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Gauge theories with time dependent couplings and their cosmological duals

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We consider the $N = 4$ super Yang-Mills theory in flat $3 + 1$-dimensional space-time with a time dependent coupling constant which vanishes at $t = 0$, like $g^2_{\rm YM} = t^p$. In an analogous quantum mechanics toy model we find that the response is singular. The energy diverges at $t = 0$, for a generic state. In addition, if $p > 1$ the phase of the wave function has a wildly oscillating behavior, which does not allow it to be continued past $t = 0$. A similar effect would make the gauge theory singular as well, though nontrivial effects of renormalization could tame this singularity and allow a smooth continuation beyond $t = 0$. The gravity dual in some cases is known to be a time dependent cosmology which exhibits a spacelike singularity at $t = 0$. Our results, if applicable in the gauge theory for the case of the vanishing coupling, imply that the singularity is a genuine sickness and does not admit a meaningful continuation. When the coupling remains nonzero and becomes small at $t = 0$, the curvature in the bulk becomes of order string scale. The gauge theory now admits a time evolution beyond this point. In this case, a finite amount of energy is produced which possibly thermalizes and leads to a black hole in the bulk.

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I. INTRODUCTION AND SUMMARY

The resolution of singularities is an outstanding problem in the study of gravity. The gauge theory/gravity correspondence [1] provides a nonperturbative framework for the study of gravity, and one would hope that it can shed some light on this question.

With this motivation some cosmological solutions which admit natural gauge theory duals have been constructed and studied recently [2–6]. These solutions can be thought of as deformations of $AdS_5 \times S^5$ (in Poincaré coordinates) and are dual to strongly coupled $\mathcal{N} = 4$ Yang-Mills theory in flat $3 + 1$-dimensional space-time subjected to external time dependent or null sources. For other classes of solutions of this type, see [7,8]. A related approach to gauge theory duals of cosmological singularities has been pursued in [9] and more recently in [10]. Ideas about finding signatures of spacelike singularities inside black holes in the dual gauge theory are described in [11].

In this paper our interest is in the time dependent cases. These have been further studied in [4]. Here, the only source is a time dependent coupling of the gauge theory. At early times, the 't Hooft coupling in the gauge theory is large and varies slowly. The AdS/CFT correspondence tells us that the gravity dual is a non-normalizable deformation of $AdS_5 \times S^5$ sourced by the dilaton field. As time evolves, the gauge theory state evolves in response to the time dependent coupling. On the gravity side, the background evolves according to the supergravity equation of motion, subject to appropriate boundary conditions. In particular the dilaton starts back reacting leading to a nontrivial metric.

Our main interest will be in situations in which the dilaton, e^{Φ}, vanishes at time $t = 0$. In the corresponding gravity solution a spacelike singularity appears at $t = 0$ which extends all the way to the boundary.[1] One would like to know if this singularity is a genuine sickness in the theory or if it merely signals a breakdown of the supergravity approximation. Since the bulk theory has a boundary dual which is formulated in a precise fashion, we can ask this question in the dual theory. One would like to know if the boundary theory is sick at the singularity or if it allows for a continuation past the point where the dilaton vanishes.[2]

In this paper we try to analyze this question in some detail. Prompted by the cosmological solutions we ask the following general question first: Consider the $\mathcal{N} = 4$ Yang-Mills theory subjected to an external time dependent dilaton. We take the dilaton to be of the form,

$$e^{\Phi} = (-t)^p, \qquad t \leq 0, \tag{1}$$

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[1]However, the 4D metric as seen in the gauge theory is flat, after a conformal transformation, as discussed in Sec. V.

[2]More precisely, the supergravity approximation breaks down before the singularity forms, once the curvature gets to be of order string scale. Using the boundary theory we would like to find out if there is a continuation past this region.

so that it vanishes as $t \rightarrow 0^-$. In Eq. (1), the index p can take any positive real value. In such a situation the question we ask is whether the response of the gauge theory to this external source is singular or not, as $t \rightarrow 0^-$.

It is useful to first consider a quantum mechanics model which has an analogous coupling to the dilaton. We find that the response in this quantum mechanical system is singular, for all values of $p > 0$, in the following sense: The time dependent dilaton pumps energy into the system, and as $t \rightarrow 0^-$ we find that the energy diverges. The nature of the singularity does depend on the index p. For $p \geq 1$, when the variation of the dilaton is more rapid, the wave function of the system acquires a time dependent phase factor which becomes wildly oscillating and diverges as $t \rightarrow 0^-$. As a result the wave function of the system (in the Schrodinger picture) does not have a well-defined limit as $t \rightarrow 0^-$. In contrast for $p < 1$ the phase factor does not diverge and the wave function has a well-defined limit as $t \rightarrow 0^-$. Even so, the energy diverges as $t \rightarrow 0^-$ (this also happens when $p \geq 1$).

The result that for $p \geq 1$ the wave function becomes singular, without a well-defined limit, holds regardless of the state of the system. On the other hand, the conclusion that the energy diverges is true for a generic state. For special states in which an appropriate matrix element vanishes, the energy can remain finite, as we discuss below.[3]

The analysis in the quantum mechanics model is most conveniently carried out in the Schrodinger picture. The conclusions stated above follow from the fact that near $t = 0$ the potential energy term in the Schrodinger equation dominates the time evolution and the Kinetic energy term is subdominant.

One can carry out a similar analysis in the gauge theory. Once again if we assume that the potential energy dominates near $t = 0$ one finds that the behavior is completely analogous to that in the quantum mechanics model discussed above. The energy diverges as $t \rightarrow 0^-$ and for $p > 1$ the wave function acquires a wildly oscillating time dependent phase which does not have a well-defined limit as $t \rightarrow 0^-$.

