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Solving the homogeneous Boltzmann equation with arbitrary scattering kernel

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With applications in astroparticle physics in mind, we generalize a method for the solution of the nonlinear, space-homogeneous Boltzmann equation with an isotropic distribution function to arbitrary matrix elements. The method is based on the expansion of the scattering kernel in terms of two cosines of the "scattering angles." The scattering functions used by previous authors in particle physics for matrix elements in the Fermi approximation are retrieved as lowest order results in this expansion. The method is designed for the unified treatment of reactive mixtures of particles obeying different scattering laws, including the quantum statistical terms for blocking or stimulated emission, in possibly large networks of Boltzmann equations. Although our notation is the relativistic one, as it is used in astroparticle physics, the results can also be applied in the classical case.

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I. INTRODUCTION

Nonequilibrium processes in astroparticle physics, such as big bang nucleosynthesis (BBN), neutrino decoupling and more speculative ones, like baryogenesis through leptogenesis or the freeze-out of hypothetical relic particles [1–7], are usually computed through the solution of the corresponding coupled system of Boltzmann equations [8–12] describing the time evolution of the one-particle distribution functions $f^i(t, k)$. Usually, in cosmology, it is anticipated that the relevant particle distribution functions are in exact kinetic equilibrium and of Maxwell-Boltzmann type. These assumptions, together with others, allow the Boltzmann equation to be linearized and integrated, which leads to coupled sets of chemical rate equations (mostly themselves dubbed Boltzmann equations in this context). This procedure drastically simplifies the numerical computation of the particle abundances, such that even the approximate solution of very large networks of Boltzmann equations, as in the case of BBN, becomes possible. However, in doing so, one loses the spectral information contained in the definition of the distribution functions and other fundamental properties of the Boltzmann equation are neglected as well. It is well known that the solution of the full Boltzmann equations can lead to relevant corrections to the equilibrium results in several cases [13–15]. In the era of precision cosmology the inclusion of such nonequilibrium effects gains in importance. Regarding the use of classical kinetic theory for the description of phenomena in the (very) early Universe there are concerns, originating in the belief that these calculations should be performed in the framework of nonequilibrium quantum field theory. These concerns are supported by recent results, revealing differences between the two approaches for simple toy models, at least in extreme nonequilibrium situations; see e.g. [16–18]. However, it seems natural to attempt to include quantum effects in modified effective kinetic equations. Boltzmann equations will continue to play an important role in cosmology at least at the relatively low energies of neutrino decoupling or nucleosynthesis, where the standard calculations give already quite good results.

In general, a network of Boltzmann equations can be written as

$$L[f^i] = \sum_l C^{il}[f^1, \ldots, f^i, \ldots, f^N], \tag{1}$$

where there is one equation for each of the N participating particle species ($i = 1 \ldots N$) and one collision term C^{il} for each interaction with particles of the same and of other species. L denotes the Liouville operator divided by the relativistic on-shell energy E^i_k of a species i particle,[1] most commonly in Minkowski space-time,

$$L[f^i](k) = \frac{\partial f^i(t, k)}{\partial t}, \tag{2}$$

or in Robertson-Walker space-time,

$$L[f^i](k) = \frac{\partial f^i(t, k)}{\partial t} - Hk\frac{\partial f^i(t, k)}{\partial k}, \tag{3}$$

with Hubble rate $H = \dot{a}/a$. By writing the collision integrals as $C^{il}[f^1, \ldots, f^i, \ldots, f^N]$, we have formally taken the possibility of multiparticle scattering processes into account. Usually only decays, inverse decays and $2 - 2$ scattering processes, $a + b \leftrightarrow A + B$, are considered. For the latter ones the collision integral reads

[1]In our notation L and C^{il} are not Lorentz-invariant, but $E^i_k L[f^i]$ and $E^i_k C^{il}$ are.

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$$C^{al}[f^a f^b f^A f^B](k) = \frac{1}{2E_k^a} \iiint \frac{d^3p}{(2\pi)^3 2E_p^b} \frac{d^3q}{(2\pi)^3 2E_q^A} \frac{d^3r}{(2\pi)^3 2E_r^B} (2\pi)^4 \delta^4(k+p-q-r)|\mathcal{M}|^2 \times [(1-\xi^a f_k^a)(1-\xi^b f_p^b) f_q^A f_r^B - f_k^a f_p^b (1-\xi^A f_q^A)(1-\xi^B f_r^B)], \tag{4}$$

where we have used the shorthand notation $f_k^i = f^i(t, k)$ and ξ^i to specify the quantum statistics of particle species i, i.e. $\xi^i = +1$ for Fermi-Dirac, $\xi^i = -1$ for Bose-Einstein and $\xi^i = 0$ for Maxwell-Boltzmann statistics. $|\mathcal{M}|^2$ denotes the invariant matrix element squared and averaged over initial and final spin states. Note that we take $|\mathcal{M}|^2$ to include possible symmetrization factors of $1/2$ for identical particles in the initial or final state. There is a vast number of different methods for the solution of the Boltzmann equation (mostly applied in different fields of physics), out of which only a few exploit the homogeneity and isotropy as imposed by the cosmological principle. So-called direct integration methods, where the collision term (4) is integrated numerically, seem to be most advantageous because they are characterized by high precision. This is desirable for applications in cosmology since one wants to keep track of only small deviations from equilibrium. The direct numerical solution by means of these deterministic methods is numerically expensive, mainly because of the multiple integrals in the collision terms. Currently, for networks involving a few species, it is feasible only subsequent to the successful reduction of those integrals exploiting the isotropy and homogeneity of the distribution functions.

In the present paper a technique for this reduction of the collision integral is presented which generalizes previous results in high energy and astrophysics [13,19] for matrix elements in the Fermi approximation to (in principle) arbitrary matrix elements, relying on a series expansion of the matrix element. The resulting reduced Boltzmann equation contains only a twofold integral over the magnitudes of the postcollisional momenta. The method is applicable to Boltzmann equations with and without quantum statistical terms and can be used independently of the dispersion relation; i.e. it can be used for massive and massless relativistic particles as well as for nonrelativistic ones. The loss and gain terms can be treated collectively or independently. Thus the method represents an approach to treat reactive mixtures of all kinds of particles with different interactions, in a unified manner.

The outline is as follows: In Sec. II we show how the nine-dimensional collision integral for $2-2$ scattering processes can be reduced to a two-dimensional one, integrating out the energy and momentum conserving δ functions. (Collision integrals for decays and inverse decays can be integrated in the same way.) In doing so, a certain angular integral over the matrix element arises. In Sec. III we establish a simple numerical model for the reduced Boltzmann equation. The integral of Sec. II is solved by expanding the matrix element in terms of the cosines of two "scattering angles"; see Sec. IV. In Sec. V we derive a formula, suitable for numerical integration of the full matrix element, which we employ in the last section to demonstrate the convergence of the series towards the exact result for a simple example. We conclude in Sec. VII.

II. REDUCTION OF THE COLLISION INTEGRAL

Omitting the superscripts denoting the particle species[2] in Eq. (4) we can write the collision integral as

$$C[f](k) = \frac{1}{2E_k} \int (2\pi)^4 \delta(E_k + E_p - E_q - E_r) \times \delta^3(\mathbf{k}+\mathbf{p}-\mathbf{q}-\mathbf{r}) |\mathcal{M}|^2 F[f] \prod_{\mathbf{v}=\mathbf{p},\mathbf{q},\mathbf{r}} \frac{d^3 v}{(2\pi)^3 2E_v}, \tag{5}$$

where we introduced

$$F[f] = (1-\xi^k f_k)(1-\xi^p f_p) f_q f_r - f_k f_p (1-\xi^q f_q)(1-\xi^r f_r). \tag{6}$$

E_v denotes the relativistic energy of the particles "v" on the mass shell, i.e. $E_v = \sqrt{\mathbf{v}^2 + m_v^2}$, with three-momentum $\mathbf{v}$ and mass m_v. We write the three-dimensional δ function as the Fourier transform of unity and switch to spherical coordinates:

$$\delta^3(\mathbf{k}+\mathbf{p}-\mathbf{q}-\mathbf{r}) = \int e^{i\boldsymbol{\lambda}(\mathbf{k}+\mathbf{p}-\mathbf{q}-\mathbf{r})} \frac{d^3\lambda}{(2\pi)^3}.$$

The collision term (5) then becomes

$$C[f](k) = \frac{1}{64\pi^3 E_k} \int \delta(E_k + E_p - E_q - E_r) \times F[f] D(k, p, q, r) \frac{pdp}{E_p} \frac{qdq}{E_q} \frac{rdr}{E_r}. \tag{7}$$

Here we have defined D as

$$D(k,p,q,r) = \frac{pqr}{8\pi^2} \int d\Omega_p \int d\Omega_q \int d\Omega_r \times \delta^3(\mathbf{k}+\mathbf{p}-\mathbf{q}-\mathbf{r})|\mathcal{M}|^2 = \frac{pqr}{64\pi^5} \int \lambda^2 d\lambda \int e^{i\boldsymbol{\lambda}\mathbf{k}} d\Omega_\lambda \int e^{i\boldsymbol{\lambda}\mathbf{p}} d\Omega_p \times \int e^{-i\boldsymbol{\lambda}\mathbf{q}} d\Omega_q \int e^{-i\boldsymbol{\lambda}\mathbf{r}} d\Omega_r |\mathcal{M}|^2. \tag{8}$$

Note that this definition renders $D(k, p, q, r)$ a dimension-

[2]From here on we will always use the momenta k, p, q and r in connection with only one particle species, such that it serves as a label for the species at the same time. We also use the convention $v = |\mathbf{v}|$ if the distinction from the four-momentum is clear from the context.

less quantity. Because of the presence of the δ function we expect that the result is nonzero only if $q + r > |k - p|$ and $k + p > |q - r|$, because the equation $\mathbf{k} + \mathbf{p} = \mathbf{q} + \mathbf{r}$ does not have a solution otherwise, for whatever combination of the solid angles Ω_p, Ω_q and Ω_r with respect to $\mathbf{k}$. Therefore, the result will be proportional to[3]

$$\begin{aligned}\Theta(k, p, q, r) &\equiv \Theta(q + r - |k - p|)\Theta(k + p - |q - r|)\\ &= \Theta(\min(k + p, q + r)\\ &\quad - \max(|k - p|, |q - r|)). \end{aligned} \tag{9}$$

Since this factor is either 0 or 1, we can always multiply the intermediate results by this term without changing the final one.

After computing $D(k, p, q, r)$ we can proceed with the integration of the remaining energy δ function in Eq. (7):

$$\begin{aligned} C[f](k) = \frac{1}{64\pi^3 E_k} \iint \Theta(E_p - m_p)F[f]\\ \times D(k, p, q, r)\frac{qdq}{E_q}\frac{rdr}{E_r}, \end{aligned} \tag{10}$$

where $p = \sqrt{E_p^2 - m_p^2}$ and $E_p = E_q + E_r - E_k$. The Heaviside functions prevent us from integrating over combinations of q and r which are kinematically forbidden. Thus we have reduced the collision integral to a two-dimensional one, suitable for numerical integration. However, all of the work is now hidden in the definition of $D = D(k, p, q, r)$ which is characteristic for the scattering model, i.e. for the matrix element of the underlying theory for the scattering process under consideration. For completeness we present the analogous calculation for $C^{1\leftrightarrow 2}$-like collision integrals in Appendix A.

