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Perturbations of spacetime around a stationary rotating cosmic string

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We consider the metric perturbations around a stationary rotating Nambu-Goto string in Minkowski spacetime. By solving the linearized Einstein equations, we study the effects of azimuthal frame-dragging around the rotation axis and linear frame-dragging along the rotation axis, the Newtonian logarithmic potential, and the angular deficit around the string as the potential mode. We also investigate gravitational waves propagating off the string and propagating along the string, and show that the stationary rotating string emits gravitational waves toward the directions specified by discrete angles from the rotation axis. Waveforms, polarizations, and amplitudes which depend on the direction are shown explicitly.

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I. INTRODUCTION

The phase transition of vacuum in the early universe is one of the most important topics of cosmology and elementary particle physics. It is well known that topological defects are necessarily created due to the spontaneous symmetry breaking of vacuum states [1] (see also [2–4]). Among the several types of topological defects, cosmic strings are possible to survive until the present stage of the Universe and to be observed by the gravitational effects. Alternatively, it is pointed out that fundamental strings and/or *D*-strings can play a role of cosmic strings [5–9]. There is no doubt that detection of cosmic strings in the present stage of the Universe is important and challenging work.

The gravitational waves from cosmic strings is one of the targets of ongoing experiments for searching gravitational waves due to recent technological advances, e.g., LIGO, LISA, VIRGO, TAMA300, GEO600, and so on [10–14], and also theoretical research has been established. For example, there are many works on the gravitational waves produced by oscillating loop cosmic strings [15,16], by an infinitely long string with a helicoidal standing wave [17], and by colliding wiggles on a straight string [18,19]. Damour and Vilenkin [20,21] discussed the gravitational wave bursts from cusps of the cosmic string.

A conical spacetime generated around a straight string makes undistorted double images of a distant source. The gravitational lensing caused by the cosmic strings is studied extensively [22]. Recently, a variety of gravitational lensing, weak lensing [23], lensing by string loops [24], and lensing by strings with small-scale structure, [25] was studied.

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It is known that reconnection probability for gauge theory strings is essentially 1 [26]. Such the strings evolve in a scale invariant way (see [3] and references therein). In contrast, regarding the cosmic strings in the framework of the superstring theory, the reconnection probability is suppressed sufficiently <1[6–9]. Evolution of such strings may differ from that of gauge strings. If the strings are practically stable, we could expect that they survive finally in the stationary states in the present stage of the Universe.

Starting from the pioneering work by Burden and Tassie [27], there are many works on the stationary rotating strings [28,29]. In our previous study [30], we reformulate the stationary rotating strings as an example of the cohomogeneity-one strings [31,32]. Because of the geometrical symmetry of the strings, it is easy to treat them as gravitational sources in the frame work of general relativity. In this paper, we investigate the gravitational fields around a stationary rotating string by solving the linearized Einstein equations toward detection of the strings in the Universe. The Newtonian logarithmic potential and angular deficit are obtained as the potential mode. Furthermore, two effects of frame-dragging are shown: azimuthal dragging around the rotation axis and linear dragging along the rotation axis. We also study the gravitational waves propagating off the strings and propagating along the strings (travelling waves). Characteristic properties of waveforms, polarization, and directions of emission are also discussed.

This paper is organized as follows. In Sec. II, we briefly review the stationary rotating strings following Ref. [30]. In Sec. III, we formulate linear perturbations of the metric around a stationary rotating string. We obtain solutions to the linearized Einstein equations explicitly then discuss a potential mode in Sec. IV, and the gravitational wave modes in Sec. V. The traveling wave modes are discussed in Sec. VI. Finally, we summarize in Sec. VII. We use the sign convention $-+++$ for the metric, and units in which $c = G = 1$.

II. SOLUTIONS OF STATIONARY ROTATING STRINGS

A. Stationary rotating Nambu-Goto strings in Minkowski spacetime

We consider cosmic strings which are described by the Nambu-Goto action,

$$S_{\rm NG} = -\mu \int_\Sigma d^2\zeta \sqrt{-\gamma}, \tag{2.1}$$

where Σ is a timelike two-dimensional world surface embedded in a target spacetime $\mathcal{M}$ with the metric $g_{\mu\nu}$, $\zeta^a(\zeta^0 = \tau, \zeta^1 = \sigma)$ are coordinates on Σ, γ is the determinant of the induced metric γ_{ab} on Σ, and a constant μ denotes the string tension. Varying the action (2.1) by the coordinates of $\mathcal{M}$, $x^\mu(\mu = 0, 1, 2, 3)$, we obtain the Nambu-Goto equations:

$$\frac{1}{\sqrt{-\gamma}} \partial_a(\sqrt{-\gamma}\gamma^{ab}\partial_b x^\mu) + \Gamma^\mu_{\nu\lambda}\gamma^{ab}\partial_a x^\nu \partial_b x^\lambda = 0, \tag{2.2}$$

where $\Gamma^\mu_{\nu\lambda}$ is the Christoffel symbol associated with $g_{\mu\nu}$.

When the world surface of a string Σ is tangent to a Killing vector field in a target spacetime $\mathcal{M}$, i.e., cohomogeneity-one string, the Nambu-Goto equation (2.2) can be reduced to a geodesic equation in an appropriate three-dimensional metric [28,30,31]. Here, we concentrate on stationary rotating strings, which belong to a class of the cohomogeneity-one strings. We briefly review the solutions of stationary rotating strings in Minkowski spacetime according to [30].

In Minkowski spacetime with the metric by the cylindrical coordinate system,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + d\rho^2 + \rho^2 d\varphi^2 + dz^2, \tag{2.3}$$

the Killing vector field ξ which describes the stationary rotation around the z axis with a constant angular velocity Ω is

$$\xi = \partial_t + \Omega\partial_\varphi. \tag{2.4}$$

We consider a world surface Σ of a stationary rotating string which is tangent to ξ. The solutions are characterized by two dimensionless parameters l and q, and explicit forms are given by

$$\begin{aligned} t(\tau) &= \tau, \\ \rho(\sigma)^2 &= \tfrac{1}{2}\{(\rho^2_{\rm max} + \rho^2_{\rm min}) - (\rho^2_{\rm max} - \rho^2_{\rm min})\cos(2\Omega\sigma)\}, \\ \varphi(\tau, \sigma) &= \Omega\tau + \bar\varphi(\sigma), \qquad z(\sigma) = q\sigma, \end{aligned} \tag{2.5}$$

where $\bar\varphi(\sigma)$ is implicitly given by

$$\frac{2l}{\Omega^2}\tan(\bar\varphi(\sigma) - \varphi_0 + l|\Omega|\sigma) = (\rho^2_{\rm max} + \rho^2_{\rm min})\tan\left(|\Omega|\sigma + \frac{\pi}{4}\right) - (\rho^2_{\rm max} - \rho^2_{\rm min}), \tag{2.6}$$

and $\rho_{\rm min}$, $\rho_{\rm max}$ are defined by

$$\begin{aligned} \rho^2_{\rm min} &= \frac{1}{2\Omega^2}(1 + l^2 - q^2 \\ &\quad - \sqrt{(1 + l + q)(1 + l - q)(1 - l + q)(1 - l - q)}), \\ \rho^2_{\rm max} &= \frac{1}{2\Omega^2}(1 + l^2 - q^2 \\ &\quad + \sqrt{(1 + l + q)(1 + l - q)(1 - l + q)(1 - l - q)}). \end{aligned} \tag{2.7}$$

The constant φ_0 has been fixed for convenience as

$$\tan\varphi_0 = -\frac{\Omega^2\rho^2_{\rm min}}{l}, \tag{2.8}$$

in order that $\bar\varphi = 0$ when $\sigma = 0$.

The range of l and q are limited for the stationary rotating strings as

$$|l| + |q| \le 1. \tag{2.9}$$

We do not consider the case $q = 0$ in which the Killing vector ξ becomes null at the end points of the stationary string. Changes of sign of parameters l, q, and Ω can be interpreted as reflection of the space and time. Then, we consider, hereafter, the case

$$l \ge 0, \qquad q > 0, \qquad \Omega > 0, \quad \text{and} \quad l + q \le 1. \tag{2.10}$$

In the stationary rotating string solutions (2.5), (2.6), (2.7), and (2.8), we use the parameters τ and σ which respect the Killing vector ξ, i.e., $\xi = \frac{\partial}{\partial\tau}$. In contrast, using the conformal flat gauge, which is normally used, Burden gave a clear expression for the solutions [33].

We show here typical shapes of stationary rotating strings. First we consider the case $l + q = 1$ $(q \neq 0)$. The solutions are given by

$$\rho = \frac{\sqrt{l}}{\Omega}, \qquad \varphi = \Omega(t + z). \tag{2.11}$$

In this case, a snapshot of string becomes a helix as shown in Fig. 1; we call these "helical strings."

Second we consider the case $l = 0$, $q \neq 0$. The solution can be described by

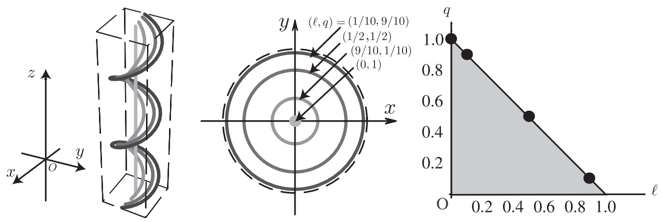

FIG. 1. Helical strings: $l + q = 1$ $(q \neq 0)$. The three-dimensional snapshots are given in the left panel, and the projection of strings to the x-y plane are given in the middle. The dashed circle in the middle figure represents the light cylinder $\rho = 1/\Omega$. The parameters on the l-q plane are plotted in the right panel.

$$x = \frac{\sqrt{1 - q^2}}{\Omega} \sin\left(\frac{\Omega z}{q}\right) \cos\Omega t,$$
$$y = \frac{\sqrt{1 - q^2}}{\Omega} \sin\left(\frac{\Omega z}{q}\right) \sin\Omega t, \tag{2.12}$$

where $x := \rho \cos\varphi$, $y := \rho \sin\varphi$. The strings, we call "planar", are confined in a rotating plane. Snapshots of the planar strings are shown in the first row of Fig. 2.

Third we consider the case $l + q \leq 1$ $(l \neq 0,\ q \neq 0)$. We show the shapes of strings in Fig. 2 for $l = 1/5$, $1/3$, and $1/2$, respectively.

