsilon_{m\kappa}\epsilon_{s\mu}\tilde{\epsilon}_{r\beta}\int dl_0\,\delta(l_0+\Delta q)\{\tilde{T}_i(l_0,\mathbf{p})[|\mathbf{q}|+|\mathbf{q}+\mathbf{p}|+l_0]\epsilon^{\sigma i0\kappa}+2\tilde{T}_0(l_0,\mathbf{p})q_i\epsilon^{\sigma i0\kappa}\}$$

$$\times\{\tilde{T}_j(p_0,\mathbf{p})[|\mathbf{q}|+|\mathbf{q}+\mathbf{p}|-p_0]\epsilon^{\mu j0\beta}-2\tilde{T}_0(p_0,\mathbf{p})q_j\epsilon^{\mu j0\beta}\},$$

$$= \tilde{\epsilon}_{n\sigma}\epsilon_{m\kappa}\epsilon_{s\mu}\tilde{\epsilon}_{r\beta}\{\tilde{T}_i(-\Delta q,\mathbf{p})[|\mathbf{q}|+|\mathbf{q}+\mathbf{p}|-\Delta q]\epsilon^{\sigma i0\kappa}+2\tilde{T}_0(-\Delta q,\mathbf{p})q_i\epsilon^{\sigma i0\kappa}\}$$

$$\times\{\tilde{T}_j(p_0,\mathbf{p})[|\mathbf{q}|+|\mathbf{q}+\mathbf{p}|-p_0]\epsilon^{\mu j0\beta}-2\tilde{T}_0(p_0,\mathbf{p})q_j\epsilon^{\mu j0\beta}\},$$

$$= \tilde{\epsilon}_{n\sigma}\epsilon_{m\kappa}\epsilon_{s\mu}\tilde{\epsilon}_{r\beta}\{-2\tilde{T}_i(\Delta q,\mathbf{p})|\mathbf{q}|\epsilon^{\sigma i0\kappa}+2\tilde{T}_0(\Delta q,\mathbf{p})q_i\epsilon^{\sigma i0\kappa}\}\{\tilde{T}_j(p_0,\mathbf{p})[|\mathbf{q}|+|\mathbf{q}+\mathbf{p}|-p_0]\epsilon^{\mu j0\beta}-2\tilde{T}_0(p_0,\mathbf{p})q_j\epsilon^{\mu j0\beta}\},$$

$$= \tilde{\epsilon}_{n\sigma}\epsilon_{m\kappa}\epsilon_{s\mu}\tilde{\epsilon}_{r\beta}\{-2\tilde{T}_\lambda(\Delta q,\mathbf{p})q_\rho\epsilon^{\sigma\lambda\rho\kappa}\}\{\tilde{T}_j(p_0,\mathbf{p})[|\mathbf{q}|+|\mathbf{q}+\mathbf{p}|-p_0]\epsilon^{\mu j0\beta}-2\tilde{T}_0(p_0,\mathbf{p})q_j\epsilon^{\mu j0\beta}\}, \tag{A17}$$

where in arriving at the final expression above we have used the assumed behavior of $T_\mu(x)$ under a time reversal transformation Eq. (37). Now perform the p_0 integral present in the first term of Eq. (A16) and arrive at

$$\tilde{\epsilon}_{n\sigma}\epsilon_{m\kappa}\epsilon_{s\mu}\tilde{\epsilon}_{r\beta}\int dp_0\,\delta(p_0-\Delta q)\{\tilde{T}_i(l_0,\mathbf{p})[|\mathbf{q}|+|\mathbf{q}+\mathbf{p}|+l_0]\epsilon^{\sigma i0\kappa}+2\tilde{T}_0(l_0,\mathbf{p})q_i\epsilon^{\sigma i0\kappa}\}$$

$$\times\{\tilde{T}_j(p_0,\mathbf{p})[|\mathbf{q}|+|\mathbf{q}+\mathbf{p}|-p_0]\epsilon^{\mu j0\beta}-2\tilde{T}_0(p_0,\mathbf{p})q_j\epsilon^{\mu j0\beta}\}$$

$$= \tilde{\epsilon}_{n\sigma}\epsilon_{m\kappa}\epsilon_{s\mu}\tilde{\epsilon}_{r\beta}\{\tilde{T}_i(l_0,\mathbf{p})[|\mathbf{q}|+|\mathbf{q}+\mathbf{p}|+l_0]\epsilon^{\sigma i0\kappa}+2\tilde{T}_0(l_0,\mathbf{p})q_i\epsilon^{\sigma i0\kappa}\}\{2\tilde{T}_\nu(\Delta q,\mathbf{p})q_\alpha\epsilon^{\mu\nu\alpha\beta}\}.$$