The computation of D is easily carried out for matrix elements squared with simple angular dependence such as the constant $[|\mathcal{M}|^2 = \text{const}]$, for matrix elements in the Fermi approximation $[|\mathcal{M}|^2 \propto (k\cdot p)(q\cdot r), (k\cdot p)$, including renamings of the momenta therein] and for resonant processes in the narrow width approximation $[|\mathcal{M}|^2 \propto \delta(s - m_X^2)$, where m_X is the mass of the particle in the intermediate state]. However, in particle physics, one encounters matrix elements squared with a more complicated structure, such as products of tree-level s-, t- and u-channel contributions, for which the integrals D are in general unknown.

III. A SIMPLE NUMERICAL MODEL

In this section we establish a simple numerical model to benefit from the reduced form of the collision integral. For simplicity we assume a single particle species undergoing $2 - 2$ scattering processes only. The system (1) then acquires the form

$$L[f] = C[f], \tag{11}$$

with $C[f]$ from Eq. (10) and $L[f]$ from either Eq. (3) or (2).

In case the Liouville operator of the system is of the first kind with the extra term

$$- Hk\frac{\partial f^i(t, k)}{\partial k}, \tag{12}$$

as compared to Eq. (2), which accounts for the expansion of the Universe,[4] we first introduce the transformed variables

$$x = Ma(t), \qquad \tilde{k} = ka(t), \tag{13}$$

with some suitable mass scale M and cosmic scale factor $a(t)$. The Liouville operator in these new coordinates has the simpler form

$$L[f] = Hx\frac{\partial f(x, \tilde{k})}{\partial x}. \tag{14}$$

In what follows we omit the tilde over the transformed momenta and time, but it is important to remember that, in this case, the collision integrals need to be expressed in terms of the transformed variables as well.

Now we divide the physical relevant part of the positive real axis of momenta; i.e. we consider only momenta up to a maximum of $k_{\max}$, into a set of M disjoint (not necessarily equidistant) intervals Δk_i and choose a k_i for each interval with $k_i \in \Delta k_i$ $(i = 1 \ldots M)$.

Then we make the approximation

$$\int_{\Delta k_i} f(t, k)dk \simeq f(t, k_i)\Delta k_i \equiv f_i\Delta k_i. \tag{15}$$

By integrating the left-hand side of (11) over the interval Δk_l we obtain

$$\begin{aligned} L_l &= \int_{\Delta k_l}\frac{\partial f_k}{\partial t}dk \simeq \frac{\partial f_l}{\partial t}\Delta k_l\\ \text{or}\quad L_l &= \int_{\Delta k_l} Hx\frac{\partial f_k}{\partial x}dk \simeq Hx\frac{\partial f_l}{\partial x}\Delta k_l. \end{aligned} \tag{16}$$

For the right-hand side, we find

$$\begin{aligned} C_l = \frac{1}{64\pi^3 E_{k_l}} \sum_{\substack{i,j\\ E_p\geq m_p, p\leq k_{\max}}}^{M} &[(1 - \xi f_l)(1 - \xi f_p)f_i f_j\\ &- f_l f_p(1 - \xi f_i)(1 - \xi f_j)]D(k_l, p, k_i, k_j)\\ &\times \frac{k_i\Delta k_i}{E_{k_i}}\frac{k_j\Delta k_j}{E_{k_j}}, \end{aligned} \tag{17}$$

where $p = \sqrt{(E_{k_i} + E_{k_j} - E_{k_l})^2 - m_p^2}$.

This way, we turned the reduced Boltzmann equation into a coupled set of M ordinary differential equations for the discrete variables f_l:

[3]Throughout we use the Heaviside step function Θ in the half-maximum convention (which will become relevant later).

[4]It is this term which prevents the Maxwell-Jüttner distribution function, in general, from being an exact equilibrium solution for the Boltzmann equation in Robertson-Walker space-time.

$$L_l = C_l \qquad (l = 1 \ldots M). \tag{18}$$

Because of the finite momentum cutoff $k_{\max}$, this method is not conservative; i.e. energy and total particle number are not conserved inherently. In order to make the method energy and particle number conserving, the cutoff has to be chosen high enough.

The number of equations, or grid points M, depends on the specific problem and the required accuracy. In any case Eq. (18) will represent a large system of differential equations. The amount of numerical work for the evaluation of the collision integral is of order $\mathcal{O}(M^3)$.

Note that this straightforward discretization of the Boltzmann equation serves for illustration purposes mainly. The details of possible numerical implications can be more difficult.

D can in principle be computed numerically (see Sec. V) and tabulated once and for all on the grid. For cosmological problems, however, the momenta remain to be scaled according to the time-dependent connection (13), so that D needs to be reevaluated in every time step. (A possible exception is the one of ultrarelativistic particles for which the scale invariance of D can be exploited.) This shows that the entire method depends on analytic expressions for D.

IV. D^{nm} EXPANSION OF THE SCATTERING KERNEL

In this section a method is presented for the computation of D for matrix elements which can be expanded into a convergent series in the cosines of two scattering angles. Subsequent to this expansion of the matrix element, the angle integrals can be carried out analytically for the individual terms. As will be shown, the resulting (angle-integrated) series will be pointwise absolute convergent if the series of the coefficients in the expansion is pointwise absolute convergent. Hence, the truncated series can serve as an approximation of the exact value of D in this case.

A. Expansion of the kernel

The matrix element squared $|\mathcal{M}|^2$ will in general depend on Lorentz-invariant combinations of the four-momenta of the in- and outgoing particles, usually the Mandelstam variables s, t, and u:

$$\begin{aligned} s &= (k+p)^2,\\ t &= (k-q)^2 = m_k^2 + m_q^2 - 2E_kE_q + 2|\mathbf{k}||\mathbf{q}|\cos(\theta_{kq}),\\ u &= (k-r)^2 = m_k^2 + m_r^2 - 2E_kE_r + 2|\mathbf{k}||\mathbf{r}|\cos(\theta_{kr}). \end{aligned} \tag{19}$$

In the following we take t and u as the independent variables and s is expressed by[5]

$$s = \sum_{i=k,p,q,r} m_i^2 - t - u. \tag{20}$$

In order to evaluate Eq. (8) for arbitrary matrix elements we expand $|\mathcal{M}|^2$ in terms of $\cos(\theta_{kq})$ and $\cos(\theta_{kr})$:

$$|\mathcal{M}|^2 = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{nm}(\cos\theta_{kq})^n(\cos\theta_{kr})^m. \tag{21}$$

Note that the coefficients A_{nm} can depend on the magnitudes of the momenta. Upon integration of Eq. (8) we can then write

$$D(k,p,q,r) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{nm}(k,p,q,r)D^{nm}(k,p,q,r), \tag{22}$$

assuming that the series converges for all relevant k, p, q and r (the momenta are still restricted by energy conservation).

In order to give a meaning to Eq. (22) we need to compute the integral

$$\begin{aligned} D^{nm}(k,p,q,r) &= \frac{pqr}{8\pi^2}\int d\Omega_p \int d\Omega_q \int d\Omega_r \\ &\quad \times \delta^3(\mathbf{k}+\mathbf{p}-\mathbf{q}-\mathbf{r})(\cos\theta_{kq})^n(\cos\theta_{kr})^m \\ &= \frac{pqr}{64\pi^5}\int \lambda^2 d\lambda \int e^{i\boldsymbol{\lambda}\mathbf{k}} d\Omega_\lambda \int e^{i\boldsymbol{\lambda}\mathbf{p}} d\Omega_p \\ &\quad \times \int e^{-i\boldsymbol{\lambda}\mathbf{q}}(\cos\theta_{kq})^n d\Omega_q \\ &\quad \times \int e^{-i\boldsymbol{\lambda}\mathbf{r}}(\cos\theta_{kr})^m d\Omega_r. \end{aligned} \tag{23}$$

Because of this definition the D^{nm}'s are dimensionless, scale-invariant functions; i.e. $D^{nm}(\alpha k, \alpha p, \alpha q, \alpha r) = D^{nm}(k,p,q,r)$ for any $\alpha \neq 0$. They are fully generic as they do not depend on the matrix element. From the first line of Eq. (23) we can infer that, for given k, p, q, and r, the lowest order function $D^{0,0}(k,p,q,r)$ represents an upper bound for all higher order functions $D^{nm}(k,p,q,r)$.

Before investigating further the general solution, we compute D^{nm} for this simplest case, corresponding to $|\mathcal{M}|^2 = 1$, for which only the zeroth-order term $D^{0,0}$ is needed.

We can evaluate all solid angle integrals which $|\mathcal{M}|^2$ does not depend on, in Eq. (23), using

$$\int e^{\pm i\boldsymbol{\lambda}\mathbf{p}} d\Omega_p = \frac{4\pi}{\lambda p}\sin(\lambda p). \tag{24}$$

Thus $D^{0,0}$ simplifies to[6]

[5]It can be advantageous to use different variables, in which case the results of this section can be adapted easily.

[6]Without loss of generality we assume $k, q, r > 0$ in the following. The cases $k = 0$, $q = 0$, and $r = 0$ can be understood in the limiting sense.

$$D^{0,0}(k,p,q,r) = \frac{4}{k\pi}\int_0^\infty \sin(\lambda k)\sin(\lambda p)\sin(\lambda q) \times \sin(\lambda r)\lambda^{-2}d\lambda. \quad (25)$$

Using the addition theorems for sine and cosine, the result for this integral is found to be

$$\begin{aligned} D^{0,0}(k,p,q,r) = \frac{1}{4k}(&|k-p+q+r| - |k+p-q-r| \\ &+ |k+p+q-r| + |k-p-q-r| \\ &+ |k+p-q+r| - |k-p-q+r| \\ &- |k-p+q-r| - |k+p+q+r|), \end{aligned} \quad (26)$$

or equivalently

$$\begin{aligned} D^{0,0}(k,p,q,r) = \frac{1}{2k}(&R(k-p+q+r) \\ &- R(k+p-q-r) + R(k+p+q-r) \\ &+ R(k-p-q-r) + R(k+p-q+r) \\ &- R(k-p-q+r) - R(k-p+q-r) \\ &- R(k+p+q+r)), \end{aligned} \quad (27)$$

where we introduced the ramp function:

$$R(x) = x\Theta(x) = \begin{Bmatrix} x & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{Bmatrix}. \quad (28)$$

Now, we multiply Eq. (27) by the term $\Theta(k,p,q,r)$ from Eq. (9), which we can always do, and infer that $(k-p+q+r) \geq 0$, $(k+p-q+r) \geq 0$, $(k+p+q-r) \geq 0$ and $(k-p-q-r) \leq 0$ for all values of k, p, q and r for which the prefactor is nonzero (none of the momenta can be greater than the sum of the other three). Obviously, the term $(k+p+q+r)$ is always positive. Introducing the abbreviations

$$\begin{aligned} c_1 &= k+p-q-r, \\ c_2 &= k-p+q-r, \\ c_3 &= k-p-q+r \quad \text{and} \\ R_1 &= R(c_1), \qquad R_2 = R(c_2), \qquad R_3 = R(c_3) \end{aligned} \quad (29)$$

for the remaining combinations with indefinite sign, we find for Eq. (27) the compact form:

$$D^{0,0}(k,p,q,r) = \frac{1}{2k}\Theta(k,p,q,r)(-R_1 - R_2 - R_3 + 2k). \quad (30)$$

From this it is obvious that $D^{0,0}(k,p,q,r) \leq 1$ everywhere. Since the smallest value, with all of the R_i's being positive, is $\Theta(k,p,q,r)(-k+p+q+r)/(2k) \geq 0$, it is also obvious that $D^{0,0}(k,p,q,r) \geq 0$. Remembering the note from above, we conclude that $|D^{nm}(k,p,q,r)| \leq D^{0,0}(k,p,q,r) \leq 1$ for all k, p, q and r. This property guarantees pointwise absolute convergence of the series Eq. (22) if the series of the coefficients in the expansion (21),

$$\sum_{n=0}^\infty \sum_{m=0}^\infty A_{nm}(k,p,q,r),$$

is pointwise absolute convergent.

