Soft-collinear factorization and zero-bin subtractions

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We study the Sudakov form factor for a spontaneously broken gauge theory using a (new) Δ-regulator. To be well defined, the effective theory requires zero-bin subtractions for the collinear sectors. The zero-bin subtractions depend on the gauge boson mass M and are not scaleless. They have both finite and $1/\epsilon$ contributions and are needed to give the correct anomalous dimension and low-scale matching contributions. We also demonstrate the necessity of zero-bin subtractions for soft-collinear factorization. We find that after zero-bin subtractions the form factor is the sum of the collinear contributions *minus* a soft mass-mode contribution, in agreement with a previous result of Idilbi and Mehen in QCD. This appears to conflict with the method-of-regions approach, where one gets the sum of contributions from different regions.

I. INTRODUCTION

Soft-collinear effective theory (SCET) [1,2] is a field theory which describes the interactions of energetic particles with small invariant mass. SCET was originally developed for QCD processes, but has recently [3] been extended to broken gauge theories with massive gauge bosons. This allows one to compute electroweak corrections to standard model processes at high energies, and to sum electroweak Sudakov logarithms [4,5].

Applications of SCET to electroweak processes require evaluating collinear and soft Feynman graphs with massive gauge bosons. These graphs are not well defined, even in dimensional regularization with an off-shellness, and require additional regularization. In Refs. [3–5], the graphs were evaluated with an analytic regulator [6,7]; the individual diagrams depend on the analytic regulator parameters, but the total amplitude is regulator independent. The analytic regulator has some unpleasant properties with regards to gauge invariance and factorization, two essential ingredients of SCET. We propose a convenient new regulator, called the Δ-regulator, which can be implemented directly on the level of the SCET Lagrangian without the need to define further prescriptions for computing diagrams. This regulator is similar to using a mass, and unlike off-shellness, it regulates diagrams with massive gauge bosons.

The Δ-regulator is used to compute the Sudakov form factor using SCET for a spontaneously broken $SU(2)$ gauge theory with a common gauge boson mass M. This form factor was computed previously in Ref. [3] using an analytic regulator. We discuss the factorization structure of the effective theory using the Δ-regulator. As noted previously [8,9] in the framework of QCD, the amplitudes only factorize when the collinear sectors are defined including zero-bin subtractions [10], to avoid double-counting the soft momentum region. We show in the broken $SU(2)$ gauge theory that the usual construction of collinear gauge interactions into collinear Wilson lines, while true at tree-level, is valid at the loop-level only when the collinear sector is defined with zero-bin subtractions.

Recently Idilbi and Mehen [11,12] (see also Ref. [13]) reemphasized the necessity for zero-bin subtractions [10]. They studied deep inelastic scattering and showed that the correct total amplitude has the form $I_n + I_{\bar{n}} - I_s$, the sum of the n-collinear, $\bar{n}$-collinear, *minus* the soft contributions, rather than the naive expectation $I_n + I_{\bar{n}} + I_s$. The sign change of I_s arises because the collinear contributions have to be properly thought of as zero-bin subtracted, $I_n \rightarrow I_n - I_s$. This converts the second (incorrect) form of the result into the first. In the case of deep inelastic scattering the effective theory graphs are scaleless, and so vanish in dimensional regularization. The net effect of the zero-bin subtractions is therefore to simply convert $1/\epsilon$ infrared divergent poles into $1/\epsilon$ ultraviolet divergent poles. This might lead one to think of the zero-bin subtraction as an academic issue. However, the conversion is necessary to obtain the correct form of the anomalous dimensions. In the case of a broken $SU(2)$ gauge theory we also find that the zero-bin subtractions are mandatory. In contrast to deep inelastic scattering, the effective theory graphs depend on the gauge boson mass M, and are no longer scaleless. As a result, the zero-bin subtractions are necessary not just to convert infrared divergences into ultraviolet ones, but also to correctly obtain the finite parts of the diagram.

The article is organized as follows: We start out with a discussion of the full theory and the SCET formalism in Sec. II, and in Sec. III, we discuss how Wilson line regularization breaks factorization. The Δ-regulator is introduced in Sec. IV. The effective theory computation, zero-bin subtractions, gauge dependence and momentum regions are discussed in Sec. V. The conclusions are given

in Sec. VI. Some technical details are relegated to the Appendix.

II. FORMALISM

The theory we consider is a $SU(2)$ spontaneously broken gauge theory, with a Higgs in the fundamental representation, where all gauge bosons have a common mass, M. This is the theory used in many previous computations [14–18], and allows us to compare with previous results. It is convenient, as in Ref. [18], to write the group theory factors using C_F, C_A, T_F.[1]

The physical quantity of interest is the Sudakov form factor $F(Q^2)$ in the Euclidean region,

$$F(Q^2) = \langle p_2|\bar{\psi}\Gamma\psi|p_1\rangle, \qquad (1)$$

where $Q^2 = -(p_2 - p_1)^2 \gg M^2$ and Γ is a generic Dirac structure. In SCET, $F(Q^2)$ is computed in three steps: (i) matching from the full gauge theory to SCET at $\mu = Q$ (high-scale matching) (ii) running in SCET between Q and M and (iii) integrating out the gauge bosons at $\mu = M$ (low-scale matching). The high-scale matching computation is given in Ref. [3]. The SCET computation of the running and low-scale matching is discussed in this article. All computations are done to leading order in SCET power counting, i.e. neglecting M^2/Q^2 power corrections.

