Rotation of linear polarization plane and circular polarization from cosmological pseudoscalar fields

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We discuss the rotation of the linear polarization plane and the production of circular polarization generated by a cosmological pseudoscalar field. We compute analytically and numerically the propagation of the Stokes parameters from the last scattering surface for an oscillating and a monotonic decreasing pseudoscalar field. For the models studied in this paper, we show the comparison between the widely used approximation in which the linear polarization rotation angle is constant in time and the exact result.

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I. INTRODUCTION

In 1977 R. Peccei and H. Quinn [1] suggested a solution to the strong *CP*-problem of QCD introducing a new symmetry breaking at a given energy scale f_a. The boson associated with this broken global symmetry was called *axion*. All the physical properties of this pseudoscalar field strongly depend on the energy scale f_a at which the new symmetry is broken: the particle mass and the coupling constants with other particles are inversely proportional to f_a. Pseudo-Goldstone bosons also arise in many particle physics scenarios [2].

Axions and, in general, other pseudoscalar particles are among the most favored particle physics candidates for the cold dark matter (CDM) [3–5]. They interact with photons according to the Lagrangian:

$$\mathcal{L}_{\rm int} = -\frac{g_\phi}{4}\phi F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{1}$$

where g_ϕ is the coupling constant, $F^{\mu\nu}$ is the electromagnetic tensor, and $\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ its dual. Many constraints on axion derive from this interaction with photons: laboratory experiments (photon-axion conversion experiments) and astrophysical arguments (stellar evolution of red giants) constrain g_ϕ to be small. One of the most stringent experimental bound ($g_\phi < 8.8 \times 10^{-20}$ eV^{-1} for $m_a < 0.02$ eV) is obtained by the CAST experiment [6] constraining the axion-photon conversion for solar axions. This limit supersedes the one obtained from the duration of the helium burning time in horizontal-branch stars in globular clusters: $g_\phi \lesssim 10^{-19}$ eV^{-1} [4,7].

In this paper we wish to study in detail the coupling of such a pseudoscalar field with photons. The interaction in Eq. (1) modifies the polarization of an electromagnetic wave propagating along intervening magnetic fields, or through a slowly varying background field ϕ [8]. Here we are interested in the second case, which does not require the presence of a magnetic field (note that in the first case the polarization is also modified in absence of axions, an effect known as Faraday rotation). We consider the time-dependent pseudoscalar condensate as dark matter or part of it and study the impact of its time derivative on the polarization of the photons. As a consequence of its coupling with a pseudoscalar field, the plane of linear polarization of light is rotated (*cosmological birefringence*) [9,10].

In the case of cosmic microwave background (CMB) photons, we pay attention to the rotation along the path between the last scattering surface (LSS) and the observer, modifying the polarization pattern generated by Thomson scattering at LSS [11]. This rotation induced by the pseudoscalar interaction modifies the gradient and curl of the polarization pattern (E and B following [12]), creating B modes from E modes. The parity violating nature of the interaction generates nonzero parity-odd correlators (TB and EB) which would be otherwise vanishing for the standard Gaussian cosmological case [13,14]. In particular the TB power spectrum may be very useful to constrain the coupling constant g_ϕ between photons and pseudoscalars, since it is larger than the auto and cross power spectra in polarization; in general, these nonstandard correlators are already constrained by present data sets [15–17].

We study two representative examples for the dynamics of a pseudo-Goldstone field behaving as dark matter (see [18] for a pseudoscalar field model of dark energy): the oscillating and a monotonic decreasing behavior. In the latest case we study analytically the problem, whereas in the former numerically and analytically. The case of a field growing linearly in time has been studied in [19]. We compare the polarization power spectra obtained describing the rotation of linear polarization with a time-dependent angle with the ones obtained considering a constant rotation angle.

Our paper is organized as follows. We write the relevant equations for the electromagnetic gauge potential coupled to a pseudoscalar field in Sec. II. We review the Stokes parameters for a monochromatic electromagnetic plane wave and the Boltzmann equation for CMB photons coupled to pseudoscalars in Sec. III. In Sec. IV we write the Stokes parameters in terms of the left and right polarizations gauge potential and solve the differential equations for the latter for oscillating behavior of the pseudoscalar field. In a similar way Sec. V is dedicated to the monotonic behavior of the pseudoscalar field. In Sec. VI we test the constant rotation angle approximation. We conclude in Sec. VII. We work in units where the speed of light is equal to one ($c = 1$).

II. ELECTRODYNAMICS COUPLED TO A PSEUDOSCALAR FIELD

The Lagrangian density $\mathcal{L}$ for the photons and the pseudoscalar field ϕ is [20] (following the notation of [21]):

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi) - \frac{g_\phi}{4}\phi F_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{2}$$

The Euler-Lagrange equations resulting from this Lagrangian are:

$$\Box\phi \equiv \nabla_\mu\nabla^\mu\phi = \frac{dV}{d\phi} + \frac{g_\phi}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{3}$$

$$\nabla_\mu F^{\mu\nu} = -g_\phi(\nabla_\mu\phi)\tilde{F}^{\mu\nu}, \tag{4}$$

$$\nabla_\mu\tilde{F}^{\mu\nu} = 0. \tag{5}$$

Using the definition of the electromagnetic tensor $F^{\mu\nu} \equiv \nabla^\mu A^\nu - \nabla^\nu A^\mu$ Eq. (4) becomes:

$$\Box A_\nu - \nabla_\nu(\nabla_\mu A^\mu) - R^\mu_\nu A_\mu = -\frac{g_\phi}{2}(\nabla_\mu\phi)\epsilon^\mu{}_\nu{}^{\rho\sigma}F_{\rho\sigma}. \tag{6}$$

The complete antisymmetric tensor contain the determinant of the metric g and $[\cdots]$ guarantees antisymmetry in the four indexes [22]:

$$\epsilon_{\alpha\beta\gamma\delta} = \sqrt{-g}[\alpha\beta\gamma\delta], \tag{7}$$

$$\epsilon^{\alpha\beta\gamma\delta} = -(\sqrt{-g})^{-1}[\alpha\beta\gamma\delta]. \tag{8}$$

For a spatially flat Friedmann-Robertson-Walker universe the metric is:

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2 = a^2(\eta)[-d\eta^2 + d\mathbf{x}^2], \tag{9}$$

where t is the cosmic time, η is conformal time, and $\mathbf{x}$ denotes the space coordinates. We consider a plane wave propagating along $\hat{\mathbf{n}}$ in Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$). If $\hat{\mathbf{n}}$ is aligned with the z axis and neglecting the spatial variation of the pseudoscalar field $\phi = \phi(\eta)$, the two relevant components of Eq. (6) are:

$$A''_x(\eta, z) - \frac{\partial^2 A_x(\eta, z)}{\partial z^2} = g_\phi\phi'\frac{\partial A_y(\eta, z)}{\partial z}, \tag{10}$$

$$A''_y(\eta, z) - \frac{\partial^2 A_y(\eta, z)}{\partial z^2} = -g_\phi\phi'\frac{\partial A_x(\eta, z)}{\partial z}. \tag{11}$$

Defining Fourier transform as $\tilde{A}_{x,y}(k, \eta) = (2\pi)^{-1} \times \int e^{ikz}A_{x,y}(\eta, z)dz$ the previous equations become:

$$\tilde{A}''_x(k, \eta) + k^2\tilde{A}_x(k, \eta) + g_\phi\phi' ik\tilde{A}_y(k, \eta) = 0, \tag{12}$$

$$\tilde{A}''_y(k, \eta) + k^2\tilde{A}_y(k, \eta) - g_\phi\phi' ik\tilde{A}_x(k, \eta) = 0. \tag{13}$$

where k is the Fourier conjugate of z. These equations can be decoupled introducing $\tilde{A}_\pm(k, \eta) = \tilde{A}_x(k, \eta) \pm i\tilde{A}_y(k, \eta)$, left and right components of the electromagnetic vector potential:

$$\tilde{A}''_\pm(k, \eta) + [k^2 \pm g_\phi\phi' k]\tilde{A}_\pm(k, \eta) = 0. \tag{14}$$

III. STANDARD REVIEW OF STOKES PARAMETERS AND BOLTZMANN EQUATION

A. Stokes parameters

The complex electric field vector for a plane wave propagating along $\hat{\mathbf{z}}$ direction at a point (x, y) in some transverse plane $z = z_0$ is:

$$\mathbf{E} = (E_x(t), E_y(t)) = [\hat{\mathbf{e}}_x\varepsilon_x(t)e^{i\varphi_x(t)} + \hat{\mathbf{e}}_y\varepsilon_y(t)e^{i\varphi_y(t)}]e^{-ikt}, \tag{15}$$

where the physical quantity is the real part of $\mathbf{E}$. For a spatially flat Friedmann-Robertson-Walker metric the relation between the electromagnetic tensor and the physical fields is:

$$F_{\mu\nu} = a(\eta)\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{16}$$

In general we consider quasimonochromatic waves: the amplitudes ($\varepsilon_x(t)$ and $\varepsilon_y(t)$) and the phases ($\varphi_x(t)$ and $\varphi_y(t)$) are slowly varying functions of time respect to the inverse frequency of the wave.