If l is a rational number, projection of the string on the x-y plane becomes a closed curve. For $l = a/b$ (a, b are relatively prime integer), the closed curve consists of N_l elements, where N_l is defined by

$$N_l = \frac{2b}{\mathrm{GCD}[2b, (b - a)]}. \tag{2.13}$$

Here, $\mathrm{GCD}[a, b]$ denotes the greatest common divisor of a, b. The curve wraps around the center in the x-y plane M_l times until the curve returns to the starting point, where M_l is given by

$$M_l = \frac{1 - l}{2} N_l, \tag{2.14}$$

that is,

$$\bar{\varphi}(\sigma + N_l \sigma_{\mathrm{p}}) = \bar{\varphi}(\sigma) + 2\pi M_l, \tag{2.15}$$

where $\sigma_{\mathrm{p}} := \pi/\Omega$ is the periodicity of ρ given by (2.5). The strings with rational l are periodic in z with the period

$$Z_{\mathrm{p}} = \pi N_l q/\Omega. \tag{2.16}$$

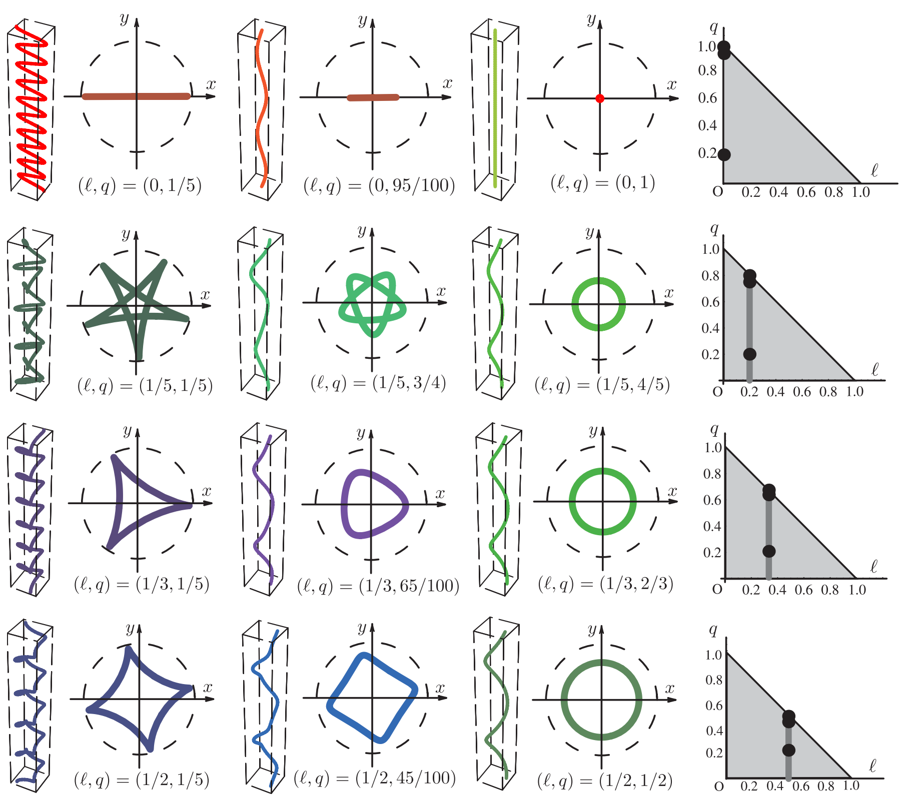

FIG. 2 (color online). Three-dimensional snapshots and projections of string are shown in the case $l = 0$, $1/5$, $1/3$, and $1/2$ as the same as Fig. 1.

B. Energy, momentum, and angular momentum

The string energy-momentum tensor $T^{\mu\nu}$ is given by [3]

$$\sqrt{-g}T^{\mu\nu}(x^\lambda) = -\mu \int d^2\zeta \Theta^{\mu\nu}(\zeta^c)\delta^{(4)}(x^\lambda - x^\lambda(\zeta^c)), \tag{2.17}$$

$$\Theta^{\mu\nu} = \sqrt{-\gamma}\gamma^{ab}\partial_a x^\mu \partial_b x^\nu, \tag{2.18}$$

where $x^\lambda(\zeta^c)$ is the solution of Σ. In the inertial reference system (2.3), the explicit form of $\Theta^{\mu\nu}(\zeta^c)$, which depend only on σ, are shown in the Appendix.

We define the string energy E, the angular momentum J, and the momentum along the rotation axis P. We consider infinitely long strings with periodic structure, i.e., l is assumed to be a rational number, then we define E, J, and P for one period, $z \sim z + Z_{\rm p}$ as

$$E := \int_{\rho_{\rm min}}^{\rho_{\rm max}} d\rho \int_0^{2\pi} d\varphi \int_0^{Z_{\rm p}} dz \sqrt{-g}T^t{}_\nu(-\partial_t)^\nu = \mu \int_0^{N_l\sigma_{\rm p}} d\sigma \Theta^t{}_t(\sigma), \tag{2.19}$$

$$J := \int_{\rho_{\rm min}}^{\rho_{\rm max}} d\rho \int_0^{2\pi} d\varphi \int_0^{Z_{\rm p}} dz \sqrt{-g}T^t{}_\nu(\partial_\varphi)^\nu = -\mu \int_0^{N_l\sigma_{\rm p}} d\sigma \Theta^t{}_\varphi(\sigma), \tag{2.20}$$

$$P := \int_{\rho_{\rm min}}^{\rho_{\rm max}} d\rho \int_0^{2\pi} d\varphi \int_0^{Z_{\rm p}} dz \sqrt{-g}T^t{}_\nu(\partial_z)^\nu = -\mu \int_0^{N_l\sigma_{\rm p}} d\sigma \Theta^t{}_z(\sigma). \tag{2.21}$$

We calculate these quantities as

$$E = \frac{\pi\mu}{|\Omega|} N_l(1 - l^2), \tag{2.22}$$

$$J = \frac{\pi\mu}{2\Omega|\Omega|} N_l(1 - l^2 - q^2), \tag{2.23}$$

$$P = -\frac{\pi\mu}{\Omega} N_l l q. \tag{2.24}$$

Here, we take care of the sign of Ω, l, and q in (2.22), (2.23), and (2.24). We can also define the averaged values of these quantities per unit length of z as

$$\langle E\rangle := E/Z_{\rm p} = \mu \frac{1 - l^2}{|q|}, \tag{2.25}$$

$$\langle J\rangle := J/Z_{\rm p} = \frac{\mu}{\Omega} \frac{1 - l^2 - q^2}{2|q|}, \tag{2.26}$$

$$\langle P\rangle := P/Z_{\rm p} = -\mu l \,\mathrm{sign}(\Omega q). \tag{2.27}$$

These quantities are applicable also for the strings with irrational l.

The effective line density $\tilde{\mu}$, and effective tension $\tilde{\mathcal{T}}$ for the stationary rotating strings are defined [3] in the reference system where the averaged value of momentum $\langle P\rangle$ vanishes. We obtain these quantities explicitly as

$$\tilde{\mu} = \frac{\mu}{2|q|}\Big[1 - l^2 + q^2 + \sqrt{(1-q-l)(1-q+l)(1+q-l)(1+q+l)}\Big],$$
$$\tilde{\mathcal{T}} = \frac{\mu}{2|q|}\Big[1 - l^2 + q^2 - \sqrt{(1-q-l)(1-q+l)(1+q-l)(1+q+l)}\Big]. \tag{2.28}$$

In general, it holds that $\tilde{\mu}\tilde{\mathcal{T}} = \mu^2$ and $\tilde{\mu} \ge \tilde{\mathcal{T}}$. In the case of helical strings, there exists no inertial reference system such that $\langle P\rangle$ vanishes because a single wave moves with the velocity of light along the rotation axis.

III. GRAVITATIONAL PERTURBATIONS

A. Mode decomposition

We consider metric perturbations $h_{\mu\nu}$ produced by a stationary rotating string in the Minkowski spacetime with the metric $\eta_{\mu\nu}$. We solve the linearized Einstein equations

$$\Box\psi_{\mu\nu} = -16\pi T_{\mu\nu}, \tag{3.1}$$

where $T_{\mu\nu}$ are given by (2.17), and $\psi_{\mu\nu}$ is defined by

$$\psi_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}h^\alpha_\alpha. \tag{3.2}$$

We have used the Lorenz gauge condition $\partial^\mu\psi_{\mu\nu} = 0$ in (3.1).

We assume, here and henceforth, the parameter l to be a rational number. In this case, the stationary rotating string solutions (2.5) have periodicity in z with the period $Z_{\rm p}$ given by (2.16). Then, $T_{\mu\nu}$ in (2.17) have the following periodicities:

$$T_{\mu\nu}(t, \rho, \varphi, z) = T_{\mu\nu}(t + 2\pi/\Omega, \rho, \varphi, z), \tag{3.3}$$

$$T_{\mu\nu}(t, \rho, \varphi, z) = T_{\mu\nu}(t, \rho, \varphi + 2\pi, z), \tag{3.4}$$

$$T_{\mu\nu}(t, \rho, \varphi, z) = T_{\mu\nu}(t, \rho, \varphi, z + Z_{\rm p}). \tag{3.5}$$

Thus, we can expand $T_{\mu\nu}$ in a Fourier series as

$$T_{\mu\nu}(t, \rho, \varphi, z) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} e^{-i\omega_n t}e^{im\varphi}e^{ik_s z}\tilde{T}^{(n,m,s)}_{\mu\nu}(\rho), \tag{3.6}$$

where

$$\omega_n := \Omega n, \qquad k_s := \frac{2\pi}{Z_{\rm p}} s = \frac{2}{N_l} \frac{\Omega}{q} s, \tag{3.7}$$

and n, m, s are integers.

By using (2.17), we obtain the Fourier coefficients as

$$\tilde{T}^{(n,m,s)}_{\mu\nu}(\rho) = \frac{\Omega}{(2\pi)^2 Z_{\rm p}} \int_0^{2\pi/\Omega} dt \int_0^{2\pi} d\varphi \int_0^{Z_{\rm p}} dz \times e^{i\omega_n t} e^{-im\varphi} e^{-ik_s z} T_{\mu\nu}(t, \rho, \varphi, z) \tag{3.8}$$

$$= -\frac{\mu \delta_{nm}}{2\pi Z_{\rm p}} \int_0^{N_l \sigma_p} d\sigma e^{-i(k_s q\sigma + m\bar{\varphi}(\sigma))} \times \frac{1}{\rho} \Theta_{\mu\nu}(\sigma) \delta(\rho - \rho_{\rm st}(\sigma)), \tag{3.9}$$

where $\rho_{\rm st}(\sigma)$ is the string solution given by (2.5). Because of δ_{nm} in (3.9), nonvanishing coefficients are specified by $(n, m) = (n, s)$, then we introduce $\tilde{T}^{(n,s)}_{\mu\nu} := \tilde{T}^{(n,n,s)}_{\mu\nu}$.

We can also expand the metric perturbations $\psi_{\mu\nu}$ related to (3.6) in a Fourier series as

$$\psi_{\mu\nu}(t, \rho, \varphi, z) = \sum_{n=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} e^{-i\omega_n t} e^{in\varphi} e^{ik_s z} \tilde{\psi}^{(n,s)}_{\mu\nu}(\rho). \tag{3.10}$$

Using (3.6) and (3.10), we can reduce (3.1) to a set of the ordinary differential equations with respect to ρ for each Fourier mode labeled by (n, s).