However, in the field theory case, we have not been able to establish that the approximation leading to the conclusions above definitely holds. Differences with the quantum mechanics model arise due to the infinite number of modes in field theory. These have to be dealt with carefully by regulating the theory after introducing a cutoff and incorporating the effects of renormalization in this cutoff theory. We have not carried out this procedure adequately to determine whether the approximation mentioned above of the potential energy dominating holds, and our conclusions about the gauge theory response being singular are therefore tentative and not definite. As we discuss below, higher loop effects could sum up to render the kinetic term important and tame the singularity, resulting in finite energy production and a smooth continuation of the wave functional beyond $t = 0$. Studying this further will require both a better understanding of the calculational aspects of renormalization in the time dependent gauge theories at hand, and a better understanding of the conceptual issues involved in incorporating these effects of renormalization in the Schrodinger picture which we use in this paper.[4]

The AdS cosmologies described in [2,4] correspond to the value $p = \sqrt{3}$ in Eq. (1). From the discussion above it follows that, if our approximation of a dominant potential energy continues to hold in the gauge theory, the resulting singularity is a genuine sickness which does not admit a well-defined continuation. Since $p > 1$ in this case, the wave function in the gauge theory description becomes wildly oscillating without a well-defined limit as $t \rightarrow 0^-$. Also the energy should diverge as $t \rightarrow 0^-$.

Our analysis shows that the singularity in the bulk arises due to two related reasons. First, the dilaton vanishes, resulting in the string frame curvature blowing up. Second, an infinite amount of energy is dumped into the system by the vanishing dilaton, resulting in a singular back reaction.

We do not know the explicit form of the bulk solutions whose boundaries are conformally flat for values of p, other than $\sqrt{3}$. However such solutions should exist since we are specifying the dilaton field on the boundary for all times. It is worth pointing out that the singular behavior we find is not tied *per se* to the nonanalyticity of the dilaton as $t \rightarrow 0$, and occurs for all values of $p > 0$, integer and noninteger. Rather, the singular behavior is related to the rate at which the dilaton vanishes. As was mentioned above, with our approximations, the behavior is more singular for $p \geq 1$, when the dilaton vanishes more rapidly, than it is for $p < 1$.

A few more comments about the analysis in the gauge theory are also worth making. At first sight one might think that when the dilaton vanishes the gauge theory becomes weakly coupled and can be analyzed in perturbation theory. This turns out not to be true. Starting from a generic state, one finds that the time dependent dilaton excites the fields to large enough values so that the cubic and quartic interaction terms are non-negligible near the point where the dilaton vanishes and perturbation theory is not valid. This is the essential reason why the analysis gets complicated. Based on the quantum mechanics model we find that the Schrodinger picture is particularly useful in analyzing the resulting behavior. A Wentzel-Kramers-Brillouin–like approximation can be formulated in this picture in the vicinity of the vanishing dilaton. This allows the leading behavior near the singularity to be analyzed without having

[3]Fluctuations in the energy will diverge in these special states, even if the expectation value of the energy remains finite. However such fluctuations are suppressed at leading order in N.

[4]We thank David Gross and the referee for emphasizing these points to us.

to resort to the perturbation theory. As was mentioned above, in the gauge theory, we have not been able to establish that this approximation is indeed correct and thus our conclusions should be taken to be indicative rather than definitive.

It is worth emphasizing that if the potential energy term continues to dominate near the singularity, our conclusions about the gauge theory in the presence of a coupling which truly vanishes are valid at finite N and finite $g_{\rm YM}^2 N$ and are therefore a result about the dual closed string theory in the presence of stringy and/or quantum corrections. As we will see, the behavior of the wave functional near the time of vanishing coupling is essentially determined by the coupling constant and not the details of the Lagrangian.

It follows from our analysis that to find cosmological solutions which are not sick the dilaton's behavior at the boundary has to be modified so that it does not vanish. Once this is done, for a smoothly varying dilaton, one does not expect the gauge theory to be singular.[5] In a situation where the dilaton profile is chosen to be constant in the far future, reaching a value such that the 't Hooft coupling in the boundary theory is large, one expects that the supergravity solution becomes a black hole in the far future. This is based on the expectation that the dual gauge theory will generically have some nonvanishing energy density in the far future and this energy will eventually thermalize. It is worth noting that if this expectation is met, the space-time which is highly curved when the dilaton is small, eventually evolves to a smoothly varying space-time outside the black hole horizon. The fate of the theory will be similar if renormalization effects tame the gauge theory even in the case where the coupling truly vanishes.

An important question which we cannot address here is how the thermalization process depends on the dilaton time profile. To answer this question one needs a better understanding of the system when the 't Hooft coupling is of order one, for example.

One might wonder if the formation of the black hole can be avoided in a situation where the dilaton profile is time reversal invariant. In such a case states should exist which evolve in a time reversal invariant manner. Classically, in such a state all velocities have to vanish at $t = 0$; quantum mechanically, the wave function has to be real at $t = 0$. Starting from $t = 0$ in such a state and evolving into the future one expects that a black hole will typically form, if some net energy is input into the system. Thus we do not expect the absence of a black hole in such states, rather these states will correspond to starting with a black hole in the far past and ending with one in the far future. In some situations, this conclusion might be avoided, but we do not understand at the moment how to identify them.[6]

The breakdown of perturbation theory near $t = 0$, in the time dependent case, is quite different from what happens in the null-dependent solutions which were studied in [2,3,5]. In the null case, the effects we describe in this paper are absent, there is no particle production and perturbation theory is possibly applicable for correlators of fields at light front times when the coupling is small.