The Stokes parameters I, Q, U, and V are defined as:

$$I \equiv \frac{1}{a^2}(\langle E^*_x(t)E_x(t)\rangle + \langle E^*_y(t)E_y(t)\rangle), \tag{17}$$

$$Q \equiv \frac{1}{a^2}(\langle E^*_x(t)E_x(t)\rangle - \langle E^*_y(t)E_y(t)\rangle), \tag{18}$$

$$U \equiv \frac{1}{a^2}(\langle E^*_x(t)E_y(t)\rangle + \langle E^*_y(t)E_x(t)\rangle) = \frac{2}{a^2}\langle\varepsilon_x\varepsilon_y\cos(\varphi_x - \varphi_y)\rangle, \tag{19}$$

$$V \equiv -\frac{i}{a^2}(\langle E_x^*(t)E_y(t)\rangle - \langle E_y^*(t)E_x(t)\rangle) = \frac{2}{a^2}\langle \varepsilon_x \varepsilon_y \sin(\varphi_x - \varphi_y)\rangle. \quad (20)$$

where $\langle \cdots \rangle$ denote the *ensemble average*, the average over all possible realizations of a given quasimonochromatic wave. For a pure monochromatic wave ensemble averages can be omitted and the wave is completely polarized:

$$I^2 - Q^2 - U^2 - V^2 = 0. \quad (21)$$

The parameter I gives the total intensity of the radiation, Q and U describe linear polarization and V circular polarization. Linear polarization can also be characterized through a vector of modulus:

$$P_L \equiv \sqrt{Q^2 + U^2}, \quad (22)$$

and an angle θ, defined as:

$$\theta \equiv \frac{1}{2}\arctan\frac{U}{Q}. \quad (23)$$

It is important to underline that I and V are physical observables, since they are independent on the particular orientation of the reference frame in the plane perpendicular to the direction of propagation $\hat{\mathbf{n}}$, while Q and U depend on the orientation of this basis [23]. After a rotation of the reference frame of an angle θ ($R(\theta)$) they transform according to:

$$Q \overset{R(\theta)}{\longrightarrow} Q\cos(2\theta) + U\sin(2\theta), \quad U \overset{R(\theta)}{\longrightarrow} -Q\sin(2\theta) + U\cos(2\theta). \quad (24)$$

Also linear polarization, like total intensity and circular polarization, can be described through quantities independent on the orientation of the reference frame in the plane perpendicular to the direction of propagation of the wave. In the context of CMB anisotropies, the linear polarization vector field is usually described in terms of a gradient-like component (*E mode*) and of a curl-like component (*B mode*).

In a similar way it is possible to describe the electric vector field in the $x - y$ plane through a superposition of left and right circular polarized waves defining:

$$\hat{\mathbf{e}}_+ \equiv \frac{\hat{\mathbf{e}}_x + i\hat{\mathbf{e}}_y}{\sqrt{2}} \quad \text{and} \quad \hat{\mathbf{e}}_- \equiv \frac{\hat{\mathbf{e}}_x - i\hat{\mathbf{e}}_y}{\sqrt{2}}. \quad (25)$$

In this new basis:

$$I \equiv \frac{1}{a^2}(\langle E_+^*(t)E_+(t)\rangle + \langle E_-^*(t)E_-(t)\rangle), \quad (26)$$

$$Q \equiv \frac{1}{a^2}(\langle E_+^*(t)E_-(t)\rangle + \langle E_-^*(t)E_+(t)\rangle) = \frac{2}{a^2}\langle \varepsilon_+ \varepsilon_- \cos(\varphi_+ - \varphi_-)\rangle, \quad (27)$$

$$U \equiv -\frac{i}{a^2}(\langle E_+^*(t)E_-(t)\rangle - \langle E_-^*(t)E_+(t)\rangle) = \frac{2}{a^2}\langle \varepsilon_+ \varepsilon_- \sin(\varphi_+ - \varphi_-)\rangle, \quad (28)$$

$$V \equiv \frac{1}{a^2}(\langle E_+^*(t)E_+(t)\rangle - \langle E_-^*(t)E_-(t)\rangle). \quad (29)$$

The relation between the vector potential and the electric field for a wave propagating in a charge-free region is:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mathbf{A}'}{a}, \quad (30)$$

According to definition given in the previous section the Stokes parameters in terms of the vector potential are:

$$I = \frac{1}{a^4}(\langle A_+'^* A_+' \rangle + \langle A_-'^* A_-' \rangle), \quad (31)$$

$$Q = \frac{1}{a^4}(\langle A_+'^* A_-' \rangle + \langle A_-'^* A_+' \rangle) = \frac{2}{a^4}\Re(\langle A_+'^* A_-' \rangle), \quad (32)$$

$$U = -\frac{i}{a^4}(\langle A_+'^* A_-' \rangle - \langle A_-'^* A_+' \rangle) = \frac{2}{a^4}\Im(\langle A_+'^* A_-' \rangle), \quad (33)$$

$$V = \frac{1}{a^4}(\langle A_+'^* A_+' \rangle - \langle A_-'^* A_-' \rangle). \quad (34)$$

As we shall see in more detail in the following section, the coupling to a cosmological pseudoscalar field induce a physical time-dependent rotation of the plane of linear polarization along the line of sight, described by:

$$Q' = 2\theta'(\eta)U \quad \text{and} \quad U' = -2\theta'(\eta)Q, \quad (35)$$

whose solution is:

$$Q = Q_i \cos 2\theta + U_i \sin 2\theta, \quad U = -Q_i \sin 2\theta + U_i \cos 2\theta. \quad (36)$$

where Q_i, U_i are the Stokes parameters at initial time which would be otherwise unchanged in absence of the interaction with the pseudoscalar field.

B. Boltzmann equation and cosmological birefringence

In the Boltzmann equations for linear polarization of the radiation density contrast averaged over momenta contains a mixing term:

$$2\theta' = g\phi', \quad (37)$$

due to the pseudoscalar interaction [11]; the Boltzmann equation for spin-2 functions $Q \pm iU$ is:

$$\Delta'_{Q\pm iU}(k,\eta) + ik\mu\Delta_{Q\pm iU}(k,\eta) = -n_e\sigma_T a(\eta)\Big[\Delta_{Q\pm iU}(k,\eta) + \sum_m \sqrt{\frac{6\pi}{5}}\,{}_{\pm 2}Y_2^m S_P^{(m)}(k,\eta)\Big] \mp i2\theta'(\eta)\Delta_{Q\pm iU}(k,\eta). \quad (38)$$

where μ is the cosine of the angle between the CMB photon direction and the Fourier wave vector, n_e is the number density of free electrons, σ_T is the Thomson cross section, sY_2^m are spherical harmonics with spin-weight s, and $S_P^{(m)}(k,\eta)$ is the source term for generating linear polarization reported in [24] ($m = 0, \pm 1, \pm 2$ corresponds, respectively, to scalar, vector, and tenor perturbations):

$$S_P^{(m)}(k,\eta) = \Delta_{T2}^{(m)}(k,\eta) + 12\sqrt{6}\Delta_{+,2}^{(m)}(k,\eta) + 12\sqrt{6}\Delta_{-,2}^{(m)}(k,\eta). \quad (39)$$

$\Delta_{Tl}^{(m)}$ and $\Delta_{\pm,l}^{(m)}$ are the Fourier transforms of the coefficients of the following series:

$$\Delta_T(\mathbf{x},\hat{\mathbf{n}},\eta) = \sum_{lm}(-i)^l\sqrt{4\pi(2l+1)}\Delta_{Tl}^{(m)}(\mathbf{x},\eta)Y_l^m(\hat{\mathbf{n}}), \quad (40)$$

$$\Delta_{Q\pm iU}(\mathbf{x},\hat{\mathbf{n}},\eta) = \sum_{lm}(-i)^l\sqrt{4\pi(2l+1)}\Delta_{\pm,\ell}^{(m)}(\mathbf{x},\eta)\,{}_{\pm 2}Y_l^m(\hat{\mathbf{n}}), \quad (41)$$

Note that Eq. (38) corrects some typos in Eq. (1) of Ref. [18].

The quantity $\Delta_{Q\pm iU}$ is related to the rotation invariant polarization fields Δ_E and Δ_B through the spin raising ($\eth$) and lowering ($\bar{\eth}$) operators:

$$\Delta_E \equiv -\frac{1}{2}(\bar{\eth}^2\Delta_{Q+iU} + \eth^2\Delta_{Q-iU}), \quad (42)$$

$$\Delta_B \equiv -\frac{i}{2}(\bar{\eth}^2\Delta_{Q+iU} - \eth^2\Delta_{Q-iU}). \quad (43)$$

Following the line of sight strategy for scalar perturbations we obtain, in agreement with Ref. [18]:

$$\Delta_T(k,\eta_0) = \int_{\eta_{\rm rec}}^{\eta_0} d\eta g(\eta)S_T(k,\eta)j_\ell(k\eta_0 - k\eta), \quad (44)$$

$$\Delta_E(k,\eta_0) = \int_{\eta_{\rm rec}}^{\eta_0} d\eta g(\eta)S_P^{(0)}(k,\eta)\frac{j_\ell(k\eta_0 - k\eta)}{(k\eta_0 - k\eta)^2} \times \cos[2\theta(\eta)], \quad (45)$$

$$\Delta_B(k,\eta_0) = \int_{\eta_{\rm rec}}^{\eta_0} d\eta g(\eta)S_P^{(0)}(k,\eta)\frac{j_\ell(k\eta_0 - k\eta)}{(k\eta_0 - k\eta)^2} \times \sin[2\theta(\eta)]. \quad (46)$$

where $g(\eta)$ is the visibility function, $S_T(k,\eta)$ is the source term for temperature anisotropies, and j_ℓ is the spherical Bessel function. The polarization C_ℓ auto- and cross-spectra are given by:

$$C_\ell^{EE} = (4\pi)^2\frac{9(\ell+2)!}{16(\ell-2)!}\int k^2dk[\Delta_E(k,\eta_0)]^2, \quad (47)$$

$$C_\ell^{BB} = (4\pi)^2\frac{9(\ell+2)!}{16(\ell-2)!}\int k^2dk[\Delta_B(k,\eta_0)]^2, \quad (48)$$

$$C_\ell^{EB} = (4\pi)^2\frac{9(\ell+2)!}{16(\ell-2)!}\int k^2dk\Delta_E(k,\eta_0)\Delta_B(k,\eta_0), \quad (49)$$

$$C_\ell^{TE} = (4\pi)^2\sqrt{\frac{9(\ell+2)!}{16(\ell-2)!}}\int k^2dk\Delta_T(k,\eta_0)\Delta_E(k,\eta_0), \quad (50)$$

$$C_\ell^{TB} = (4\pi)^2\sqrt{\frac{9(\ell+2)!}{16(\ell-2)!}}\int k^2dk\Delta_T(k,\eta_0)\Delta_B(k,\eta_0). \quad (51)$$