In case of massless particles Eq. (30) can be simplified further to (using energy conservation)

$$\begin{aligned} D^{0,0}(k,p,q,r) &= \frac{1}{k}\Theta(k,p,q,r)(k - R(q-k) - R(r-k)) \\ &= \frac{1}{2k}\Theta(k,p,q,r)(q + r - |q-k| - |r-k|). \end{aligned} \quad (31)$$

B. The integrals D^{nm}

We now turn to the computation of D^{nm} for general n and m. In Eq. (23) the Fourier integrals

$$\int e^{\pm i\boldsymbol{\lambda}\mathbf{q}}(\cos\theta_{kq})^n d\Omega_q = \int e^{\pm i\boldsymbol{\lambda}\mathbf{q}}(\hat{\mathbf{k}}\,\hat{\mathbf{q}})^n d\Omega_q \quad (32)$$

on the unit sphere can be expressed as a finite series of spherical Bessel functions of the first kind (see Appendix B for the derivation):

$$\int e^{\pm i\boldsymbol{\lambda}\eta}(\hat{\mathbf{k}}\,\hat{\eta})^n d\Omega_{\hat{\eta}} = 4\pi(\pm i)^n \sum_{\alpha=0}^{\lfloor n/2\rfloor} a_{n,\alpha}\frac{j_{n-\alpha}(\eta\lambda)}{(2\eta\lambda)^\alpha}(\hat{\mathbf{k}}\,\hat{\boldsymbol{\lambda}})^{n-2\alpha}, \quad (33)$$

with numeric coefficients

$$a_{n,\alpha} = \frac{(-1)^\alpha n!}{\alpha!(n-2\alpha)!}. \quad (34)$$

Inserting Eq. (33) into Eq. (23) we find

$$\begin{aligned} D^{nm} = \frac{pqr}{\pi^2}\int d\lambda\lambda^2 &\sum_{\alpha=0}^{\lfloor n/2\rfloor}\sum_{\beta=0}^{\lfloor m/2\rfloor}(-i)^{n+m}a_{n,\alpha}a_{m,\beta}(2q\lambda)^{-\alpha} \\ &\times (2r\lambda)^{-\beta}j_0(p\lambda)j_{n-\alpha}(q\lambda)j_{m-\beta}(r\lambda) \\ &\times \int e^{i\boldsymbol{\lambda}\mathbf{k}}(\cos\theta_{k\lambda})^{n+m-2(\alpha+\beta)}d\Omega_\lambda, \end{aligned} \quad (35)$$

where the inner integral is again of type (33), such that we arrive at

$$\begin{aligned} D^{nm} = \frac{4pqr}{\pi}&\sum_{\alpha=0}^{\lfloor n/2\rfloor}\sum_{\beta=0}^{\lfloor m/2\rfloor}\sum_{l=0}^{\lfloor (n+m/2)-(\alpha+\beta)\rfloor}(-1)^{\alpha+\beta} \\ &\times a_{n,\alpha}a_{m,\beta}a_{n+m-2(\alpha+\beta),l}(2q)^{-\alpha}(2r)^{-\beta}(2k)^{-l} \\ &\times I(\alpha+\beta+l; n+m-2(\alpha+\beta)-l, 0, n \\ &- \alpha, m-\beta; k,p,q,r), \end{aligned} \quad (36)$$

with

$$I(n; l_1, l_2, l_3, l_4; k,p,q,r) = \int_0^\infty \lambda^{2-n}j_{l_1}(k\lambda)j_{l_2}(p\lambda) \times j_{l_3}(q\lambda)j_{l_4}(r\lambda)d\lambda. \quad (37)$$

Unfortunately, the remaining integral over four spherical

Bessel functions is known to represent a mathematical problem itself. Because of the rapidly oscillating integrand it is also difficult to access by numerical methods.

From Rayleigh's formula (B4) it can be inferred that the integrand of Eq. (37) can always be decomposed into products of four sine and cosine functions and an inverse power of λ:

$$\lambda^{-m}\mathrm{trig}_1(k\lambda)\mathrm{trig}_2(p\lambda)\mathrm{trig}_3(q\lambda)\mathrm{trig}_4(r\lambda), \tag{38}$$

where $\mathrm{trig}_i(x\lambda)$ is either $\sin(x\lambda)$ or $\cos(x\lambda)$. However, it has a nonintegrable singularity at $\lambda = 0$ if the number of sines exceeds m. In principle, the problem can be circumvented by performing a Laurent series expansion of the integrand in Eq. (38) and subtracting the divergent part. The finite part can then be computed for all possible combinations of the indices. Since we expect a finite overall result for Eq. (36), the different divergent parts in the sum need to cancel.

Here, we make a different approach which is based on an explicit expression for the integral

$$I(0; l_1, l_2, l_3, l_4; k, p, q, r) = \int_0^\infty \lambda^2 j_{l_1}(k\lambda)j_{l_2}(p\lambda) \times j_{l_3}(q\lambda)j_{l_4}(r\lambda)d\lambda, \tag{39}$$

which is valid, provided that there exists an integer number L which satisfies the conditions

$$|l_1 - l_2| \le L \le l_1 + l_2 \wedge |l_3 - l_4| \le L \le l_3 + l_4, \quad l_1 + l_2 + L \quad \text{and} \quad l_3 + l_4 + L \,\text{even}. \tag{40}$$

In order to bring the integrals $I(\alpha + \beta + l; n + m - 2(\alpha + \beta) - l, 0, n - \alpha, m - \beta; k, p, q, r)$ into the form of Eq. (39), we apply the recurrence relation for spherical Bessel functions [20]:

$$j_n(z) = \frac{z}{2n+1}(j_{n-1}(z) + j_{n+1}(z)). \tag{41}$$

Applying this relation r times with respect to $j_n(x\lambda)$ yields a sequence of spherical Bessel functions of order $n - r$, $n - r + 2 \ldots n + r - 2$, $n + r$ and an overall prefactor $(x\lambda)^r$. Therefore, we apply Eq. (41) l times with respect to $j_{n+m-2(\alpha+\beta)-l}(k\lambda)$, α times with respect to $j_{n-\alpha}(q\lambda)$ and β times with respect to $j_{m-\beta}(r\lambda)$ in Eq. (35). This leads to a set of integrals of type $I(0; l_1, l_2, l_3, l_4; k, p, q, r)$, where $0 \le n + m - 2(\alpha + \beta + l) \le l_1 \le n + m - 2(\alpha + \beta)$, $l_2 = 0$, $0 \le n - 2\alpha \le l_3 \le n$ and $0 \le m - 2\beta \le l_4 \le m$. These integrals can be evaluated according to Ref. [21] if the conditions (40) on the indices are met.

Since different authors found different, relevant expressions for integrals involving three Bessel functions [22–24], we repeat here the derivation for integrals involving four spherical Bessel functions from the former ones.

With help of the closure relation for spherical Bessel functions (B5), integrals of the form (39) can be reduced to integrals of three spherical Bessel functions:

$$\begin{aligned}I(0; l_1, l_2, l_3, l_4; k, p, q, r) &= \frac{2}{\pi}\int_0^\infty d\lambda\lambda^2 \int_0^\infty z^2 j_{l_1}(kz) \\ &\quad\times j_L(\lambda z)j_{l_2}(pz)dz \int_0^\infty z'^2 j_{l_1}(qz') \\ &\quad\times j_L(\lambda z')j_{l_2}(rz')dz' \\ &= \frac{2}{\pi}\int_0^\infty d\lambda\lambda^2 I(l_1, L, l_2; k, \lambda, p) \\ &\quad\times I(l_3, L, l_4; q, \lambda, r),\end{aligned} \tag{42}$$

defining

$$I(l_1, l_2, l_3; k, p, q) = \int_0^\infty \lambda^2 j_{l_1}(k\lambda)j_{l_2}(p\lambda)j_{l_3}(q\lambda)d\lambda. \tag{43}$$

Inserting the expression (B7) for $I(l_1, l_2, l_3; k, p, q)$ found by Mehrem, Londergan, and Macfarlane [21] and performing the integration (after inserting the explicit representation (B8) for the Legendre polynomials) yields

$$\begin{aligned}I(0; l_1, l_2, l_3, l_4; k, p, q, r) &= (-1)^L \frac{\pi i^{l_1 - l_2 + l_3 - l_4}}{8kpqr}\sqrt{(2l_2+1)(2l_4+1)}\Theta(k, p, q, r)\left(\frac{k}{p}\right)^{l_2}\left(\frac{q}{r}\right)^{l_4}\begin{pmatrix} l_1 & l_2 & L \\ 0 & 0 & 0\end{pmatrix}^{-1}\begin{pmatrix} l_3 & l_4 & L \\ 0 & 0 & 0\end{pmatrix}^{-1} \\ &\quad\times \sum_{n=0}^{l_2}\sum_{n'=0}^{l_4}\left(\binom{2l_2}{2n}\binom{2l_4}{2n'}\right)^{1/2}\sum_{l=|l_1-l_2+n|}^{l_1+l_2-n}\sum_{l'=|l_3-l_4+n'|}^{l_3+l_4-n'}(2l+1)(2l'+1)\begin{pmatrix} l_1 & l_2 - n & l \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l_3 & l_4 - n' & l' \\ 0 & 0 & 0\end{pmatrix} \\ &\quad\times \begin{pmatrix} L & n & l \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} L & n' & l' \\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix} l_1 & l_2 & L \\ n & l & l_2 - n\end{Bmatrix}\begin{Bmatrix} l_3 & l_4 & L \\ n' & l' & l_4 - n'\end{Bmatrix}\frac{J(k, p, q, r; n, n', l, l')}{k^n q^{n'}},\end{aligned} \tag{44}$$

where

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\end{pmatrix} \quad \text{and} \quad \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6\end{Bmatrix} \tag{45}$$

denote the Wigner $3j$ and $6j$ symbols, respectively (these are purely numeric factors related to the Clebsch-Gordan coefficients and Racah's W coefficients, respectively [25]) and with

$$J(k,p,q,r;n,n',l,l') = \sum_{s=0}^{[l/2]}\sum_{t=0}^{[l'/2]} A_{l,s}A_{l',t}(2k)^{2s-l}(2q)^{2t-l'} \sum_{\mu=0}^{l-2s}\sum_{\nu=0}^{l'-2t} (l-2s\mu)(l'-2t\nu)(k^2-p^2)^{l-2s-\mu}(q^2-r^2)^{l'-2t-\nu} \times U_{n+n'+2\mu+2\nu+2s+2t-l-l'+1}(k,p,q,r), \quad (46)$$

where we have defined

$$U_\alpha(k,p,q,r) = \frac{\min(k+p,q+r)^\alpha - \max(|p-k|,|r-q|)^\alpha}{\alpha}. \quad (47)$$

The expected prefactor $\Theta(k,p,q,r)$ in Eq. (44) stems from the integration of the Heaviside step function in Eq. (B7).