We discussed cosmological solutions above with $p = \sqrt{3}$, which were studied in [2,4]. These correspond to symmetric Kasner-like solutions, where all spatial directions expand at the same rate, or to Friedmann-Robertson-Walker (FRW) solutions. There are other asymmetric Kasner solutions which were also constructed in [2,4]. The dual gauge theory in these cases does not live in flat space but instead in a space-time with a curvature singularity, which occurs at the time coincident with the bulk singularity. Since the boundary metric is nondynamical, it is difficult to see how time evolution on the boundary can be continued past the singularity in such circumstances leading to the conclusions that the boundary hologram is sick in these cases as well.

The behavior of the cosmological solutions near the singularity has some degree of universality. For example in the symmetric Kasner and FRW solutions, mentioned above, the spatial curvature is different but this feature is irrelevant near the singularity where the evolution of the space-time is determined by the diverging dilaton stress tensor. One expects this to be a more general feature—some differences among solutions should become irrelevant leading to the same behavior near the singularity. Our conclusions obtained from studying the gauge theory dual to the cosmological solutions, within our approximations, do not require any particular state, and apply quite generally. One therefore expects these conclusions to hold for all solutions with the same common behavior near the singularity, e.g., for all conformally flat four-dimensional metrics as discussed in Sec. VI.

To explore this further we discuss Bianchi-IX-type cosmologies in the presence of the dilaton towards the end of this paper. These solutions are constructed by taking solutions of 4D gravity coupled to dilaton and embedding them in 5D AdS space as discussed in [2,4]. Interestingly, once the dilaton is excited only a finite number of Belinski, Lifshitz, and Khlalatnikov (BKL) oscillations occur between Kasner regimes. With each oscillation the importance of the dilaton stress energy grows, at the expense of the spatial curvature. Eventually all solutions enter a Kasner regime, where all directions shrink, and the spatial

[5] We are assuming here that the gauge theory does not have any phase transitions as the dilaton is varied. If this assumption is wrong the response of the gauge theory to a smoothly varying dilaton need not be nonsingular. Such a phase transition can be avoided by working on S^3, at finite N. We thank S. Wadia for emphasizing this point to us.

[6] The case where the boundary theory is on S^3 rather than R^3 is more promising in this respect, since the formation of a black hole then requires the temperature to be bigger than that of the Hawking-Page transition.

curvature is relatively unimportant. In a whole family of solutions this final Kasner regime corresponds to the symmetric Kasner solution with $p = \sqrt{3}$, mentioned above. For other solutions the final Kasner regime is one where the three spatial directions shrink in an asymmetric manner. The holographic dual for these cosmologies can be constructed. Our analysis of the singularity in the symmetric and asymmetric Kasner solutions then applies to all these solutions. It will be interesting to explore these solutions and their holographic duals further.

This paper is organized as follows. In Sec. II we discuss the gauge theory and introduce a toy field theory which captures the essential features of the gauge theory in the dilaton background. We also introduce a quantum mechanics model which has many features in common with these field theories. The quantum mechanics model is then analyzed in considerable detail in Sec. III. The implications for the toy field theory and the gauge theory are discussed in Sec. IV. The connections to the cosmological solutions are discussed in Sec. V. Other Kasner and BKL-like solutions are discussed in Sec. VI.

Several important details are in the appendices. Appendix A discusses the coupling of the dilaton to the Yang-Mills theory. Appendix B discusses the time dependent harmonic oscillator. Subleading corrections to the energy, in the vicinity of the vanishing dilaton, are discussed in Appendix C. Particle production in the quadratic approximation, for a modified nonvanishing dilaton profile, is discussed in Appendix D. The behavior of the Yang-Mills theory in the presence of a dilaton which varies with Milne time is studied in Appendix E. Finally, some discussion about the universal behavior near singularities and about BKL cosmologies in the presence of the dilaton, is contained in Appendix F.

II. THE GAUGE THEORY AND A TOY MODEL

We will consider the $N = 4$ gauge theory defined on a flat 3 + 1-dimensional space-time, which is regarded as the Poincaré patch boundary of an AdS cosmology—e.g., the ones discussed in detail in [4]. The bulk dilaton is equal to the coupling constant of the Yang-Mills theory,

$$e^{\Phi(t)} = g_{\rm YM}^2. \tag{2}$$

Thus the varying dilaton gives rise to a varying Yang-Mills coupling.

After suitable field redefinitions the Lagrangian for the $\mathcal{N} = 4$ theory takes the form

$$L = \mathrm{Tr}\left\{-\frac{1}{4e^{\Phi}}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu X^a)^2 - \frac{1}{4}e^{\Phi}([X^a, X^b])^2 - i\bar{\Psi}\Gamma^\mu D_\mu\Psi + e^{\Phi/2}\bar{\Psi}\Gamma^a[X^a, \Psi]\right\}. \tag{3}$$

There are six scalars, X^a, $a = 1, \cdots 6$. And 4 two-component Weyl fermions of $SO(1, 3)$, which have been grouped together as one Majorana Weyl fermion of $SO(1, 9)$. The gamma matrices Γ^μ, $\mu = 0, 1, \cdots 3$, and Γ^a, $a = 1, \cdots 6$, together form the 10 gamma matrices of $SO(1, 9)$. The scalars and fermions transform as the adjoint of $SU(N)$. The covariant derivative of the scalars is

$$D_\mu X^a = \partial_\mu - i[A_\mu, X^a], \tag{4}$$

and similarly for the fermionic fields. The field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]. \tag{5}$$

In Appendix A we discuss the Lagrangian above, especially the dilaton couplings, in more detail. Let us make one comment here. It is sometimes stated that the dilaton couples to the Lagrangian of the $\mathcal{N} = 4$ theory [12]. The more correct statement is that the dilaton couples to the on shell Lagrangian [13]. This differs from the operator obtained from Eq. (3) but only by a total derivative term involving the scalars.