The SCET fields and Lagrangian depend on two null four-vectors, n and $\bar{n}$, with $n = (1, \mathbf{n})$ and $\bar{n} = (1, -\mathbf{n})$, where $\mathbf{n}$ is a unit vector, so that $\bar{n} \cdot n = 2$. In the Sudakov problem, one works in the Breit frame, with n chosen to be along the p_2 direction, so that $\bar{n}$ is along the p_1 direction. In the Breit frame, the momentum transfer q has no time component, $q^0 = 0$, so that the particle is back-scattered (see Fig. 1). The light-cone components of a four-vector p are defined by $p^+ \equiv n \cdot p$, $p^- \equiv \bar{n} \cdot p$, and $p_\perp$, which is orthogonal to n and $\bar{n}$, so that

$$p^\mu = \frac{1}{2}n^\mu(\bar{n} \cdot p) + \frac{1}{2}\bar{n}^\mu(n \cdot p) + p_\perp^\mu. \qquad (2)$$

In our problem, $p_1^- = p_{1\perp} = p_2^+ = p_{2\perp} = 0$, and $Q^2 = p_1^+ p_2^-$. A fermion moving in a direction close to n is described by the n-collinear SCET field $\xi_{n,p}(x)$, where p is a label momentum, and has components $\bar{n} \cdot p$ and $p_\perp$ [1,2]. It describes particles (on- or off-shell) with energy $2E \sim \bar{n} \cdot p$, and $p^2 \ll Q^2$. The total momentum of the field $\xi_{n,p}(x)$ is $p + k$, where k is the residual momentum of order $Q\lambda^2$ contained in the Fourier transform of x. Note that the label momentum p only contributes to the minus and $\perp$ components of the total momentum.

The massive gauge fields are represented by several distinct fields in the effective theory: n-collinear fields $A_{n,p}(x)$ and $\bar{n}$-collinear fields $A_{\bar{n},p}(x)$ with labels, and so-

FIG. 1.   Breit frame kinematics.

called mass-mode fields $A_s(x)$ [19,20] to which we do not give any label. This is analogous to the label conventions for soft and ultrasoft fields introduced in NRQCD [21]. The n-collinear field contains massive gauge bosons with momentum near the n-direction, and momentum scaling $\bar{n} \cdot p \sim Q$, $n \cdot p \sim Q\lambda^2$, $p_\perp \sim Q\lambda$, and the $\bar{n}$-collinear field contains massive gauge bosons moving near the $\bar{n}$-direction, with momentum scaling $n \cdot p \sim Q$, $\bar{n} \cdot p \sim Q\lambda^2$, $p_\perp \sim Q\lambda$. Here we have $\lambda \sim M/Q$, where $\lambda \ll 1$ is the power counting parameter used for the EFT expansion. The mass-mode field contains massive gauge bosons with all momentum components scaling as $Q\lambda \sim M$. The effective theory discussed here is SCET_{EW} studied in Refs. [3–5], and is similar to SCET_I, but with weak-scale mass-modes instead of the ultrasoft modes familiar from QCD. If we would consider the broken $SU(2)$ together with QCD, the effective theory would have additional n- and $\bar{n}$-collinear massless gluons and ultrasoft massless gluons, as in SCET_I. The n- and $\bar{n}$-collinear massless gluons fields would have the momentum scaling of the n- and $\bar{n}$-collinear massive gauge fields of the broken $SU(2)$. The ultrasoft gluon fields would have the momentum scaling $p^\mu \sim Q\lambda^2$ with $p^2 \sim M^4/Q^2$. At $\mu = M$ the n- and $\bar{n}$-collinear massive gauge fields and the mass-modes can be integrated out, leaving a common massless SCET_I theory for $\mu < M$. Such a situation is realized in the $SU(3) \times SU(2) \times U(1)$ electroweak theory [3,4].

The interactions of the mass-mode fields with the collinear fields are described by mass-mode S Wilson lines, whose definition is identical to the Y Wilson lines that arise for massless ultrasoft modes in massless SCET_I upon the ultrasoft field redefinition. The difference is that the mass-mode Wilson lines contain mass-mode gauge fields rather than ultrasoft massless gauge fields. Thus the effective field theory current for the broken $SU(2)$ has the form[2]

$$J(\omega, \bar{\omega}, \mu) = [\bar{\xi}_{n,\omega} W_n S_n^\dagger \Gamma S_{\bar{n}} W_{\bar{n}}^\dagger \xi_{\bar{n},\bar{\omega}}](0), \qquad (3)$$

where

$$
\begin{aligned}
S_n^\dagger(x) &= \text{P} \exp\left[ig \int_0^\infty ds\, n \cdot A_s(ns + x) \right], \\
S_{\bar{n}}(x) &= \bar{\text{P}} \exp\left[-ig \int_0^\infty ds\, \bar{n} \cdot A_s(\bar{n}s + x) \right].
\end{aligned}
\qquad (4)
$$

More details can be found in Ref. [19].

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[1]Note that the results only hold for $C_A = 2$, since for an $SU(N)$ group with $N > 2$, a fundamental Higgs does not break the gauge symmetry completely.

[2]In the presence of additional ultrasoft massless gauge fields the effective theory current would have the form $J(\omega, \bar{\omega}, \mu) = [\bar{\xi}_{n,\omega} W_n S_n^\dagger Y_n^\dagger \Gamma Y_{\bar{n}} S_{\bar{n}} W_{\bar{n}}^\dagger \xi_{\bar{n},\bar{\omega}}](0)$, with ultrasoft Y Wilson lines.

III. FACTORIZATION AND COLLINEAR WILSON LINES

Consider a high-energy scattering process with two or more particles, in the n_i direction, $i = 1, \ldots, r$ (see Fig. 2). n_i-collinear gauge bosons, which have momentum parallel to particle i, can interact with particle i, or with the other particles $j \neq i$. The coupling of n_i-collinear gauge bosons to particle i is included explicitly in the SCET Lagrangian. The particle-gauge interactions are identical to those in the full theory, and there is no simplification on making the transition to SCET. However, if an n_i-collinear gauge boson interacts with a particle j not in the n_i-direction, then particle j becomes off-shell by an amount of order Q, and the intermediate particle j propagators can be integrated out, giving a Wilson line interaction in SCET. The form of these operators was derived in Refs. [2,22], and gives the Wilson line interaction $W_{n_i}^\dagger \xi_{n_i}$, where W_{n_i} is a Wilson line in the $\bar{n}_i$ direction in the same representation as ξ_{n_i}. This is easy to see in processes with only two collinear particles. But even in complicated scattering processes with more than two collinear particles the Wilson line interaction still has the form $W_{n_i}^\dagger \xi_{n_i}$. Gluon emission from all particles other than the n_i-collinear particle combine (using the fact that the operator is a gauge singlet) to give a Wilson line in the representation of the n_i-collinear particle.