From Eq. (B7) the result (44) also inherits the restrictions on the indices of the Bessel functions $|l_1 - l_2| \leq L \leq l_1 + l_2 \wedge |l_3 - l_4| \leq L \leq l_3 + l_4$. Note that, in general, Eq. (44) can be evaluated for different values of L and with different mappings of the indices l_1, l_2, l_3 and l_4, leading to different equivalent results.

All sums in Eqs. (36), (44), and (46) are finite, so they can be used to determine the functions D^{nm}. In order to demonstrate the usefulness of this result we apply it explicitly to the cases of $D^{0,0}$ and $D^{2,0}$. Evaluating (36) for $n = m = 0$ we find

$$D^{0,0} = \frac{4pqr}{\pi} I(0;0,0,0,0;k,p,q,r). \quad (48)$$

Applying Eq. (44) leads immediately to

$$D^{0,0} = \frac{\Theta(k,p,q,r)U_1(k,q,p,r)}{2k} = \frac{\Theta(k,p,q,r)(\min(p+r,k+q) - \max(|-q+k|,|-r+p|))}{2k}. \quad (49)$$

Distinguishing the eight cases with $\mathrm{sgn}(c_i) = \pm 1$, this can be shown to be equivalent to Eq. (30).

A more sophisticated example is $D^{2,0}$. Again, from Eq. (36), we find

$$D^{2,0} = \frac{8pqr}{\pi}\left(\frac{1}{2}I(0;2,0,2,0;k,p,q,r) - \frac{I(-1;1,0,2,0;k,p,q,r)}{2k} + \frac{I(-1;0,0,1,0;k,p,q,r)}{2q}\right). \quad (50)$$

We apply the recurrence relation (41) to the second and third terms with respect to $j_1(k\lambda)$ and $j_1(q\lambda)$, respectively. This leads to

$$D^{2,0} = \frac{8pqr}{\pi}\left(\frac{1}{6}I(0;0,0,0,0;k,p,q,r) + \frac{1}{3}I(0;2,0,2,0;k,p,q,r)\right). \quad (51)$$

The terms involving $I(0;0,0,2,0;k,p,q,r)$ did cancel exactly and both of the remaining integrals can be evaluated according to Eq. (44) (with the unique choice of $L = 0$ and $L = 2$ for the first and the second integral, respectively), giving after some algebra:

$$D^{2,0} = \frac{\Theta(k,p,q,r)}{8q^2k^3}((k^2+q^2)^2U_1(k,q,p,r) - 2(k^2+q^2)U_3(k,q,p,r) + U_5(k,q,p,r)). \quad (52)$$

Equation (52) as well as all other functions $D^{nm}(k,p,q,r)$ can be brought into the compact form

$$D^{n,m}(k,p,q,r) = A\frac{\Theta(k,p,q,r)}{k^{n+m+1}q^n r^m}(B_1R_1 + B_2R_2 + B_3R_3 + C). \quad (53)$$

We computed the numeric prefactor A and the momentum-dependent coefficients $B_1(k,p,q,r)$, $B_2(k,p,q,r)$, $B_3(k,p,q,r)$ and $C(k,p,q,r)$ for $n+m \leq 16$. They are listed in Appendix C for D^{nm} with $n+m \leq 5$.[7] The coefficients B_i and C are homogeneous multivariate polynomials of degree $2(n+m)$ and $2(n+m)+1$ in k, p, q, and r. The number of elementary operations, necessary to evaluate D^{nm}, increases with increasing n and m; however, their shape permits considerable optimization, especially when many D^{nm}'s are to be computed. In addition, when dealing with networks of Boltzmann equations, they can be used for all matrix elements in the system.

Figures 1 and 2 show $D^{2,0}(2.0, p = q_i + r - 2.0, q_i, r)$ plotted against r (all momenta are in relative units) for various values of q_i. For $r > k$ the graph of $D^{2,0}$ becomes constant. This can be understood by the observation that $U_\alpha(k, q+r-k, q, r)$ is independent of r for $r > k$. The same relation holds for all other $D^{n,0}$'s (and a corresponding one for $D^{0,m}$). Figures 3 and 4 show similar plots for $D^{4,3}$ which do not have this property. Figure 5 shows a surface plot of $D^{4,3}$ for fixed k, as a function of q and r, $D^{4,3}(2.0, p = q + r - 2.0, q, r)$.

In general the shape of the graphs varies strongly with varying indices n and m. All functions possess a kink at $r = k$, because

[7] They become too lengthy for greater indices to be presented here.

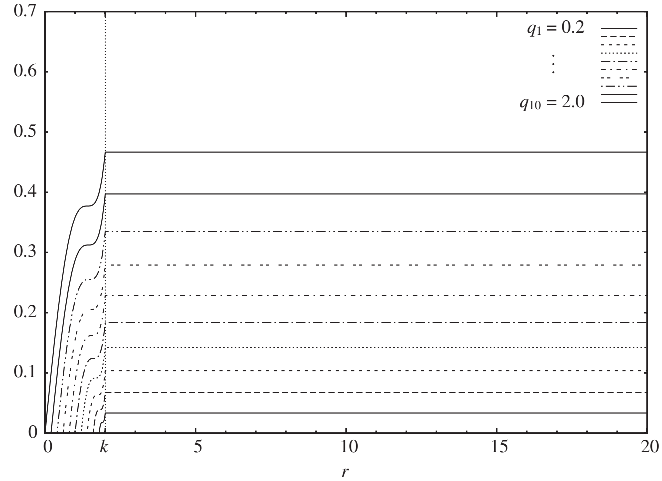

FIG. 1. $D^{2,0}(2.0, p = q_i + r - 2.0, q_i, r)$ for $q_i \leq k$. The flattening for $r > k$ is common to all $D^{n,0}$. All momenta are in relative units.

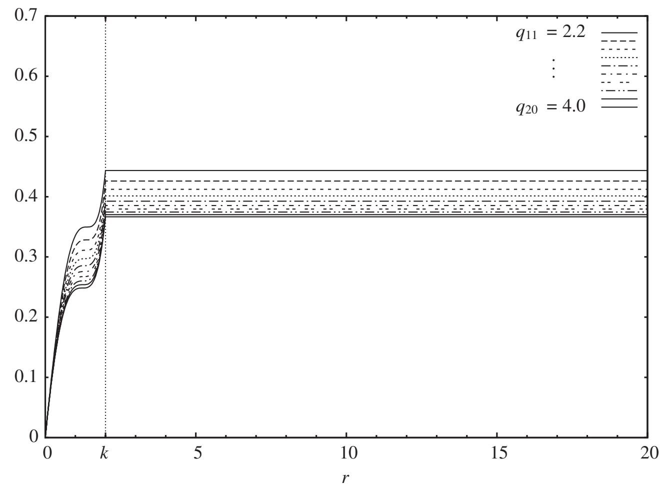

FIG. 2. This plot continues Fig. 1, $D^{2,0}(2.0, p = q_i + r - 2.0, q_i, r)$ for $q_i > k$.

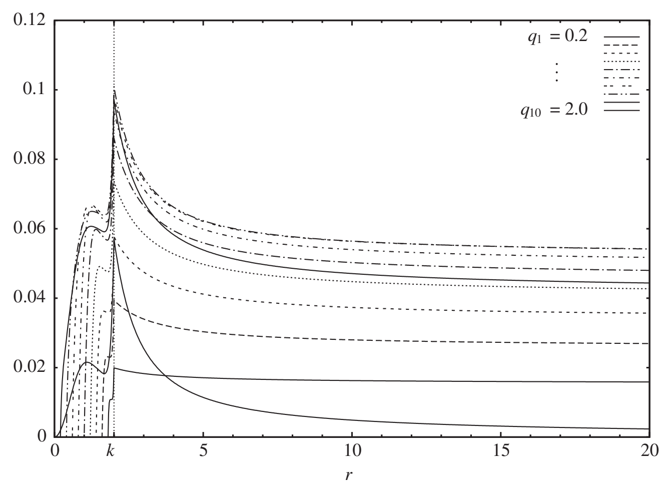

FIG. 3. $D^{4,3}(2.0, p = q_i + r - 2.0, q_i, r)$ for $q_i \leq k$. The kink at $r = k$ is common to all $D^{nm}(k, q + r - k, q, r)$. For $r < k - q$ we have $D^{nm} = 0$. All momenta are in relative units.

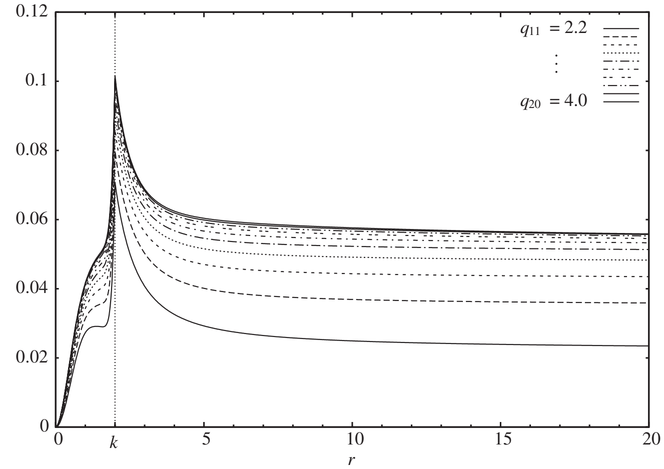

FIG. 4. $D^{4,3}(2.0, p = q_i + r - 2.0, q_i, r)$ for $q_i > k$.

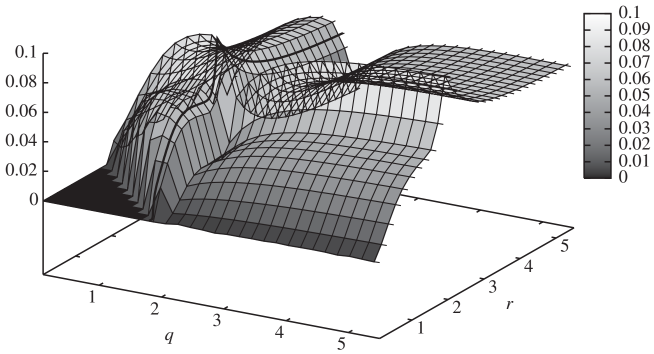

FIG. 5. $D^{4,3}(2.0, p = q + r - 2.0, q, r)$. Figures 3 and 4 correspond to cuts with $q = \text{const}$ (the thick line corresponds to $q_9 = 1.8$).

$$D^{n,m}(k, p = q + r - k, q, r) = A \frac{\Theta(k, p, q, r)}{k^{n+m+1} q^n r^m} (2B_2 R(k - r) + 2B_3 R(k - q) + C), \quad (54)$$

and the properties $D^{nm}(k, p, q, r) = 0$ for $r < k - q$ $[\Theta(k, p, q, r) =^{p=q+r-k} \Theta(q + r - k) = \Theta(p)]$ and $\lim_{r\to 0+} D^{nm}(k, p, q, r) = 0$ for k, p and q are held constant. The latter can be inferred from $|D^{nm}(k, p, q, r)| \leq D^{0,0}(k, p, q, r)$ and $\lim_{r\to 0+} D^{0,0}(k, p, q, r) = 0$, which is obvious from Eq. (49). Similar relations hold for the q dependence (with k and r fixed) and in the case of massive particles.

V. NUMERICAL INTEGRATION OF *D*

In this section we derive a formula for $D(k, p, q, r)$, suitable for numerical integration of arbitrary matrix elements. This method is independent of the one presented in the previous section and can be used to test the accuracy of results obtained by truncating the D^{nm} expansion. Again, we assume that the angular dependence of $|\mathcal{M}|^2$ is given in terms of $\cos(\theta_{kq})$ and $\cos(\theta_{kr})$, i.e. in terms of the momentum transfer t and u. A possible dependence on s can be expressed in terms of t and u exploiting energy and momentum conservation [Eq. (20)].