At the singularity, e^{Φ} goes to zero. As a result, the prefactor in the gauge kinetic energy term in Eq. (3) blows up. This is the essential complication which must be dealt with in our analysis of the time dependent situation at hand. To study it further it is useful to introduce a toy model consisting of a single scalar field $\tilde{X}$ with Lagrangian

$$L = -\frac{1}{e^{\Phi}}\left[\frac{1}{2}(\partial\tilde{X})^2 + \tilde{X}^4\right]. \tag{6}$$

Note that the quartic term is "right side up," so that for a constant dilaton this model would have a stable minimum. We will see that this model captures all the essential features of the real problem of interest.

In fact it is useful to simplify the model further and to consider a quantum mechanics system, with action,

$$S = \int dt \frac{1}{e^{\Phi}}\left[\frac{1}{2}\dot{\tilde{X}}^2 - \frac{1}{2}\omega_0^2\tilde{X}^2 - \tilde{X}^4\right]. \tag{7}$$

Starting from the field theory Eq. (6) such a system would arise if we keep only one Fourier mode of momentum $\vec{k}$ with $\omega_0^2 = k^2$, and only the quartic self-interactions of this mode.

It will be useful in the discussion below to first consider the quantum mechanics system and then return to the field theories Eqs. (6) and (3).

III. ANALYSIS OF QUANTUM MECHANICS MODEL

We now turn to analyzing the quantum mechanics model Eq. (7) further.

It is useful to begin by carrying out a field redefinition which gives rise to a variable with a canonical kinetic energy term and analyzing the system in terms of this new variable. We define the variable X to be

$$X = e^{-\Phi/2}\tilde{X}. \tag{8}$$

Up to a surface term the Lagrangian is

$$L = \frac{1}{2}\dot{X}^2 - \frac{1}{2}\omega^2(t)X^2 - e^{\Phi}X^4. \qquad (9)$$

Here $\omega^2(t)$ is a time dependent angular frequency that arises due to the time dependent dilaton and is given by

$$\omega^2(t) = -\left[\frac{1}{4}(\dot{\Phi})^2 - \frac{1}{2}\ddot{\Phi}\right] + \omega_0^2, \qquad (10)$$

where the dot superscript indicates a time derivative of the dilaton. For a dilaton dependence

$$e^{\Phi} = (-t)^p, \qquad t < 0, \qquad (11)$$

we have

$$\omega^2(t) = -\frac{\alpha^2}{t^2} + \omega_0^2 \qquad (12)$$

with,

$$\alpha^2 = \frac{p}{4}(p+2). \qquad (13)$$

Note that for sufficiently small time, ω^2 becomes negative, and the variable X has a tachyonic mass term [negative $(\text{mass})^2$], which arises due to the time dependent dilaton. In fact ω^2 diverges as $t \to 0$, this will be important in the subsequent analysis.

Let us briefly outline the detailed analysis that follows. Since the X^4 term in Eq. (9) is multiplied by e^{Φ}, one might at first expect that this term is not important near $t = 0$. Accordingly, we first neglect it and analyze the resulting quadratic theory. It turns out that due to the diverging tachyonic mass the system quite generally gets driven to $|X| \to \infty$ as $t \to 0$. As a result, the quantum mechanical description in terms of the X variable is not complete. If we want to know what happens as $t \to 0$, and beyond, one needs additional information about the behavior at $X \to \infty$. The diverging value of X also means that the quartic term cannot be neglected.

At this stage it is worth remembering that the field redefinition, Eq. (8), is singular at $t = 0$. In fact, we find that the rate at which X diverges is exactly balanced by the rate at which the dilaton vanishes, leaving $\tilde{X}$ to be finite, as $t \to 0$. This motivates us to study the system in terms of the $\tilde{X}$ variable. Although, as mentioned above, the analysis in terms of the X variable has already revealed that the quartic term cannot be neglected, we ignore it at first, to gain some understanding of the system. Our analysis shows that the system is singular in a manner we have described in Sec. I. We then incorporate the quartic terms and find that all the essential conclusions about the singular nature of the response go through unchanged.

A. The X description

As was mentioned above, to begin we drop the quartic term in Eq. (9). this gives rise to the quadratic action

$$S = \int dt \frac{1}{2}[\dot{X}^2 - \omega(t)^2 X^2]. \qquad (14)$$

The quadratic theory can then be analyzed in standard fashion by expanding the field X in terms of normal modes.

From Eq. (14) it follows that the corresponding operator $\hat{X}$ in the Heisenberg picture, satisfies the equation

$$\ddot{\hat{X}} + \omega^2(t)\hat{X} = 0. \qquad (15)$$

Let us define $f(t)$ to be

$$f(t) = \sqrt{\frac{\pi\omega_0}{2}}\sqrt{-t}H_\nu^1(-\omega_0 t), \qquad (16)$$

where

$$\nu = \frac{p+1}{2}, \qquad (17)$$

and $H_\nu^{(1)}(x)$ is the Hankel function which asymptotically, as $x \to \infty$, behaves like

$$H_\nu^{(1)}(x) \to \sqrt{\frac{2}{\pi x}}e^{[i(x-(\nu+(1/2))(\pi/2)]}. \qquad (18)$$

Then it follows from the standard properties of Hankel functions that $f(t)$ satisfies the equation

$$\ddot{f(t)} + \omega^2(t)f(t) = 0, \qquad (19)$$

with boundary condition

$$f(t) \to e^{-i\omega_0 t}, \qquad t \to -\infty. \qquad (20)$$

The solution to Eq. (15) now is

$$\hat{X} = \frac{1}{\sqrt{2\omega_0}}[\hat{a}f(t) + (\hat{a})^\dagger f^*(t)]. \qquad (21)$$