The Feynman rules for multiple gauge emission of n_i-collinear gluons from particle j gives factors of the form

$$\frac{\epsilon \cdot n_j}{k \cdot n_j}. \tag{5}$$

The n_i-collinear gauge field has momentum k and polarization ϵ in the n_i-direction at leading order in SCET power counting, so the above expression can be replaced by

$$\frac{\epsilon \cdot n_j}{k \cdot n_j} \to \frac{n_i \cdot n_j}{n_i \cdot n_j} \frac{\epsilon \cdot \bar{n}_i}{k \cdot \bar{n}_i} = \frac{\epsilon \cdot \bar{n}_i}{k \cdot \bar{n}_i} \tag{6}$$

using the leading (first) term in Eq. (2) for the decomposition of both k and ϵ. This expression is independent of n_j. As a result, when one combines emission from all particles which are not in the n_i-direction, the term in Eq. (6) can be factored out, and the color matrices combined to form a single Wilson line in the $\bar{n}_i$ direction. This is the basis for
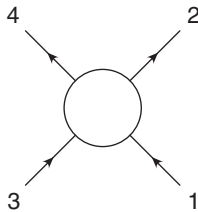

FIG. 2. A scattering amplitude with four external particles defining four collinear directions n_1, n_2, n_3, and n_4.

factorization in SCET, since n_i-collinear interactions are independent of the dynamics of all particles not in the n_i-direction.

Unfortunately, Eq. (6), while valid at tree-level, cannot be used for loop diagrams. The reason is that loop diagrams require a regulator for the Wilson lines. For example, with analytic regularization, Eq. (6) becomes

$$\frac{\epsilon \cdot n_j}{(k \cdot n_j)^{1+\delta}} \to \frac{n_i \cdot n_j}{(n_i \cdot n_j)^{1+\delta}} \frac{\epsilon \cdot \bar{n}_i}{(k \cdot \bar{n}_i)^{1+\delta}}$$

$$= \frac{1}{(n_i \cdot n_j)^\delta} \frac{\epsilon \cdot \bar{n}_i}{(k \cdot \bar{n}_i)^{1+\delta}} \tag{7}$$

and the j dependence no longer cancels. Thus the identities which allowed one to combine all the n_i-collinear emissions into a single Wilson line in the $\bar{n}_i$ direction no longer hold.

In this paper, we regulate the Wilson lines using the Δ-regulator, which also introduces i-dependence into Eq. (6), and naively breaks factorization. We will see that after zero-bin subtraction, the j-dependence cancels, and factorization is restored.

IV. Δ REGULATOR

The Δ-regulator for particle i is given by replacing the propagator denominators by

$$\frac{1}{(p_i + k)^2 - m_i^2} \to \frac{1}{(p_i + k)^2 - m_i^2 - \Delta_i}. \tag{8}$$

This regulator can be implemented at the level of the Lagrangian, since it corresponds to a shift in the particle mass. The on-shell condition remains $p_i^2 = m_i^2$.

In SCET, the collinear propagator denominators have the replacement of Eq. (8). If particle j interacts with an n_i-collinear gluon and becomes off-shell, then

$$\frac{1}{(p_j + k)^2 - m_j^2 - \Delta_j} \to \frac{1}{\frac{1}{2}(\bar{n}_i \cdot k)(\bar{n}_j \cdot p_j)(n_i \cdot n_j) - \Delta_j}, \tag{9}$$

where k is n-collinear, and Eq. (6) becomes

$$\frac{\epsilon \cdot n_j}{k \cdot n_j} \to \frac{\epsilon \cdot \bar{n}_i}{k \cdot \bar{n}_i - \delta_{j,n_i}}, \qquad \delta_{j,n_i} \equiv \frac{2\Delta_j}{(n_i \cdot n_j)(\bar{n}_j \cdot p_j)}. \tag{10}$$

The denominator in Eq. (5) gets shifted by δ_{j,n_i}, as can be seen from the denominator of Eq. (9). The Wilson lines in the Δ-regulator method will be regulated using Eq. (10). As a result, in the multiparticle case, it is not possible to combine n_i-collinear gluon emission off the various particles into a single Wilson line in the $\bar{n}_i$ direction, since δ_{j,n_i} depends on the particle j. However, we will see that after zero-bin subtraction, the j-dependence drops out, and n_i-collinear gluon emission can be combined into a single Wilson line.

While $\delta_{i,n}$ and Δ_i are related by Eq. (10), it is useful to retain both variables during the computation.

V. CALCULATION IN THE EFFECTIVE THEORY

The one-loop effective theory vertex graphs are the n-collinear, $\bar{n}$-collinear and mass-mode graphs, shown in Fig. 3. The n-collinear diagram reads

$$I_n = -2ig^2 C_F f_\epsilon \int \frac{d^d k}{(2\pi)^d} \frac{1}{[-\bar{n} \cdot k]} \frac{\bar{n} \cdot (p_2 - k)}{[(p_2 - k)^2]} \frac{1}{k^2 - M^2}, \tag{11}$$

with $f_\epsilon = (4\pi)^{-\epsilon} \mu^{2\epsilon} e^{\epsilon \gamma_E}$. Since the gauge boson is massive, this integral is divergent even if p_2 is off-shell in $d = 4 - 2\epsilon$ dimensions. This can be seen as follows: Integrate over k^+ by contours and perform the substitution $k^- = zp_2^-$. Because the poles of k^+ lie in the same half-plane for $k^- > p_2^-$ and $k^- < 0$, one obtains (keeping $p_2^+ \neq 0$ to regulate the integral)

$$I_n = -2a\mu^{2\epsilon} e^{\epsilon \gamma_E} \Gamma(\epsilon) \int_0^1 dz \frac{1-z}{z}$$
$$\times [M^2(1-z) - p_2^2 z(1-z)]^{-\epsilon} \tag{12}$$

where $a = C_F \alpha/(4\pi)$. For $z \to 0$ this integral diverges as long as the gauge boson is massive $M \neq 0$, even if $p_2^2 \neq 0$. For massless gauge bosons, $M = 0$, and the factor $z^{-\epsilon}$ from the $p_2^2 \neq 0$ term regulates the integral when the fermion is off-shell.