Next change the integration variable $l_0\rightarrow -p_0$ remaining in the first term of Eq. (A16) and the factor above becomes

$$\tilde{\epsilon}_{n\sigma}\epsilon_{m\kappa}\epsilon_{s\mu}\tilde{\epsilon}_{r\beta}\{-\tilde{T}_i(p_0,\mathbf{p})[|\mathbf{q}|+|\mathbf{q}+\mathbf{p}|-p_0]\epsilon^{\sigma i0\kappa}+2\tilde{T}_0(p_0,\mathbf{p})q_i\epsilon^{\sigma i0\kappa}\}\{2\tilde{T}_\nu(\Delta q,\mathbf{p})q_\alpha\epsilon^{\mu\nu\alpha\beta}\}, \tag{A18}$$

where we have again used Eq. (37). Following some relabeling of spacetime and polarization indices, the factor (A18) becomes identical to the factor (A17). Thus Eq. (A16) can be expressed as

$$D_{uv} = -(2ig)^2\epsilon_{m\kappa}\epsilon_{s\mu}\int\frac{dp}{2|\mathbf{q}+\mathbf{p}|}\tilde{T}_\lambda(\tilde{p})q_\rho\epsilon^{\sigma\lambda\rho\kappa}\tilde{T}_\nu(p)(2q+\tilde{p})_\alpha\epsilon^{\mu\nu\alpha\beta}\tilde{\epsilon}_{n\sigma}\tilde{\epsilon}_{r\beta}(\delta_{nr}\delta_{us}\rho_{mv}(\mathbf{q})+\delta_{rn}\delta_{vs}\rho_{um}(\mathbf{q})$$

$$-\delta_{us}\delta_{mv}\rho_{rn}(\mathbf{q}+\mathbf{p})-\delta_{um}\delta_{sv}\rho_{nr}(\mathbf{q}+\mathbf{p})), \tag{A19}$$

where we have defined the 4-vector $\tilde{p}=(\Delta q,\mathbf{p})$ and recall that $\Delta q=|\mathbf{q}+\mathbf{p}|-|\mathbf{q}|$. The linear combinations necessary to describe the evolution of the independent polarization degrees of freedom are

$$D_{11}\pm D_{22} = -(2ig)^2\int\frac{dp}{2|\mathbf{q}+\mathbf{p}|}\tilde{T}_\lambda(\tilde{p})q_\rho\epsilon^{\sigma\lambda\rho\kappa}\tilde{T}_\nu(p)(2q+\tilde{p})_\alpha\{(\tilde{\epsilon}_{1\sigma}\tilde{\epsilon}_{1\beta}+\tilde{\epsilon}_{2\sigma}\tilde{\epsilon}_{2\beta})[2\epsilon_{1\kappa}\epsilon_{1\mu}\rho_{11}(\mathbf{q})\pm 2\epsilon_{2\kappa}\epsilon_{2\mu}\rho_{22}(\mathbf{q})$$

$$+(\epsilon_{2\kappa}\epsilon_{1\mu}\pm\epsilon_{1\kappa}\epsilon_{2\mu})[\rho_{12}(\mathbf{q})+\rho_{21}(\mathbf{q})]]-(\epsilon_{1\kappa}\epsilon_{1\mu}\pm\epsilon_{2\kappa}\epsilon_{2\mu})[2\tilde{\epsilon}_{1\beta}\tilde{\epsilon}_{1\sigma}\rho_{11}(\mathbf{q}+\mathbf{p})+2\tilde{\epsilon}_{2\beta}\tilde{\epsilon}_{2\sigma}\rho_{22}(\mathbf{q}+\mathbf{p})$$

$$+(\tilde{\epsilon}_{1\beta}\tilde{\epsilon}_{2\sigma}+\tilde{\epsilon}_{2\beta}\tilde{\epsilon}_{1\sigma})[\rho_{12}(\mathbf{q}+\mathbf{p})+\rho_{21}(\mathbf{q}+\mathbf{p})]]\}, \tag{A20}$$

$$D_{12} \pm D_{21} = -(2ig)^2 \int \frac{dp}{2|\mathbf{q} + \mathbf{p}|} \tilde{T}_\lambda(\tilde{p}) q_\rho \epsilon^{\sigma\lambda\rho\kappa} \tilde{T}_\nu(p)(2q + \tilde{p})_\alpha \{ (\tilde{\epsilon}_{1\sigma}\tilde{\epsilon}_{1\beta} + \tilde{\epsilon}_{2\sigma}\tilde{\epsilon}_{2\beta})[\epsilon_{1\kappa}\epsilon_{2\mu}[\rho_{11}(\mathbf{q}) \pm \rho_{11}(\mathbf{q})]$$

$$+ \epsilon_{2\kappa}\epsilon_{1\mu}[\rho_{22}(\mathbf{q}) \pm \rho_{22}(\mathbf{q})] + (\epsilon_{1\kappa}\epsilon_{1\mu} + \epsilon_{2\kappa}\epsilon_{2\mu})[\rho_{12}(\mathbf{q}) \pm \rho_{21}(\mathbf{q})]] - (\epsilon_{2\kappa}\epsilon_{1\mu} \pm \epsilon_{1\kappa}\epsilon_{2\mu})$$

$$\times [\tilde{\epsilon}_{1\beta}\tilde{\epsilon}_{1\sigma}[\rho_{11}(\mathbf{q} + \mathbf{p}) \pm \rho_{11}(\mathbf{q} + \mathbf{p})] + \tilde{\epsilon}_{2\beta}\tilde{\epsilon}_{2\sigma}[\rho_{22}(\mathbf{q} + \mathbf{p}) \pm \rho_{22}(\mathbf{q} + \mathbf{p})] + (\tilde{\epsilon}_{1\beta}\tilde{\epsilon}_{2\sigma} \pm \tilde{\epsilon}_{1\sigma}\tilde{\epsilon}_{2\beta})$$

$$\times [\rho_{12}(\mathbf{q} + \mathbf{p}) \pm \rho_{21}(\mathbf{q} + \mathbf{p})]]\}. \tag{A21}$$