The momentum conjugate to $\hat{X}$ is

$$\hat{P} = \dot{\hat{X}} = \frac{1}{\sqrt{2\omega_0}}[\hat{a}\dot{f}(t) + (\hat{a})^\dagger \dot{f}^*(t)]. \qquad (22)$$

The operators $\hat{X}$ and $\hat{P}$ satisfy the canonical commutation relation iff, $\hat{a}$, $\hat{a}^\dagger$, satisfy the standard relation

$$[\hat{a}, \hat{a}^\dagger] = 1. \qquad (23)$$

Classical solutions to Eq. (15) take the form

$$X(t) = \sqrt{(-t)}[AJ_\nu(-t) + BN_\nu(-t)], \qquad (24)$$

where J_ν, N_ν, stand for the Bessel and Neumann functions. The constants A, B are determined by the initial conditions, at $t \to -\infty$. For generic initial conditions, $B \neq 0$, and as $t \to 0^-$,

$$X(t) \sim (-t)^{(1/2)-\nu} = (-t)^{-(p/2)} \to \infty. \qquad (25)$$

Here we have used Eq. (17), and the fact that $p > 0$. Thus we see that due to the negative and diverging value of ω^2 near the singularity, a generic trajectory gets driven out to infinite values of the position coordinate.

Classical states correspond to coherent states in the quantum theory. We see that the center of the wave packet for a generic coherent state runs away to infinity due to the tachyonic mass term. This shows that further data is needed to make the quantum theory in terms of the variable X, and Lagrangian, Eq. (14), well defined. This additional data should specify what happens to the wave packet once it gets to large values of the X coordinate.

It is also illuminating to calculate the wave function of the ground state in terms of the X description. As $t \to -\infty$, the system becomes a conventional harmonic oscillator with constant angular frequency ω_0. Consider the state specified by the condition,

$$\hat{a}|0\rangle = 0, \tag{26}$$

which we will refer to as the ground state. We are interested in asking how expectation values in this state evolve with time. We have been working in the Heisenberg picture above. It is useful to answer this question by constructing the wave function in the Schrodinger picture. The resulting time dependence of the wave function carries information about the time dependent expectation values for all operators in this state.

As discussed in Appendix B, the Schrodinger picture wave function for the ground state is given by

$$\psi(x, t) = \frac{A}{\sqrt{f^*(t)}} e^{i[(\dot{f}/f)^*(x^2/2)]}, \tag{27}$$

where A is a time independent constant which is fixed by requiring that the state has unit norm. Two features of the resulting behavior of this wave function near $t = 0$ are worth commenting upon.

First, as discussed in Appendix B, it follows that the probability density $|\psi(x, t)|^2$ is given by

$$|\psi(x, t)|^2 = \frac{|A|^2}{|f|} e^{-[(\omega_0 x^2)/|f|^2]}. \tag{28}$$

It follows from Eqs. (16) and (17), and the properties of Hankel functions that

$$|f|^2 \sim (-t)^{-p} \to \infty, \qquad t \to 0^-. \tag{29}$$

Thus the probability distribution in X become infinity spread out, as $t \to 0$. We saw above that for generic coherent states the center of the wave packet runs off to infinite X. The vacuum is a nongeneric coherent state, for which this does not happen. The expectation value of $\langle \hat{X} \rangle$ vanishes in this state, this corresponds to A, B in Eq. (24) both vanishing. However we see now that even for this state the spreading of the wave function, which is a quantum effect, makes the wave function sensitive to large X.

Second, the exponential factor in Eq. (27) gives rise to a phase factor,

$$e^{i[(\dot{f}/f)^*(x^2/2)]} \sim e^{[i(px^2/4t)]}. \tag{30}$$

This phase factor oscillates "wildly" near $t = 0$. As a result $\psi(x, t)$ does not have a well-defined limit as $t \to 0^-$. This feature will be crucial in the subsequent analysis that follows.

We have neglected the quartic interaction term of Eq. (9) in the analysis above. We now see from Eq. (25) that X diverges, as $t^{-p/2}$, and thus $e^{\Phi} X^4$ goes like, $t^p t^{-2p} \to \infty$, as $t \to 0$, and also diverges. This means that despite the vanishing dilaton the quartic term cannot be regarded as a small perturbation.

It is worth recalling now that the theory we started with was formulated in terms of the $\tilde{X}$ variable, Eq. (6). The change of variables from X to $\tilde{X}$ is in fact singular at $t = 0$ where the dilaton vanishes. Moreover, as we will see below, $\tilde{X}$ is in fact finite, as $t \to 0$, since the rate at which X diverges in Eq. (8) is exactly the rate at which $e^{-\Phi/2}$ also blows up. It is therefore worth constructing a description directly in terms of the $\tilde{X}$ variable. The resulting behavior at finite $\tilde{X}$ will also provide information about what happens at infinite X, which is the additional data we seek. We turn to this next.

B. The $\tilde{X}$ description

As was mentioned above, to gain some understanding in the $\tilde{X}$ variable we first begin by neglecting the quartic term. The action in terms of this variable then takes the form

$$S = \int dt e^{-\Phi} \frac{1}{2} [\dot{\tilde{X}}^2 - \omega_0^2 \tilde{X}^2]. \tag{31}$$

Classical solutions take the form

$$\tilde{X}(t) = e^{\Phi/2} X(t) = e^{\Phi/2} \sqrt{(-t)} [A J_\nu(-t) + B N_\nu(-t)]. \tag{32}$$

It is easy to check from Eqs. (17) and (11) that as $t \to 0$,

$$\tilde{X}(t) \sim B e^{\Phi/2} N_\nu(-t) \to \text{constant}. \tag{33}$$

Thus classical trajectories do not reach $|\tilde{X}| \to \infty$, but instead are at finite values as $t \to 0$.