Introducing the Δ-regulator, the n-collinear diagram becomes

$$I_n = -2ig^2 C_F f_\epsilon \int \frac{d^d k}{(2\pi)^d} \frac{1}{[-\bar{n} \cdot k - \delta_1]}$$
$$\times \frac{\bar{n} \cdot (p_2 - k)}{[(p_2 - k)^2 - \Delta_2]} \frac{1}{k^2 - M^2}. \tag{13}$$

Doing the integrations in exactly the same way as described above, one obtains for the n-collinear integral with an on-shell external fermion $p_2^2 = 0$

$$I_n = -2a\Gamma(\epsilon)(M^2)^{-\epsilon} \mu^{2\epsilon} e^{\epsilon \gamma_E} \int_0^1 dz \frac{(1-z)^{1-\epsilon}}{z + \delta_1/p_2^-}$$
$$= a\left[\left(\frac{2}{\epsilon} - 2\mathsf{L_M}\right)(1 + \log(\delta_1/p_2^-)) - \frac{\pi^2}{3} + 2\right]. \tag{14}$$

Note that $\Delta_{1,2}$ and $\delta_1 \equiv \delta_{1,n}$, $\delta_2 \equiv \delta_{2,\bar{n}}$ are regulator

parameters, and are set to zero unless they are needed to regulate any divergence. The regulator parameters are defined using Eq. (10),

$$\delta_1 \equiv \delta_{1,n} = 2\frac{\Delta_1}{(n \cdot \bar{n})(n \cdot p_1)} = \frac{\Delta_1}{p_1^+},$$
$$\delta_2 \equiv \delta_{2,\bar{n}} = 2\frac{\Delta_2}{(n \cdot \bar{n})(\bar{n} \cdot p_2)} = \frac{\Delta_2}{p_2^-}. \tag{15}$$

We recall that $n_1 = \bar{n}$ because p_1 is in the $\bar{n}$ direction, and similarly for δ_2. The terms δ_1 and δ_2 transform under boosts as the $-$ and $+$ component of a vector, and $\mathsf{L_M}$, $\mathsf{L_Q}$ are defined as

$$\mathsf{L_M} = \log\frac{M^2}{\mu^2}, \qquad \mathsf{L_Q} = \log\frac{Q^2}{\mu^2}. \tag{16}$$

Equation (13) depends on Δ_2 and δ_1, which regulate the collinear and Wilson line propagators, respectively. Δ_2 is not needed to regulate a divergence in the integral, so the result in Eq. (14) only depends on the Wilson line regulator δ_1. The n-collinear graph depends on the scale Q via the regulator dependence,

$$\log\frac{\delta_1}{p_2^-} = \log\frac{\Delta_1}{p_1^+ p_2^-} = \log\frac{\Delta_1}{Q^2}. \tag{17}$$

The n-collinear particle momentum is p_2, but the n-collinear graph Eq. (14) depends on particle 1 via its dependence on the regulator Δ_1 for particle 1. This leads to a violation of factorization in the collinear sector, since the n-collinear graph depends on the properties of the other particles. In the multiparticle case, this means that the Wilson lines for all the other particles cannot be combined into a single Wilson line in the $\bar{n}$ direction—the loop contributions from the other particles each depend on their own regulator δ_i, and the different contributions cannot be combined into a single amplitude.

The n-collinear wave function renormalization graph is identical to that in the full theory. It does not need any Δ-regularization, and reads

$$W_n = a\left[\frac{1}{\epsilon} - \frac{1}{2} - \mathsf{L_M}\right]. \tag{18}$$

The normalization convention is such that one gets a contribution of $-W_n/2$ for each external n-collinear fermion.

The $\bar{n}$-collinear integral $I_{\bar{n}}$ can be obtained from I_n by replacing p_2^- by p_1^+ and δ_1 by δ_2,
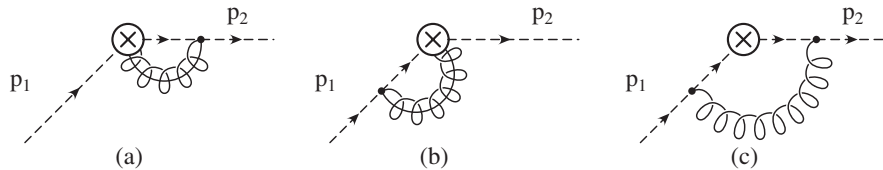

FIG. 3.   Diagrams of order $O(\alpha)$ in the effective theory. Wave function diagrams are not shown. The dashed line denotes a fermion, the spring denotes a mass-mode gauge boson and the spring with a line denotes a collinear gauge boson.

$$I_{\bar{n}} = a\left[\left(\frac{2}{\epsilon} - 2L_M\right)(1 + \log(\delta_2/p_1^+)) - \frac{\pi^2}{3} + 2\right], \quad (19)$$

and the $\bar{n}$-collinear wave function renormalization is $W_{\bar{n}} = W_n$.

For the mass-mode diagram, one finds

$$I_s = -2ig^2 C_F f_\epsilon \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - M^2} \frac{1}{-n \cdot k - \delta_2}$$

$$\times \frac{1}{-\bar{n} \cdot k - \delta_1}$$

$$= a\left[-\frac{2}{\epsilon^2} + \frac{2}{\epsilon} \log\frac{\delta_1\delta_2}{\mu^2} + L_M^2 - 2L_M \log\frac{\delta_1\delta_2}{\mu^2} + \frac{\pi^2}{6}\right].$$
$$(20)$$

Again, we only keep the leading terms in δ_i, and the integral depends on both δ_1 and δ_2. The mass-mode wave function contribution vanishes, $W_s = 0$, since $n^2 = \bar{n}^2 = 0$.

A. Zero-bin subtractions

In the effective theory, the gauge boson fields of the full theory are split up into several different fields which fluctuate over different scales. In order to avoid double-counting of the mass-modes, one has to subtract the contribution from the collinear fields with vanishing label momenta [10]. The zero-bin subtraction for Eq. (11), which amounts to taking the soft limit in the integrand of the collinear integral, is

$$I_{n\emptyset} = -2ig^2 C_F f_\epsilon \int \frac{d^d k}{(2\pi)^d} \frac{1}{[-\bar{n} \cdot k - \delta_1]}$$

$$\times \frac{1}{[-n \cdot k - \Delta_2/p_2^-]} \frac{1}{k^2 - M^2}, \quad (21)$$

which is the same as the integral Eq. (20), with $\delta_2 \rightarrow \Delta_2/p_2^-$. One needs to retain Δ_2 to regulate the singularity, since the k^2 term in the collinear propagator has been expanded out. Subtracting this from the collinear integral yields

$$I_n - I_{n\emptyset} = a\left[\frac{2}{\epsilon^2} - \frac{2}{\epsilon} \log\frac{\Delta_2}{\mu^2} + \frac{2}{\epsilon} - 2\left(1 - \log\frac{\Delta_2}{\mu^2}\right)L_M \right.$$

$$\left. - L_M^2 - \frac{\pi^2}{2} + 2\right]. \quad (22)$$

This combination only depends on the gauge boson mass and the regulator of the collinear fermion, Δ_2. The zero-bin subtraction $W_{n\emptyset}$ for the wave function renormalization W_n vanishes.