We orientate the coordinate system such that the z axis points in the direction of $\mathbf{k}$. We can then write

$$\cos(\theta_{\lambda\mathbf{v}}) = \hat{\lambda}\cdot\hat{\mathbf{v}} = \cos\theta_\lambda\cos\theta_\mathbf{v} + \sin\theta_\lambda\sin\theta_\mathbf{v}\cos(\phi_\lambda - \phi_v),$$
$$\cos(\theta_{kq}) = \cos\theta_\mathbf{q},$$
$$\cos(\theta_{kr}) = \cos\theta_\mathbf{r}. \qquad (55)$$

Using Eq. (33) for the Ω_p integration we can write Eq. (8) as

$$D(k,p,q,r) = \frac{pqr}{16\pi^4}\int d^3\lambda e^{i\boldsymbol{\lambda}\mathbf{k}}\frac{\sin(\lambda p)}{\lambda p} \times \int e^{-i\boldsymbol{\lambda}\mathbf{q}}d\Omega_q\int e^{-i\boldsymbol{\lambda}\mathbf{r}}d\Omega_r|\mathcal{M}|^2. \qquad (56)$$

We can then perform the integration over ϕ_q and ϕ_r since $|\mathcal{M}|^2$ does not depend on these angles:

$$\begin{aligned}\int e^{-i\boldsymbol{\lambda}\mathbf{r}}d\Omega_r|\mathcal{M}|^2 &= \int_{-1}^{1} e^{-i\lambda r\cos\theta_{\boldsymbol{\lambda}}\cos\theta_\mathbf{r}}d\cos\theta_\mathbf{r}\int_0^{2\pi} e^{-i\lambda r\sin\theta_{\boldsymbol{\lambda}}\sin\theta_\mathbf{r}\cos(\phi_\lambda-\phi_r)}|\mathcal{M}|^2 d\phi_r \\ &= \int_{-1}^{1} e^{-i\lambda r\cos\theta_{\boldsymbol{\lambda}}\cos\theta_\mathbf{r}}|\mathcal{M}|^2 d\cos\theta_\mathbf{r}\int_0^{\pi} 2\cos(\lambda r\sin\theta_{\boldsymbol{\lambda}}\sin\theta_\mathbf{r}\cos(\phi_r))d\phi_r \\ &= 2\pi\int_{-1}^{1} e^{-i\lambda r\cos\theta_{\boldsymbol{\lambda}}\cos\theta_\mathbf{r}}J_0(\lambda r\sin\theta_{\boldsymbol{\lambda}}\sin\theta_\mathbf{r})|\mathcal{M}|^2 d\cos\theta_\mathbf{r},\end{aligned} \qquad (57)$$

where we have taken into account the fact that the inner integral does not depend on ϕ_λ, since the integration is over 2π and the odd symmetry of the imaginary part of this integral with respect to ϕ_r. The remaining integral was recognized as the integral definition of J_0, the Bessel function of the first kind of order zero; see e.g. [20].

Inserting Eq. (57) into Eq. (56) gives

$$D(k,p,q,r) = \frac{pqr}{4\pi^2}\int_0^\infty \lambda^2\frac{\sin(\lambda p)}{\lambda p}I(k,p,q,r,\lambda)d\lambda, \qquad (58)$$

with

$$I(k,p,q,r,\lambda) = \int e^{i\lambda k\cos\theta_{\boldsymbol{\lambda}}}d\Omega_\lambda\int_{-1}^{1} d\cos\theta_\mathbf{q}e^{-i\lambda q\cos\theta_{\boldsymbol{\lambda}}\cos\theta_\mathbf{q}} \times \int_{-1}^{1} d\cos\theta_\mathbf{r}e^{-i\lambda r\cos\theta_{\boldsymbol{\lambda}}\cos\theta_\mathbf{r}} \times J_0(\lambda q\sin\theta_{\boldsymbol{\lambda}}\sin\theta_\mathbf{q}) \times J_0(\lambda r\sin\theta_{\boldsymbol{\lambda}}\sin\theta_\mathbf{r})|\mathcal{M}|^2. \qquad (59)$$

For the product of the two Bessel functions we may use the relation (B3). Inserting it into Eq. (59) and interchanging the order of integration, we find

$$I(k,p,q,r,\lambda) = 2\pi\int_{-1}^{1}\int_{-1}^{1} d\cos\theta_\mathbf{q}d\cos\theta_\mathbf{r}|\mathcal{M}|^2 \times I'(k,p,q,r,\lambda,\theta_\mathbf{q},\theta_\mathbf{r}), \qquad (60)$$

with

$$I'(k,p,q,r,\lambda,\theta_\mathbf{q},\theta_\mathbf{r}) = \frac{1}{\pi}\int_0^\pi\int_0^\pi e^{i\lambda(k-q\cos\theta_\mathbf{q}-r\cos\theta_\mathbf{r})\cos\theta_{\boldsymbol{\lambda}}} \times J_0\Big(\lambda\sin\theta_{\boldsymbol{\lambda}}\sqrt{(q\sin\theta_\mathbf{q})^2+(r\sin\theta_\mathbf{r})^2-2qr\sin\theta_\mathbf{q}\sin\theta_\mathbf{r}\cos(x)}\Big)\sin\theta_{\boldsymbol{\lambda}}dxd\theta_\lambda. \qquad (61)$$

Again interchanging the order of the integrals and exploiting the odd symmetry of the imaginary part of the exponential, we get

$$I'(k,p,q,r,\lambda,\theta_\mathbf{q},\theta_\mathbf{r}) = \frac{1}{\pi}\int_0^\pi\int_0^\pi \cos(\lambda(k-q\cos\theta_\mathbf{q}-r\cos\theta_\mathbf{r})\cos\theta_{\boldsymbol{\lambda}}) \times J_0\Big(\lambda\sin\theta_{\boldsymbol{\lambda}}\sqrt{(q\sin\theta_\mathbf{q})^2+(r\sin\theta_\mathbf{r})^2-2qr\sin\theta_\mathbf{q}\sin\theta_\mathbf{r}\cos(x)}\Big)\sin\theta_{\boldsymbol{\lambda}}d\theta_\lambda dx. \qquad (62)$$

Now, we apply the integral (B2) to arrive at

$$I'(k,p,q,r,\lambda,\theta_\mathbf{q},\theta_\mathbf{r}) = \frac{2}{\pi}\int_0^\pi\frac{\sin(\lambda\sqrt{f(x)})}{\lambda\sqrt{f(x)}}dx, \qquad (63)$$

with (omitting the dependence on the other variables for brevity)

$$f(\cos x) = (k-q\cos\theta_\mathbf{q}-r\cos\theta_\mathbf{r})^2+(q\sin\theta_\mathbf{q})^2 + (r\sin\theta_\mathbf{r})^2-2qr\sin\theta_\mathbf{q}\sin\theta_\mathbf{r}\cos x, \qquad (64)$$

considering $0\le\theta_q$ and $\theta_r\le\pi$ as parameters. If we interpret x as the polar angle enclosed by $\mathbf{q}$ and $\mathbf{r}$, we can write

$$\begin{aligned} f(\cos x) &= (\mathbf{k}-\mathbf{q})^2 + (\mathbf{k}-\mathbf{r})^2 - (\mathbf{q}-\mathbf{r})^2 - k^2 + q^2 + r^2 \\ &= (\mathbf{k}-\mathbf{q})^2 + (\mathbf{k}-\mathbf{r})^2 - (\mathbf{q}-\mathbf{r})^2 - k^2 + q^2 + r^2 \\ &\quad + (E_k + E_p - E_q - E_r)^2 \\ &= s + t + u - \sum_i m_i^2 + p^2, \end{aligned} \tag{65}$$

under the assumption of energy and momentum conservation.

Since $qr\sin\theta_{\mathbf{q}}\sin\theta_{\mathbf{r}} \geq 0$ for all θ_q and θ_r, the function $f(x)$ takes its minimum for $\cos x = 1$:

$$\begin{aligned} f(1) &= (k - q\cos\theta_{\mathbf{q}} - r\cos\theta_{\mathbf{r}})^2 + (q\sin\theta_{\mathbf{q}} - r\sin\theta_{\mathbf{r}})^2 \\ &\geq 0. \end{aligned} \tag{66}$$

I' is therefore well defined.

Inserting both (60) and (63) into Eq. (58) and again exchanging the order of integration, we find

$$\begin{aligned} D(k,p,q,r) &= \frac{pqr}{\pi^2}\int_{-1}^{1}\int_{-1}^{1} d\cos\theta_{\mathbf{q}}\, d\cos\theta_{\mathbf{r}} |\mathcal{M}|^2 \\ &\quad \times \int_0^{\pi} dx \int_0^{\infty} \lambda^2 \frac{\sin(\lambda p)}{\lambda p}\frac{\sin(\lambda\sqrt{f(\cos x)})}{\lambda\sqrt{f(\cos x)}} d\lambda. \end{aligned} \tag{67}$$

In the rightmost integral we recognize the closure relation for spherical Bessel functions (B5) for $n = 0$.

This leads to

$$D(k,p,q,r) = \frac{qr}{2\pi p}\int_{-1}^{1}\int_{-1}^{1} d\cos\theta_{\mathbf{q}}\, d\cos\theta_{\mathbf{r}} |\mathcal{M}|^2 I'', \tag{68}$$

where we defined

$$\begin{aligned} I'' &= \int_0^{\pi} \delta(p - \sqrt{f(\cos x)})dx = \int_{-1}^{1}\frac{\delta(p-\sqrt{f(y)})}{\sqrt{1-y^2}}dy \\ &= \frac{2p\Theta(F(\theta_q,\theta_r))}{\sqrt{F(\theta_q,\theta_r)}}, \end{aligned} \tag{69}$$

with

$$\begin{aligned} F(\theta_q,\theta_r) &= (p^2 - f(1))(f(-1) - p^2) \\ &= (2qr\sin\theta_{\mathbf{q}}\sin\theta_{\mathbf{r}})^2 - [(k - q\cos\theta_{\mathbf{q}} - r\cos\theta_{\mathbf{r}})^2 \\ &\quad + (q\sin\theta_{\mathbf{q}})^2 + (r\sin\theta_{\mathbf{r}})^2 - p^2]^2. \end{aligned} \tag{70}$$

Because of Eq. (65) the δ function in Eq. (69) ensures energy and momentum conservation.

The final expression for D reads

$$D(k,p,q,r) = \frac{qr}{\pi}\int_A \frac{|\mathcal{M}|^2}{\sqrt{F(\theta_q,\theta_r)}} d\cos\theta_{\mathbf{q}}\, d\cos\theta_{\mathbf{r}}, \tag{71}$$

where the domain of integration A is given by $-1 \leq \cos\theta_q, \cos\theta_r \leq 1$ and $F(\theta_q,\theta_r) > 0$. Note that the term $1/\sqrt{F(\theta_q,\theta_r)}$ is absent in an analogous but erroneous [26] expression in [1].