From the relation, Eqs. (8) and (21), it follows that the operator $\hat{\tilde{X}}$ in the quantum theory is

$$\hat{\tilde{X}} = \frac{e^{\Phi/2}}{\sqrt{2\omega_0}} [\hat{a} f(t) + \hat{a}^\dagger f^*(t)]. \tag{34}$$

The ground state satisfies the condition given in[7] Eq. (27). The resulting wave function in the Schrodinger picture (see Appendix B) is

[7]Note that for the dilaton dependence given in Eq. (89), the Lagrangian in terms of the $\tilde{X}$ variable does not reduce to that of a standard harmonic oscillator as $t \to -\infty$. This is in contrast to the solution, Eq. (84), where we do get a standard harmonic oscillator, as $t \to -\infty$, as discussed in Appendix E. It is this latter case that is better defined in any case, as was discussed in Sec. III. The ground state in this latter case, which becomes the vacuum of the standard harmonic oscillator in the far past, behaves similarly to the vacuum state consider here near the singularity.

$$\psi(\tilde{x},t) = \frac{A}{\sqrt{f^*(t)e^{\Phi/2}}} e^{i\{[(\dot{f}/f)^* + (\dot{\Phi}/2)](e^{-\Phi}\tilde{x}^2)/2\}}. \quad (35)$$

The probability to find the system between $\tilde{x}$, $\tilde{x} + d\tilde{x}$ is $|\psi(\tilde{x},t)|^2$, and is given by

$$|\psi(\tilde{x},t)|^2 = \frac{|A|^2}{\sqrt{|f|^2 e^{\Phi}}} e^{-[(\omega_0\tilde{x}^2)/(|f|^2 e^{\Phi})]}. \quad (36)$$

From Eqs. (29) and (11), it follows that $|f|^2 e^{\Phi}$ goes to a constant as $t \to 0$. Thus $|\psi(\tilde{x},t)|^2$ becomes a well-defined smooth Gaussian function in the limit $t \to 0$. The absolute value of the wave function, $|\psi(\tilde{x},t)|$ thus has a smooth limit as $t \to 0^-$. Contrast this with the phase of the wave function. As discussed in Appendix B,

$$e^{-\Phi}\left[\left(\frac{\dot{f}}{f}\right)^* + \frac{\dot{\Phi}}{2}\right] \to 1/(-t)^{p-1}, \quad (37)$$

as $t \to 0^-$, and thus the phase of the wave function goes like

$$e^{i[(\dot{f}/f)^* + (\dot{\Phi}/2)](e^{-\Phi}\tilde{x}^2)/2} \to e^{i[(C\tilde{x}^2)/(-t)^{p-1}]}, \quad (38)$$

where C is a constant. Note that for $p \geq 1$ the phase factor diverges.[8] The result is that the wave function, Eq. (35), does not have a well-defined limit as $t \to 0^-$. In terms of expectation values, this divergence results in the expectation value for $\hat{\tilde{P}}^2$ blowing up. One finds that

$$\langle \hat{\tilde{P}}^2 \rangle \sim (-t)^{2(1-p)} \to \infty. \quad (39)$$

We have considered the ground state wave function above. In the subsection that follows, we will give a general argument for why the same diverging phase factor arises for the wave function of any state. This general argument will also include quartic terms.

C. General analysis of wave function near $t = 0$

The behavior of wave function can be analyzed quite generally in the vicinity of $t = 0$. It is easy enough to carry out the analysis in the full quantum mechanics system, Eq. (7), including the quartic interaction terms.

The Schrodinger equation takes the form

$$-\frac{e^{\Phi}}{2}\partial_{\tilde{x}}^2\psi + e^{-\Phi}V(\tilde{x})\psi = i\partial_t\psi, \quad (40)$$

where the potential $V(\tilde{x})$ is

$$V(\tilde{x}) = \frac{1}{2}\omega_0^2\tilde{x}^2 + \tilde{x}^4. \quad (41)$$

Since e^{Φ} vanishes near the singularity let us begin by assuming that the potential energy term on the left-hand side of Eq. (40) dominates; we will verify below that this assumption is self-consistently true. This gives

$$e^{-\Phi}V(\tilde{x})\psi \simeq i\partial_t\psi, \quad (42)$$

which can be easily solved to give

$$\psi(x,t) = e^{-iG(t)V(\tilde{x})}\psi_0(x), \quad (43)$$

where

$$G(t) = \int dte^{-\Phi} = -\frac{(-t)^{1-p}}{(1-p)}, \quad (44)$$

and $\psi_0(x)$ is a time independent integration constant. Here we have used Eq. (11) for the dilaton. For $p > 1$, we see that $G(t)$ diverges at the singularity, leading to a diverging phase factor in the wave function. This divergence is a general feature, independent of the initial state $\psi_0(x)$. In the quadratic case where

$$V(\tilde{x}) = \frac{1}{2}\omega_0^2\tilde{x}^2, \quad (45)$$

we see that this phase factor agrees with what was obtained in the exact solution for the ground state, Eq. (38).

To check the self-consistency of our assumptions let us evaluate the contribution due to the kinetic energy term in Eq. (40) on the solution, Eq. (43). It is useful to analyze the two cases $p > 1$ and $p < 1$ separately;[9] in both cases, we see below that the kinetic energy term is subdominant compared to the potential energy term.