There are two very important differences between the zero-bin subtracted result $I_n - I_{n\emptyset}$ and the unsubtracted result I_n: The zero-bin subtracted integral no longer depends on the hard scale Q, and it depends only on the regulator Δ_2 for the n-collinear particle, rather than on the

regulator Δ_1 for the other particle. This means it depends on the regulator of the collinear particle rather than the regulator of the Wilson line, and it implies that zero-bin subtraction restores factorization in the effective theory. The hard scale has been factored out of the collinear contribution. In addition, in the multiparticle case, since the collinear graphs only depend on the n-collinear particle regulator (Δ_2 in our problem), the Wilson line contributions from all the other particles can be combined into a single Wilson line in the $\bar{n}$ direction, as was naively true at tree-level. This is because the zero-bin subtracted collinear graph does not need a regulator for the Wilson line.

The final result of the effective theory vertex computation is

$$(I_n - I_{n\emptyset}) + (I_{\bar{n}} - I_{\bar{n}\emptyset}) + I_s - \frac{1}{2}(W_n - W_{n\emptyset})$$

$$- \frac{1}{2}(W_{\bar{n}} - W_{\bar{n}\emptyset}) - W_s$$

$$= a\left[\frac{2}{\epsilon^2} + \frac{2}{\epsilon} \log\frac{\delta_1\delta_2\mu^2}{\Delta_1\Delta_2} + \frac{3}{\epsilon} - L_M^2\right.$$

$$\left. - 2L_M \log\frac{\delta_1\delta_2\mu^2}{\Delta_1\Delta_2} - 3L_M - \frac{5\pi^2}{6} + \frac{9}{2}\right], \quad (23)$$

where we have added the zero-bin subtracted collinear graphs and the mass-mode graph. This result has to be independent of the regulators. Indeed, using Eq. (15) and (23), can be simplified to

$$a\left[\frac{2}{\epsilon^2} - \frac{2}{\epsilon} L_Q + \frac{3}{\epsilon} - L_M^2 + 2L_Q L_M - 3L_M - \frac{5\pi^2}{6} + \frac{9}{2}\right]. \quad (24)$$

This is the correct effective theory result, and when combined with the matching computation at Q [23] correctly reproduces the known full-theory computation of the form-factor.

Note that without zero-bin subtractions, the effective theory result would be

$$I_n + I_{\bar{n}} + I_s - \frac{1}{2}W_n - \frac{1}{2}W_{\bar{n}} - W_s$$

$$= a\left[-\frac{2}{\epsilon^2} + \frac{2}{\epsilon} \log\frac{\Delta_1^2\Delta_2^2}{Q^6\mu^2} + \frac{3}{\epsilon} + L_M^2\right.$$

$$\left. - 2L_M \log\frac{\Delta_1^2\Delta_2^2}{Q^6\mu^2} - 3L_M - \frac{\pi^2}{2} + \frac{9}{2}\right]. \quad (25)$$

This is incorrect, and does not reproduce the full-theory result when the matching condition at Q is included. The $1/\epsilon$ singularities, which are ultraviolet, do not give the correct anomalous dimension $\gamma = a(4L_Q - 6)$ for the current in SCET. The result is also not independent of the regulator. Idilbi and Mehen [11,12] have previously arrived at the same conclusions for QCD, where the gauge boson is massless. However, the necessity of zero-bin subtractions becomes more obvious with a massive gauge boson, since

the effective theory graphs are no longer scaleless due to the gauge boson mass. One can see that Eq. (25) also does not give the correct finite parts of the diagram.

Using the Δ-regulator, we have seen that the vertex corrections are $(I_n - I_{n\emptyset}) + (I_{\bar{n}} - I_{\bar{n}\emptyset}) + I_s$ after including zero-bin subtractions. Also, $I_{n\emptyset} = I_{\bar{n}\emptyset} = I_s$, so the vertex correction can be written as $I_n + I_{\bar{n}} - I_s$. Recently, Idilbi and Mehen [11,12] showed in their study of deep inelastic scattering in QCD, that the combination $I_n + I_{\bar{n}} - I_s$ does not need any additional regulator beyond dimensional regularization. The same result continues to hold for broken $SU(2)$ with massive gauge bosons, where the role of the ultrasoft contribution is adopted by the mass-mode contribution I_s. Thus the integrand obtained by combining $I_n + I_{\bar{n}} - I_s$ does not need any Δ-regulator either, and can be evaluated explicitly to give

$$I_n + I_{\bar{n}} - I_s = a\left[\frac{2}{\epsilon^2} - \frac{2}{\epsilon}\mathsf{L_Q} + \frac{4}{\epsilon} - \mathsf{L_M}^2 \right.$$
$$\left. + 2\mathsf{L_Q}\mathsf{L_M} - 4\mathsf{L_M} - \frac{5\pi^2}{6} + 4\right]. \quad (26)$$

When combined with the wave function graphs this gives the correct amplitude Eq. (24). The details of the computation are given in the Appendix. The form of the expression, $I_n + I_{\bar{n}} - I_s$ in broken $SU(2)$—and similarly in QCD—is counter-intuitive if one is used to thinking about effective field theories using the method of regions. This is because one has to subtract the mass-mode/ultrasoft region from the sum of the collinear regions to get the correct amplitude.

The above discussion shows that in practice one can identify the respective zero-bin contribution of the collinear integrals $I_{n\emptyset}$ and $I_{\bar{n}\emptyset}$ with the mass-mode integral I_s. Doing this identification also in the case where the integrals are done separately with the regulator, one does not need any relations between Δ_i and δ_i. Instead one regulates the collinear Wilson lines and propagators in the soft graphs with δ_i, and Eq. (22) is instead given by the same expression with $\delta_2 p_2^-$ in place of Δ_2.