Ten components of linearized Einstein equations (3.1) are classified into three types: scalar type ($m = n$), vector type ($m = n \pm 1$), and tensor type ($m = n \pm 2$). Equations for these types have the following form:

$$\text{scalar type}: \quad \mathcal{L}^{(n,s)}_n \tilde{\psi}^{(n,s)}_{\rm S}(\rho) + 16\pi \tilde{T}^{(n,s)}_{\rm S}(\rho) = 0, \tag{3.11}$$

$$\text{vector type}: \quad \mathcal{L}^{(n,s)}_{n\pm 1} \tilde{\psi}^{(n,s)}_{\rm V\pm}(\rho) + 16\pi \tilde{T}^{(n,s)}_{\rm V\pm}(\rho) = 0, \tag{3.12}$$

$$\text{tensor type}: \quad \mathcal{L}^{(n,s)}_{n\pm 2} \tilde{\psi}^{(n,s)}_{\rm T\pm}(\rho) + 16\pi \tilde{T}^{(n,s)}_{\rm T\pm}(\rho) = 0, \tag{3.13}$$

where the differential operator $\mathcal{L}^{(n,s)}_m$ with respect to ρ is defined by

$$\mathcal{L}^{(n,s)}_m = \frac{1}{\rho} \frac{d}{d\rho}\left(\rho \frac{d}{d\rho}\right) + \left(\kappa_{ns}^2 - \frac{m^2}{\rho^2}\right) \tag{3.14}$$

with

$$\kappa_{ns}^2 := \omega_n^2 - k_s^2. \tag{3.15}$$

The members of $(\tilde{\psi}^{(n,s)}_{\rm S}, \tilde{\psi}^{(n,s)}_{\rm V\pm}, \tilde{\psi}^{(n,s)}_{\rm T\pm})$ and $(\tilde{T}^{(n,s)}_{\rm S}, \tilde{T}^{(n,s)}_{\rm V\pm}, \tilde{T}^{(n,s)}_{\rm T\pm})$ are defined by

$$\begin{aligned}
\tilde{\psi}^{(n,s)}_{\rm S} &= \{\tilde{\psi}^{(n,s)}_{tt}, \tilde{\psi}^{(n,s)}_{zz}, \tilde{\psi}^{(n,s)}_{tz}, (\tilde{\psi}^{(n,s)}_{\rho\rho} + \tilde{\psi}^{(n,s)}_{\varphi\varphi}/\rho^2)\},\\
\tilde{\psi}^{(n,s)}_{\rm V\pm} &= \{(\tilde{\psi}^{(n,s)}_{t\rho} \pm i\tilde{\psi}^{(n,s)}_{t\varphi}/\rho), (\tilde{\psi}^{(n,s)}_{\rho z} \pm i\tilde{\psi}^{(n,s)}_{\varphi z}/\rho)\},\\
\tilde{\psi}^{(n,s)}_{\rm T\pm} &= \{(\tilde{\psi}^{(n,s)}_{\rho\rho} - \tilde{\psi}^{(n,s)}_{\varphi\varphi}/\rho^2 \pm 2i\tilde{\psi}^{(n,s)}_{\rho\varphi}/\rho)\},
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
\tilde{T}^{(n,s)}_{\rm S} &= \{\tilde{T}^{(n,s)}_{tt}, \tilde{T}^{(n,s)}_{zz}, \tilde{T}^{(n,s)}_{tz}, (\tilde{T}^{(n,s)}_{\rho\rho} + \tilde{T}^{(n,s)}_{\varphi\varphi}/\rho^2)\},\\
\tilde{T}^{(n,s)}_{\rm V\pm} &= \{(\tilde{T}^{(n,s)}_{t\rho} \pm i\tilde{T}^{(n,s)}_{t\varphi}/\rho), (\tilde{T}^{(n,s)}_{\rho z} \pm i\tilde{T}^{(n,s)}_{\varphi z}/\rho)\},\\
\tilde{T}^{(n,s)}_{\rm T\pm} &= \{(\tilde{T}^{(n,s)}_{\rho\rho} - \tilde{T}^{(n,s)}_{\varphi\varphi}/\rho^2 \pm 2i\tilde{T}^{(n,s)}_{\rho\varphi}/\rho)\},
\end{aligned} \tag{3.17}$$

respectively.

At the infinity, because $m^2/\rho^2 \to 0$, (3.15) means the dispersion relation of the gravitational waves, where κ_{ns} and k_s can be regarded as the radial and the z-axis components of the wave vector, respectively.

B. Green's function method

All of Eqs. (3.11), (3.12), and (3.13) have the same form of

$$\mathcal{L}^{(n,s)}_m \tilde{\psi}^{(n,s)}(\rho) + 16\pi \tilde{T}^{(n,s)}(\rho) = 0, \tag{3.18}$$

where the indices S, V $\pm$, T$\pm$ are suppressed. The ordinary differential equations (3.18) of the Sturm-Liouville type are formally solvable by using Green's function method. (See [34], for example.)

Introducing Green's function $G^{ns}_m(\rho, \rho')$ which satisfies

$$\mathcal{L}^{(n,s)}_m G^{ns}_m(\rho, \rho') = -\frac{1}{\rho} \delta(\rho - \rho'), \tag{3.19}$$

we can express the solutions $\tilde{\psi}^{(n,s)}$ of (3.18) as

$$\tilde{\psi}^{(n,s)}(\rho) = \int_0^\infty d\rho' G^{ns}_m(\rho, \rho') 16\pi \rho' \tilde{T}^{(n,s)}(\rho'). \tag{3.20}$$

Using (3.9) for the scalar, vector, and tensor types of $\tilde{T}^{(n,s)}$, we can write $\tilde{\psi}^{(n,s)}$ as

$$\tilde{\psi}^{(n,s)}(\rho) = -\frac{8\mu}{qN_l\sigma_{\rm p}} \int_0^{N_l\sigma_{\rm p}} d\sigma G^{ns}_m(\rho, \rho_{\rm st}(\sigma)) \Theta(\sigma) \times \exp(-ik_s q\sigma - in\bar{\varphi}(\sigma)), \tag{3.21}$$

where $\Theta := \{\Theta_{\rm S}, \Theta_{\rm V\pm}, \Theta_{\rm T\pm}\}$ in the right-hand side takes the same combination of $\Theta_{\mu\nu}$ as (3.17). The coefficients $\tilde{\psi}^{(n,s)}$ should satisfy

$$\tilde{\psi}^{(-n,-s)}(\rho) = (\tilde{\psi}^{(n,s)}(\rho))^*, \tag{3.22}$$

so that the metric perturbations $h_{\mu\nu}$ are real, where $*$ means the complex conjugate.

C. Nonvanishing (n, s) modes

For the stationary rotating strings with rational l, the product $G^{ns}_m(\rho, \rho_{\rm st}(\sigma))\Theta(\sigma)$ in (3.21) is periodic in σ with the period $\sigma_{\rm p}$ as

$$G_m^{ns}(\rho, \rho_{\rm st}(\sigma + \sigma_{\rm p}))\Theta(\sigma + \sigma_{\rm p}) = G_m^{ns}(\rho, \rho_{\rm st}(\sigma))\Theta(\sigma), \tag{3.23}$$

because of the periodicity of $\rho_{\rm st}(\sigma)$ in (2.5). At the same time, from (2.15) the exponential factor in (3.21) varies as

$$\exp(-ik_s q(\sigma + N_l\sigma_{\rm p}) - in\bar{\varphi}(\sigma + N_l\sigma_{\rm p})) \tag{3.24}$$

$$= \exp(-ik_s q\sigma - in\bar{\varphi}(\sigma))\exp(-2\pi i(s + nM_l)). \tag{3.25}$$

Here, we introduce a function $\Phi(\sigma)$ by

$$\Phi(\sigma) = (k_s q\sigma + n\bar{\varphi}(\sigma))/L_{ns}, \tag{3.26}$$

where

$$L_{ns} := s + nM_l \tag{3.27}$$

is an integer specified by mode indices n and s for a stationary rotating string. The function $\Phi(\sigma)$ is monotonic in σ and varies as

$$\Phi(\sigma + N_l\sigma_{\rm p}) = \Phi(\sigma) + 2\pi. \tag{3.28}$$

Then, Eq. (3.21) leads to

$$\begin{aligned}\tilde{\psi}^{(n,s)}(\rho) &\propto \int_0^{N_l\sigma_{\rm p}} d\sigma G_m^{ns}(\rho, \rho_{\rm st}(\sigma))\Theta(\sigma) \\ &\quad\times \exp(-iL_{ns}\Phi(\sigma)) \\ &= \int_0^{2\pi} d\Phi \frac{d\sigma}{d\Phi} G_m^{ns}(\rho, \rho_{\rm st}(\sigma(\Phi)))\Theta(\sigma(\Phi)) \\ &\quad\times \exp(-iL_{ns}\Phi),\end{aligned} \tag{3.29}$$

where we have changed the integration variable σ by Φ. Since $d\bar{\varphi}/d\sigma$ is periodic with the period $\sigma_{\rm p}$, $d\Phi/d\sigma$ is also periodic in σ with the same period. Therefore, we can see that $d\sigma/d\Phi$, $\rho_{\rm st}(\sigma(\Phi))$, and $\Theta(\sigma(\Phi))$ have a periodicity in Φ with the period $2\pi/N_l$, and then, we can obtain a Fourier series

$$\frac{d\sigma}{d\Phi} G_m^{ns}(\rho, \rho_{\rm st}(\sigma(\Phi)))\Theta(\sigma(\Phi)) = \sum_j a_j \exp(ijN_l\Phi), \tag{3.30}$$

where a_j are Fourier coefficients labeled by an integer j. Inserting this into (3.29) we have

$$\tilde{\psi}^{(n,s)}(\rho) \propto \int_0^{2\pi} d\Phi \sum_j a_j \exp(i(jN_l - L_{ns})\Phi). \tag{3.31}$$

Therefore, for the combination (n, s) of nonvanishing $\tilde{\psi}^{(n,s)}(\rho)$, there should exist an integer j which satisfies

$$L_{ns} = s + nM_l = jN_l. \tag{3.32}$$

Especially, in the case of helical strings, $l + q = 1$, $q \neq 0$, because $G_m^{ns}(\rho, \rho_{\rm st}(\sigma))\Theta(\rho_{\rm st}(\sigma))$ in (3.21) is constant with respect to σ, the nonvanishing (n, s) modes is specified by the condition

$$L_{ns} = s + nM_l = 0. \tag{3.33}$$

D. Explicit forms of Green's functions

We obtain the explicit form of Green's functions G_m^{ns} here. We consider three cases with respect to the sign of κ_{ns}^2 defined by (3.15).