In the approximation in which $\theta = \bar{\theta}$, with $\bar{\theta}$ constant in time, Eqs. (45) and (46) simplify since $\cos[2\bar{\theta}]$, $\sin[2\bar{\theta}]$ can be extracted from the integral along the line of sight and:

$$\Delta_E^{\rm obs} = \Delta_E(\theta = 0)\cos(2\bar{\theta}), \quad (52)$$

$$\Delta_B^{\rm obs} = \Delta_E(\theta = 0)\sin(2\bar{\theta}), \quad (53)$$

and the power spectra are given by [11,15]:

$$C_\ell^{EE,\rm obs} = C_\ell^{EE}\cos^2(2\bar{\theta}), \quad (54)$$

$$C_\ell^{BB,\rm obs} = C_\ell^{EE}\sin^2(2\bar{\theta}), \quad (55)$$

$$C_\ell^{EB,\rm obs} = \frac{1}{2}C_\ell^{EE}\sin(4\bar{\theta}), \quad (56)$$

$$C_\ell^{TE,\rm obs} = C_\ell^{TE}\cos(2\bar{\theta}), \quad (57)$$

$$C_\ell^{TB,\rm obs} = C_\ell^{TE}\sin(2\bar{\theta}). \quad (58)$$

The expression for $\bar{\theta}$ to insert in Eqs. (52)–(58) is:

$$\bar{\theta} = \frac{g_\phi}{2}[\phi(\eta_0) - \phi(\eta_{\rm rec})]. \quad (59)$$

Several limits on the constant rotation angle $\bar{\theta}$ have been already obtained using current observation of CMBP (see Table I).

This time-independent rotation angle approximation is an operative approximation which allows to write Eqs. (53), is clearly inconsistent since for $\theta =$ const the term proportional to θ' in the Boltzmann Eq. (38) vanishes

TABLE I. Constraints on linear polarization rotation $\bar{\theta}$ in the constant angle approximation.

Data set	$\bar{\theta}(2\sigma)$[deg]
WMAP3 and Boomerang (B03) [15]	$-13.7 < \bar{\theta} < 1.9$
WMAP3 [16]	$-8.5 < \bar{\theta} < 3.5$
WMAP5 [17]	$-5.9 < \bar{\theta} < 2.4$
QUaD [25]	$-1.2 < \bar{\theta} < 3.9$

and therefore there is no rotation of the linear polarization plane. See Figs. 6 and 7 for a comparison of this approximation with a full Boltzmann description of the birefringence effect for a dynamical pseudoscalar field.

IV. COSINE-TYPE POTENTIAL

In this section we assume that dark matter is given by massive axions, ϕ is governed by the potential [3]:

$$V(\phi) = m^2 \frac{f_a^2}{N^2}\left(1 - \cos\frac{\phi N}{f_a}\right), \tag{60}$$

where N is the color anomaly of the Peccei-Quinn symmetry. Here we are interested in the regime where the axion field oscillates near the minimum of the potential (for simplicity we shall consider $N = 1$ in the following): $\phi/f_a \ll 1$ and the potential can be approximated with $V(\phi) \simeq m^2\phi^2/2$. In this case $\phi(t)$ satisfies the equation:

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0. \tag{61}$$

When $m > 3H$ the scalar field begins to oscillate, and the solution in a matter dominated universe ($\dot{a}/a = 2/3t$) is [26]:

$$\phi(t) = t^{-1/2}[c_1 J_{1/2}(mt) + c_2 J_{-1/2}(mt)] \overset{mt\gg 1}{\simeq} \frac{\phi_0}{mt}\sin(mt), \tag{62}$$

where the time-independent coefficients of the Bessel functions c_1, c_2 depend on the initial conditions.

The averaged energy and pressure densities associated with the field are:

$$\bar{\rho}_\phi = \frac{\overline{\dot{\phi}^2}}{2} + \frac{1}{2}m^2\bar{\phi}^2 \overset{mt\gg 1}{\simeq} \frac{\phi_0^2}{2t^2}\left[1 + \mathcal{O}\left(\frac{1}{mt}\right)^2\right], \tag{63}$$

$$\bar{p}_\phi = \frac{\overline{\dot{\phi}^2}}{2} - \frac{1}{2}m^2\bar{\phi}^2 \overset{mt\gg 1}{\simeq} \frac{\phi_0^2}{2t^2} \times \mathcal{O}\left(\frac{1}{mt}\right)^2, \tag{64}$$

where $^-$ denotes the average over an oscillation period of the axion condensate. Note that we are implicitly assuming that the pseudoscalar field is homogeneous. In the context of axion physics, this means that in our observable universe we have just one value for the misalignment angle, which means that the PQ symmetry has occurred before or during inflation.

We fix the constant ϕ_0 comparing ρ_ϕ with the energy density in a matter dominated universe:

$$\rho_M = \frac{3H^2M_{\rm pl}^2}{8\pi} = \frac{M_{\rm pl}^2}{6\pi t^2} \Rightarrow \phi_0 = \frac{M_{\rm pl}}{\sqrt{3\pi}}, \tag{65}$$

$$\phi(t) \simeq \frac{M_{\rm pl}}{\sqrt{3\pi}mt}\sin(mt), \tag{66}$$

where $M_{\rm pl} \simeq 1.22 \times 10^{19}$ GeV is the Planck mass.

Using the relation between cosmic and conformal time in a universe of matter:

$$t = \frac{\eta_0}{3}\left(\frac{\eta}{\eta_0}\right)^3, \tag{67}$$

we find the following approximation for $\phi(\eta)$:

$$\phi(\eta) \simeq \sqrt{\frac{3}{\pi}}\frac{M_{\rm pl}}{m\eta_0(\frac{\eta}{\eta_0})^3}\sin\left[m\frac{\eta_0}{3}\left(\frac{\eta}{\eta_0}\right)^3\right], \tag{68}$$

and

$$\phi'(\eta) \simeq \sqrt{\frac{3}{\pi}}\frac{M_{\rm pl}}{\eta}\left\{\cos\left[m\frac{\eta_0}{3}\left(\frac{\eta}{\eta_0}\right)^3\right] - \frac{3\eta_0^2}{m\eta^3}\sin\left[m\frac{\eta_0}{3}\left(\frac{\eta}{\eta_0}\right)^3\right]\right\}. \tag{69}$$

If m is not too small the value of $\mathcal{H} \equiv a'/a$ obtained with the scalar field density in the Friedmann equation coincides with that of a matter dominated universe $\mathcal{H} = 2/\eta$ once the average through oscillations is performed [27] (see Fig. 1). The derivative can be replaced in Eq. (14) for the evolution of the gauge potential:

$$\tilde{A}''_\pm(k,\eta) + k^2[1 \pm \Delta(\eta; g_\phi, m, k, \eta_0)]\tilde{A}_\pm(k,\eta) = 0, \tag{70}$$

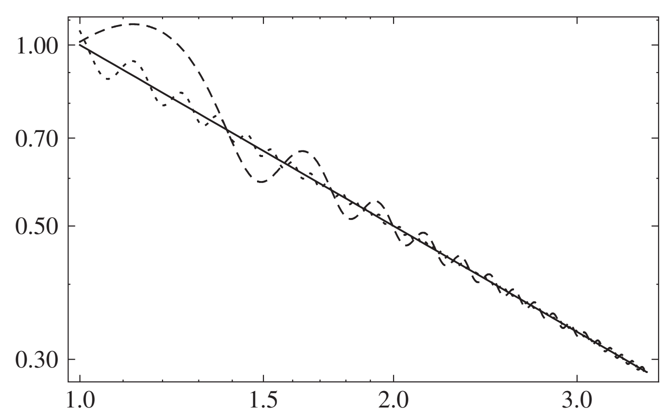

FIG. 1. Evolution of $\mathcal{H}/\mathcal{H}_{\rm rec}$ in function of conformal time for $m = 10^{-28}$ eV (dashed line), $m = 5 \times 10^{-27}$ eV (dotted line) and for a matter dominated universe (continuous line), from recombination ($\eta_{\rm rec}$) to $3.5\eta_{\rm rec}$. Present time corresponds to $\eta_0 = \eta_{\rm rec}\sqrt{1 + z_{\rm rec}} \simeq 33.18\eta_{\rm rec}$.

defined the function:

$$\Delta(\eta; g_\phi, m, k, \eta_0) \equiv \sqrt{\frac{3}{\pi}}\frac{g_\phi M_{\rm pl}}{k\eta}\left\{\cos\left[m\frac{\eta_0}{3}\left(\frac{\eta}{\eta_0}\right)^3\right] - \frac{3\eta_0^2}{m\eta^3}\sin\left[m\frac{\eta_0}{3}\left(\frac{\eta}{\eta_0}\right)^3\right]\right\}. \quad (71)$$

This term, induced by axion-photon coupling, oscillates with frequency proportional to the mass of the axion and its amplitude decreases with time.

In the next two subsections we study analytically and numerically Eq. (70) for different values of the parameters m and g_ϕ; we exclude the region where the mass of the pseudoscalar field is so small that the field starts to oscillate after equivalence ($m < 3H_{\rm eq}$), and the region corresponding to a PQ symmetry broken at energies higher than Planck scale ($f_a > M_{\rm pl}$): see Fig. 2.

A. Adiabatic solution

Adiabatic solutions of Eq. (70) are:

$$\tilde{A}_s = \frac{1}{\sqrt{2\omega_s}} e^{\pm i\int \omega_s d\eta}, \quad (72)$$

where $\omega_s(\eta) = k\sqrt{1 \pm \frac{g_\phi \phi'(\eta)}{k}} = k\sqrt{1 \pm \Delta(\eta)}$ and $s = \pm$.