B. Momentum regions

In this section we discuss the various momentum regions that contribute to the computation of the Sudakov form factor paying particular attention to the role of the zero-bin subtractions to the contributions from the n- and $\bar{n}$-collinear regions. The momentum regions which contribute to the Sudakov form factor are illustrated in Fig. 4. The hard contribution is the high-scale matching at Q, and the remaining contributions are given by the effective theory. The effective theory contributions are located along the curve $k^2 = M^2$. The n-collinear contribution arises from $k^- \sim p_2^- \sim Q$, so that $k^+ \sim M^2/Q$ and the $\bar{n}$-collinear contribution arises from $k^+ \sim p_1^+ \sim Q$, $k^- \sim M^2/Q$. The ultrasoft region with $k^+ \sim M^2/p_2^- \sim M^2/Q$
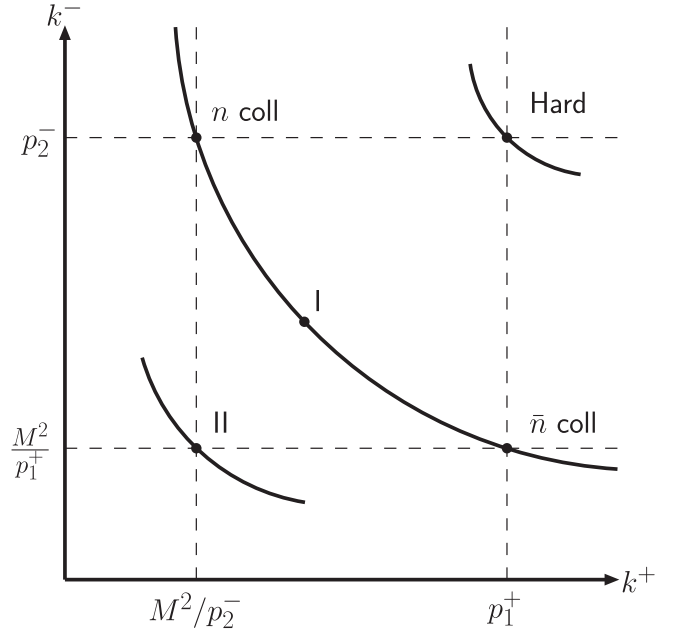

FIG. 4. Modes which contribute to the effective theory. Mass-modes are shown as I. Additional massless gluons contribute modes shown as II.

and $k^- \sim M^2/p_1^+ \sim M^2/Q$ is not on the $k^2 = M^2$ hyperbola, and does not contribute to the amplitude. Interestingly, it now turns out that the mass-mode region, with all components of k of order M, does not contribute to the amplitude either.[3] As shown in Ref. [3], the contributions to the amplitude from the mass-mode region vanish if the analytic regulator of Eq. (7) is employed, and only the n- and $\bar{n}$-collinear regions give nonzero results. As we show below, the same is true also when the Δ-regulator is used. However, in this case, the argument is more subtle and requires that zero-bin contributions to the n- and $\bar{n}$-collinear pieces are properly accounted for.

While the analytic regulator does not introduce any new dimensionful parameters, the Δ-regulator introduces the new dimensionful scales $\Delta_{1,2}$, and the picture changes because of contributions from unphysical regulator-dependent regions, which are absent in the total amplitude. This has been noted previously [24,25]. The n-collinear graph I_n, Eq. (14), gets contributions from $k^- \sim p_2^- \sim Q$, $k^+ k^- \sim M^2$ and $k^- \sim \delta_1$, $k^+ k^- \sim M^2$. This is shown as regions A and C in Fig. 5. The $\bar{n}$-collinear graph gets contributions from regions D and B.

The zero-bin integral $I_{n\emptyset}$ gets contributions from the region with $k^+ \sim \Delta_2/p_2^-$ and from $k^- \sim \delta_1$, i.e. from regions B and C. Thus, for the zero-bin subtracted collinear integral $I_n - I_{n\emptyset}$, region C cancels, and the resulting con-

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[3]It is important to include the mass-mode contribution. The mass-mode region does not contribute, because of a cancellation between the collinear modes and mass-modes in the mass-mode region. See below.
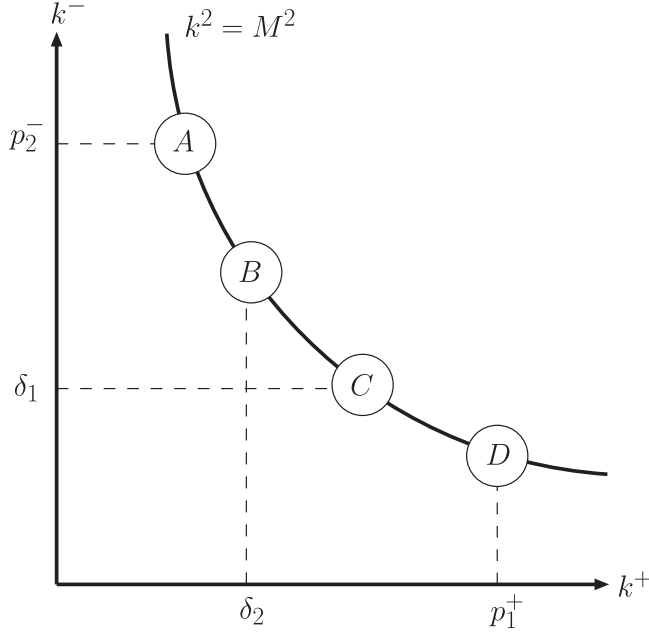
FIG. 5.   Momentum regions which contribute to the effective theory integrals. A and D are collinear, and B and C are regulator-dependent mass-mode regions.

tributions are from regions A and B. Similarly, $I_{\bar{n}} - I_{\bar{n}\emptyset}$ gets contributions from D and C. On the other hand, the mass-mode graphs get a contribution from the region $k^- \sim \delta_1$, $k^+ k^- \sim M^2$ and from $k^+ \sim \delta_2$, $k^+ k^- \sim M^2$, i.e. regions B and C. We now see that the total amplitude only gets contributions from A and D. The additional regions B and C introduced by the regulator drop out, as they should. To achieve the cancellation of the unphysical regions B and C it is essential to account for the zero-bin subtractions for the collinear regions.