The expression (71) for D has only a two-dimensional integral and is by far superior to Eq. (56) with respect to numerical integration. The integrand may have singular points at the boundary of A. Hence the routines for numerical integration must be chosen adequately. Usually, for a numerical method, it is sufficient to know $D(k, p = \sqrt{(E_q + E_r - E_k)^2 - m_p^2}, q, r)$ for a finite set of momenta $\{k_i, q_j, r_l\}$ on a grid. Therefore it is possible, in principle, to tabulate D through numerical integration of Eq. (71). As pointed out above, for applications in cosmology, this relation is only of restricted use, since the momenta or, equivalently, the particle masses are scaled in each step of the time evolution, so that the values of $D(k_i, p, q_j, r_l)$ need to be recomputed permanently.[8]

VI. CONVERGENCE OF THE METHOD

To demonstrate the application and convergence of the method described above, we apply it to a (hypothetical) matrix element involving tree-level t- and u-channel contributions. We compare the approximate analytical result according to the truncated series expansion of Sec. IV with the exact numerical result from the previous section.

Without specifying a theory we take the matrix element to be (for simplicity we assume that all in- and outgoing particles are massless)

$$\begin{aligned} |\mathcal{M}|^2 &= \left(\frac{g}{{m_X}^2 - t} + \frac{g}{{m_X}^2 - u}\right)^2 \\ &= \left(\frac{g}{2kq}\frac{1}{(a - \cos\theta_{\mathbf{q}})} + \frac{g}{2kr}\frac{1}{(b - \cos\theta_{\mathbf{r}})}\right)^2, \end{aligned} \tag{72}$$

with $a = 1 + {m_X}^2/(2kq) > 1$, $b = 1 + {m_X}^2/(2kr) > 1$ and some mass m_X of the intermediate state and a coupling g with mass dimension 2.

A Taylor series expansion in $\cos\theta_{\mathbf{q}}$ and $\cos\theta_{\mathbf{r}}$ leads to

$$\begin{aligned} |\mathcal{M}|^2 &= \frac{g^2}{4k^2q^2}\frac{1}{a^2}\left(1 + \frac{2\cos\theta_{\mathbf{q}}}{a} + \cdots\right) + \frac{g^2}{4k^2r^2}\frac{1}{b^2} \\ &\quad \times \left(1 + \frac{2\cos\theta_{\mathbf{r}}}{b} + \cdots\right) + \frac{g^2}{2k^2qr}\frac{1}{ab}\left(1 + \frac{\cos\theta_{\mathbf{q}}}{a}\right. \\ &\quad + \frac{\cos\theta_{\mathbf{r}}}{b} + \frac{(\cos\theta_{\mathbf{q}})^2}{a^2} + \frac{\cos\theta_{\mathbf{r}}\cos\theta_{\mathbf{q}}}{ab} \\ &\quad \left. + \frac{(\cos\theta_{\mathbf{r}})^2}{b^2} + \cdots\right). \end{aligned} \tag{73}$$

[8] Also note that the numerical evaluation of the twofolded integral (71), with the required precision for the solution of the Boltzmann equation, itself can be challenging, depending on the matrix element. The memory consumption for the storage of D is bounded from above by $M^3 \times 8$ bytes for double precision, where $M \sim \mathcal{O}(100)$ is the dimension of the system; see Eq. (17). The true amount of necessary memory (and CPU power) is smaller and depends on the masses, since D can be nonzero only if $\int_A d\cos\theta_{\mathbf{q}}\, d\cos\theta_{\mathbf{r}} > 0$.

The angle-integrated matrix element $D(k, p, q, r)$ is found by substituting every appearance of $(\cos\theta_{\mathbf{q}})^n(\cos\theta_{\mathbf{r}})^m$ by D^{nm}:

$$|\mathcal{M}|^2 = \frac{g^2}{4k^2q^2}\frac{1}{a^2}\left(1 + \frac{2D^{1,0}}{a} + \cdots\right) + \frac{g^2}{4k^2r^2}\frac{1}{b^2} \times \left(1 + \frac{2D^{0,1}}{b} + \cdots\right) + \frac{g^2}{2k^2qr}\frac{1}{ab}\left(1 + \frac{D^{1,0}}{a} + \frac{D^{0,1}}{b} + \frac{D^{2,0}}{a^2} + \frac{D^{1,1}}{ab} + \frac{D^{0,2}}{b^2} + \cdots\right). \tag{74}$$

Setting $\cos\theta_{\mathbf{q}} = 1$ and $\cos\theta_{\mathbf{r}} = 1$ yields the series of coefficients which converges to

$$|\mathcal{M}|^2|_{\cos\theta_{\mathbf{q}}=\cos\theta_{\mathbf{r}}=1} = \left(\frac{g}{2kq}\frac{1}{(a-1)} + \frac{g}{2kr}\frac{1}{(b-1)}\right)^2. \tag{75}$$

According to what has been said below Eq. (30) about the convergence of the series, $|\mathcal{M}|^2|_{\cos\theta_{\mathbf{q}}=\cos\theta_{\mathbf{r}}=1} \times D^{0,0}(k, p, q, r)$ represents an upper bound to $D(k, p, q, r)$. This property can be used as a convergence test for the series in general. On the other hand, in the low energy limit, Eq. (72) can be expanded in terms of inverse powers of ${m_X}^2$:

$$|\mathcal{M}|^2 \simeq \frac{g^2}{m_X^4}\left(4 + 4\frac{t}{{m_X}^2} + 4\frac{u}{{m_X}^2} + 5\frac{t^2}{{m_X}^4} + 2\frac{tu}{{m_X}^4} + 5\frac{u^2}{{m_X}^4}\right). \tag{76}$$

This corresponds to the Fermi approximation and includes only $(\cos\theta_{\mathbf{q}})^n(\cos\theta_{\mathbf{r}})^m$ terms with $n + m \leq 2$ at this order

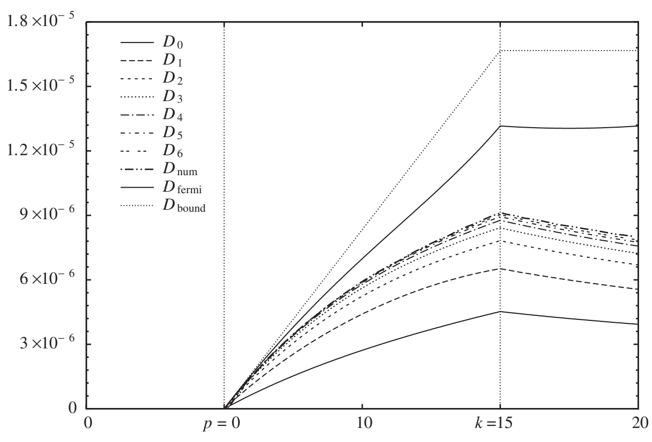

FIG. 6. $D(k, q + r - k, q, r)$ with $m_X = 20.0$, $k = 15.0$, and $q = 10.0$. Exact numerical result $D_{\rm num}$, the result in large m_X limit $D_{\rm Fermi}$, and successive analytic approximations $D_0 \ldots D_6$ (D_{n+m} corresponds to an expansion of $|\mathcal{M}|^2$ up to order $n + m$ in the cosines).

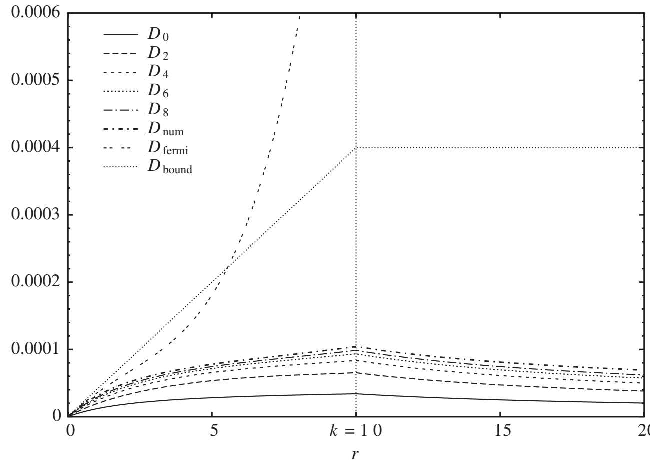

FIG. 7. $D(k, q + r - k, q, r)$ with $m_X = 10.0$, $k = 10.0$, and $q = 15.0$. Exact numerical result $D_{\rm num}$, the result in large m_X limit $D_{\rm Fermi}$, and successive analytic approximations $D_0, D_2 \ldots D_8$.

(i.e. these terms can be integrated as in the previous literature).

Figures 6 and 7 show the graphs of the exact numerical result [Eq. (71)], the graphs for the "Fermi approximation" [Eq. (76)], the graphs of the theoretical upper bound [Eq. (75)] and the graphs corresponding to successive approximations according to Eq. (74) for two sets of parameters. Figures 6 and 7 show that for parameters, for which the expansion in inverse powers of ${m_X}^2$ naturally fails, the approximation by the truncated D^{nm} expansion gives quite good results.

The rate of convergence of successive D_i's towards the exact result depends on the momenta since the coefficients in the expansion (73) are momentum-dependent. In this case it becomes worse for ${m_X}^2/(2kq) \ll 1$ and ${m_X}^2/(2kr) \ll 1$.

VII. CONCLUSION

In state-of-the-art computations in astroparticle physics, usually all species apart from the neutrinos, which experience only weak interactions, are assumed to be in exact kinetic equilibrium, and heterogeneous (at best) networks of Boltzmann and rate equations are solved instead of the full system of kinetic equations. The number of species involved in realistic systems is large and requires a unified treatment of the different particles and interactions.

A method for the solution of the space-homogeneous Boltzmann equation (with isotropic distribution function) for general scattering laws was presented here. Thereafter the straightforward discrete version of the equation was presented. In doing so, it was assumed that the collision integral can be reduced to a twofold one by integration of the angular part. To perform this integration two methods were presented. The first one relies on the expansion of the matrix element in terms of cosines of two scattering angles.

For the separate terms in this expansion the full angular integration was carried out. The functions $D^{0,0}$, $D^{0,1}$ and $D^{0,2}$, corresponding to matrix elements in the Fermi approximation, were used in previous papers to compute nonequilibrium corrections to the neutrino-distribution functions and have been obtained in the lowest order of the expansion. In this case existing implementations might profit from our more compact notation. The second method results in a twofold integral suitable for numerical integration. Although it will be of restricted practical importance in cosmology, it is useful to test the quality of the approximation obtained by the first method.

Our starting point was the relativistic form of the Boltzmann equation, as encountered in astroparticle physics. Nevertheless, the method can be used for the nonrelativistic equation as well. In any case, it allows for the full angular integration of the scattering kernel, reducing the collision integral from effective dimension 5 to dimension 2. The only prerequisite is that the matrix element can be expanded into a series of the scattering angles and that this series converges rapidly enough. The quantum statistical terms for blocking and stimulated emission can be carried along.

In the introduction we mentioned that, at high densities and temperatures, modifications of the Boltzmann equation might become necessary (if these are sufficient at all). One such modification, which has been suggested on various occasions, is the inclusion of higher order scattering processes. Since the representation of the angular integral in terms of spherical Bessel functions (35) and (36) and the integral (44), by means of Eq. (42), can be generalized for higher order processes (in which case more than four Bessel functions appear in the integrals), the method, described above, can in principle be used to reduce the corresponding collision integrals from dimension $3(n-1)$ to $n-2$ (n is the number of particles involved).

The functions D^{nm}, though of very simple structure, can become lengthy for higher orders. Moreover, due to the presence of very different relaxation time scales the system of ordinary differential equations, corresponding to the numerical method presented in Sec. III, tends to behave stiffly. Therefore, in possible implementations, careful optimization for efficiency and stability is necessary.