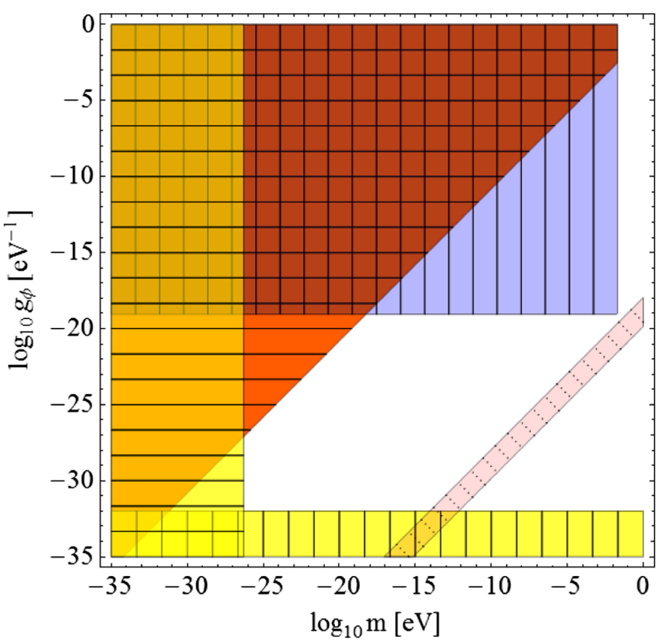

FIG. 2 (color online). Plane ($\log_{10}m$ [eV], $\log_{10}g_\phi$ [eV^{-1}]): region excluded by CAST [6] (blue with vertical lines), region where $|\theta_A(\Omega_{\rm MAT} = 0.3, m, g_\phi)| > 10$ deg (red region with horizontal lines), (m, g_ϕ) values expected in main QCD axion models (red with dots), region where the mass of the pseudoscalar field is too small in order to explain dark matter ($m < 3H_{\rm eq}$) (yellow with horizontal lines), and region where PQ symmetry is broken at energies higher than Planck scale ($f_a > M_{\rm pl}$) (yellow with vertical lines).

The second derivative respect to conformal time is:

$$\tilde{A}_s'' = \tilde{A}_s\left(-\omega_s^2 + \frac{3\omega_s'^2}{4\omega_s^2} - \frac{\omega_s''}{2\omega_s^3}\right). \quad (73)$$

The adiabatic solution (72) is a good approximation for the vector potential when the terms $\frac{3\omega_s'^2}{4\omega_s^2}$ and $\frac{\omega_s''}{2\omega_s^3}$ are small compared to ω_s^2:

$$\frac{3\omega_s'^2}{4\omega_s^4} = \frac{3\Delta'^2}{16k^2(1\pm\Delta)^3} \ll 1, \quad (74)$$

$$\frac{\omega_s''}{2\omega_s^3} = \frac{\pm 2(1\pm\Delta)\Delta'' - \Delta'^2}{8k^2(1\pm\Delta)^3} \ll 1. \quad (75)$$

If both conditions are satisfied and $\Delta \ll 1$:

$$\tilde{A}_\pm \simeq \frac{1}{\sqrt{2k(1\pm\Delta/4)}}\exp\left[\pm ik\left(\eta \pm \frac{1}{2}\int \Delta d\eta\right)\right] = \frac{1}{\sqrt{2k(1\pm\pi g_\phi \phi'k)}}\exp[\pm i(k\eta \pm 2\pi g\phi)]. \quad (76)$$

In the adiabatic regime the coupling between photons and axions produces a frequency independent shift between the two polarized waves, which corresponds to a rotation of the plane of linear polarization:

$$\theta_{\rm adiabatic} = \frac{g_\phi}{2}[\phi(\eta_0) - \phi(\eta_{\rm rec})]. \quad (77)$$

This result agrees with the one obtained in Ref. [8], which therefore holds in the adiabatic regime. More important than this, $\theta_{\rm adiabatic} = \bar{\theta}$, i.e. Eq. (77) agree with the rotation angle which is approximated by Eq. (59) in the Boltzmann Sec. III B. This agreement is not a coincidence and shows the usefulness of studying the gauge potential as done in this section: the estimate based on the adiabatic approximation of the rotation angle due to cosmological birefringence can be also obtained by studying the gauge potential A_s.

Typically $\phi(\eta_{\rm rec}) \gg \phi(\eta_0)$; from last scattering to now $\bar{\rho} \simeq m^2\bar{\phi}^2$ so, in a matter dominated universe:

$$\bar{\phi}(\eta) \simeq \sqrt{\frac{3}{8\pi}}\frac{M_{\rm pl}\mathcal{H}(\eta)}{m} \simeq \sqrt{\frac{3}{2\pi}}\frac{M_{\rm pl}}{m\eta_0}\left(\frac{\eta_0}{\eta}\right)^3. \quad (78)$$

An estimate of the angle $\theta_{\rm adiabatic}$ is:

$$\theta_{\rm adiabatic} \simeq g_\phi\sqrt{\frac{3}{8\pi}}\frac{M_{\rm pl}}{m\eta_0}[(1+z_{\rm rec})^{3/2} - 1]. \quad (79)$$

Note the dependence of $\theta_{\rm adiabatic}$ on the coupling constant and on the mass of the pseudoscalar field: for fixed g_ϕ the effect is larger for smaller masses.

The amplitude of the electromagnetic field changes according to:

$$|\tilde{\mathbf{E}}|^2 = \frac{|\tilde{\mathbf{A}}'|^2}{a^2} \simeq \frac{\omega_s}{2a^2}, \tag{80}$$

so the degree of circular polarization evolves according [28,29]:

$$\tilde{\Pi}_C = \frac{|\tilde{A}'_+|^2 - |\tilde{A}'_-|^2}{|\tilde{A}'_+|^2 + |\tilde{A}'_-|^2} = \frac{\sqrt{1+\Delta} - \sqrt{1-\Delta}}{\sqrt{1+\Delta} + \sqrt{1-\Delta}} \simeq \frac{\Delta}{2} = \frac{2\pi g_\phi \phi'}{k}. \tag{81}$$

B. CMBP constraints on the (m, g_ϕ) plane

In a flat universe dominated by dust ($w = 0$) plus a component with $w = -1$ (cosmological constant) the evolution of the scale factor in terms of cosmic time is [30]:

$$a(t) = \left(\frac{\Omega_{\rm MAT}}{1-\Omega_{\rm MAT}}\right)^{1/3} \sinh\left[\frac{3}{2}\sqrt{1-\Omega_{\rm MAT}} H_0 t\right]^{2/3}, \tag{82}$$

where $\Omega_{\rm MAT}$ is the density parameter for matter nowadays. The Hubble parameter is:

$$H = H_0\sqrt{1-\Omega_{\rm MAT}} \coth\left(\frac{3}{2}\sqrt{1-\Omega_{\rm MAT}} H_0 t\right). \tag{83}$$

The pseudoscalar field evolves according to:

$$\phi(t) \overset{mt\gg 1}{\simeq} \frac{\phi_0}{[\sinh(\frac{3}{2}\sqrt{1-\Omega_{\rm MAT}} H_0 t)]} \times \sin\left[mt\sqrt{1-(1-\Omega_{\rm MAT})\left(\frac{3H_0}{2m}\right)^2}\right]. \tag{84}$$

The energy density is:

$$\rho_\phi = \frac{\bar{\dot{\phi}}^2}{2} + \frac{1}{2}m^2\bar{\phi}^2 \overset{mt\gg 1}{\simeq} \frac{m^2\phi_0^2}{2[\sinh(\frac{3}{2}\sqrt{1-\Omega_{\rm MAT}} H_0 t)]^2} \propto a^{-3}. \tag{85}$$

Assuming that the axionlike particles contribute to the cold dark matter density $\rho_{\phi,0} = \Omega_{\rm MAT}\rho_{\rm CR,0}$ (where $\rho_{\rm CR,0}$ is the critical density) we can estimate ϕ_0:

$$\phi_0 = \sqrt{\frac{3(1-\Omega_{\rm MAT})}{\pi}}\frac{H_0 M_{\rm pl}}{2m}. \tag{86}$$

Therefore the evolution of the pseudoscalar field as a function of cosmic time is:

$$\phi(t) = \sqrt{\frac{3\Omega_{\rm MAT}}{\pi}}\frac{H_0 M_{\rm pl}}{2ma^{3/2}(t)} \times \sin\left[mt\sqrt{1-(1-\Omega_{\rm MAT})\left(\frac{3H_0}{2m}\right)^2}\right]. \tag{87}$$

Note how this equations reduces to Eq. (66) in a matter dominated universe: $\Omega_{\rm MAT} = 1$, $H_0/(2a^{3/2}) = 1/(3t)$. The linear polarization plane, from last scattering surface, rotates according to:

$$\theta(t) = \frac{g_\phi}{2}[\phi(t) - \phi(t_{\rm rec})]. \tag{88}$$

The Boltzmann equation contains the derivative of the rotation angle respect to of conformal time (cf. Equation (38)), so we need the relation between cosmic and conformal time. For a particular model with $\Omega_{\rm MAT} = 0.3$ it is possible to fit numerically the relation between cosmic and conformal time from last scattering to nowadays:

$$t \simeq \frac{\eta_0}{3.45}\left(\frac{\eta}{\eta_0}\right)^{3.09}. \tag{89}$$

Replacing this expression in Eq. (87) we obtain the evolution of the pseudoscalar field as a function of conformal time $\phi = \phi(\eta)$.

The linear polarization angle is not constant in time, but it oscillates with varying amplitude. If the field represents a fraction $\Omega_{\rm MAT}$ of the universe energy density, then the amplitude of these oscillations is:

$$\theta_A(\Omega_{\rm MAT}, m, g_\phi) = \frac{1}{4}\sqrt{\frac{3\Omega_{\rm MAT}}{\pi}}\frac{g_\phi M_{\rm pl} H_0}{m}\left(\frac{1}{a_0^{3/2}} - \frac{1}{a_{\rm rec}^{3/2}}\right) \simeq \frac{1}{4}\sqrt{\frac{3\Omega_{\rm MAT}}{\pi}}\frac{g_\phi M_{\rm pl} H_0}{m} z_{\rm rec}^{3/2}. \tag{90}$$

Fixed $\Omega_{\rm MAT}$, it is possible to constraint a certain region of the (m, g_ϕ)-plane requiring $\theta_A(\Omega_{\rm MAT}, m, g_\phi)$ to be smaller of a certain angle, typically of the order of few degrees (see Table I). The excluded region considering current limits on CMB birefringence is shown in Fig. 2.