First, we consider the case $\kappa_{ns}^2 < 0$. If we require the regularity both at the center and at the infinity, the operator (3.14) with negative κ_{ns}^2 allows damping solutions to (3.18) with the length scale $|\kappa_{ns}^{-1}|$. Green's functions in this case have the form

$$\begin{aligned}G_m^{ns}(\rho, \rho') &= I_m(|\kappa_{ns}|\rho)K_m(|\kappa_{ns}|\rho')\theta(\rho' - \rho) \\ &\quad + K_m(|\kappa_{ns}|\rho)I_m(|\kappa_{ns}|\rho')\theta(\rho - \rho'),\end{aligned} \tag{3.34}$$

where the functions $\theta(x)$ is the Heaviside step function, and $I_m(x)$ and $K_m(x)$ are the modified Bessel functions,

$$I_m(x) = i^{-m}J_m(ix), \qquad K_m(x) = (\pi/2)i^{m+1}H_m^{(1)}(ix), \tag{3.35}$$

and J_m and $H_m^{(1)}$ are the Bessel function and the Hankel function of the first kind, respectively.

Next, in the case $\kappa_{ns}^2 = 0$, because the scale vanishes in the operator (3.14), the solutions to (3.18) have long tails. Green's functions are

$$G_m^{ns}(\rho, \rho') = \begin{cases} -\{\ln(\rho'/\rho_0)\theta(\rho' - \rho) + \ln(\rho/\rho_0)\theta(\rho - \rho')\} & \text{for } m = 0, \\ \frac{1}{2|m|}\{(\rho/\rho')^{|m|}\theta(\rho' - \rho) + (\rho'/\rho)^{|m|}\theta(\rho - \rho')\} & \text{for } m \neq 0. \end{cases} \tag{3.36}$$

In the case of $m = 0$, we have introduced a constant ρ_0 as the boundary instead of the infinity such that $G_0^{ns} \to 0$ in the limit $\rho \to \rho_0$.

Finally, in the case $\kappa_{ns}^2 > 0$, the operator (3.14) allows wave solutions to (3.18). The scale κ_{ns}^{-1} gives the wave length of the solutions. Green's functions take the form of

$$\begin{aligned}G_m^{ns}(\rho, \rho') &= \frac{\pi}{2} i\{J_m(\kappa_{ns}\rho)H_m(\kappa_{ns}\rho')\theta(\rho' - \rho) \\ &\quad + H_m(\kappa_{ns}\rho)J_m(\kappa_{ns}\rho')\theta(\rho - \rho')\}.\end{aligned} \tag{3.37}$$

Here, H_m are defined by

$$H_m(x) = \begin{cases} H_m^{(1)}(x) & \text{for } \omega_n > 0, \\ -H_m^{(2)}(x) & \text{for } \omega_n < 0, \end{cases} \tag{3.38}$$

where $H_m^{(1)}$ and $H_m^{(2)}$ denote the Hankel functions of first and second kind, respectively. This definition guarantees that the solutions describe the outgoing waves at the infinity in any case of ω_n.

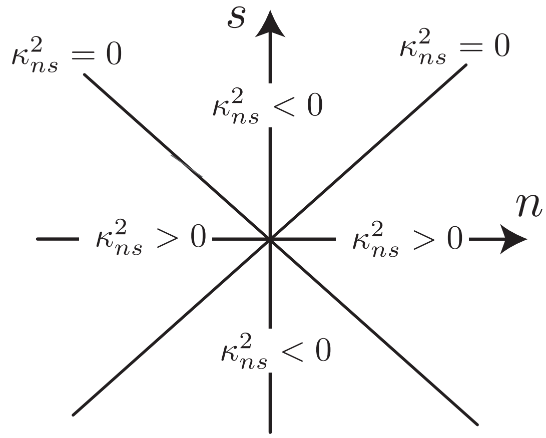

FIG. 3. Three cases of κ^2_{ns} in the n-s plane.

E. Potential mode and wave modes

In the previous subsection, Green's functions are constructed in three different cases: $\kappa^2_{ns} < 0$, $\kappa^2_{ns} = 0$, and $\kappa^2_{ns} > 0$, respectively. These three cases correspond to the regions on the n-s plane as

$$|s| > \left|\frac{qN_l}{2}n\right| \quad \text{for } \kappa^2_{ns} < 0, \tag{3.39}$$

$$|s| = \left|\frac{qN_l}{2}n\right| \quad \text{for } \kappa^2_{ns} = 0, \tag{3.40}$$

$$|s| < \left|\frac{qN_l}{2}n\right| \quad \text{for } \kappa^2_{ns} > 0, \tag{3.41}$$

which are shown in Fig. 3.

The two lines which denote $\kappa^2_{ns} = 0$ in the n-s plane are given by

$$s = \pm \frac{qN_l}{2} n. \tag{3.42}$$

The inclinations of the lines, which depend on l and q, have the maximum absolute value M_l when $q = 1 - l$ for given l.

Here, we divide the metric perturbation into four parts, namely, short range force modes $h^{\rm Short}_{\mu\nu}$, stationary potential mode $h^{\rm Pot}_{\mu\nu}$, traveling wave modes $h^{\rm TW}_{\mu\nu}$, and gravitational wave modes $h^{\rm GW}_{\mu\nu}$ as

$$h_{\mu\nu} = h^{\rm Short}_{\mu\nu} + h^{\rm Pot}_{\mu\nu} + h^{\rm TW}_{\mu\nu} + h^{\rm GW}_{\mu\nu}, \tag{3.43}$$

where

$$h^{\rm Short}_{\mu\nu}(t, \rho, \varphi, z) := \sum_{\substack{n>0 \\ |s|>(qN_l/2)n}} [\exp(-i\omega_n t + in\varphi + ik_s z)\tilde{h}^{(n,s)}_{\mu\nu}(\rho) + ({\rm c.c.})], \tag{3.44}$$

$$h^{\rm Pot}_{\mu\nu}(t, \rho, \varphi, z) := \tilde{h}^{(0,0)}_{\mu\nu}(\rho), \tag{3.45}$$

$$h^{\rm TW}_{\mu\nu}(t, \rho, \varphi, z) := \sum_{\substack{n>0 \\ |s|=(qN_l/2)n}} [\exp(-i\omega_n t + in\varphi + ik_s z)\tilde{h}^{(n,s)}_{\mu\nu}(\rho) + ({\rm c.c.})], \tag{3.46}$$

$$h^{\rm GW}_{\mu\nu}(t, \rho, \varphi, z) := \sum_{\substack{n>0 \\ |s|<(qN_l/2)n}} [\exp(-i\omega_n t + in\varphi + ik_s z)\tilde{h}^{(n,s)}_{\mu\nu}(\rho) + ({\rm c.c.})] \tag{3.47}$$

[(c.c.) denotes complex conjugate]. The summations in (3.44), (3.46), and (3.47) are taken over pairs (n, s) which satisfy the condition (3.32) or (3.33). These (n, s) are shown in Fig. 4 as dots in the n-s plane.

The modes in $\kappa^2_{ns} < 0$ $(|s| > (qN_l/2)n)$, given by Green's function (3.34), describe the gravitational field in a short range around the string, and exponentially decrease in the region $\rho \gg 1/|\Omega|$; we name these "short-range modes." Since a distant observer hardly accesses the short-range modes, we do not discuss these further.

The mode of $(n, s) = (0, 0)$ is clearly time independent. The metric components of this mode represents the Newtonian potential, the angular deficit, and the effects of frame-dragging. The modes in $\kappa^2_{ns} = 0$ describe waves propagating along the z axis, i.e., along the rotating string. These waves are named "traveling waves" following Ref. [35]. The modes in $\kappa^2_{ns} > 0$ are gravitational waves propagating toward distant observers from the string. The facts noted above will be in successive sections.

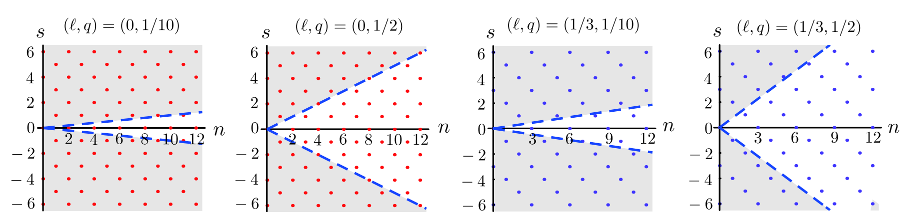

FIG. 4 (color online). The pairs (n, s) of nonvanishing modes specified by (3.32) are shown by dots for $l = 0$ and $l = 1/3$ cases, as examples. The dots in the shadowed regions are short-range modes, while the dots in the unshaded region represent the gravitational wave modes. The dots on the border, thick broken lines, are traveling wave modes, and $(n, s) = (0, 0)$ is the gravitational potential mode.

IV. POTENTIAL MODE

The mode $(n, s) = (0, 0)$ describes time-independent long-range potential. The components $\tilde{\psi}^{(0,0)}_{\mu\nu}$ given by (3.21) have the following form:

$$\tilde{\psi}^{(0,0)}_{\rm S}(\rho) = -\frac{8\mu}{qN_l\sigma_{\rm p}}\int_0^{N_l\sigma_{\rm p}} d\sigma G^{00}_0(\rho, \rho_{\rm st}(\sigma))\Theta_{\rm S}(\sigma),$$
$$\tilde{\psi}^{(0,0)}_{\rm V\pm}(\rho) = -\frac{8\mu}{qN_l\sigma_{\rm p}}\int_0^{N_l\sigma_{\rm p}} d\sigma G^{00}_{\pm1}(\rho, \rho_{\rm st}(\sigma))\Theta_{\rm V\pm}(\sigma),$$
$$\tilde{\psi}^{(0,0)}_{\rm T\pm}(\rho) = -\frac{8\mu}{qN_l\sigma_{\rm p}}\int_0^{N_l\sigma_{\rm p}} d\sigma G^{00}_{\pm2}(\rho, \rho_{\rm st}(\sigma))\Theta_{\rm T\pm}(\sigma), \tag{4.1}$$

where Green's functions are given by (3.36). After some calculations, explicit forms of $h^{\rm Pot}_{\mu\nu}(\rho) = h^{(0,0)}_{\mu\nu}(\rho)$ in the far region are given as

$$h^{\rm Pot}_{tt} = h^{\rm Pot}_{zz} = -\frac{4\mu}{q}(1 - l^2 - q^2)\ln\left(\frac{\rho}{\rho_0}\right), \tag{4.2}$$

$$h^{\rm Pot}_{tz} = -8\mu l\ln\left(\frac{\rho}{\rho_0}\right), \tag{4.3}$$

$$h^{\rm Pot}_{\rho\rho} = -\frac{4\mu}{q}(1 - l^2 + q^2)\ln\left(\frac{\rho}{\rho_0}\right) - \frac{\mu}{q}((1 - l^2 - q^2)^2 - 4l^2q^2)\frac{1}{(\Omega\rho)^2}, \tag{4.4}$$

$$\frac{h^{\rm Pot}_{\varphi\varphi}}{\rho^2} = -\frac{4\mu}{q}(1 - l^2 + q^2)\ln\left(\frac{\rho}{\rho_0}\right) + \frac{\mu}{q}((1 - l^2 - q^2)^2 - 4l^2q^2)\frac{1}{(\Omega\rho)^2}, \tag{4.5}$$

$$\frac{h^{\rm Pot}_{t\varphi}}{\rho} = -\frac{2\mu}{q}(1 - l^2 - q^2)\frac{1}{\Omega\rho}, \tag{4.6}$$

$$\frac{h^{\rm Pot}_{z\varphi}}{\rho} = -4\mu l\frac{1}{\Omega\rho}, \tag{4.7}$$

$$h^{\rm Pot}_{t\rho} = h^{\rm Pot}_{z\rho} = h^{\rm Pot}_{\rho\varphi}/\rho = 0. \tag{4.8}$$

Although we assume l to be a rational number, the expressions of $h^{\rm Pot}_{\mu\nu}$ given above are also valid for irrational l.