If we define the quantity

$$(2\pi)^3 \delta^{(3)}(0) \mathcal{B}_1^{\mu\kappa}(q) \equiv \int \frac{dp}{2|\mathbf{q} + \mathbf{p}|} \tilde{T}_\lambda(\tilde{p}) q_\rho \epsilon^{\sigma\lambda\rho\kappa} \tilde{T}_\nu(p)$$
$$\times (2q + \tilde{p})_\alpha \epsilon^{\mu\nu\alpha\beta} \tilde{\epsilon}_{n\sigma} \delta_{nr} \tilde{\epsilon}_{r\beta} \tag{A22}$$

and the integral operator

$$(2\pi)^3 \delta^{(3)}(0) \mathcal{B}_2[q; \rho_{uv}] \equiv \epsilon_{m\kappa} \epsilon_{s\mu} \int \frac{dp}{2|\mathbf{q} + \mathbf{p}|} \tilde{T}_\lambda(\tilde{p}) q_\rho$$
$$\times \epsilon^{\sigma\lambda\rho\kappa} \tilde{T}_\nu(p)(2q + \tilde{p})_\alpha$$
$$\times \epsilon^{\mu\nu\alpha\beta} \tilde{\epsilon}_{n\sigma} \tilde{\epsilon}_{r\beta}$$
$$\times (\delta_{us}\delta_{mv}\rho_{rn}(\mathbf{q} + \mathbf{p})$$
$$+ \delta_{um}\delta_{sv}\rho_{nr}(\mathbf{q} + \mathbf{p})), \tag{A23}$$

Equation (A19) can be expressed in the general form

$$D_{uv} = -(2ig)^2 (2\pi)^3 \delta^{(3)}(0) \{ \epsilon_{m\kappa}(\epsilon_{u\mu}\rho_{mv}(\mathbf{q})$$
$$+ \epsilon_{v\mu}\rho_{um}(\mathbf{q})) \mathcal{B}_1^{\mu\kappa}(q) - \mathcal{B}_2[q; \rho_{uv}(q)]\}. \tag{A24}$$

APPENDIX B: SECOND-ORDER CALCULATION

In this Appendix we explicitly calculate the relevant quantities describing the evolution of the photon density matrix due to processes which are second order in the interaction Hamiltonian Eq. (34). The second-order scattering matrix operator is

$$\hat{S}^{(2)} = -\frac{1}{2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' T\{\hat{H}_{\text{int}}^{(1)}(t)\hat{H}_{\text{int}}^{(1)}(t')\},$$
$$\equiv -i \int_{-\infty}^{\infty} dt \hat{H}_{\text{int}}^{(2)}(t), \tag{B1}$$

where $\hat{H}_{\text{int}}^{(1)}(t)$ is the first-order interaction Hamiltonian, Eq. (A1). We will denote the interaction Hamiltonian operator which has a nonzero overlap with a single photon lying in both the initial and final scattering states as $^2H_{\text{int}}(t)$. Applying Wick's theorem to simplify the time ordered product in Eq. (B1) gives (ignoring vacuum terms)

$$-i\hat{H}^{(2)}(t) = -\frac{1}{2}(2g)^2 \int d^3\mathbf{x} d^4y \, \epsilon^{\mu\nu\alpha\beta} \epsilon^{\rho\sigma\lambda\kappa} T_\nu(x) T_\sigma(y) : \left(\overset{\frown}{\hat{A}_\mu(x)} \partial_\alpha \hat{A}_\beta(x) \overset{\frown}{\hat{A}_\rho(y)} \partial_\lambda \hat{A}_\kappa(y) \right.$$

$$\left. + \overset{\frown}{\hat{A}_\mu(x)} \partial_\alpha \hat{A}_\beta(x) \hat{A}_\rho(y) \partial_\lambda \overset{\frown}{\hat{A}_\kappa(y)} + \hat{A}_\mu(x) \partial_\alpha \overset{\frown}{\hat{A}_\beta(x)} \overset{\frown}{\hat{A}_\rho(y)} \partial_\lambda \hat{A}_\kappa(y) + \hat{A}_\mu(x) \partial_\alpha \overset{\frown}{\hat{A}_\beta(x)} \hat{A}_\rho(y) \partial_\lambda \overset{\frown}{\hat{A}_\kappa(y)} \right) :, \tag{B2}$$

where all partial derivatives are understood as acting solely on the function immediately to the right; the variable being differentiated is in the argument of this function. We denote the contraction of two operators $\hat{A}$ and $\hat{B}$ by

$$\overset{\frown}{\hat{A}\hat{B}}.$$

To simplify Eq. (B2), we use the free-theory photon propagator in the Feynman gauge,

$$D_{\mu\nu}(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{-ig_{\mu\nu}e^{-ik\cdot(x-y)}}{k^2 + i\epsilon}. \tag{B3}$$

(See Appendix C below for a demonstration of gauge invariance, where we explicitly consider a different gauge-fixed photon propagator). In order to deal with the derivative couplings, we interpret the time ordering as T^* ordering; specifically, we require that derivative couplings act outside of the time ordering operation [46]. For convenience, define the operators

$$\hat{A}_\mu^+(x, p) = \hat{a}_s(p)\epsilon_{s\mu}(p)e^{-ip\cdot x}, \tag{B4}$$

$$\hat{A}_\mu^-(x, p) = \hat{a}_s^\dagger(p)\epsilon_{s\mu}^*(p)e^{ip\cdot x}. \tag{B5}$$