C. Gauge dependence

So far, we have been working in Feynman gauge, $\xi = 1$. Let us now analyze the gauge dependence of the different parts of the effective theory calculation by using a general R_ξ gauge with the gauge boson propagator

$$i\Delta_{\alpha\beta}(k) = \frac{1}{k^2 - M^2}\left[g_{\alpha\beta} + (\xi - 1)\frac{k_\alpha k_\beta}{k^2 - \xi M^2}\right]. \quad (27)$$

In the full theory, the new ξ dependent contribution to the vertex graph, $I^{(\xi)}$, stemming from the second part of the gauge boson propagator is

$$I^{(\xi)} = \frac{\alpha_s}{4\pi}C_F\Gamma J$$

$$J = -16\pi^2 i(\xi - 1)f_\epsilon \int \frac{d^d k}{(2\pi)^d}\frac{1}{k^2 - M^2}\frac{1}{k^2 - \xi M^2}$$

$$= \frac{\xi - 1}{\epsilon_{\text{UV}}} - (\xi - 1)\log\frac{M^2}{\mu^2} + (\xi - 1) - \xi\log\xi. \quad (28)$$

The full-theory vertex graph gets shifted, $I_V \rightarrow I_V + aJ$. There is a similar shift in the full-theory wave function contribution, $W \rightarrow W + aJ$ so that the on-shell S-matrix element $I_V - W$ is ξ-independent. The high-scale matching coefficient is the full-theory result with all infrared scales set to zero, and so is gauge invariant.

In terms of the method of regions, J only has contributions from the mass-mode region where $k^+ \sim k^- \sim k^\perp \sim M$, given ξ is counted as order $O(1)$. Therefore, one might not expect this additional piece to show up in the collinear vertex diagrams in the effective theory. However, doing the calculation of the additional parts $I_n^{(\xi)}$ and $I_{\bar{n}}^{(\xi)}$ of the collinear integrals yields (see Ref. [2] for the Feynman rules)

$$I_n^{(\xi)} = -ig^2 C_F f_\epsilon \int \frac{d^d k}{(2\pi)^d}\left[n^\alpha - \frac{\gamma_\perp^\alpha \not{k}_\perp}{\bar{n}\cdot(p_2 - k)}\right]$$

$$\times \frac{\bar{n}\cdot(p_2 - k)}{(p_2 - k)^2 - \Delta_2}\Gamma\frac{1}{-\bar{n}\cdot k - \delta_1}\bar{n}^\beta\frac{1}{k^2 - M^2}$$

$$\times \left[(\xi - 1)\frac{k_\alpha k_\beta}{k^2 - \xi M^2}\right]$$

$$= \frac{\alpha}{4\pi}C_F\Gamma J, \quad (29)$$

and similarly for $I_{\bar{n}}^{(\xi)}$. Note that for Eq. (29) we have adopted $p_2^+ = p_2^\perp = 0$.

For the mass-mode diagram, the additional piece reads

$$I_s^{(\xi)} = -ig^2 C_F f_\epsilon \int \frac{d^d k}{(2\pi)^d}\frac{1}{-n\cdot k - \delta_2}\Gamma\frac{1}{-\bar{n}\cdot k - \delta_1}$$

$$\times \frac{1}{k^2 - M^2}\left[(\xi - 1)\frac{(n\cdot k)(\bar{n}\cdot k)}{k^2 - \xi M^2}\right]$$

$$= \frac{\alpha}{4\pi}C_F\Gamma J. \quad (30)$$

The collinear and soft wave function renormalization graphs also get shifted by J, $W_n \rightarrow W_n + aJ$, $W_{\bar{n}} \rightarrow W_{\bar{n}} + aJ$, $W_s \rightarrow W_s + aJ$.

Without accounting for zero-bin subtractions, the effective theory result $I_n + I_{\bar{n}} + I_s - (W_n/2 + W_{\bar{n}}/2 + W_s)$ would get shifted by $aJ + aJ + aJ - (aJ/2 + aJ/2 + aJ) = aJ$, and is not gauge invariant. With zero-bin subtractions, however, the effective theory collinear vertex graph is $I_n - I_{n\emptyset}$, which is gauge invariant, since both terms shift by aJ. Similarly, the collinear wave function graph is $W_n - W_{n\emptyset} = W_n - W_s$, which is also gauge invariant, since again both terms are shifted by aJ. Thus, the collinear vertex and wave function contributions are each separately gauge invariant. The mass-mode contributions I_s and W_s are each shifted by aJ, so the net soft contribution $I_s - W_s$ is gauge invariant as well.

Thus zero-bin subtractions are also necessary to maintain gauge invariance of the two collinear and the mass-mode sectors of the effective theory, as required by factorization.

VI. CONCLUSIONS

SCET with massive gauge bosons requires an additional regulator on top of the common dimensional regularization to obtain well-defined expressions for individual Feynman diagrams. In this work we have proposed the Δ-regulator to regularize the singularity from the Wilson line propagators. Using the Δ-regulator, the effective theory only gives the correct result for the scattering amplitude if zero-bin subtractions for the n- and $\bar{n}$-collinear contributions are included. For the Sudakov form factor in a broken $SU(2)$ gauge theory with a common gauge boson mass M, the total amplitude then is $I_n + I_{\bar{n}} - I_s$ as a result of the subtractions. Here I_n and $I_{\bar{n}}$ refer to the n- and $\bar{n}$-collinear diagrams, and I_s refers to the contributions of the mass-mode graphs related to gluons with momenta $k^+ \sim k^- \sim k^\perp \sim M$. This result is in analogy to previous results obtained for unbroken gauge theories with massless gauge bosons [11,12]. The result appears to contradict the method-of-regions approach, where one has to add up the contributions from all different regions.

We have demonstrated that one needs to subtract the mass-mode region from the sum of the collinear regions to avoid double-counting, and that gauge invariance of the effective theory is only maintained if the zero-bin subtractions are accounted for.