It is found that $h^{\rm Pot}_{t\varphi}$ denotes the azimuthal frame-dragging caused by the angular momentum of the string, and $h^{\rm Pot}_{tz}$ does dragging along the z axis caused by the linear momentum along the rotation axis of the string. In the case of planar strings, $l = 0$, we see that $\langle P\rangle = 0$ from (2.27) and that there is no dragging along the z axis from (4.3). If we transform the inertial reference frame $(t, \rho, \varphi, z) \to (\tilde{t}, \tilde{\rho}, \tilde{\varphi}, \tilde{z})$ by the Lorentz boost such that $\langle P\rangle = 0$ as shown in Ref. [30], the dragging along z axis disappears. In this frame, the logarithmic terms of $h^{\rm Pot}_{\mu\nu}$ give the metric in the form

$$ds^2 = -(1 + 4(\tilde{\mu} - \tilde{\mathcal{T}})\ln(\tilde{\rho}/\rho_0))d\tilde{t}^2 + (1 - 4(\tilde{\mu} - \tilde{\mathcal{T}})\ln(\tilde{\rho}/\rho_0))d\tilde{z}^2 + (1 - 4(\tilde{\mu} + \tilde{\mathcal{T}})\ln(\tilde{\rho}/\rho_0))(d\tilde{\rho}^2 + \tilde{\rho}^2d\tilde{\varphi}^2). \tag{4.9}$$

Using the coordinate transformation,

$$r = (1 + 2(\tilde{\mu} + \tilde{\mathcal{T}})(1 - \ln(\tilde{\rho}/\rho_0))\tilde{\rho}, \qquad \phi = (1 - 2(\tilde{\mu} + \tilde{\mathcal{T}}))\tilde{\varphi}, \tag{4.10}$$

and ignoring $\mathcal{O}((\tilde{\mu} + \tilde{\mathcal{T}})^2)$ terms, the metric of the $\tilde{t} =$ const and $\tilde{z} =$ const surface becomes flat metric

$$ds^2 = dr^2 + r^2d\phi^2. \tag{4.11}$$

Since the range of ϕ is $0 \leq \phi < 2\pi(1 - 2(\tilde{\mu} + \tilde{\mathcal{T}}))$, the flat surface is the conical space with angular deficit $4\pi(\tilde{\mu} + \tilde{\mathcal{T}})$ [3].

Alternatively, using the coordinate

$$\bar{r} = (1 + 4\tilde{\mathcal{T}}(1 - \ln(\tilde{\rho}/\rho_0))\tilde{\rho}, \qquad \bar{\phi} = (1 - 4\tilde{\mathcal{T}})\tilde{\varphi}, \tag{4.12}$$

we rewrite the metric (4.9) as

$$ds^2 = -(1 + 2\Psi(\tilde{\rho}))d\tilde{t}^2 + (1 - 2\Psi(\tilde{\rho}))(d\bar{r}^2 + \bar{r}^2d\bar{\phi}^2 + d\tilde{z}^2) \tag{4.13}$$

where

$$\Psi(\tilde{\rho}) = 2(\tilde{\mu} - \tilde{\mathcal{T}})\ln(\tilde{\rho}/\rho_0). \tag{4.14}$$

This metric means that the stationary rotating string produces the Newtonian logarithmic potential Ψ around it [3].

In general, the stationary rotating string in the frame of $\langle P\rangle = 0$ yields the logarithmic potential, the angular deficit, and the azimuthal frame-dragging in φ. It should be noted, as an exceptional case, that the dragging along the rotation axis, the z axis, can not be erased for the helical strings because there is no reference frame such that $\langle P\rangle = 0$. In addition, the Newtonian potential vanishes, and the angular deficit, $8\pi\mu$, is the same value as the straight string.

V. GRAVITATIONAL WAVE MODES

In this section, we consider the metric perturbations propagating away from a string to a distant observer, i.e., the gravitational wave modes $h^{\rm GW}_{\mu\nu}$ given in (3.47), where the summation is taken over all (n, s) satisfying (3.32) and (3.41). Fourier components of metric perturbations $\tilde{h}^{(n,s)}_{\mu\nu}(\rho)$, equivalently $\tilde{\psi}^{(n,s)}_{\mu\nu}(\rho)$, are given by (3.21) where Green's functions are (3.37).

First, we define the physical modes of polarization, plus modes and cross modes. Next, we show that the gravitational waves can be emitted to several discrete directions. Finally, we present waveforms of the gravitational waves emitted to the possible directions by using numerical calculations.

A. Plus modes and cross modes

Here, we fix the gauge freedom of propagating modes in the vacuum. We use the transverse traceless (TT) gauge conditions:

$$h^{TT}_{t\mu} = 0, \qquad \partial^i h^{TT}_{ij} = 0, \qquad h^{TTi}{}_i = 0. \tag{5.1}$$

The metric perturbations satisfying TT conditions, $h^{\rm TT}_{ij}$, are invariant under gauge transformations. Using the fact that the Riemann tensor, which is gauge invariant, is expressed by $h^{\rm TT}_{ij}$ in the linear order, we can obtain the TT modes by integration of

$$\begin{aligned}\partial_t^2 h^{\rm TT}_{ij} &= -2R_{itjt}\\ &= -(\partial_t\partial_j h_{it} + \partial_i\partial_t h_{tj} - \partial_i\partial_j h_{tt} - \partial_t\partial_t h_{ij}),\end{aligned} \tag{5.2}$$

where $h_{\mu\nu}$ in the right-hand side are solutions of the wave equation. (See Sec. 35.4 of [36].)

In the cylindrical coordinate, $\tilde{h}^{(n,s)\rm TT}_{ij}$ can be obtained as

$$\begin{aligned}
\tilde{h}^{(n,s)\rm TT}_{\rho\rho} &= \frac{1}{2}\left(\tilde{\psi}^{(n,s)}_{tt} + \tilde{\psi}^{(n,s)}_{\rho\rho} - \frac{\tilde{\psi}^{(n,s)}_{\varphi\varphi}}{\rho^2} - \tilde{\psi}^{(n,s)}_{zz}\right) - \frac{2i}{\omega_n}\partial_\rho\tilde{\psi}^{(n,s)}_{t\rho} - \frac{1}{2\omega_n^2}\partial_\rho^2\left(\tilde{\psi}^{(n,s)}_{tt} + \tilde{\psi}^{(n,s)}_{\rho\rho} + \frac{\tilde{\psi}^{(n,s)}_{\varphi\varphi}}{\rho^2} + \tilde{\psi}^{(n,s)}_{zz}\right),\\
\frac{\tilde{h}^{(n,s)\rm TT}_{\rho\varphi}}{\rho} &= \frac{\tilde{\psi}^{(n,s)}_{\rho\varphi}}{\rho} - \frac{i}{\omega_n}\left(\partial_\rho - \frac{1}{\rho}\right)\left(\frac{\tilde{\psi}^{(n,s)}_{t\varphi}}{\rho}\right) + \frac{1}{\Omega\rho}\left\{\tilde{\psi}^{(n,s)}_{t\rho} - \frac{i}{2\omega_n}\left(\partial_\rho - \frac{1}{\rho}\right)\left(\tilde{\psi}^{(n,s)}_{tt} + \tilde{\psi}^{(n,s)}_{\rho\rho} + \frac{\tilde{\psi}^{(n,s)}_{\varphi\varphi}}{\rho^2} + \tilde{\psi}^{(n,s)}_{zz}\right)\right\},\\
\tilde{h}^{(n,s)\rm TT}_{\rho z} &= \tilde{\psi}^{(n,s)}_{\rho z} - \frac{i}{\omega_n}\partial_\rho\tilde{\psi}^{(n,s)}_{tz} + \frac{k_s}{\omega_n}\left\{\tilde{\psi}^{(n,s)}_{t\rho} - \frac{i}{2\omega_n}\partial_\rho\left(\tilde{\psi}^{(n,s)}_{tt} + \tilde{\psi}^{(n,s)}_{\rho\rho} + \frac{\tilde{\psi}^{(n,s)}_{\varphi\varphi}}{\rho^2} + \tilde{\psi}^{(n,s)}_{zz}\right)\right\},\\
\frac{\tilde{h}^{(n,s)\rm TT}_{\varphi\varphi}}{\rho^2} &= \frac{1}{2}\left(\tilde{\psi}^{(n,s)}_{tt} - \tilde{\psi}^{(n,s)}_{\rho\rho} + \frac{\tilde{\psi}^{(n,s)}_{\varphi\varphi}}{\rho^2} - \tilde{\psi}^{(n,s)}_{zz}\right) + \frac{2}{\Omega\rho}\left\{\frac{\tilde{\psi}^{(n,s)}_{t\varphi}}{\rho} - \frac{i}{n}\tilde{\psi}^{(n,s)}_{t\rho}\right\}\\
&\quad + \frac{1}{2(\Omega\rho)^2}\left(1 - \frac{\rho}{n^2}\partial_\rho\right)\left(\tilde{\psi}^{(n,s)}_{tt} + \tilde{\psi}^{(n,s)}_{\rho\rho} + \frac{\tilde{\psi}^{(n,s)}_{\varphi\varphi}}{\rho^2} + \tilde{\psi}^{(n,s)}_{zz}\right),\\
\frac{\tilde{h}^{(n,s)\rm TT}_{\varphi z}}{\rho} &= \frac{\tilde{\psi}^{(n,s)}_{\varphi z}}{\rho} + \frac{k_s}{\omega_n}\left(\frac{\tilde{\psi}^{(n,s)}_{t\varphi}}{\rho}\right) + \frac{1}{\Omega\rho}\left\{\tilde{\psi}^{(n,s)}_{tz} + \frac{k_s}{2\omega_n}\left(\tilde{\psi}^{(n,s)}_{tt} + \tilde{\psi}^{(n,s)}_{\rho\rho} + \frac{\tilde{\psi}^{(n,s)}_{\varphi\varphi}}{\rho^2} + \tilde{\psi}^{(n,s)}_{zz}\right)\right\},\\
\tilde{h}^{(n,s)\rm TT}_{zz} &= \frac{1}{2}\left(\tilde{\psi}^{(n,s)}_{tt} - \tilde{\psi}^{(n,s)}_{\rho\rho} - \frac{\tilde{\psi}^{(n,s)}_{\varphi\varphi}}{\rho^2} + \tilde{\psi}^{(n,s)}_{zz}\right) + \frac{2k_s}{\omega_n}\tilde{\psi}^{(n,s)}_{tz} + \frac{1}{2}\left(\frac{k_s}{\omega_n}\right)^2\left(\tilde{\psi}^{(n,s)}_{tt} + \tilde{\psi}^{(n,s)}_{\rho\rho} + \frac{\tilde{\psi}^{(n,s)}_{\varphi\varphi}}{\rho^2} + \tilde{\psi}^{(n,s)}_{zz}\right).
\end{aligned} \tag{5.3}$$