Equation (B2) then becomes

$$-i\hat{H}^{(2)}(t) = -\frac{1}{2}(2g)^2 \int d^3\mathbf{x} d^4y \int \frac{d^4k}{(2\pi)^4} \frac{d^3\mathbf{p}_1}{(2\pi)^3 2p_1^0} \frac{d^3\mathbf{p}_2}{(2\pi)^3 2p_2^0} \epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\lambda\kappa} T_\nu(x) T_\sigma(y)\left(\frac{-ie^{-ik\cdot(x-y)}}{k^2 + i\epsilon}\right)$$

$$:[g_{\mu\rho}(-ip_{1\alpha})(A_\beta^+(x, p_1) - A_\beta^-(x, p_1))(-ip_{2\lambda})(A_\kappa^+(y, p_2) - A_\kappa^-(y, p_2)) + (ik_\lambda)g_{\mu\kappa}(-ip_{1\alpha})(A_\beta^+(x, p_1)$$

$$- A_\beta^-(x, p_1))(A_\rho^+(y, p_2) + A_\rho^-(y, p_2)) + (-ik_\alpha)g_{\beta\rho}(A_\mu^+(x, p_1) + A_\mu^-(x, p_1))(-ip_{2\lambda})(A_\kappa^+(y, p_2) - A_\kappa^-(y, p_2))$$

$$+ (-ik_\alpha)(ik_\lambda)g_{\beta\kappa}(A_\mu^+(x, p_1) + A_\mu^-(x, p_1))(A_\rho^+(y, p_2) + A_\rho^-(y, p_2))]:. \tag{B6}$$

Picking out the nonvanishing overlap of $\hat{H}^{(2)}(t)$ on single-photon initial and final scattering states and calling this $^2\hat{H}_{\text{int}}(t)$ gives

$$^2\hat{H}_{\text{int}}(t) = -\frac{i}{2}(2g)^2 \int d^3\mathbf{x} d^4y \int dk d\mathbf{p}_1 d\mathbf{p}_2 \epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\lambda\kappa} T_\nu(x) T_\sigma(y)\left(\frac{-ie^{-ik\cdot(x-y)}}{k^2 + i\epsilon}\right)[g_{\mu\rho}(-ip_{1\alpha})(-ip_{2\lambda})$$

$$\times (-\hat{A}_\kappa^-(y, p_2)\hat{A}_\beta^+(x, p_1) - \hat{A}_\beta^-(x, p_1)\hat{A}_\kappa^+(y, p_2)) + g_{\mu\kappa}(ik_\lambda)(-ip_{1\alpha})(\hat{A}_\rho^-(y, p_2)\hat{A}_\beta^+(x, p_1) - \hat{A}_\beta^-(x, p_1)\hat{A}_\rho^+(y, p_2))$$

$$+ g_{\beta\rho}(-ik_\alpha)(-ip_{2\lambda})(-\hat{A}_\kappa^-(y, p_2)\hat{A}_\mu^+(x, p_1) + \hat{A}_\mu^-(x, p_1)\hat{A}_\kappa^+(y, p_2)) + g_{\beta\kappa}(-ik_\alpha)(ik_\lambda)(\hat{A}_\rho^-(y, p_2)\hat{A}_\mu^+(x, p_1)$$

$$+ \hat{A}_\mu^-(x, p_1)\hat{A}_\rho^+(y, p_2))], \tag{B7}$$

where we have again used the shorthand notation for the integral measures defined in Appendix A After some relabeling of spacetime indices, Eq. (B7) becomes

$$^2\hat{H}_{\text{int}}(t) = -\frac{i}{2}(2g)^2 \int d^3\mathbf{x} d^4y \int dk d\mathbf{p}_1 d\mathbf{p}_2 \epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\lambda\kappa} T_\nu(x) T_\sigma(y)(-ig_{\mu\rho})\left[\left(\frac{e^{-ik\cdot(x-y)}}{k^2 + i\epsilon}\right)(p_{1\alpha}(p_2 - k)_\lambda\right.$$

$$\left. - k_\alpha(p_2 - k)_\lambda)\hat{A}_\kappa^-(y, p_2)\hat{A}_\beta^+(x, p_1) + \left(\frac{e^{-ik\cdot(y-x)}}{k^2 + i\epsilon}\right)(p_{1\alpha}(p_2 - k)_\lambda - k_\alpha(p_2 - k)_\lambda)\hat{A}_\beta^-(x, p_1)\hat{A}_\kappa^+(y, p_2)\right], \tag{B8}$$

using the property $D_{\mu\nu}(x - y) = D_{\mu\nu}(y - x)$. Now plug in the Eqs. (B4) and (B5), express $T_\mu(x)$ in terms of its Fourier transform, and perform the $\int d\mathbf{x}$ and $\int dy$ integrals to get

$$^2\hat{H}_{\text{int}}(t) = -\frac{i(2\pi)^7}{2}(2g)^2 \int dk d\mathbf{p}_1 d\mathbf{p}_2 \frac{\epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\lambda\kappa}\tilde{T}_\nu(l_1)\tilde{T}_\sigma(l_2)}{k^2 + i\epsilon}(-ig_{\mu\rho})(p_{1\alpha}(p_2 - k)_\lambda - k_\alpha(p_2 - k)_\lambda)$$

$$\times [e^{-i((p_1)^0 - (l_1)^0 + k^0)t}\delta^3(\mathbf{p}_1 - l + k)\delta^4(p_2 + l_2 + k)\epsilon_{s\kappa}^*(p_2)\epsilon_{r\beta}(p_1)\hat{a}_s^\dagger(p_2)\hat{a}_r(p_1)$$

$$+ e^{i((p_1)^0 + l_1^0 + k_0)t}\delta^3(\mathbf{p}_1 + l + k)\delta^4(p_2 - l_2 + k)\epsilon_{s\beta}^*(p_1)\epsilon_{r\kappa}(p_2)\hat{a}_s^\dagger(p_1)\hat{a}_r(p_2)]. \tag{B9}$$