Fixed a particular value for the pseudoscalar field mass and for its coupling with photons we can also estimate how the polarization angular power spectra are modified by a rotation of the linear polarization plane. We modified the source term for linear polarization in the public Boltzmann code CAMB [31] following Eqs. (45) and (46). The linear polarization rotation angle is given by Eq. (88) and the evolution of the pseudoscalar field by Eq. (87). The new power spectra are compared with the standard unrotated ones in Fig. 3 fixed $m = 10^{-22}$ eV and $g_\phi = 10^{-20}$ eV^{-1}.

In Sec. VI we compare the power spectra modified version of CAMB obtained starting by Eqs. (45) and (46) which takes into account the time dependence of the pseudoscalar field in the integral along the line of sight with the approximated spectra obtained following Eqs. (55)–(59).

C. Comments for axion cosmology

For axions the coupling constant with photons g_ϕ and the energy scale f_a at which the new symmetry is broken are related [4]:

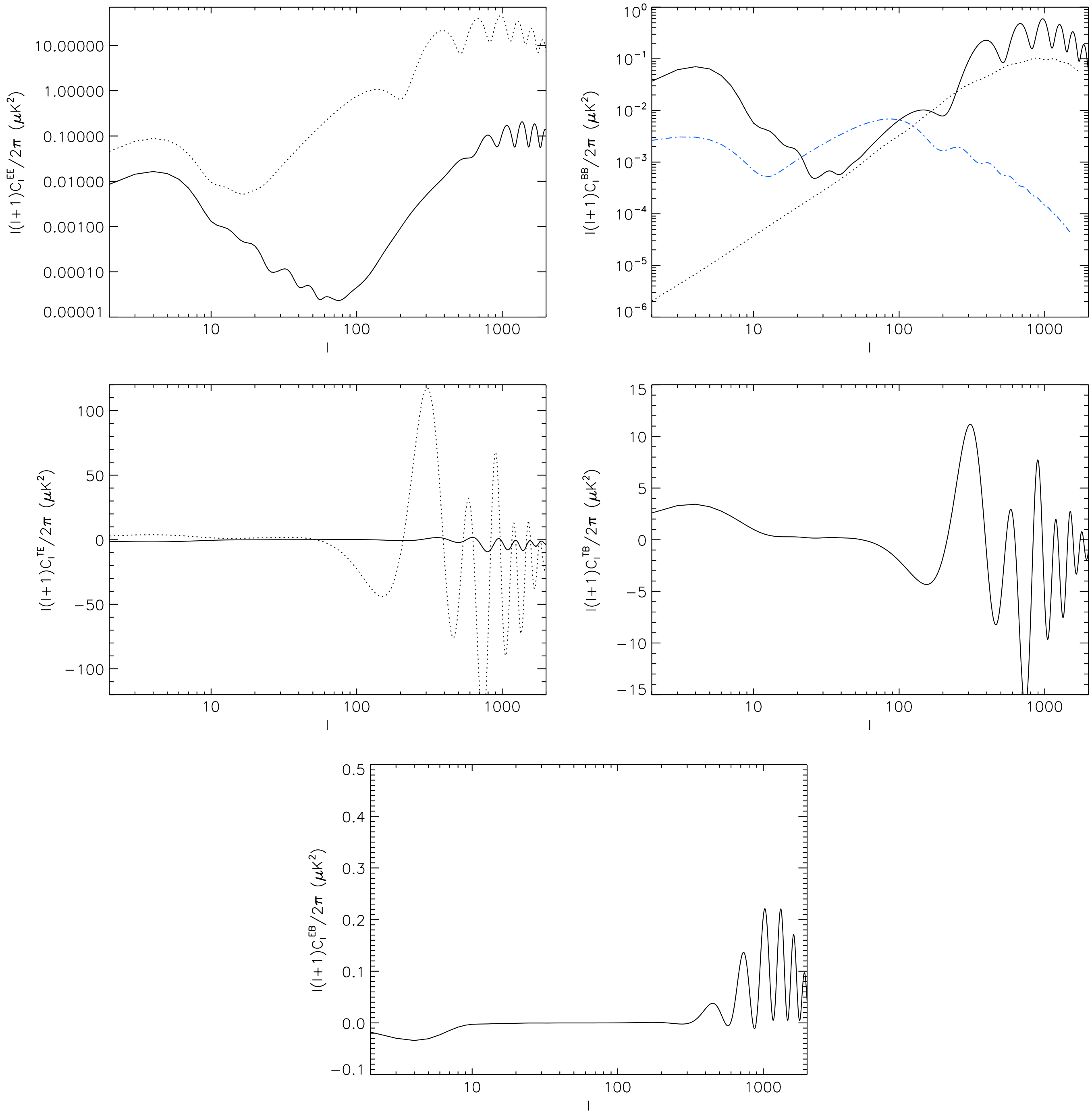

FIG. 3 (color online). *EE* (a), *BB* (b), *TE* (c), *TB* (d), and *EB* (e) angular power spectra for $m = 10^{-22}$ eV and $g_\phi = 10^{-20}$ eV^{-1} (black solid line), the black dotted line is the standard case in which there is no coupling between photons and pseudoscalars ($\theta = 0$). For the *BB* power spectrum (b) we plot for comparison also the polarization signal induced by gravitational lensing (black dotted line), and primordial *BB* signal if $r = 0.1$ (blue dot-dashed line). The cosmological parameters of the flat ΛCDM model used here are $\Omega_b h^2 = 0.022$, $\Omega_c h^2 = 0.123$, $\tau = 0.09$, $n_s = 1$, $A_s = 2.3 \times 10^{-9}$, $H_0 = 100h\ \mathrm{km\,s^{-1}\,Mpc^{-1}} = 72\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$.

$$|g_\phi| = \frac{\alpha_{EM}}{2\pi f_a}\frac{3}{4}\xi \quad \text{with} \quad 0.1 \lesssim \xi \lesssim 1, \qquad (91)$$

where the value for ξ depends on the particular model considered for the axion. By using this relation a limit on the coupling constant is turned into a limit on the energy of symmetry breaking.

The critical density associated with the misalignment production of axions strongly depends on the initial misalignment angle associated with the axion field Θ_i through the following relation [3,4]:

$$\Omega_{\rm mis}h^2 \sim 0.23 \times 10^{\pm 0.6}\left(\frac{f_a}{10^{12}\ \mathrm{GeV}}\right)^{1.175}\Theta_i^2 F(\Theta_i), \quad (92)$$

where h encodes the actual value of the Hubble parameter ($H_0 = 100h$ km s^{-1} Mpc^{-1}) and $F(\Theta_i)$ accounts for anharmonic effects if $\Theta_i \gg 1$. The demand $\Omega_{\rm mis} \le \Omega_{\rm DM}$ provides an upper bound on $f_a^{1.175}\Theta_i^2$ (assuming $F(\Theta_i) \simeq 1$) [26,32,33]:

$$f_a \Theta_i^{1.7} \le 2 \times 10^{11\div 12}\ \mathrm{GeV}. \quad (93)$$

This condition becomes also an upper bound for f_a under the assumption that inflation occurred before the breaking of PQ-symmetry ($f_a \le f_{\rm INF}$) [3]: in this scenario different regions have different values for Θ_i, so averaging over all observable universe the value of Θ_i in equation can replaced by its *rms* value ($\pi/\sqrt{3}$) and the limit $f_a \le 10^{11\div 12}$ GeV is obtained. As can be seen from Fig. 2, CAST disfavors values of $g_\phi \sim 10^{-11\div -12}$ GeV^{-1} with a mass up to 0.02 eV. Note however that our calculation cannot be applied directly to this case since we assume ϕ homogeneous in our universe, whereas it is not if the PQ symmetry breaking occurs after inflation: although taking into account space inhomogeneities were a second order effect in cosmological perturbation theory, cosmological birefringence might be larger than the one computed in this paper.

Our calculations apply without modifications to the case in which inflation occurs after PQ-symmetry breaking: the initial misalignment angle Θ_i is homogeneous throughout our universe and can be much smaller than $\pi/\sqrt{3}$. Such possibility allow the scale of PQ-symmetry breaking f_a to be much higher than $10^{11\div 12}$ GeV and is motivated by anthropic considerations [34–37]. These smaller values of g_ϕ can be constrained by present data in CMB polarization in a much better way than CAST, in particular, for small masses.

V. EXPONENTIAL POTENTIAL

We consider in this section a pseudoscalar field with an exponential potential:

$$V(\phi) = V_0 \exp(-\lambda\kappa\phi), \quad (94)$$

with $\kappa^2 \equiv 8\pi G$. Theoretical motivations to this exponential potential are certainly weaker than the ones for the potential presented in Eq. (60). However it is interesting to show how the kinematics of the pseudoscalar field is important for the resulting spectra of CMB anisotropies in polarization. Whereas the time derivative of the pseudoscalar field in the previous case contains oscillations about a vanishing value (see Eq. (69)), we study here a case where the behavior in time is monotonous.

It is known [38] that exponential potential with $\lambda^2 > 3(1 + w_{\rm F})$ leads to a component which tracks the dominant background fluid with equation of state $p_\phi = w_\phi \rho_\phi$. In order to satisfy the nucleosynthesis bound we choose $\lambda = 4.5$. During the matter dominated era the scalar field behaves as:

$$\rho_\phi = \frac{\dot\phi^2}{2} + V_0\exp(-\lambda\kappa\phi) = f\rho_{\rm MAT} \equiv f\frac{\rho_{\rm MAT,0}}{a^3}, \quad (95)$$

$$P_\phi = \frac{\dot\phi^2}{2} - V_0\exp(-\lambda\kappa\phi), \quad (96)$$

where $\rho_{\rm MAT} = \rho_{\rm DM} + \rho_{\rm baryons} + \rho_\phi$.