In the large distance limit, the wave vector of a (n, s) mode in the normalized orthogonal frame $(\hat{t}, \hat{\rho}, \hat{\varphi}, \hat{z})$ is expressed as

$$\hat{k}^{(n,s)}_{\hat{\mu}} = (-\omega_n, \kappa_{ns}, 0, k_s), \tag{5.4}$$

because the $\hat{\varphi}$ component of the wave vector becomes small as $1/\rho$ in the far region. Then, the (n, s) mode propagates in the direction specified by the angle $\theta_{s/n}$ from the rotation axis which is defined by

$$\cos\theta_{s/n} = \frac{k_s}{\omega_n} = \frac{2}{N_l q}\frac{s}{n}. \tag{5.5}$$

The direction $\theta_{s/n} = \pi/2$ is perpendicular to the z axis, i.e., perpendicular to the string.

Here, we introduce a new normal frame $(\hat{t}, \hat{\eta}, \hat{\varphi}, \hat{\zeta})$ at the observer, such that the direction of the wave vector coincides with $\hat{\eta}$, i.e.,

$$-\hat{k}^{(n,s)}_{\hat{t}} = \hat{k}^{(n,s)}_{\hat{\eta}} = \omega_n. \tag{5.6}$$

The new basis is defined for each (n, s) explicitly as

$$\hat{\eta} = \sin\theta_{s/n}\hat{\rho} + \cos\theta_{s/n}\hat{z}, \tag{5.7}$$

$$\hat{\zeta} = -\cos\theta_{s/n}\hat{\rho} + \sin\theta_{s/n}\hat{z}. \tag{5.8}$$

By the use of this frame the components of metric perturbations (5.3) are given by

$$\tilde{h}^{(n,s)\mathrm{TT}}_{\hat{\eta}\hat{\eta}} = \left(\frac{\kappa_{ns}}{\omega_n}\right)^2 \tilde{h}^{(n,s)\mathrm{TT}}_{\rho\rho} + 2\left(\frac{k_s\kappa_{ns}}{\omega_n^2}\right)\tilde{h}^{(n,s)\mathrm{TT}}_{\rho z} + \left(\frac{k_s}{\omega_n}\right)^2 \tilde{h}^{(n,s)\mathrm{TT}}_{zz},$$

$$\tilde{h}^{(n,s)\mathrm{TT}}_{\hat{\zeta}\hat{\zeta}} = \left(\frac{k_s}{\omega_n}\right)^2 \tilde{h}^{(n,s)\mathrm{TT}}_{\rho\rho} - 2\left(\frac{k_s\kappa_{ns}}{\omega_n^2}\right)\tilde{h}^{(n,s)\mathrm{TT}}_{\rho z} + \left(\frac{\kappa_{ns}}{\omega_n}\right)^2 \tilde{h}^{(n,s)\mathrm{TT}}_{zz}, \qquad \tilde{h}^{(n,s)\mathrm{TT}}_{\hat{\varphi}\hat{\varphi}} = \frac{\tilde{h}^{(n,s)\mathrm{TT}}_{\varphi\varphi}}{\rho^2},$$

$$\tilde{h}^{(n,s)\mathrm{TT}}_{\hat{\eta}\hat{\zeta}} = \left\{1 - 2\left(\frac{k_s}{\omega_n}\right)^2\right\}\tilde{h}^{(n,s)\mathrm{TT}}_{\rho z} - \left(\frac{k_s\kappa_{ns}}{\omega_n^2}\right)(\tilde{h}^{(n,s)\mathrm{TT}}_{\rho\rho} - \tilde{h}^{(n,s)\mathrm{TT}}_{zz}), \qquad \tilde{h}^{(n,s)\mathrm{TT}}_{\hat{\eta}\hat{\varphi}} = \left(\frac{\kappa_{ns}}{\omega_n}\right)\frac{\tilde{h}^{(n,s)\mathrm{TT}}_{\rho\varphi}}{\rho} + \left(\frac{k_s}{\omega_n}\right)\frac{\tilde{h}^{(n,s)\mathrm{TT}}_{\varphi z}}{\rho}, \tag{5.9}$$

$$\tilde{h}^{(n,s)\mathrm{TT}}_{\hat{\varphi}\hat{\zeta}} = -\left(\frac{k_s}{\omega_n}\right)\frac{\tilde{h}^{(n,s)\mathrm{TT}}_{\rho\varphi}}{\rho} + \left(\frac{\kappa_{ns}}{\omega_n}\right)\frac{\tilde{h}^{(n,s)\mathrm{TT}}_{\varphi z}}{\rho}.$$

It can be shown that $\tilde{h}^{(n,s)\mathrm{TT}}_{\hat{\eta}\hat{\eta}}$, $\tilde{h}^{(n,s)\mathrm{TT}}_{\hat{\eta}\hat{\varphi}}$, and $\tilde{h}^{(n,s)\mathrm{TT}}_{\hat{\eta}\hat{\zeta}}$ are vanishing by using the wave equation, i.e., (3.15), and TT-gauge condition (5.1). For convenience, we define the two modes of polarizations: the plus mode $\tilde{h}^{(n,s)}_{+}$, and the cross mode $\tilde{h}^{(n,s)}_{\times}$, as

$$\tilde{h}^{(n,s)}_{+}(\rho) = \tilde{h}^{(n,s)\mathrm{TT}}_{\hat{\varphi}\hat{\varphi}}(\rho) = -\tilde{h}^{(n,s)\mathrm{TT}}_{\hat{\zeta}\hat{\zeta}}(\rho), \tag{5.10}$$

$$\tilde{h}^{(n,s)}_{\times}(\rho) = \tilde{h}^{(n,s)\mathrm{TT}}_{\hat{\varphi}\hat{\zeta}}(\rho). \tag{5.11}$$

B. Directions of gravitational wave emission

Let us consider a set of pairs (n, s) which give the same ratio s/n under the conditions (3.32) and (3.41). For a stationary rotating string with fixed l and q, all (n, s) modes in the set are emitted in the same direction $\theta_{s/n}$ defined by (5.5) [37]. The lowest number of n and corresponding s in the set, say (n_0, s_0), is the fundamental mode of gravitational wave emitted to the direction $\theta_{s/n}$. The overtone modes are specified by the indices which are multiplications of (n_0, s_0) by positive integers larger than 1. For example, mode indices of the fundamental mode and the overtone modes for each direction are shown in the following table in the cases of string with $(l,q)=(0,1/2)$ and $(1/3, 1/2)$.

$(l, q) = (0, 1/2)$ case:

$\cos\theta_{s/n}$	Fundamental mode (n_0, s_0)	Overtone modes (n, s)
0	(2, 0)	(4, 0), (6, 0) $\cdots$
$\pm 2/3$	(3, ± 1)	(6, ± 2), (9, ± 3) $\cdots$
$\pm 2/5$	(5, ± 1)	(10, ± 2), (15, ± 3) $\cdots$
$\pm 2/7$	(7, ± 1)	(14, ± 2), (21, ± 3) $\cdots$
$\pm 6/7$	(7, ± 3)	(14, ± 6), (21, ± 9) $\cdots$
$\vdots$	$\vdots$	$\vdots$

$(l, q) = (1/3, 1/2)$ case:

$\cos\theta_{s/n}$	Fundamental mode (n_0, s_0)	Overtone modes (n, s)
2/3	(2, 1)	(4, 2), (6, 3) $\cdots$
0	(3, 0)	(6, 0), (9, 0) $\cdots$
$-1/3$	(4, -1)	(8, -2), (12, -3) $\cdots$
4/15	(5, 1)	(10, 2), (15, 3) $\cdots$
$-8/15$	(5, -2)	(10, -4), (15, -6) $\cdots$
$-2/3$	(6, -3)	(12, -6), (18, -9) $\cdots$
$-4/21$	(7, -1)	(14, -2), (21, -3) $\cdots$
$\vdots$	$\vdots$	$\vdots$

If the direction $\theta_{s/n}$ is fixed, the gravitational wave is given by superposition as

$$h^{(s/n)}_{+,\times}(t, \rho, \varphi, z) = \sum{}' [\exp(-in\{\Omega(t - \cos\theta_{s/n} z) - \varphi\}) \times \tilde{h}^{(n,s)}_{+,\times}(\rho) + (\mathrm{c.c.})], \tag{5.12}$$

where the summation Σ' is taken over the fundamental mode with frequency $n_0\Omega$ and overtone modes for given $\theta_{s/n}$. As will be shown later, the amplitude of the mode with large n is highly suppressed, then only several discrete directions are effective for gravitational wave emission. The discreteness of the directions is analogous to the diffraction by gratings. This effect comes from the periodic structures of strings. Because the stationary rotating strings considered here have infinite length along the rotation axis, then a distant observer detects gravitational waves coming from discrete directions specified by $\theta_{s/n}$.

In the case of the helical strings $M_l = qN_l/2$ then nonvanishing (n, s) modes specified by (3.33) leads to $\kappa^2_{ns} = 0$. Therefore, the helical strings do not emit the gravitational wave away from the strings.

C. Waveforms

The amplitude of gravitational waves behave as $1/\sqrt{\rho}$ at the far region because the source string is assumed to be infinitely long. Then, it is convenient to factorize a nondimensional quantity $\mu/\sqrt{\Omega\rho}$ as

$$h^{(s/n)}_{+,\times}(t, \rho, \varphi, z) = \left(\frac{\mu}{\sqrt{\Omega\rho}}\right)\hat{h}^{(s/n)}_{+,\times}(t, \rho, \varphi, z), \tag{5.13}$$

equivalently,

$$\tilde{h}^{(n,s)}_{+,\times}(\rho) = \left(\frac{\mu}{\sqrt{\Omega\rho}}\right)\hat{\tilde{h}}^{(n,s)}_{+,\times}(\rho). \tag{5.14}$$

By this rescaling, the amplitudes of $\hat{h}^{(s/n)}_{+,\times}$ are independent of μ, Ω, and ρ in the far region.