For $p > 1$ the leading contribution to the kinetic energy comes when the spatial derivatives act on the phase factor, and not on $\psi_0(x)$. This gives

$$-\frac{e^{\Phi}}{2}\partial_{\tilde{x}}^2\psi \simeq \frac{e^{\Phi}}{2}[G(t)^2V'(\tilde{x})^2 + iG(t)V'']\psi. \quad (46)$$

Since $G(t)$ diverges the dominant contribution comes from the first term on the right-hand side leading to

$$-\frac{e^{\Phi}}{2}\partial_{\tilde{x}}^2\psi \sim \frac{e^{\Phi}}{2}[G(t)^2V'(\tilde{x})^2]\psi \sim t^2e^{-\Phi}V'(\tilde{x})^2\psi, \quad (47)$$

where we have used the behavior for $G(t)$ in Eq. (44). Comparison with the potential energy term in Eq. (40) shows that this contribution is suppressed by an extra power of t^2.

For $p < 1$ the leading contribution comes when the spatial derivatives act on $\psi_0(x)$. This gives

$$-\frac{e^{\Phi}}{2}\partial_{\tilde{x}}^2\psi \simeq -\frac{e^{\Phi}}{2}e^{-iG(t)V(\tilde{x})}\psi_0(x)''. \quad (48)$$

Comparing with the potential energy term in Eq. (40) we see that this term is suppressed by an extra power of t^{2p}.

From the point of view of the bulk dual cosmology, we are especially interested in the question of whether the

[8] For $p = 1$, the divergence goes like $\log(-t)$. This follows from standard properties of Bessel functions, and also from the general discussion in the next subsection.

[9] A similar analysis can also be carried out when $p = 1$.

state can be continued past $t = 0$, with the dilaton varying, for example, like

$$e^{\Phi} = |t|^p. \quad (49)$$

We have found above that the wave function for a general state in the quantum mechanics model does not have a well-defined limit as $t \to 0^-$, when $p > 1$. This means it is not meaningful to ask about its continuation for $t > 0$ in this case. To obtain this continuation one would need to impose that the wave function at $t = 0$ is continuous, i.e., meets the condition

$$\psi(\tilde{x}, t = 0^-) = \psi(\tilde{x}, t = 0^+). \quad (50)$$

This condition cannot be imposed if $\lim_{t\to 0^-} \psi(\tilde{x}, t)$ does not exist.

D. The energy blows up at $t \to 0$

We continue with our general analysis of the wave function in the quantum mechanics system in this subsection and find that for a generic state, and all values of $p > 0$, the energy at $t \to 0^-$ diverges. We will work below with a general potential $V(x)$.

The Hamiltonian operator H is given by the left-hand side of the Schrodinger equation, Eqs. (40) and (59). We have argued above that the kinetic energy contribution is subdominant to the potential energy, near $t = 0$, so that

$$\langle H\rangle \simeq e^{-\Phi}\langle V\rangle. \quad (51)$$

The expectation value of the potential,

$$\langle V\rangle = \int d\tilde{x} V(\tilde{x})\psi^*(\tilde{x}, t)\psi(\tilde{x}, t). \quad (52)$$

Substituting for the wave function from Eq. (43) we see that the phase factor drops out so that $\langle V\rangle$ near $t = 0$ is given by

$$\langle V\rangle = \int d\tilde{x} V(\tilde{x})|\psi_0(\tilde{x})|^2, \quad (53)$$

and is time independent. This means the leading time dependence in $\langle V\rangle$ comes from the prefactor $e^{-\Phi}$ in front in Eq. (51), leading to the conclusion that the energy diverges as

$$\langle H\rangle \to (-t)^{-p} \quad (54)$$

when $t \to 0$. Note that this conclusion holds for all $p > 0$.

This conclusion can be avoided if the state is such that $\langle V\rangle$ vanishes. This issue is examined further in some detail in Appendix C. The conclusion, after analyzing subleading corrections which could also have been potentially divergent contributions, is the following: Unless $p > 2$, in which case a divergent contribution to the energy arises from the kinetic energy term, the subdominant contributions to the energy do not diverge as $t \to 0$. Thus, requiring that $\langle V\rangle$ vanishes is enough to ensure that the energy stays finite.

Now even if $\langle V\rangle$ vanishes we should note that the expectation value of $\langle H^2\rangle$ will diverge. From the discussion above it follows that

$$\langle H^2\rangle \simeq e^{-2\Phi}\langle V^2\rangle. \quad (55)$$

In general in a state where $\langle V\rangle$ vanishes, $\langle V^2\rangle$ will not vanish. This means that even in those special states where the expectation value of the energy stays finite as $t \to 0$, the fluctuations about this finite mean value will diverge[10]

IV. ANALYSIS IN FIELD THEORY

In the previous section we have analyzed the quantum mechanical model, Eq. (7), extensively. Here we return to field theory, first discussing the toy model field theory, Eq. (6), and then turning to the deformed $\mathcal{N} = 4$ field theory, Eq. (3).

The lessons from the study of the quantum mechanics model can be directly applied to the field theory, Eq. (6). Carrying out the field redefinition in the full field theory gives rise to a Lagrangian, up to a surface term

$$L = -\frac{1}{2}(\partial X)^2 - m^2(t)X^2 - e^{\Phi}X^4 \quad (56)$$

with m^2 being a tachyonic time dependent mass

$$m^2 = -\frac{\alpha^2}{t^2} \quad (57)$$

with

$$\alpha^2 = \frac{p}{4}(p + 2), \quad (58)$$

which diverges as $t \to 0$.

The first lesson which carries over from quantum mechanics to field theory is that contrary to what one might have guessed at first, the quartic term cannot be neglected near $t = 0$. The second lesson is that the variable X is not so convenient to work with and the analysis is more conveniently carried out in terms of the original variables $\tilde{X}$. The third lesson is that the analysis is conveniently carried out in terms of the Schrodinger picture. This last lesson is not easy to apply in field theory, since typically the Schrodinger picture has not been used in this context. Nevertheless with the experience of the quantum mechanics model in mind we will in this subsection analyze the field theory in the Schrodinger picture; various caveats will be discussed in the next subsection.