For $\lambda = 4.5$ the contribution of the pseudoscalar field to universe energy density is shown in Fig. 4. The value of Ω_ϕ changes with time, but it is almost constant ($\Omega_\phi \simeq \Omega_{\phi,0} = 0.148$) from recombination ($\log a_{\rm rec} \simeq -7$) to nowadays.

The derivative of the pseudoscalar field respect to conformal time is proportional to $a^{-1/2}$ and the evolution of the scale factor in the matter dominated phase is $a(\eta) = (\eta/\eta_0)^2$ so:

$$\phi' = \sqrt{f\rho_{\rm MAT,0}}\frac{\eta_0}{\eta}. \quad (97)$$

Substituting this relation in Eq. (14) we obtain the following expression for the evolution of the electromagnetic potential:

$$\tilde A''_\pm + \left(k^2 \pm g_\phi\sqrt{f\rho_{\rm MAT,0}}\frac{\eta_0}{\eta}k\right)\tilde A_\pm = 0. \quad (98)$$

This is a particular differential equation, called Coulomb wave equation; defining $q_\pm \equiv \mp g_\phi\sqrt{f\rho_{\rm MAT,0}}\eta_0/2 = \mp q$ and $x \equiv k\eta$ it becomes:

$$\frac{d^2\tilde A_\pm}{dx^2} + \left(1 - \frac{2q_\pm}{x}\right)\tilde A_\pm = 0. \quad (99)$$

The solution of this particular equation can be written in terms of regular ($F_0(q,x)$) and irregular ($G_0(q,x)$)

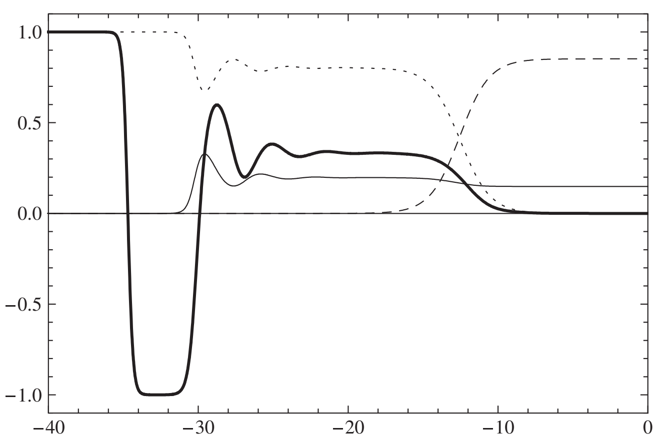

FIG. 4. For $\lambda = 4.5$ Dashed line: $\Omega_{\rm DM} + \Omega_{\rm baryons}$, dotted line: $\Omega_{\rm RAD}$, thin continuous line: Ω_ϕ, thick continuous line: w_ϕ, in terms of the natural logarithm of the scale factor (from $\log a \simeq -40$ to nowadays $\log a_0 = 0$). Here $\Omega_{\rm DM,0} + \Omega_{\rm baryons,0} = 0.852$ and $\Omega_{\phi,0} = 0.148$.

Coulomb wave functions [39,40]:

$$\begin{aligned}\tilde{A}_+ &= f_+F_0(q_+, x) + g_+G_0(q_+, x)\\ &= f_+F_0(-q, x) + g_+G_0(-q, x),\\ \tilde{A}_- &= f_-F_0(q_-, x) + g_-G_0(q_-, x)\\ &= f_-F_0(q, x) + g_-G_0(q, x),\end{aligned}$$

where f_+, f_-, g_+, and $g_- \in \mathbb{C}$; in a compact notation:

$$\tilde{A}_\pm(q, x) = f_\pm F_0(\mp q, x) + g_\pm G_0(\mp q, x). \tag{100}$$

The Stokes parameters contain the derivative respect to conformal time η, so we evaluate:

$$\tilde{A}'_\pm(q, x) = k\left[f_\pm \frac{\partial F_0(\mp q, x)}{\partial x} + g_\pm \frac{\partial G_0(\mp q, x)}{\partial x}\right]. \tag{101}$$

The solution given in Eq. (100) verifies the *Wronskian condition* ($\tilde{A}_\pm \tilde{A}'^*_\pm - \tilde{A}'_\pm \tilde{A}^*_\pm = i$) if the following relation holds:

$$f^*_\pm g_\pm - f_\pm g^*_\pm = \frac{i}{k} \Rightarrow \Im(f^*_\pm g_\pm) = \frac{1}{2k}. \tag{102}$$

In the general case, when the coupling does not vanishes ($g_\phi \neq 0$), we expand the solution (100) for large value of x neglecting terms proportional to $\mathcal{O}(x^{-2})$ (see the Appendix):

$$\begin{aligned}\tilde{A}_\pm(q, x) \simeq f_\pm&\left[\frac{q^2}{2x}\cos(x \pm \alpha(q, x))\right.\\ &\left.+ \left(1 \mp \frac{q}{2x}\right)\sin(x \pm \alpha(q, x))\right]\\ &+ g_\pm\left[\left(1 \mp \frac{q}{2x}\right)\cos(x \pm \alpha(q, x))\right.\\ &\left.- \frac{q^2}{2x}\sin(x \pm \alpha(q, x))\right],\end{aligned} \tag{103}$$

where $\alpha(q, x) \equiv q\ln 2x - \arg\Gamma(1 + iq)$. The derivative respect to conformal time is:

$$\begin{aligned}\tilde{A}'_\pm(q, x) \simeq k&\left\{f_\pm\left[\left(1 \pm \frac{q}{2x}\right)\cos(x \pm \alpha(q, x))\right.\right.\\ &\left.- \frac{q^2}{2x}\sin(x \pm \alpha(q, x))\right]\\ &+ g_\pm\left[-\frac{q^2}{2x}\cos(x \pm \alpha(q, x))\right.\\ &\left.\left.- \left(1 \pm \frac{q}{2x}\right)\sin(x \pm \alpha(q, x))\right]\right\}\\ = \frac{k}{2}&\left[e^{i(x\pm\alpha(q,x))}\left(1 \pm \frac{q}{2x} + i\frac{q^2}{2x}\right)(f_\pm + ig_\pm)\right.\\ &\left.\times e^{-i(x\pm\alpha(q,x))}\left(1 \pm \frac{q}{2x} - i\frac{q^2}{2x}\right)(f_\pm - ig_\pm)\right].\end{aligned} \tag{104}$$

In general both forward moving waves ($\tilde{A}_\pm \propto e^{-ik\eta}$) and backward moving waves ($\tilde{A}_\pm \propto e^{ik\eta}$) must be taken into account for propagation of light in a medium. Chosen a particular value for the constants $f_\pm$ and $g_\pm$ that verifies the Wronskian relation (102) the evolution of polarization is fixed.

If we assume, according with [41,42], that the photon pseudoscalar conversion is a small effect due to low energy of CMB photons, the production of backward moving waves can be neglected (see [43] for the use of this approximation). The Eq. (104) setting $f_\pm = -ig_\pm$ becomes:

$$\tilde{A}'_\pm(q, x) \simeq -ikg_\pm\left(1 \pm \frac{q}{2x} - i\frac{q^2}{2x}\right)e^{-i(x\pm\alpha(q,x))}, \tag{105}$$

and in terms of the value at recombination time:

$$\begin{aligned}\tilde{A}'_\pm(q, x) \simeq \tilde{A}'_\pm(q, x_{\rm rec})&\left[1 \pm \frac{q}{2}\left(\frac{1}{x} - \frac{1}{x_{\rm rec}}\right)\right.\\ &\left.- i\frac{q^2}{2}\left(\frac{1}{x} - \frac{1}{x_{\rm rec}}\right)\right]\exp\{-i[x - x_{\rm rec} \pm \Delta\alpha]\},\end{aligned} \tag{106}$$

where we have introduced

$$\begin{aligned}\Delta\alpha \equiv \alpha(q, x) - \alpha(q, x_{\rm rec}) &= q\ln(\eta/\eta_{\rm rec})\\ &= \frac{q}{2}\ln(a/a_{\rm rec}).\end{aligned} \tag{107}$$

We observe that also in this exact case the plane of linear polarization is rotated of an angle $\Delta\alpha$ independent on k whose dependence on the difference between the present value of ϕ and the corresponding one at recombination is the same of the adiabatic approximation and of Eq. (59).

Current measures and constraints on the polarization pattern of CMB anisotropies produce an upper limit on the linear polarization rotation angle of the order of few degrees (see Table I). We now use these constraints and our analytic expression:

$$|\theta| = \frac{|q|}{2}\ln(1 + z_{\rm rec}) \simeq \frac{1}{4}\sqrt{\frac{3}{2\pi}\Omega_{\phi,0}}|g_\phi|M_{\rm pl}\ln(1 + z_{\rm rec}), \tag{108}$$

to obtain an upper bound for q, which can be turned into a upper bound on g_ϕ; if $|\theta| \lesssim 6$ deg, then:

$$|g_\phi| \lesssim 10^{-30}\ \mathrm{eV}^{-1}, \tag{109}$$

where we have assumed: $\Omega_{\phi,0} \simeq 0.148$ and $z_{\rm rec} \simeq 1100$.

The angle of linear polarization $\theta(\eta)$ appearing in Eqs. (45) and (46) can be replaced with:

$$|\theta(\eta)| \simeq \frac{1}{2}\sqrt{\frac{3}{2\pi}\Omega_{\phi,0}}|g_\phi|M_{\rm pl}\ln\left(\frac{\eta}{\eta_{\rm rec}}\right), \tag{110}$$

and the polarization power spectra are evaluated using the expression given in Sec. III B, see angular power spectra of Fig. 5.