Let us add that the expansion of the scattering kernel in terms of the cosines of the angles is not a new idea. For example, in Ref. [27] an expansion of the scattering kernel has been combined with a moment method for the nonrelativistic, inhomogeneous Boltzmann equation. The expansion of generic kernels with full integration of the angular part, in the space-homogeneous and isotropic case, seems to be new, however.

ACKNOWLEDGMENTS

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APPENDIX A: REDUCTION OF $C^{1\leftrightarrow 2}$-LIKE COLLISION INTEGRALS

For collision integrals describing the evolution of decaying particles the angle-integrated matrix element can be defined similarly to Eq. (23). In this case the matrix element is a constant and only the zeroth-order integral, corresponding to $|\mathcal{M}|^2 = 1$, has to be computed. The collision integral reads

$$C^{1\leftrightarrow 2}[f](k) = \frac{1}{2E_k}\int (2\pi)^4\delta^4(k-q-r)|\mathcal{M}|^2 F'[f] \times \frac{d^3q}{(2\pi)^3 2E_q}\frac{d^3r}{(2\pi)^3 2E_r}, \tag{A1}$$

with $F'[f] = (1-\xi^k f_k) f_q f_r - f_k(1-\xi^q f_q)(1-\xi^r f_r)$. Performing the same steps as in the main text, we derive

$$C^{1\leftrightarrow 2}[f](k) = \frac{1}{32\pi E_k}\int \Theta(E_q - m_q)F'[f]D'(k,q,r)\frac{rdr}{E_r}, \tag{A2}$$

where $E_q = E_k - E_r$, $q = \sqrt{E_q^2 - m_q^2}$ and we have defined the function D' as

$$D'(k,q,r) = \frac{qr}{8\pi^4}\int \lambda^2 d\lambda \int e^{i\boldsymbol{\lambda}\mathbf{k}}d\Omega_\lambda \int e^{-i\boldsymbol{\lambda}\mathbf{q}}d\Omega_q \times \int e^{-i\boldsymbol{\lambda}\mathbf{r}}d\Omega_r|\mathcal{M}|^2. \tag{A3}$$

For $|\mathcal{M}|^2 = 1$ we find

$$D'(k,q,r) = \frac{8}{\pi k}\int_0^\infty \sin(\lambda k)\sin(\lambda q)\sin(\lambda r)\frac{d\lambda}{\lambda} = \frac{2}{k}\Theta(k - |q-r|)\Theta((q+r)-k). \tag{A4}$$

The expressions for particles created in decays are analogous.

APPENDIX B: RELATIONS INVOLVING BESSEL FUNCTIONS OF INTEGER AND FRACTIONAL ORDER

For the reader's convenience we collect some facts about Bessel functions of the first kind J_n and spherical Bessel functions of the first kind j_n. They are related by

$$j_n(z) = \sqrt{\frac{\pi}{2z}}J_{n+1/2}(z). \tag{B1}$$

In the main text we employ the following integral of Bessel functions from Ref. [28]:

$$\int_0^\pi \cos(\beta\cos\theta)J_0(\alpha\sin\theta)\sin\theta d\theta = 2\frac{\sin(\sqrt{\alpha^2+\beta^2})}{\sqrt{\alpha^2+\beta^2}}, \tag{B2}$$

and for the product of two Bessel functions:

$$J_0(\alpha)J_0(\beta) = \frac{1}{\pi}\int_0^\pi J_0(\sqrt{\alpha^2 + \beta^2 - 2\alpha\beta\cos(x)})dx. \tag{B3}$$

Using Rayleigh's formula, the spherical Bessel functions can be computed iteratively from the sinc function:

$$j_n(z) = z^n\left(-\frac{1}{z}\frac{d}{dz}\right)^n \frac{\sin(z)}{z}. \tag{B4}$$

They satisfy the closure relation [29]

$$\frac{2z^2}{\pi}\int_0^\infty \lambda^2 j_n(\lambda z)j_n(\lambda z')d\lambda = \delta(z - z'). \tag{B5}$$

Several authors have derived expressions or algorithms for the computation of integrals involving products of three spherical Bessel functions [22–24]:

$$I(l_1, l_2, l_3; k, p, q) = \int_0^\infty \lambda^2 j_{l_1}(k\lambda)j_{l_2}(p\lambda)j_{l_3}(q\lambda)d\lambda. \tag{B6}$$

Here we cite the explicit result, found by Mehrem, Londergan, and Macfarlane [21] by relating $I(l_1, l_2, l_3; k, p, q)$ to known integrals of three spherical harmonics:

$$I(l_1, l_2, l_3; k, p, q) = \frac{\pi\Theta(q - |k - p|)\Theta(k + p - q)i^{l_1+l_2-l_3}}{4kpq}\sqrt{2l_3 + 1}(k/q)^{l_3}\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}^{-1}\sum_{n=0}^{l_3}\binom{2l_3}{2n}^{1/2}(p/k)^n \times \sum_{l=|l_1-(l_3-n)|}^{l_1+l_3-n}(2l + 1)\begin{pmatrix} l_1 & l_3 - n & l \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l_2 & n & l \\ 0 & 0 & 0 \end{pmatrix}\begin{Bmatrix} l_1 & l_2 & l_3 \\ n & l_3 - n & l \end{Bmatrix}P_l\left(\frac{k^2 + p^2 - q^2}{2kp}\right), \tag{B7}$$

where P_l denotes the Legendre polynomials with explicit representation [20]

$$P_l(x) = \sum_{i=0}^{\lfloor l/2\rfloor}\frac{(-1)^i(2l - 2i)!}{2^l i!(l - i)!(l - 2i)!}x^{l-2i}. \tag{B8}$$

Wigner's $3j$ symbol is related to the Clebsch-Gordan coefficients by [25]

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \equiv \frac{(-1)^{j_1-j_2-m_3}}{\sqrt{2j_3 + 1}}\langle j_1 m_1 j_2 m_2 | j_3 - m_3\rangle. \tag{B9}$$

Wigner's $3j$ symbols are equal to zero, unless $m_1 + m_2 + m_3 = 0$, $|m_i| \le j_i$ and $|j_1 - j_2| \le j_3 \le j_1 + j_2$.

Wigner's $6j$ symbols are related to Racah's W coefficients by

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = (-1)^{j_1+j_2+j_4+j_5}W(j_1 j_2 j_5 j_4; j_3 j_6). \tag{B10}$$

In the main text, Eq. (33), we use an expansion into spherical Bessel functions [30,31]. We start with the expression (given in [31]):

$$\int_{S^{d-1}} e^{i(\mathbf{x}|\hat\eta)}P^n(\hat\eta)d\sigma_{\hat\eta} = \left(\frac{i}{2}\right)^n\sum_{\alpha=0}^{\lfloor n/2\rfloor}\frac{(-1)^k\Gamma(d/2)}{\alpha!\Gamma(n - \alpha + d/2)} \times \tilde{j}_{n-\alpha+d/2-1}(x)(\Delta^\alpha P^n)(\mathbf{x}), \tag{B11}$$

where $\mathbf{x}, \hat\eta \in \mathbb{R}^d$, with $(\hat\eta|\hat\eta) = 1$, $\tilde{j}_\nu(z) = \Gamma(\nu + 1)\times(\frac{2}{z})^\nu J_\nu(z)$ and the integration is over the $(d - 1)$ sphere S^{d-1}. P^n is a homogeneous polynomial of degree n on $\mathbb{R}^d$ and $(.\,|,)$ denotes the inner product. For $d = 3$, we find

$$\int e^{i\mathbf{x}\hat\eta}P^n(\hat\eta)d\Omega_{\hat\eta} = \left(\frac{i}{2}\right)^n\sum_{\alpha=0}^{\lfloor n/2\rfloor}\frac{(-1)^\alpha}{\alpha!}\left(\frac{2}{x}\right)^{n-\alpha}j_{n-\alpha}(x) \times (\Delta^\alpha P^n)(\mathbf{x}). \tag{B12}$$

Choosing $P^n(\eta) = (\hat{\mathbf{k}}\,\hat\eta)^n$ we get with $(\Delta^\alpha P^n)\times(\mathbf{x}) = n!/(n - 2\alpha)!\cdot(\hat{\mathbf{k}}\mathbf{x})^{n-2\alpha}$:

$$\int e^{i\mathbf{x}\hat\eta}(\hat{\mathbf{k}}\,\hat\eta)^n d\Omega_{\hat\eta} = i^n\sum_{\alpha=0}^{\lfloor n/2\rfloor}\frac{(-1)^\alpha n!}{\alpha!(n - 2\alpha)!}\frac{j_{n-\alpha}(x)}{(2x)^\alpha} \times (\hat{\mathbf{k}}\,\hat{\mathbf{x}})^{n-2\alpha}. \tag{B13}$$

Substituting $\mathbf{x} \to \pm\boldsymbol{\lambda}\eta$ reproduces Eq. (33).

APPENDIX C: THE INTEGRALS D^{nm}

The functions $D^{nm}(k, p, q, r)$ can all be written in the form

$$D^{n,m}(k, p, q, r) = A\frac{\Theta(k, p, q, r)}{k^{n+m+1}q^n r^m}(B_1R_1 + B_2R_2 + B_3R_3 + C). \tag{C1}$$

In the following we list the coefficients A, B_1 and C, which themselves depend on the momenta, for D^{nm} with $n + m \le 5$ and $n \le m$ ($D^{n,m}$ with $n > m$ can be derived from $D^{m,n}$ by interchanging q and r). The expressions B_2 (B_3) are found by substituting in B_1 the term c_1 by c_2 (c_3) and f_i by $f_i^{q\to-q}$ ($f_i^{r\to-r}$) for all i. (R_i and c_i are defined as in the main text.)

$D^{0,0}(k, p, q, r)$:

$$A = 1/2, \qquad C = 2k, \qquad B_1 = -1, \tag{C2}$$

$D^{0,1}(k, p, q, r)$:

$$A = -1/12,$$
$$C = -4k^3,$$
$$B_1 = f_2 c_1 - {c_1}^2 + f_1,$$
$$[f_1 = 6kr, f_2 = 3k - 3r], \tag{C3}$$

$D^{0,2}(k, p, q, r)$:

$$A = \frac{1}{120},$$
$$C = 8k^3(2k^2 + 5r^2),$$
$$B_1 = f_2 c_1 + f_3 {c_1}^2 + f_4 {c_1}^3 - 3{c_1}^4 + f_1,$$
$$[f_1 = -60k^2r^2, f_2 = -60kr(k - r), \qquad f_3 = -20r^2 - 20k^2 + 60kr, f_4 = -15r + 15k], \tag{C4}$$

$D^{0,3}(k, p, q, r)$:

$$A = -\frac{1}{560},$$
$$C = -16k^5(2k^2 + 7r^2),$$
$$B_1 = f_2 c_1 + f_3 {c_1}^2 + f_4 {c_1}^3 + f_5 {c_1}^4 + f_6 {c_1}^5 - 5{c_1}^6 + f_1,$$
$$[f_1 = 280r^3k^3, f_2 = 420k^2r^2(k - r), \qquad f_3 = 140kr(2k - r)(k - 2r),$$
$$f_4 = 70(k - r)(r^2 - 5kr + k^2), \qquad f_5 = -42(2k - r)(k - 2r), f_6 = -35r + 35k], \tag{C5}$$