Next perform the k integral:

$$^2\hat{H}_{\text{int}}(t) = -\frac{i(2\pi)^3}{2}(2g)^2 \int d\mathbf{p}_1 d\mathbf{p}_2 dl_1 dl_2 \epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\lambda\kappa}\tilde{T}_\nu(l_1)\tilde{T}_\sigma(l_2)(-ig_{\mu\rho})\left[((p_1 + p_2 + l_2)_\alpha(2p_2 + l_2)_\lambda)\right.$$

$$\times e^{-i((p_1)^0 - (l_1)^0 - (p_2)^0 - (l_2)^0)t}\delta^3(\mathbf{p}_1 - l - \mathbf{p}_2 - l)\frac{\epsilon_{s\kappa}^*(p_2)\epsilon_{s\beta}(p_1)\hat{a}_s^\dagger(p_2)\hat{a}_r(p_1)}{(p_2 + l_2)^2 + i\epsilon}$$

$$\left. + ((p_1 + p_2 - l_2)_\alpha(2p_2 - l_2)_\lambda)e^{i((p_1)^0 + l_1^0 - (p_2)^0 + l_2^0)t}\delta^3(\mathbf{p}_1 + l - \mathbf{p}_2 + l)\frac{\epsilon_{s\beta}^*(p_1)\epsilon_{r\kappa}(p_2)\hat{a}_s^\dagger(p_1)\hat{a}_r(p_2)}{(p_2 - l_2)^2 + i\epsilon}\right]. \tag{B10}$$

We are now in the position to compute the commutator of $^2\hat{H}_{\text{int}}(t)$ and $\hat{\mathcal{D}}_{uv}(\mathbf{q})$ necessary for the refractive term of Eq. (27). This is given by

$$[^2\hat{H}_{\mathrm{int}}(t), \hat{\mathcal{D}}_{uv}(\mathbf{q})] = -\frac{i(2\pi)^3}{2}(2g)^2 \int d\mathbf{p}_1 d\mathbf{p}_2 dl_1 dl_2 \epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\lambda\kappa}\tilde{T}_\nu(l_1)\tilde{T}_\sigma(l_2)(-ig_{\mu\rho})e^{i(l_1^0+l_2^0)t}$$

$$\times \left[((p_1 + p_2 + l_2)_\alpha(2p_2 + l_2)_\lambda)e^{-i((p_1)^0-(p_2)^0)t}\delta^3(\mathbf{p}_1 - l - \mathbf{p}_2 - l)\frac{\epsilon^*_{s\kappa}(p_2)\epsilon_{s\beta}(p_1)}{(p_2 + l_2)^2 + i\epsilon}(2\pi)^3 2q^0 \right.$$

$$\times (\delta^3(\mathbf{q} - \mathbf{p}_1)\delta_{ur}\hat{a}^\dagger_s(p_2)\hat{a}_v(q) - \delta^3(\mathbf{q} - \mathbf{p}_2)\delta_{vs}\hat{a}^\dagger_u(q)\hat{a}_r(p_1)) + ((p_1 + p_2 - l_2)_\alpha(2p_2 - l_2)_\lambda)$$

$$\times e^{i((p_1)^0-(p_2)^0)t}\delta^3(\mathbf{p}_1 + l - \mathbf{p}_2 + l)\frac{\epsilon^*_{s\beta}(p_1)\epsilon_{r\kappa}(p_2)}{(p_2 - l_2)^2 + i\epsilon}(2\pi)^3 2q^0(\delta^3(\mathbf{q} - \mathbf{p}_2)\delta_{ur}\hat{a}^\dagger_s(p_1)\hat{a}_v(q)$$

$$\left. - \delta^3(\mathbf{q} - \mathbf{p}_1)\delta_{vs}\hat{a}^\dagger_u(q)\hat{a}_r(p_2)) \right]. \tag{B11}$$

Take the expectation value of Eq. (B11) above to get

$$\langle[^2\hat{H}_{\mathrm{int}}(t), \hat{\mathcal{D}}_{uv}(\mathbf{q})]\rangle = -\frac{i(2\pi)^3}{2}(2g)^2 \int d\mathbf{p}_1 d\mathbf{p}_2 dl_1 dl_2 \epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\lambda\kappa}\tilde{T}_\nu(l_1)\tilde{T}_\sigma(l_2)(-ig_{\mu\rho})e^{i(l_1^0+l_2^0)t}(2\pi)^6(2q^0)^2\delta^3(\mathbf{q} - \mathbf{p}_1)$$

$$\times \delta^3(\mathbf{p}_2 - \mathbf{q})\delta^3(\mathbf{p}_1 + l - \mathbf{p}_2 + l)(\delta_{ur}\rho_{sv}(\mathbf{q}) - \delta_{vs}\rho_{ur}(q))\left[(p_1 + p_2 - l_2)_\alpha(2p_2 - l_2)_\lambda \right.$$

$$\left. \times e^{-i((p_1)^0-(p_2)^0)t}\frac{\epsilon^*_{s\kappa}(p_2)\epsilon_{s\beta}(p_1)}{(p_2 + l_2)^2 + i\epsilon} + (p_1 + p_2 - l_2)_\alpha(2p_2 - l_2)_\lambda e^{i((p_1)^0-(p_2)^0)t}\frac{\epsilon^*_{s\beta}(p_1)\epsilon_{r\kappa}(p_2)}{(p_2 - l_2)^2 + i\epsilon} \right]. \tag{B12}$$