In Sec. VI we compare the power spectra modified version of CAMB obtained starting by Eqs. (45) and (46) which takes into account the time dependence of the pseudoscalar field in the integral along the line of sight

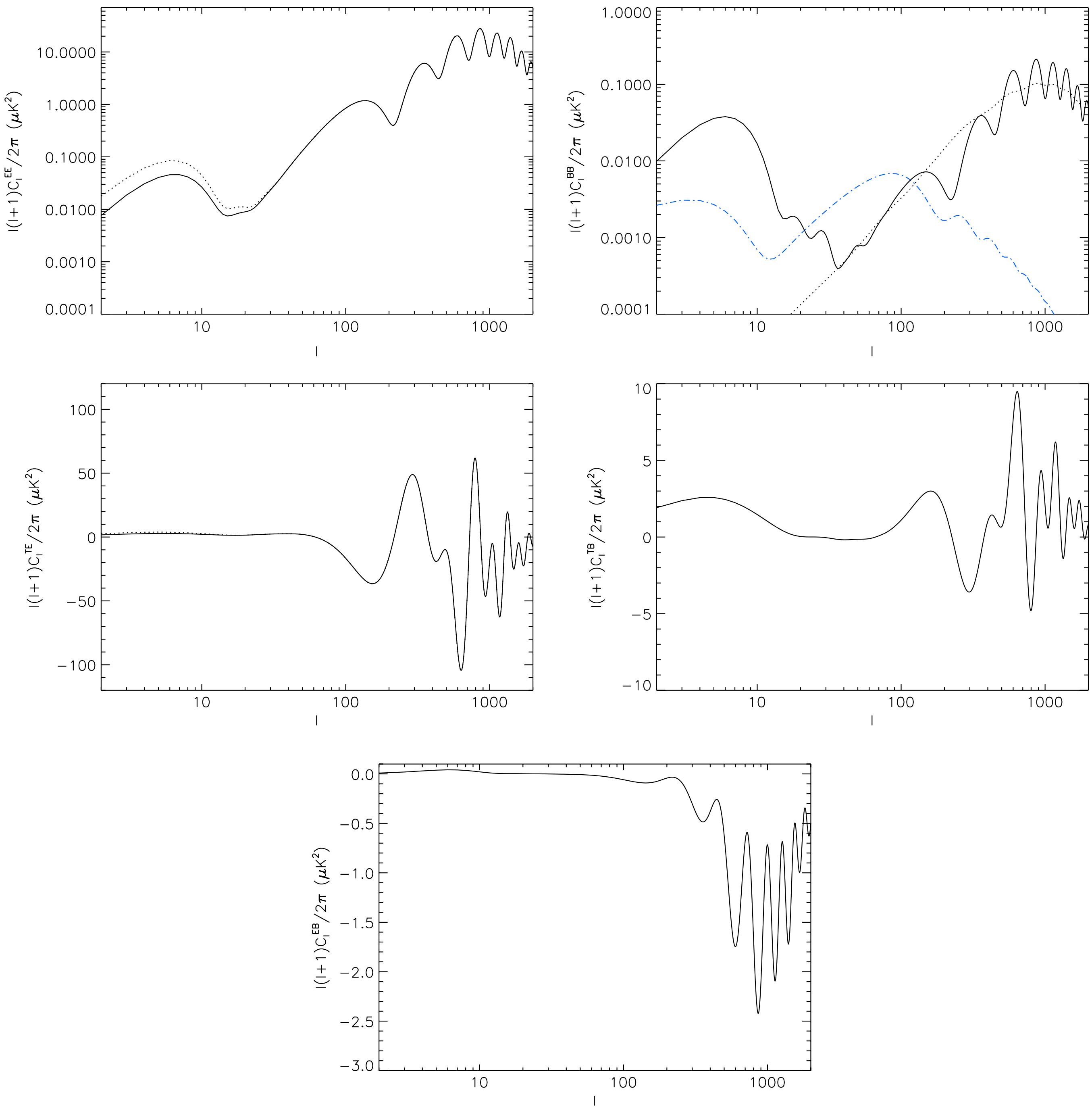

FIG. 5 (color online). *EE* (a), *BB* (b), *TE* (c), *TB* (d) and *EB* (e) angular power spectra for $g_\phi = 10^{-28}$ eV^{-1} (black solid line); the black dotted line is the standard case in which there is no coupling ($\theta = 0$). For the *BB* power spectrum (b) we plot for comparison also the polarization signal induced by gravitational lensing (black dotted line), and primordial *BB* signal if $r = 0.1$ (blue dot-dashed line). The cosmological parameters of the flat CDM model used here are $\Omega_b = 0.0462$, $\Omega_c = 0.9538$ ($\Omega_\phi \simeq 0.148$), $\tau = 0.09$, $n_s = 1$, $A_s = 2.3 \times 10^{-9}$, $H_0 = 72$ km s^{-1} Mpc^{-1}.

with the approximated spectra obtained following Eqs. (55)–(60).

VI. COMPARISON WITH CONSTANT ROTATION ANGLE APPROXIMATION

In this section we compare the angular power spectra obtained modifying the public code CAMB [31] considering the correct dynamic of the pseudoscalar field ($\theta = \theta(\eta)$) as described in Sec. III B, with the ones obtained

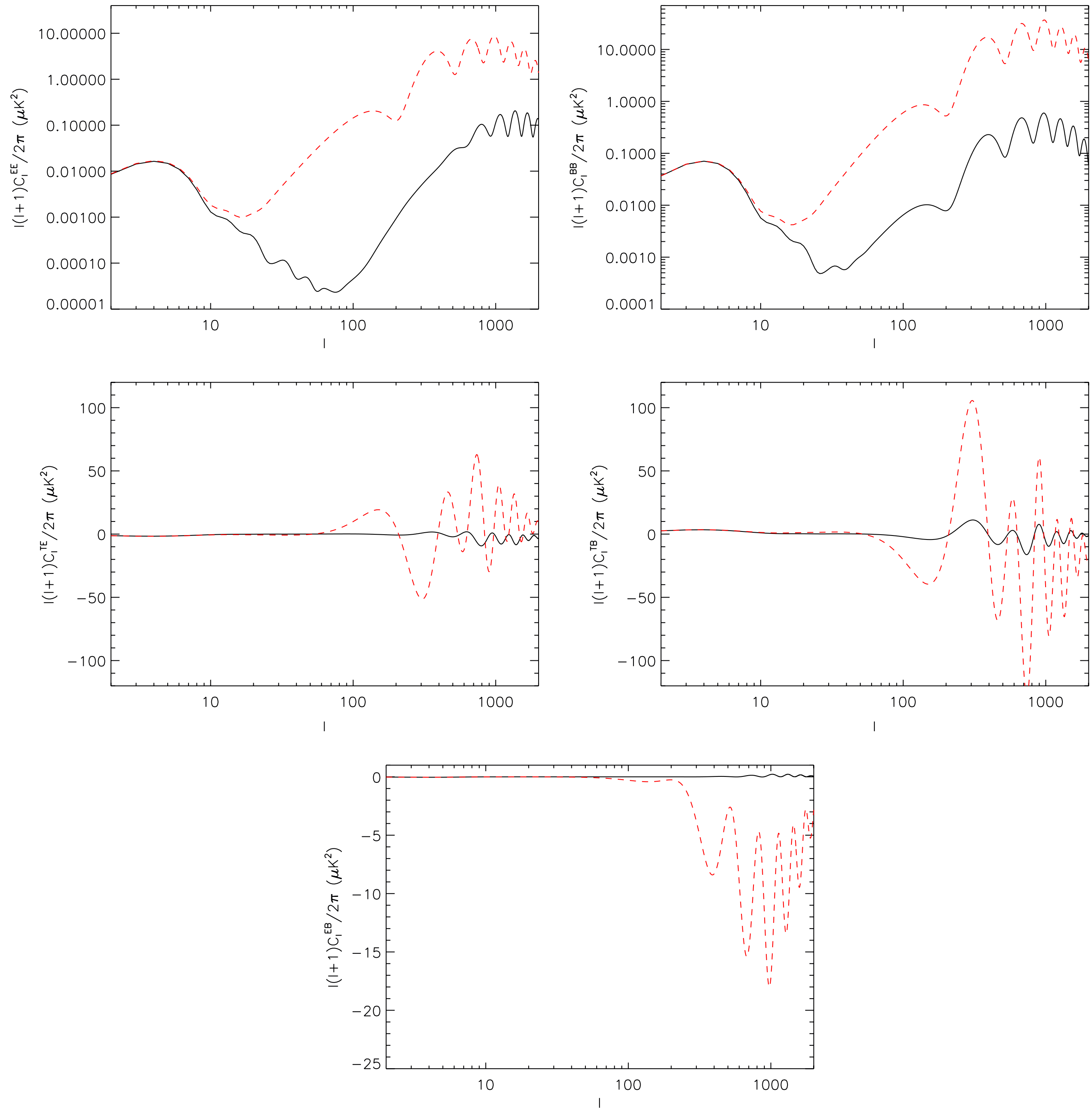

FIG. 6 (color online). EE (a), BB (b), TE (c), TB (d) and EB (e) angular power spectra for $m = 10^{-22}$ eV and $g_\phi = 10^{-20}$ eV^{-1} (black solid line) and approximating the rotation angle with the constant value $\theta_{\rm rec}$ (red dashed line). The cosmological parameters of the flat ΛCDM model used here are $\Omega_b h^2 = 0.022$, $\Omega_c h^2 = 0.123$, $\tau = 0.09$, $n_s = 1$, $A_s = 2.3 \times 10^{-9}$, $H_0 = 100h$ km s^{-1} Mpc^{-1} = 72 km s^{-1} Mpc^{-1}.

in the constant rotation angle approximation ($\theta = \text{const}$) for the two different potential considered in the previous sections: see Figs. 6 and 7.