In Figs. 5 and 6, we show the waveforms of $\hat{h}^{(s/n)}_{+,\times}$ emitted to the direction $\theta_{s/n}$ by the stationary rotating string with $(l, q) = (0, 1/2)$ (planar string), and $(l, q) = (1/3, 1/2)$, respectively. The solid lines and dashed lines in the right figures denote the waveform of the plus and cross modes, respectively.

We can see some characteristic features of the waveforms from Figs. 5 and 6. First, the waveforms of plus and cross modes are deformed from the sine curves of fundamental modes by the overtone modes. This is because the magnitudes of the overtone modes are not negligible. "Saw-teeth"-like shapes appear in the waveforms.

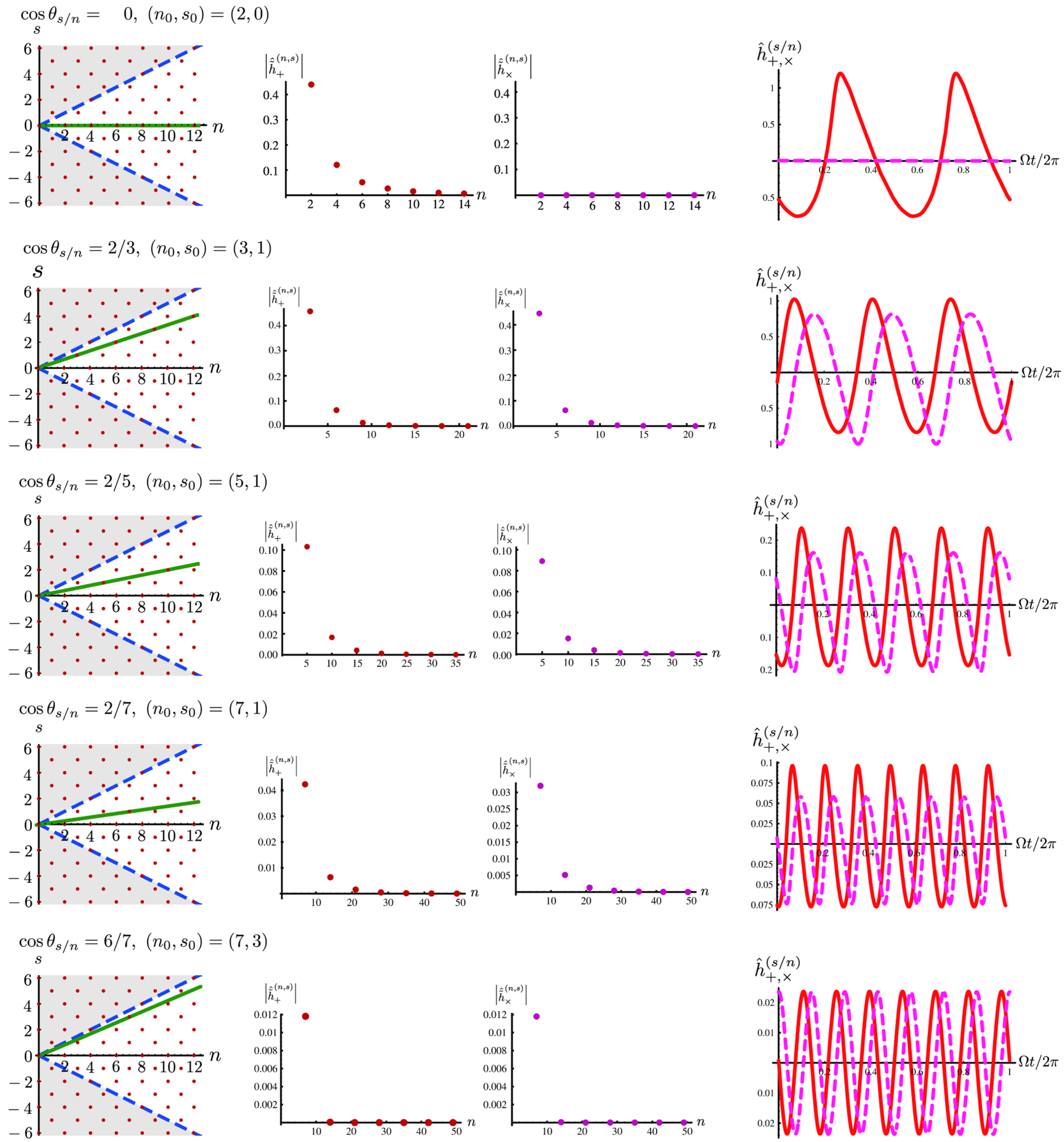

FIG. 5 (color online). Waveforms of gravitational waves emitted to $\cos\theta_{s/n} = 0, 2/3, 2/5, 2/7$, and $6/7$ from a planar string $(l, q) = (0, 1/2)$. In each row, the left panel shows the (n, s) for nonvanishing modes. The dots on the solid lines correspond to the modes which propagate in the direction of $\theta_{s/n}$, i.e., fundamental mode and overtone modes for $\theta_{s/n}$. The amplitudes $|\hat{\tilde{h}}^{(n,s)}_{+,\times}|$ of the fundamental mode and the overtone modes are shown in the middle two panels. The right panel shows the waveforms of the plus-mode (solid line) and cross modes (broken line).

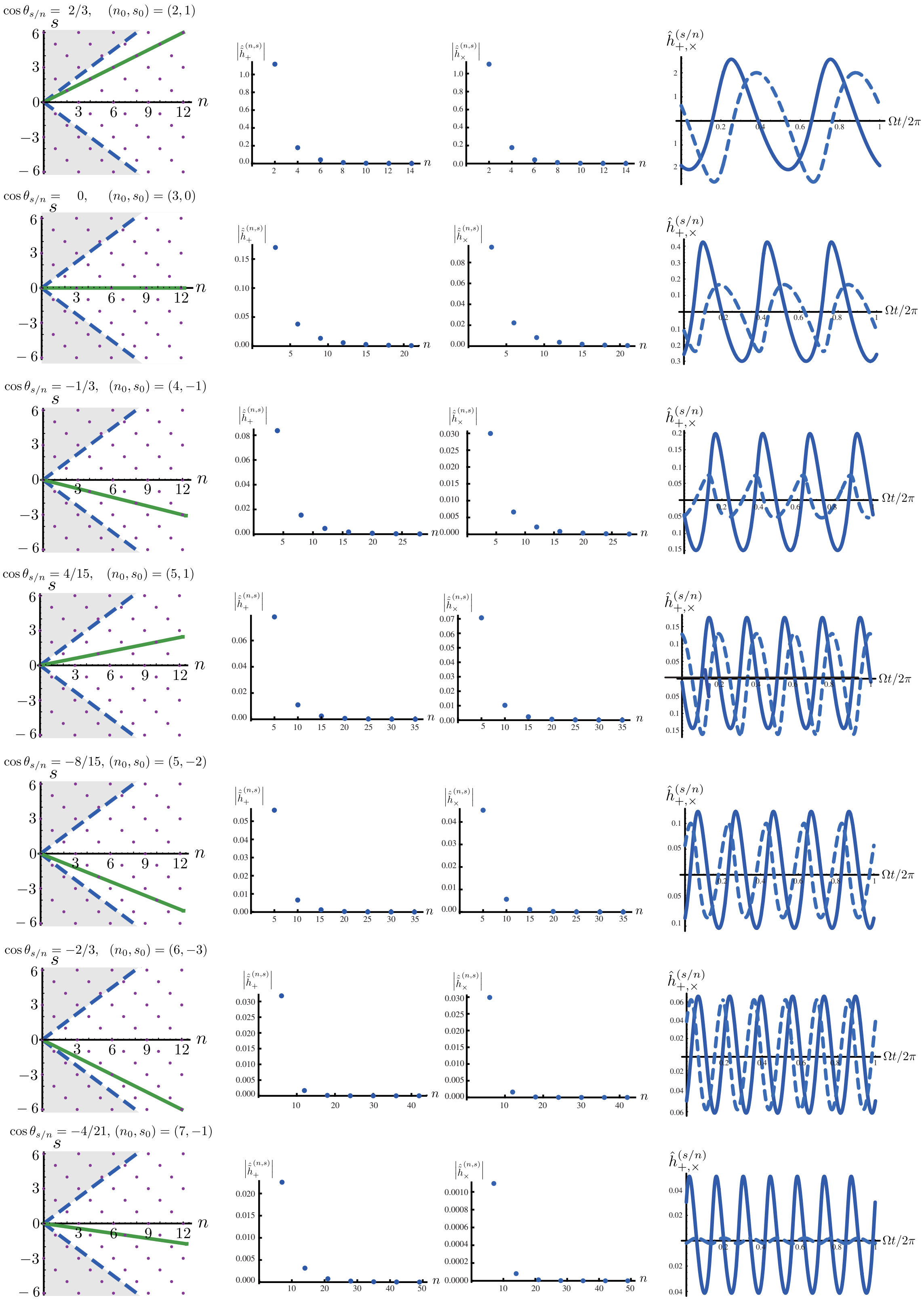

FIG. 6 (color online). Waveforms of gravitational waves emitted to the directions $\cos\theta_{s/n} = 2/3, 0, -1/3, 4/15, -8/15, -2/3$, and $-4/21$ from the string with $(l, q) = (1/3, 1/2)$.

Second, the amplitude of plus modes in each direction, $\hat{h}_+^{(s/n)}$, is determined basically by n_0 of the fundamental mode. The small n_0 gives the large amplitude and the large n_0 does the small amplitude. Third, the amplitude of cross modes, in contrast, depends on the direction $\theta_{s/n}$. The superposition of plus modes and cross modes makes "almost elliptically polarized waves". The gravitational waves are not exactly elliptically polarized because the waves are deformed from the sinusoidal form. The "ellipticity" which is given by the amplitude ratio of plus and cross modes depend on the direction $\theta_{s/n}$.

In the case of planar strings ($l = 0$), purely plus modes are emitted in the direction $\theta_{s/n} = \pi/2$, and the cross modes grow as $|\theta_{s/n}|$ becomes large. When $|\cos\theta_{s/n}|$ approaches to 1, the amplitudes of both modes become almost the same, i.e., the waves become the circular polarization. In the case of string with $(l, q) = (1/3, 1/2)$, the amplitude of cross mode is quite small in the direction $\cos\theta_{s/n} = -4/21$, and the amplitudes of both modes becomes almost the same again as $|\cos\theta_{s/n}|$ approaches to 1.