Now perform the $\mathbf{p}_1$ and $\mathbf{p}_2$ integrals, giving

$$\langle[^2\hat{H}_{\mathrm{int}}(t), \hat{\mathcal{D}}_{uv}(\mathbf{q})]\rangle = -\frac{i(2\pi)^3}{2}(2g)^2 \int dl_1 dl_2 \delta^3(l + l)\epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\lambda\kappa}\tilde{T}_\nu(l_1)\tilde{T}_\sigma(l_2)(-ig_{\mu\rho})e^{i(l_1^0+l_2^0)t}(\delta_{ur}\rho_{sv}(\mathbf{q})$$

$$\left. - \delta_{vs}\rho_{ur}(\mathbf{q}))\left[(2q + l_2)_\alpha(2q + l_2)_\lambda\frac{\epsilon^*_{s\kappa}(q)\epsilon_{s\beta}(q)}{(q + l_2)^2 + i\epsilon} + (2q - l_2)_\alpha(2q - l_2)_\lambda\frac{\epsilon^*_{s\beta}(q)\epsilon_{r\kappa}(q)}{(q - l_2)^2 + i\epsilon} \right]. \tag{B13}$$

Focus now on the following factors present in Eq. (B13):

$$\epsilon^{\mu\nu\alpha\beta}\epsilon_\mu{}^{\sigma\lambda\kappa}\tilde{T}_\nu(l_1)\tilde{T}_\sigma(l_2)\epsilon^*_{s\kappa}(q)\epsilon_{r\beta}(q)(2q \pm l_2)_\alpha(2q \pm l_2)_\lambda = 2\epsilon^{\mu\nu\alpha\beta}\epsilon_\mu{}^{\sigma\lambda\kappa}\tilde{T}_\nu(l_1)\tilde{T}_\sigma(l_2)\epsilon^*_{s\kappa}(q)\epsilon_{r\beta}(q)(2q \pm l_2)_\alpha q_\lambda,$$

where we have used $l_{[\lambda}\tilde{T}_{\sigma]}(l) = 0$ in arriving at the expression on the right. Now use the epsilon identity $\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\sigma\lambda\kappa} = -6\delta^{[\nu}_\sigma\delta^\alpha_\lambda\delta^{\beta]}_\kappa$ to perform the contractions above and arrive at

$$-2([\tilde{T}_1 \cdot \tilde{T}_2][(2q \pm l_2) \cdot q][\epsilon_r \cdot \epsilon_s] - [\tilde{T}_2 \cdot \tilde{T}_1][\epsilon_r \cdot q][(2q \pm l_2) \cdot \epsilon_s] + [\tilde{T}_2 \cdot \epsilon_r][(2q \pm l_2) \cdot \epsilon_s][\tilde{T}_1 \cdot q]$$

$$- [\tilde{T}_2 \cdot \epsilon_r][(2q \pm l_2) \cdot q][\epsilon_s \cdot \tilde{T}_1] + [(2q \pm l_2) \cdot \tilde{T}_2][q \cdot \epsilon_r][\tilde{T}_1 \cdot \epsilon_s] - [\tilde{T}_2 \cdot (2q \pm l_2)][\tilde{T}_1 \cdot q][\epsilon_r \cdot \epsilon_s]). \tag{B14}$$

To simplify this, use $\epsilon_r \cdot \epsilon_s = -\delta_{rs}$, which when contracted with $(\delta_{ur}\rho_{sv}(\mathbf{q}) - \delta_{vs}\rho_{ur}(\mathbf{q}))$ in Eq. (B13) vanishes, as well as $\epsilon_r(q) \cdot q = 0$ and $q \cdot q = 0$. The remaining terms from the expression (B14) then become

$$\mp 2[\tilde{T}(l_2^0, l) \cdot \epsilon_r]\epsilon_{s\mu}q_\nu((l_2)^\mu\tilde{T}^\nu(l_1^0, -l) - (l_2)^\nu\tilde{T}^\mu(l_1^0, -l)) = \pm 2[\tilde{T}(l_2^0, l) \cdot \epsilon_r]\epsilon_{si}q_0((l_2)^i\tilde{T}^0(l_1^0, l) + (l_2)^0\tilde{T}^i(l_1^0, l))$$

$$= \pm 2[\tilde{T}(l_2^0, l) \cdot \epsilon_r]\epsilon_{si}\tilde{T}^i(l_1^0, l)q_0(l_1^0 + l_2^0),$$

where we have used Eq. (36) as well as repeated use of $l_{[\lambda}\tilde{T}_{\sigma]}(l) = 0$. Inserting this expression back into Eq. (B13), we have