In Sec. III B we have already shown how the power spectra in the constant rotation angle approximation [Eqs. (54)–(58)] can be obtained from the general expressions [Eqs. (47)–(51)].

Power spectrum modifications obtained starting directly form the Boltzmann equations and taking into account the temporal evolution of the pseudoscalar field are usually smaller than effects predicted considering a constant rotation angle equal to the total rotation angle from last scattering to nowadays. If the cosmological pseudoscalar field

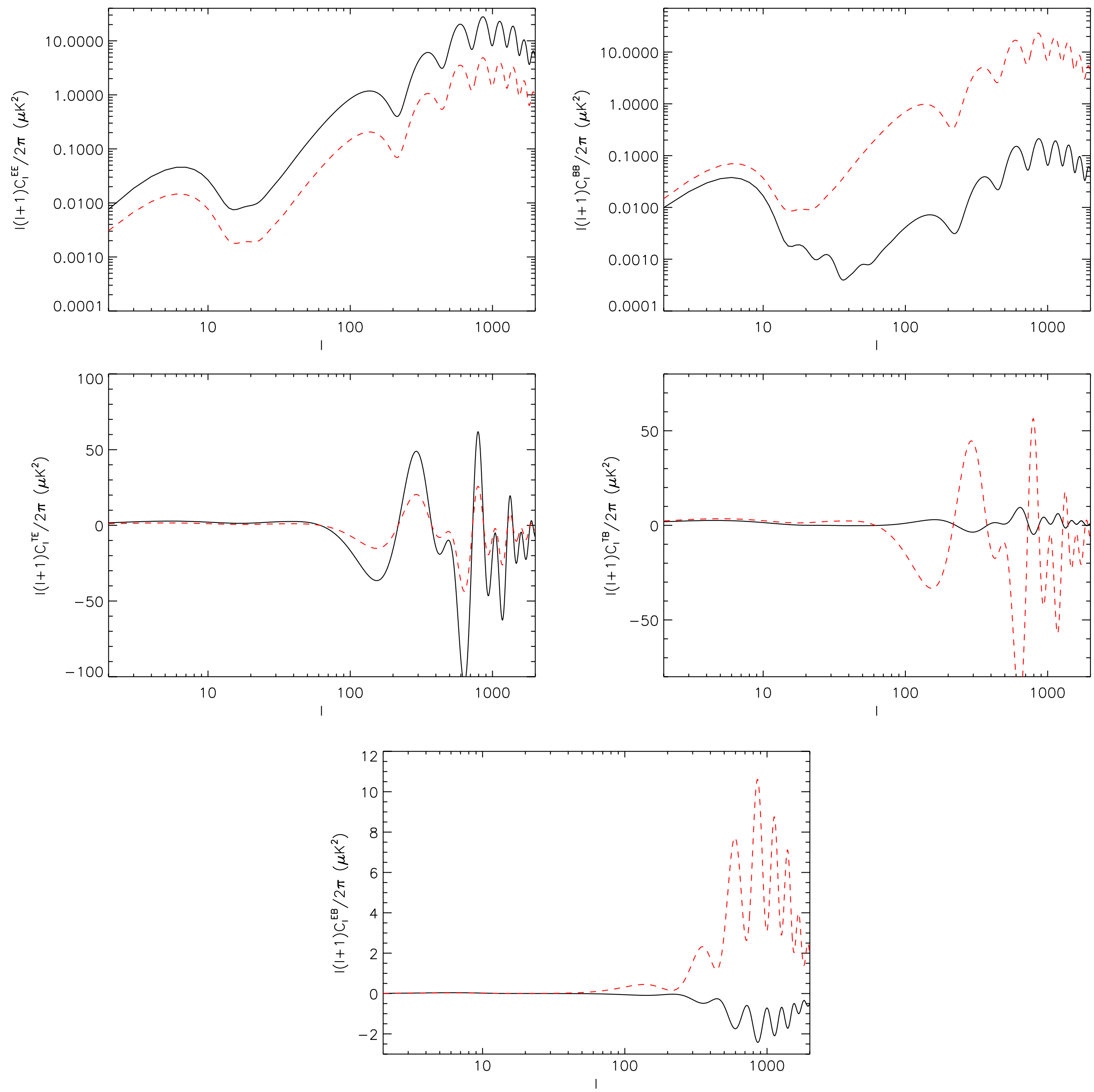

FIG. 7 (color online). *EE* (a), *BB* (b), *TE* (c), *TB* (d) and *EB* (e) angular power spectra for $g_\phi = 10^{-28}\ \text{eV}^{-1}$ (black solid line) and approximating the rotation angle with the constant value θ_{rec} (red dashed line); the black dotted line is the standard case in which there is no coupling. The cosmological parameters of the flat CDM model used here are $\Omega_b = 0.0462$, $\Omega_c = 0.9538$ ($\Omega_\phi \simeq 0.148$), $\tau = 0.09$, $n_s = 1$, $A_s = 2.3 \times 10^{-9}$, $H_0 = 72\ \text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}$.

evolves quickly, then the constant rotation angle approximation clearly leads to an overestimate of the effects.

It is important to stress that the constant rotation angle approximation is just an operative approximation. The additional term in the Boltzmann equations which rotates the linear polarization plane is (see Eq. (38)):

$$\mp i2\theta'(\eta)\Delta_{Q\pm iU}(k,\eta), \tag{111}$$

which clearly vanishes for $\theta = \text{const}$.

VII. CONCLUSIONS

We have studied the impact of a pseudoscalar field acting as dark matter on CMBP. We have shown that such pseudoscalar interaction with photons rotates the plane of linear polarization and generates circular polarization. In absence of measures for the V mode of CMBP, the existing upper limits on an isotropic TB and EB correlations can constrain the coupling constants of photons with the pseudoscalar field.

We have examined two representative examples for the dynamics of a pseudo-Goldstone field behaving as dark matter: the oscillating and the monotonic decreasing behavior. In the monotonic decreasing behavior, by neglecting backward moving waves, we have shown how present CMB observations can constrain the coupling constant g_ϕ to small values as $\mathcal{O}(10^{-30})$ eV. For the more physically motivated axion case which leads to an oscillating behavior, we have shown how constraints from CMB cosmological birefringence can become important for small masses for the axion.

We have also shown how the use of integral solution of the Boltzmann function may improve the estimate obtained by multiplying the CMB power spectrum by the suitable trigonometric functions of the rotation angle as in Eqs. (54)–(58).

ACKNOWLEDGMENTS

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APPENDIX: COULOMB WAVE EQUATION

The Coulomb wave equation is [39]:

$$\frac{d^2w}{dx^2} - \left[1 - \frac{2q}{x} - \frac{L(L+1)}{x^2}\right]w = 0, \tag{A1}$$

with $x > 0$, $-\infty < q < \infty$, L a non negative integer. Here, in order to solve Eq. (99), we are particular interested to the particular case when $L = 0$.

The solution can be written in terms of regular ($F_L(q,x)$) and irregular ($G_L(q,x)$) Coulomb wave function:

$$w = c_1F_L(q,x) + c_2G_L(q,x). \tag{A2}$$

The Coulomb functions can be expanded for large values of x [39]:

$$F_0 = g\cos\theta + f\sin\theta, \tag{A3}$$

$$G_0 = f\cos\theta - g\sin\theta, \tag{A4}$$

similarly for the first derivative respect to x

$$F_0' = g^*\cos\theta + f^*\sin\theta, \tag{A5}$$

$$G_0' = f^*\cos\theta - g^*\sin\theta, \tag{A6}$$

with $\theta \equiv x - q\ln 2x + \arg\Gamma(1+iq)$ and:

$$f = \sum_{k=0}^{\infty} f_k, \qquad g = \sum_{k=0}^{\infty} g_k, \qquad f^* = \sum_{k=0}^{\infty} f_k^*, \qquad g^* = \sum_{k=0}^{\infty} g_k^*, \tag{A7}$$

where:

$$f_0 = 1, \qquad f_{k+1} = a_kf_k - b_kg_k; \tag{A8}$$

$$g_0 = 0, \qquad g_{k+1} = a_kg_k + b_kf_k; \tag{A9}$$

$$f_0^* = 0, \qquad f_{k+1}^* = a_kf_k^* - b_kg_k^* - \frac{f_{k+1}}{x}; \tag{A10}$$

$$g_0^* = 1 - \frac{q}{x}, \qquad g_{k+1}^* = a_kg_k^* + b_kf_k^* - \frac{g_{k+1}}{x}; \tag{A11}$$

$$a_k = \frac{(2k+1)q}{2(k+1)x}, \qquad b_k = \frac{q^2 - k(k+1)}{2(k+1)x}. \tag{A12}$$

Restricting to the first order:

$$f = 1 + \frac{q}{2x} + \mathcal{O}\left(\frac{1}{x^2}\right), \tag{A13}$$

$$g = \frac{q^2}{2x} + \mathcal{O}\left(\frac{1}{x^2}\right), \tag{A14}$$

$$f^* = -\frac{q^2}{2x} + \mathcal{O}\left(\frac{1}{x^2}\right), \tag{A15}$$

$$g^* = 1 - \frac{q}{2x} + \mathcal{O}\left(\frac{1}{x^2}\right). \tag{A16}$$

Summarizing the asymptotic expansion of $F_L(q,x)$ and $F_L(q,x)$ for large values of x is:

$$F_0(q,x) \simeq \frac{q^2}{2x}\cos\theta + \left(1 + \frac{q}{2x}\right)\sin\theta, \tag{A17}$$

$$G_0(q,x) \simeq \left(1 + \frac{q}{2x}\right)\cos\theta - \frac{q^2}{2x}\sin\theta, \tag{A18}$$

and for the first derivative:

$$F_0'(q,x) \simeq \left(1 - \frac{q}{2x}\right)\cos\theta - \frac{q^2}{2x}\sin\theta, \tag{A19}$$

$$G_0'(q,x) \simeq -\frac{q^2}{2x}\cos\theta - \left(1 - \frac{q}{2x}\right)\sin\theta. \tag{A20}$$

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