Zero-bin subtractions also restore factorization between the different collinear sectors. Naively, one implements factorization by redefining the collinear fields as [2,22]

$$W_n^\dagger \xi_n \to S_n W_n^{(0)\dagger} \xi_n^{(0)} \tag{31}$$

where the mass-mode fields are in the Wilson line S_n, and no longer couple to the collinear fields in $W_n^{(0)}$ and $\xi_n^{(0)}$. This redefinition is not valid at the loop-level, because the regulator dependence of the collinear graphs breaks factorization. Factorization is restored after zero-bin subtraction, and thus the proper replacement is

$$W_n^\dagger \xi_n \to S_n [W_n^{(0)\dagger} \xi_n^{(0)}]_\emptyset \tag{32}$$

where the subscript $\emptyset$ is a reminder that the collinear sector requires zero-bin subtraction.[4]

APPENDIX: CALCULATION WITHOUT A REGULATOR

In this Appendix, we calculate the effective field amplitude including zero-bin subtractions by first adding and then performing the integration. No regulators are needed in this case, as in the massless case [11,12].

$$R = I_n + I_{\bar{n}} - I_s$$

$$= -2ig^2 C_F f_\epsilon \int \frac{d^d k}{(2\pi)^d} \left[\frac{p_1^+ - k^+}{(-p_1^+ k^- + k^2)(-k^+)(k^2 - M^2)} + \frac{p_2^- - k^-}{(-p_2^- k^+ + k^2)(-k^-)(k^2 - M^2)} - \frac{1}{(-k^+)(-k^-)(k^2 - M^2)} \right]$$

$$= -2ig^2 C_F f_\epsilon \int \frac{d^d k}{(2\pi)^d} \left[\frac{2k^2 + p_1^+ p_2^- - p_1^+ k^- - p_2^- k^+}{(-p_1^+ k^- + k^2)(-p_2^- k^+ + k^2)(k^2 - M^2)} - \frac{k^4}{(-p_1^+ k^- + k^2)(-p_2^- k^+ + k^2)(-k^+)(-k^-)(k^2 - M^2)} \right]. \tag{A1}$$

The total integral is IR finite. It can be decomposed as

$$R = -2ig^2 C_F [Q^2 I_0 - 2(p_1 + p_2)_\mu I_1^\mu + 2I_2 - I_3] \tag{A2}$$

with

$$I_0 = f_\epsilon \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k - p_1)^2 (k - p_2)^2 (k^2 - M^2)}$$

$$I_1^\mu = f_\epsilon \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu}{(k - p_1)^2 (k - p_2)^2 (k^2 - M^2)}$$

$$I_2 = f_\epsilon \int \frac{d^d k}{(2\pi)^d} \frac{k^2}{(k - p_1)^2 (k - p_2)^2 (k^2 - M^2)}$$

$$I_3 = f_\epsilon \int \frac{d^d k}{(2\pi)^d} \frac{k^2}{(-p_1^+ k^- + k^2)(-p_2^- k^+ + k^2)(-k^+)(-k^-)}. \tag{A3}$$

Since we are not interested in subleading terms in M^2/Q^2 for all these integrals, we simplify the last part of the integrand in Eq. (A1) and set $M = 0$ to obtain I_3. One finds

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[4]We note that in the presence of massless ultrasoft gauge fields the right-hand side of Eq. (32) reads $Y_n S_n [W_n^{(0)\dagger} \xi_n^{(0)}]_\emptyset$, with an ultrasoft Y Wilson line.

$$I_0 = -\frac{i}{16\pi^2}\int_0^1 dz \int_0^1 dx \frac{z}{Q^2 x(1-x)z^2 + M^2(1-z)} = -\frac{i}{16\pi^2 Q^2} J_1,$$

$$I_1^\mu = -\frac{i}{16\pi^2}\frac{1}{2}\int_0^1 dz \int_0^1 dx \frac{z^2(p_1+p_2)^\mu}{Q^2 x(1-x)z^2 + M^2(1-z)} = -\frac{i}{16\pi^2 Q^2} J_2 \frac{1}{2}(p_1+p_2)^\mu,$$

$$I_2 = M^2 I_0 + f_\epsilon \int \frac{d^d k}{(2\pi)^d}\frac{1}{(k-p_1)^2(k-p_2)^2} = M^2 I_0 + \frac{i}{16\pi^2}\left[\frac{1}{\epsilon} + \log\frac{\mu^2}{Q^2} + 2\right],$$

$$I_3 = -\frac{i}{2}\int_0^{p_2^-}\frac{dk^-}{2\pi}\frac{d^{d-2}k_\perp}{(2\pi)^{d-2}}\frac{p_1^+ k^-}{[\mathbf{k}_\perp^2 + p_1^+ k^-][(k^-)^2 p_1^+ - p_2^- \mathbf{k}_\perp^2 - p_1^+ p_2^- k^-]}$$

$$+ \frac{i}{2}f_\epsilon \int_{p_2^-}^\infty \frac{dk^-}{2\pi}\frac{d^{d-2}k_\perp}{(2\pi)^{d-2}}\frac{1}{(-p_1^+ k^- - \mathbf{k}_\perp^2)(k^-)}$$

$$= -\frac{i}{16\pi^2}\left[\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\log\frac{Q^2}{\mu^2} + \frac{1}{2}\log^2\frac{Q^2}{\mu^2} - \frac{\pi^2}{12}\right], \tag{A4}$$

with

$$J_n = \int_0^1 dz \int_0^1 dy \frac{4z^n}{z^2(1-y^2) + \lambda^2(1-z)} \tag{A5}$$

and $\lambda^2 = 4M^2/Q^2$. To calculate the integral J_1, integrate first over y and substitute $w = z + \sqrt{z^2 + \lambda^2(1-z)}$, leading to

$$J_1 = \frac{\pi^2}{3} + \frac{1}{2}\log^2\frac{Q^2}{M^2}. \tag{A6}$$

For J_2, one can simply expand in λ after the integration over y to obtain

$$J_2 = 2\log\frac{Q^2}{M^2} - 2. \tag{A7}$$

Adding everything up, one finally finds

$$R = a\left[\frac{2}{\epsilon^2} - \frac{2}{\epsilon}\mathsf{L_Q} + \frac{4}{\epsilon} - \mathsf{L_M}^2 + 2\mathsf{L_Q L_M} - 4\mathsf{L_M}\right.$$
$$\left. - \frac{5\pi^2}{6} + 4\right]. \tag{A8}$$

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