$$\langle[^2\hat{H}_{\text{int}}(t), \hat{\mathcal{D}}_{uv}(\mathbf{q})]\rangle = -(2\pi)^3(2g)^2(\delta_{ur}\rho_{sv}(\mathbf{q}) - \delta_{vs}\rho_{ur}(\mathbf{q}))\int dl_1 dl_2 \delta^3(\mathbf{l} + \mathbf{l})e^{i(l_1^0 + l_2^0)t}[\epsilon_r \cdot \tilde{T}(l_2^0, \mathbf{l})][\tilde{T}(l_1^0, \mathbf{l}) \cdot \epsilon_s]$$

$$\times [q \cdot (l_1 + l_2)]\left[\frac{1}{(l_2)^2 + 2l_2 \cdot q + i\epsilon} - \frac{1}{(l_2)^2 - 2l_2 \cdot q + i\epsilon}\right]. \tag{B15}$$

We now define a quantity $\mathcal{T}_{ij}$ by

$$(2\pi)^3 \delta^3(0)\mathcal{T}_{ij}(q) = -\int dl_1 dl_2 (2\pi)^3 \delta^3(\mathbf{l} + \mathbf{l})\tilde{T}_j(l_2^0, \mathbf{l})$$

$$\times \tilde{T}_i(l_1^0, \mathbf{l})[q \cdot (l_1 + l_2)]$$

$$\times \left[\frac{1}{(l_2)^2 + 2l_2 \cdot q + i\epsilon}\right.$$

$$\left. - \frac{1}{(l_2)^2 - 2l_2 \cdot q + i\epsilon}\right]. \tag{B16}$$

Equation (B15) can be expressed conveniently in terms of this quantity:

$$i\langle[^2\hat{H}_{\text{int}}(0), \hat{\mathcal{D}}_{uv}(\mathbf{q})]\rangle = i(2\pi)^3 \delta^3(0)(2g)^2(\delta_{ur}\rho_{sv}(\mathbf{q})$$

$$- \delta_{vs}\rho_{ur}(\mathbf{q}))\epsilon_s^\mu(q)\epsilon_r^\nu(q)\mathcal{T}_{\mu\nu}(q), \tag{B17}$$

where we have exploited the fact that $\epsilon_s^i(q)\epsilon_r^j(q)\mathcal{T}_{ij}(q) = \epsilon_s^\mu(q)\epsilon_r^\nu(q)\mathcal{T}_{\mu\nu}(q)$ since $\epsilon(q)$ is purely spatial. Explicitly, the components of $i\langle[^2\hat{H}_{\text{int}}(t), \hat{\mathcal{D}}_{uv}(\mathbf{q})]\rangle$ are given by

$$i\langle[^2\hat{H}_{\text{int}}(t), \hat{\mathcal{D}}_{11}(\mathbf{q})]\rangle = i(2\pi)^3 \delta^3(0)(2g)^2\mathcal{T}_{\mu\nu}(q)$$

$$\times (-\epsilon_1^\mu(q)\epsilon_2^\nu(q)\rho_{12}(\mathbf{q})$$

$$+ \epsilon_2^\mu(q)\epsilon_1^\nu(q)\rho_{21}(\mathbf{q})), \tag{B18}$$

$$i\langle[^2\hat{H}_{\text{int}}(t), \hat{\mathcal{D}}_{12}(\mathbf{q})]\rangle = i(2\pi)^3 \delta^3(0)(2g)^2\mathcal{T}_{\mu\nu}(q)$$

$$\times ([\epsilon_1^\mu(q)\epsilon_1^\nu(q) - \epsilon_2^\mu(q)\epsilon_2^\nu(q)]$$

$$\times \rho_{12}(\mathbf{q}) - \epsilon_2^\mu(q)\epsilon_1^\nu(q)[\rho_{11}(\mathbf{q})$$

$$- \rho_{22}(\mathbf{q})]), \tag{B19}$$

$$i\langle[^2\hat{H}_{\text{int}}(t), \hat{\mathcal{D}}_{21}(\mathbf{q})]\rangle = i(2\pi)^3 \delta^3(0)(2g)^2\mathcal{T}_{\mu\nu}(q)$$

$$\times (-[\epsilon_1^\mu(q)\epsilon_1^\nu(q)$$

$$- \epsilon_2^\nu(q)\epsilon_2^\mu(q)]\rho_{21}(\mathbf{q})$$

$$+ \epsilon_1^\mu(q)\epsilon_2^\nu(q)[\rho_{11}(\mathbf{q}) - \rho_{22}(\mathbf{q})]), \tag{B20}$$

$$i\langle[^2\hat{H}_{\text{int}}(t), \hat{\mathcal{D}}_{22}(\mathbf{q})]\rangle = i(2\pi)^3 \delta^3(0)(2g)^2\mathcal{T}_{\mu\nu}(q)$$

$$\times (\epsilon_1^\mu(q)\epsilon_2^\nu(q)\rho_{12}(\mathbf{q})$$

$$- \epsilon_2^\mu(q)\epsilon_1^\nu(q)\rho_{21}(\mathbf{q})). \tag{B21}$$

APPENDIX C: GAUGE INVARIANCE

Here we verify that the calculation in Appendix B is gauge-invariant by explicitly using a different gauge-fixed photon propagator, namely

$$D_{\mu\nu}(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{-ie^{-ik\cdot(x-y)}}{k^2 + i\epsilon}$$