$D^{0,4}(k, p, q, r)$:

$$A = \frac{1}{10080},$$
$$C = 32k^5(36k^2r^2 + 63r^4 + 8k^4),$$
$$B_1 = f_2 c_1 + f_3 {c_1}^2 + f_4 {c_1}^3 + f_5 {c_1}^4 + f_6 {c_1}^5 + f_7 {c_1}^6 + f_8 {c_1}^7 - 35{c_1}^8 + f_1,$$
$$[f_1 = -5040k^4r^4, f_2 = -10080k^3r^3(k - r), f_3 = -3360k^2r^2(3r^2 + 3k^2 - 7kr),$$
$$f_4 = -2520kr(k - r)(2r^2 - 7kr + 2k^2), f_5 = 10080k^3r - 1008r^4 + 10080kr^3 - 1008k^4 - 19656k^2r^2,$$
$$f_6 = 840(k - r)(2r^2 - 7kr + 2k^2), f_7 = 2520kr - 1080k^2 - 1080r^2, f_8 = -315r + 315k], \tag{C6}$$

$D^{0,5}(k, p, q, r)$:

$$A = -\frac{1}{44352},$$
$$C = -64k^7(44k^2r^2 + 8k^4 + 99r^4),$$
$$B_1 = f_2 c_1 + f_3 {c_1}^2 + f_4 {c_1}^3 + f_5 {c_1}^4 + f_6 {c_1}^5 + f_7 {c_1}^6 + f_8 {c_1}^7 + f_9 {c_1}^8 + f_{10} {c_1}^9 - 63{c_1}^{10} + f_1,$$
$$[f_1 = 22176r^5k^5, f_2 = 55440k^4r^4(k - r), f_3 = 18480k^3r^3(4r^2 - 9kr + 4k^2), f_4 = 55440k^2r^2(k - r)(r^2 - 3kr + k^2),$$
$$f_5 = 11088kr(-14k^3r + 2r^4 - 14kr^3 + 25k^2r^2 + 2k^4), f_6 = 1848(k - r)(2r^4 - 28kr^3 + 67k^2r^2 - 28k^3r + 2k^4),$$
$$f_7 = -99000k^2r^2 - 7920k^4 - 7920r^4 + 55440kr^3 + 55440k^3r, f_8 = 6930(k - r)(r^2 - 3kr + k^2),$$
$$f_9 = -3080k^2 - 3080r^2 + 6930kr, f_{10} = -693r + 693k], \tag{C7}$$

$D^{1,1}(k, p, q, r)$:

$$A = \frac{1}{120},$$
$$C = 4k^3(3k^2 + 5p^2 - 5q^2 - 5r^2),$$
$$B_1 = f_2c_1 + f_3{c_1}^2 + f_4{c_1}^3 - {c_1}^4 + f_1,$$
$$[f_1 = -60k^2qr, f_2 = -30k(-2qr + kq + kr), f_3 = 20kq + 20kr - 20qr - 10k^2, f_4 = -5q - 5r + 5k], \quad \text{(C8)}$$

$D^{1,2}(k, p, q, r)$:

$$A = -\frac{1}{1680},$$
$$C = -16k^5(-14q^2 + 14p^2 + 7r^2 + 4k^2),$$
$$B_1 = f_2c_1 + f_3{c_1}^2 + f_4{c_1}^3 + f_5{c_1}^4 + f_6{c_1}^5 - 3{c_1}^6 + f_1,$$
$$[f_1 = 840qr^2k^3, f_2 = 420k^2r(kr - 3qr + 2kq), f_3 = 140k(2rk^2 + 6r^2q - 3r^2k + 2qk^2 - 8rqk),$$
$$f_4 = -280qk^2 + 630rqk + 70k^3 - 280rk^2 - 210r^2q + 210r^2k, f_5 = 126kr - 126qr - 42r^2 - 56k^2 + 126kq,$$
$$f_6 = 21k - 21r - 21q], \quad \text{(C9)}$$

$D^{1,3}(k, p, q, r)$:

$$A = \frac{1}{10080},$$
$$C = 16k^5(54k^2p^2 - 63q^2r^2 - 54q^2k^2 - 63r^4 + 27k^2r^2 + 63p^2r^2 + 10k^4),$$
$$B_1 = f_2c_1 + f_3{c_1}^2 + f_4{c_1}^3 + f_5{c_1}^4 + f_6{c_1}^5 + f_7{c_1}^6 + f_8{c_1}^7 - 5{c_1}^8 + f_1,$$
$$[f_1 = -5040r^3k^4q, f_2 = -2520k^3r^2(kr - 4qr + 3kq), f_3 = -840k^2r(-4r^2k + 12r^2q + 3rk^2 - 18rqk + 6qk^2),$$
$$f_4 = -1260k(-4r^3q + 11kqr^2 - 7k^2qr + qk^3 - 3k^2r^2 + k^3r + 2kr^3),$$
$$f_5 = 6048kqr^2 + 1764k^3r + 1008kr^3 + 1764qk^3 - 1008r^3q - 252k^4 - 6804k^2qr - 2772k^2r^2,$$
$$f_6 = 1008r^2k + 2520rqk - 1008r^2q - 1134qk^2 - 1134rk^2 - 168r^3 + 294k^3,$$
$$f_7 = 360kr - 360qr - 144r^2 + 360kq - 162k^2, f_8 = 45k - 45q - 45r], \quad \text{(C10)}$$

$D^{1,4}(k, p, q, r)$:

$$A = -\frac{1}{221760},$$
$$C = -64k^7(99r^4 + 88k^2r^2 + 396p^2r^2 - 396q^2r^2 + 176k^2p^2 - 176q^2k^2 + 24k^4),$$
$$B_1 = f_2c_1 + f_3{c_1}^2 + f_4{c_1}^3 + f_5{c_1}^4 + f_6{c_1}^5 + f_7{c_1}^6 + f_8{c_1}^7 + f_9{c_1}^8 + f_{10}{c_1}^9 - 35{c_1}^{10} + f_1,$$
$$[f_1 = 110880r^4qk^5, f_2 = 55440k^4r^3(-5qr + kr + 4kq), f_3 = 18480k^3r^2(20r^2q - 5r^2k - 32rqk + 4rk^2 + 12qk^2),$$
$$f_4 = 18480k^2r(5kr^3 - 15r^3q - 8k^2r^2 + 41kqr^2 - 30k^2qr + 3k^3r + 6qk^3),$$
$$f_5 = 3696k(-15r^4k + 30r^4q - 66k^3qr + 6qk^4 - 141kr^3q + 6k^4r - 30k^3r^2 + 171k^2qr^2 + 41k^2r^3),$$
$$f_6 = 18480r^4k - 40656k^4r + 240240k^3qr + 105336k^3r^2 + 184800kr^3q - 18480r^4q - 86856k^2r^3 + 3696k^5 - 40656qk^4 - 382536k^2qr^2,$$
$$f_7 = -2640r^4 - 126720k^2qr - 26400r^3q + 118800kqr^2 - 5808k^4 + 34320k^3r + 34320qk^3 + 26400kr^3 - 54648k^2r^2,$$
$$f_8 = 14850r^2k - 14850r^2q - 15840rk^2 + 34650rqk - 15840qk^2 + 4290k^3 - 3300r^3,$$
$$f_9 = -1760k^2 + 3850kq - 1650r^2 - 3850qr + 3850kr, f_{10} = -385r - 385q + 385k], \quad \text{(C11)}$$

$D^{2,2}(k, p, q, r)$:

$$\begin{aligned}
A &= \frac{1}{3360},\\
C &= 16k^5(28q^2r^2 + 7r^4 + 22k^2p^2 + 7p^4 + 2q^2k^2 + 3k^4 - 14p^2r^2 - 14p^2q^2 + 2k^2r^2 + 7q^4),\\
B_1 &= f_2c_1 + f_3{c_1}^2 + f_4{c_1}^3 + f_5{c_1}^4 + f_6{c_1}^5 + f_7{c_1}^6 + f_8{c_1}^7 - {c_1}^8 + f_1,\\
[f_1 &= -1680q^2k^4r^2, f_2 = -1680k^3qr(-2qr + kq + kr), f_3 = -560k^2(kr - 3qr + kq)(-2qr + kq + kr),\\
f_4 &= -140k(12qr - 5kr - 5kq + 2k^2)(-qr + kr + kq),\\
f_5 &= -336q^2r^2 + 1008kqr^2 + 1008kq^2r - 476q^2k^2 + 336k^3r - 1344k^2qr + 336qk^3 - 476k^2r^2 - 56k^4,\\
f_6 &= 56(-q - r + k)(3qr - 3kq + k^2 - 3kr), f_7 = -72qr - 24q^2 - 24r^2 - 32k^2 + 72kq + 72kr,\\
f_8 &= -9q - 9r + 9k],
\end{aligned} \tag{C12}$$

$D^{2,3}(k, p, q, r)$:

$$\begin{aligned}
A &= -\frac{1}{221760},\\
C &= -32k^7(297p^4 + 198p^2r^2 - 22q^2k^2 + 297q^4 + 66k^2r^2 + 462k^2p^2 - 495r^4 + 396q^2r^2 + 41k^4 - 594p^2q^2),\\
B_1 &= f_2c_1 + f_3{c_1}^2 + f_4{c_1}^3 + f_5{c_1}^4 + f_6{c_1}^5 + f_7{c_1}^6 + f_8{c_1}^7 + f_9{c_1}^8 + f_{10}{c_1}^9 - 15{c_1}^{10} + f_1,\\
[f_1 &= 110880q^2r^3k^5, f_2 = 55440k^4qr^2(-5qr + 3kq + 2kr),\\
f_3 &= 18480k^3r(20q^2r^2 + 6k^2qr + 6q^2k^2 - 21kq^2r - 12kqr^2 + 2k^2r^2),\\
f_4 &= 27720k^2(2k^3qr + 17kq^2r^2 - 8k^2qr^2 - 10q^2r^3 + q^2k^3 + k^3r^2 + 9kr^3q - 2k^2r^3 - 8k^2q^2r),\\
f_5 &= 5544k(9k^2r^3 - 8k^3r^2 + 20q^2r^3 + 45k^2qr^2 + 2k^4r - 57kq^2r^2 - 18k^3qr - 29kr^3q + 2qk^4 - 8q^2k^3 + 41k^2q^2r),\\
f_6 &= 110880kq^2r^2 - 152460k^2qr^2 + 99792k^3qr + 37884q^2k^3 - 18480q^2r^3 - 130284k^2q^2r - 16632k^4r + 41580k^3r^2\\
&\quad - 26796k^2r^3 + 55440kr^3q + 1848k^5 - 16632qk^4,\\
f_7 &= -21780k^2r^2 - 2376k^4 - 7920r^3q + 14256qk^3 - 53856k^2qr + 39600kq^2r - 18612q^2k^2 + 47520kqr^2 + 7920kr^3\\
&\quad - 15840q^2r^2 + 14256k^3r,\\
f_8 &= 198(-q - r + k)(-25kq + 25qr + 9k^2 - 25kr + 5r^2),\\
f_9 &= -660r^2 - 748k^2 - 550q^2 + 1650kr + 1650kq - 1650qr, f_{10} = -165q + 165k - 165r].
\end{aligned} \tag{C13}$$

The D^{nm}'s do not possess any singularities inside the domain of integration, which is an almost essential feature for the numerical solution of the Boltzmann equation. Note, however, that the expressions given here are optimized neither for numerical efficiency nor for stability. Not all of the terms B_i need to be computed in every step since $R_i = R(c_i) = 0$ for $c_i \leq 0$, and quantities such as powers of the momenta, which appear several times, need to be computed only once for all D^{nm}'s. In an implementation according to the model presented in Sec. III the D^{nm}'s need to be computed only once for all matrix elements in the system (with the possible exception of C and the c_i terms including p, which is determined by energy conservation).

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