VI. TRAVELING WAVE MODES

We consider, here, the traveling wave modes $h_{\mu\nu}^{\rm TW}$ given by (3.46), where $\kappa_{ns}^2 = 0$. Since Green's function is real in this case, then $\tilde{h}_{\mu\nu}^{(n,s)}(\rho)$ are real functions of ρ with meaningless phase factors; then, $h_{\mu\nu}^{\rm TW}$ has the form of

$$h_{\mu\nu}^{\rm TW}(t,\rho,\varphi,z) = 2\sum_{n,s}{}'\{\cos(n\{\Omega(t-z)-\varphi\})\tilde{h}_{\mu\nu}^{(n,s)\rm TT}(\rho)\} + 2\sum_{n,-s}{}'\{\cos(n\{\Omega(t+z)-\varphi\}) \times \tilde{h}_{\mu\nu}^{(n,-s)\rm TT}(\rho)\}, \tag{6.1}$$

where the integers n and s are required to satisfy the conditions (3.32) and (3.40). From these two conditions, n should be a positive integer which satisfies

$$\frac{1-l\pm q}{2}n = j, \tag{6.2}$$

where j is a positive integer, and s is given by

$$s = \frac{qN_l}{2}n. \tag{6.3}$$

The condition (6.2) means the parameter q should be a rational number for appearance of traveling wave modes.

The summations in (6.1) are taken over pairs of (n, s) and $(n, -s)$ satisfying the conditions (6.2) and (6.3). For example, in the case of $(l, q) = (0, 1/2)$, the pairs $(n, \pm s)$ are $(4, \pm 2), (8, \pm 4), \cdots$. Nonzero components of $\tilde{h}_{\mu\nu}^{(n,\pm s)\rm TT}$ are

$$\tilde{h}_{\rho\rho}^{(n,\pm s)\rm TT}(\rho) = -\frac{\tilde{h}_{\varphi\varphi}^{(n,s)\rm TT}(\rho)}{\rho^2} = -\frac{2\mu}{qN_l\sigma_{\rm p}}\int_0^{N_l\sigma_{\rm p}} d\sigma[\{G_{n+2}^{n\pm s}(\rho,\rho_{\rm st}(\sigma)) \times \Theta_{\rm T+}(\rho_{\rm st}(\sigma)) + G_{n-2}^{n\pm s}(\rho,\rho_{\rm st}(\sigma)) \times \Theta_{\rm T-}(\rho_{\rm st}(\sigma))\}\exp(-in\{\pm\Omega q\sigma + \bar{\varphi}(\sigma)\})], \tag{6.4}$$

$$\frac{\tilde{h}_{\rho\varphi}^{(n,\pm s)\rm TT}(\rho)}{\rho} = -\frac{2\mu i}{qN_l\sigma_{\rm p}}\int_0^{N_l\sigma_{\rm p}} d\sigma[\{G_{n+2}^{n\pm s}(\rho,\rho_{\rm st}(\sigma)) \times \Theta_{\rm T+}(\rho_{\rm st}(\sigma)) - G_{n-2}^{n\pm s}(\rho,\rho_{\rm st}(\sigma)) \times \Theta_{\rm T-}(\rho_{\rm st}(\sigma))\}\exp(-in\{\pm\Omega q\sigma + \bar{\varphi}(\sigma)\})], \tag{6.5}$$

where Green's functions are given by (3.36).

The modes $h_{\mu\nu}^{\rm TW}$ given by (6.1) consist of the superposition of the propagating waves with circular polarization in the $(\pm z)$ direction for $\pm s$, respectively. We can understand that these modes are obtained by the limit $\kappa_{ns}^2 \to 0$ in the gravitational wave modes, that is, the direction of wave emission in this limit is $\theta_{s/n} = 0, \pi$. The metric perturbations of the modes do not propagate off the string toward the radial direction. These are related to the traveling waves discussed in Ref. [35].

In the helical string cases, from (3.33), the first line in the right-hand side of (6.1) vanishes and summation in the second line is taken over pairs $(n, -s)$, where n is a positive integer and s is given by

$$s = \frac{(1-l)N_l}{2}n = M_l n. \tag{6.6}$$

There exists only a downward gravitational wave which is accompanied with the downward string wave (2.11).

The wave length of each wave propagating along the z axis in the traveling wave modes is

$$\lambda = \frac{2\pi}{n\Omega}. \tag{6.7}$$

Then, the condition (6.3) for the appearance of the traveling wave mode means that the periodicity of the stationary rotating string, which is given by (2.16), should be the wavelength of traveling wave times the integer s, i.e.,

$$Z_{\rm p} = s\lambda. \tag{6.8}$$

This fact is consistent with the result in Ref. [38] which implies that the deformation of the string is caused by the gravitational waves propagating on the string.

VII. SUMMARY

We have studied gravitational perturbations around a stationary rotating string in Minkowski spacetime. We have solved the linearized Einstein equations with the energy-momentum tensor of the string by using the one-dimensional Green's function method. We have analyzed three long-range modes: potential mode, gravitational wave modes, and traveling wave modes.

A. Potential mode

The stationary rotating strings produce the logarithmic Newtonian potential which is in proportion to $\tilde{\mu} - \tilde{\mathcal{T}}$, where $\tilde{\mu}$ and $\tilde{\mathcal{T}}$ denote the effective line density and the effective tension of a stationary rotating "wiggly" string defined by averaging of the energy-momentum tensor along its rotation axis. The appearance of the Newtonian potential is the result of the fact that the effective line density becomes larger than the effective tension for rotating strings. There also exists angular deficit, $4\pi(\tilde{\mu} + \tilde{\mathcal{T}})$, around the string.

In addition, there are the azimuthal frame-dragging effect caused by the angular momentum of the rotating string and the linear frame-dragging along the rotation axis caused by the linear momentum $\langle P \rangle$ of the string along the rotation axis. The linear frame-dragging disappears if $\langle P \rangle = 0$ in the inertial reference frame of observer.

The helical strings are very special strings. Since $\tilde{\mu} = \tilde{\mathcal{T}}$ for the helical strings, they are not associated with the Newtonian potential, and there is an angular deficit with the same amount as the straight string case. Further, the helical strings cause the linear frame-dragging inevitably because there is no inertial reference frame such that $\langle P \rangle = 0$. The azimuthal frame-dragging and the linear frame-dragging distinguish the helical strings from the straight string.

B. Gravitational wave modes

The stationary rotating strings can emit the gravitational waves in several discrete directions. The possible directions for each string are determined by the set of parameters (l, q) which specifies the shape of string. This property, analogous to the diffraction grating, comes from the periodic structure of the strings along the rotation axis. The following depend on the directions of gravitational wave emission: fundamental frequency, waveforms, amplitude ratio between plus and cross modes, equivalently, and the ellipticity of elliptic polarization of the waves. The waveform of gravitational wave is not the sinusoidal curve but a "saw-teeth" like shape. This means that the polarization is not exactly elliptical but almost elliptical.

Since the strings are infinitely long, the amplitude of gravitational waves at the large distance is proportional to $1/\sqrt{\rho}$. Actually, infinite strings are oversimplification. But, if the description of the stationary rotating strings is applicable to a cosmological string in the long range comparable to the distance between the string and a observer, the amplitude of gravitational waves decreases more gradually than the case of point source. In this case, it would be possible to detect gravitational waves from the stationary rotating strings in the cosmological distance (e.g., $\sim 10^3$ Mpc) by the present interferometric detectors. As the result of the numerical calculations, we have obtained the following rough estimation of the gravitational wave amplitude:

$$h_{+,\times} \simeq \mathcal{O}(10^{-14})\left(\frac{\mu}{10^{-7}}\right)\left(\frac{\Omega/2\pi}{10^3\ \mathrm{Hz}}\right)^{-1/2}\left(\frac{\rho}{10^3\ \mathrm{Mpc}}\right)^{-1/2}, \tag{7.1}$$

where we choose 10^{-7} as a reference line density of the grand unified theory string, 10^3 Hz as a reference frequency, the most sensitive value of the current interferometric detectors (TAMA300, LIGO, VIRGO, and GEO600), and 10^3 Mpc as a reference cosmological distance.

C. Traveling wave modes

As with the special case of gravitational waves, traveling waves with the circular polarization propagating along the rotating string can appear. The strings play the role of wave guide then the amplitude of the gravitational wave does not decrease along the string. These waves do not propagate off the string toward distant observers, but the waves are not confined in the vicinity of the string. The amplitude of the traveling waves, described by the power or logarithmic function in the radial coordinate, gradually decreases as the distance from the string increases. Then, even for the distant observer, it would be detectable as the gravitational waves propagate parallelly to the strings.

The general stationary rotating strings lose the energy, angular momentum, and linear momentum by the gravitational wave emission. Then, the strings should evolve by the gravitational radiation. If the loss rate of these quantities are small, we can expect that the evolution occurs as the transitions in the family of the stationary rotating strings, approximately. What is the final state of strings after the gravitational wave emission? One would expect that the straight string is the final state. But, we should point out that the helical strings are also candidates for the final states. Because, they do not lose energy, angular momentum, and linear momentum by the gravitational radiation. They keep the rotation constant with traveling waves. The study on the final state of the stationary rotating strings is under investigation [39].

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APPENDIX: THE COMPONENTS OF $\Theta^{\mu\nu}$

The components of $\Theta^{\mu\nu}$ are explicitly expressed in the following:

$$\Theta^{tt} = -(1 - l^2), \qquad \Theta^{t\rho} = \frac{l\Omega}{2\rho}(\rho_{\max}^2 - \rho_{\min}^2)\sin(2|\Omega|\sigma), \qquad \rho\Theta^{t\varphi} = -\frac{\Omega^2\rho^2 - l^2}{\Omega\rho}, \qquad \Theta^{tz} = ql\,\mathrm{sign}(\Omega),$$
$$\Theta^{\rho\rho} = \frac{\Omega^2}{4\rho^2}(\rho_{\max}^2 - \rho_{\min}^2)^2\sin^2(2|\Omega|\sigma), \qquad \rho\Theta^{\rho\varphi} = \frac{l}{2\rho^2}(\rho_{\max}^2 - \rho_{\min}^2)\sin(2|\Omega|\sigma), \tag{A1}$$
$$\Theta^{\rho z} = \frac{q|\Omega|}{2\rho}(\rho_{\max}^2 - \rho_{\min}^2)\sin(2|\Omega|\sigma), \qquad \rho^2\Theta^{\varphi\varphi} = -\Omega^2\rho^2\left(1 - \frac{l^2}{\Omega^4\rho^4}\right), \qquad \rho\Theta^{\varphi z} = \frac{lq}{|\Omega|\rho}, \qquad \Theta^{zz} = q^2.$$

In these expressions, $\rho = \rho(\sigma)$ is given by (2.5).

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