$$\times \left(g_{\mu\nu} - (1 - \xi)\frac{k_\mu k_\nu}{k^2}\right). \tag{C1}$$

Isolating the contribution to $\langle[^2\hat{H}_{\text{int}}(t), \hat{\mathcal{D}}_{uv}(\mathbf{q})]\rangle$ due to the term linear in $(1 - \xi)$ gives

$$\langle[^2\hat{H}_{\text{int}}(t)_\xi, \hat{\mathcal{D}}_{uv}(\mathbf{q})]\rangle = -\frac{i(2\pi)^3}{2}(2g)^2 \int dl_1 dl_2 \delta^3(\mathbf{l} + \mathbf{l})\epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\lambda\kappa}\tilde{T}_\nu(l_1)\tilde{T}_\sigma(l_2)(i(1 - \xi))e^{i(l_1^0 + l_2^0)t}(\delta_{ur}\rho_{sv}(\mathbf{q})$$

$$- \delta_{vs}\rho_{ur}(\mathbf{q}))\left[\frac{(q + l_2)_\mu(q + l_2)_\rho}{(q + l_2)^2}(q + q + l_2)_\alpha(q + q + l_2)_\lambda\frac{\epsilon_{s\kappa}^*(q)\epsilon_{s\beta}(q)}{(q + l_2)^2 + i\epsilon}\right.$$

$$\left. + \frac{(q - l_2)_\mu(q - l_2)_\rho}{(q - l_2)^2}(q + q - l_2)_\alpha(q + q - l_2)_\lambda\frac{\epsilon_{s\beta}^*(q)\epsilon_{r\kappa}(q)}{(q - l_2)^2 + i\epsilon}\right], \tag{C2}$$

where we have performed all the integrals similar to those in arriving at Eq. (B13). Simplify the above by making repeated use of the antisymmetry of the epsilon tensor:

$$\langle[^2\hat{H}_{\text{int}}(t)_\xi, \hat{\mathcal{D}}_{uv}(\mathbf{q})]\rangle = -\frac{i(2\pi)^3}{2}(2g)^2 \int dl_1 dl_2 \delta^3(\mathbf{l}+\mathbf{l})\epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\lambda\kappa}\tilde{T}_\nu(l_1)\tilde{T}_\sigma(l_2)(i(1-\xi))e^{i(l_1^0+l_2^0)t}(\delta_{ur}\rho_{sv}(\mathbf{q})$$

$$- \delta_{vs}\rho_{ur}(\mathbf{q}))\left[\frac{(q+l_2)_\mu(q+l_2)_\rho}{(q+l_2)^2}q_\alpha q_\lambda\frac{\epsilon^*_{s\kappa}(q)\epsilon_{s\beta}(q)}{(q+l_2)^2+i\epsilon} + \frac{(q-l_2)_\mu(q-l_2)_\rho}{(q-l_2)^2}q_\alpha q_\lambda\frac{\epsilon^*_{s\beta}(q)\epsilon_{r\kappa}(q)}{(q-l_2)^2+i\epsilon}\right],$$

$$= -\frac{i(2\pi)^3}{2}(2g)^2 \int dl_1 dl_2 \delta^3(\mathbf{l}+\mathbf{l})\epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\lambda\kappa}\tilde{T}_\nu(l_1)\tilde{T}_\sigma(l_2)(i(1-\xi))e^{i(l_1^0+l_2^0)t}(\delta_{ur}\rho_{sv}(\mathbf{q})$$

$$- \delta_{vs}\rho_{ur}(\mathbf{q}))\left[\frac{(l_2)_\mu(l_2)_\rho}{(q+l_2)^2}q_\alpha q_\lambda\frac{\epsilon^*_{s\kappa}(q)\epsilon_{s\beta}(q)}{(q+l_2)^2+i\epsilon} + \frac{(l_2)_\mu(l_2)_\rho}{(q-l_2)^2}q_\alpha q_\lambda\frac{\epsilon^*_{s\beta}(q)\epsilon_{r\kappa}(q)}{(q-l_2)^2+i\epsilon}\right],$$

$$= -\frac{i(2\pi)^3}{2}(2g)^2 \int dl_1 dl_2 \delta^3(\mathbf{l}+\mathbf{l})(i(1-\xi))(\delta_{ur}\rho_{sv}(\mathbf{q}) - \delta_{vs}\rho_{ur}(\mathbf{q}))e^{i(l_1^0+l_2^0)t}(\epsilon^{\mu\nu\alpha\beta}(l_2)_\mu\tilde{T}_\nu(l_1)q_\alpha)$$

$$\times (\epsilon^{\rho\sigma\lambda\kappa}(l_2)_\rho\tilde{T}_\sigma(l_2)q_\lambda)\left[\frac{\epsilon^*_{s\kappa}(q)\epsilon_{s\beta}(q)}{(q+l_2)^2(q+l_2)^2+i\epsilon} + \frac{\epsilon^*_{s\beta}(q)\epsilon_{r\kappa}(q)}{(q-l_2)^2(q-l_2)^2+i\epsilon}\right],$$

$$= 0.$$

This expression vanishes because $\epsilon^{\rho\sigma\lambda\kappa}(l_2)_\rho\tilde{T}_\sigma(l_2) = 0$, as we have required in order to arrive at Eq. (B15). Therefore, the final evolution equations for the photon density matrix ρ will indeed be independent of the gauge parameter ξ.

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