In the Schrodinger picture in field theory the state of the system is described by a time dependent wave functional,

[10]In the next section we will apply the discussion of this and the previous subsection to the $\mathcal{N} = 4$ gauge theory. In that case, both $\langle H\rangle$, $\langle H^2\rangle$ should scale like N^2—the number of colors. This means that the fluctuations in energy will be suppressed in the large N limit. This suggests that to leading order in $1/N$, and for $p < 2$, the vanishing of $\langle V\rangle$ is sufficient to ensure that the expectation value of energy stays finite when $t \to 0$.

$\psi[\tilde{X}(x), t]$, which satisfies the Schrodinger equation

$$-\frac{1}{2}\int d^3x e^{\Phi}\frac{\delta^2\psi}{\delta\tilde{X}^2} + e^{-\Phi}V[\tilde{X}]\psi = i\partial_t\psi. \tag{59}$$

Here the potential energy, $V[\tilde{X}]$, is a functional given by

$$V[\tilde{X}] = \int d^3x\left\{\frac{1}{2}(\partial_i\tilde{X})^2 + \tilde{X}^4\right\}. \tag{60}$$

The first term on the left-hand side in Eq. (59) is the kinetic energy, and the second term is the potential energy. We see that, like in quantum mechanics, the kinetic energy term has a prefactor e^{Φ}, while the potential energy has the prefactor $e^{-\Phi}$. This suggests that the kinetic energy term is once again subdominant close to the singularity, leading to the solution

$$\psi[\tilde{X}(x^i), t] = e^{-i\{G(t)V[\tilde{X}]\}}\psi_0[\tilde{X}(x^i)]. \tag{61}$$

We see that the wave functional has a phase factor which diverges, as in the quantum mechanics case, resulting in a singular limit for the wave function if $p \geq 1$. Moreover with the kinetic energy being subdominant, near $t = 0$, the Hamiltonian is well approximated by

$$\langle H\rangle \simeq e^{-\Phi}\langle V\rangle. \tag{62}$$

As in the quantum mechanics model, the phase factor drops out in the expectation value of $\langle V\rangle$ near $t = 0$ leading to the conclusion that, in field theory as well, $\langle H\rangle$ goes like Eq. (54), and therefore blows up for all $p > 0$.

The central assumption here is that the kinetic energy is subdominant compared to the potential energy near $t = 0$. This was shown to be self-consistently true in the case of quantum mechanics. We will analyze this issue for the field theory in the next subsection.

Before proceeding let us also mention that we analyze the quadratic theory in further detail in Appendix D. We consider a dilaton profile of the form $e^{\Phi} = |t|^p$, and evolve the field theory in this background, starting from the vacuum into the far future. It is useful for this purpose to regulate the dilaton profile near $t = 0$ in a manner we make more precise in the appendix. Our conclusion is that particle production always occurs. For $p < 1$ the particle production is finite, while for $p > 1$ it becomes infinite as the regulator is taken to zero.

Turning now to the gauge theory, Eq. (3), we see that the coupling of the dilaton to the quartic scalar potential and the fermionic Yukawa terms are proportional to positive powers of e^{Φ} and can be neglected when the dilaton is very small. The fermions and scalars have canonical kinetic terms. The gauge field in contrast has a noncanonical kinetic energy term, it is the analogue of the $\tilde{X}$ field in the toy model, Eq. (6). As $t \to 0$, and the dilaton vanishes, it is this gauge kinetic energy term which will determine the behavior of the system. Accordingly, in the analysis below we focus on the pure gauge theory, without fermions and scalars, and with action

$$S = \int d^4x\left(-\frac{1}{4e^{\Phi}}\right)\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}. \tag{63}$$

The equation of motion is

$$D_\mu(e^{-\Phi}F^{\mu\nu}) = 0. \tag{64}$$

We work in Coulomb gauge, where

$$A_0 = 0. \tag{65}$$

Consider now the noninteracting theory. The equation of motion, Eq. (64), for $\nu = 0$, becomes (this is the Gauss Law constraint)

$$\partial_0(\partial^j A^j) = 0. \tag{66}$$

Thus the longitudinal part of the gauge field is time independent. We can now do an additional time independent gauge transformation to set

$$\partial_j A^j = 0. \tag{67}$$

The equation of motion, Eq. (64), with $\nu = i$, then become

$$\partial_0(e^{-\Phi}\partial^0 A^i) + e^{-\Phi}\partial_j\partial^j A^i = 0, \tag{68}$$

for the two transverse components satisfying Eq. (67). Equation (68) is exactly the equation we have for a scalar field with Lagrangian

$$L = e^{-\Phi}(\partial\tilde{X})^2. \tag{69}$$

Thus at the quadratic level, the analysis for the gauge field reduces to that of the scalar field considered above. There are two transverse components coming from each gauge field. We have neglected the color degrees of freedom above. They are easily incorporated and give $(N^2 - 1)$ degrees of freedom for each of the two transverse components.

Interactions give rise to cubic and quartic terms. In Coulomb gauge, these terms do not depend on any time derivatives. Thus, in the Hamiltonian they only contribute to the potential energy and not to the kinetic energy. The interactions can therefore be included in a way very similar to the quartic terms in the toy model for $\tilde{X}$.

In particular, we are interested in the wave function in the Schrodinger picture. In this picture the operators are time independent. The total potential energy is given by energy due to the magnetic field

$$V[A_i(x)] = \int d^3x\frac{1}{4}\mathrm{Tr}(F_{ij}F^{ij}). \tag{70}$$

Motivated by the quantum mechanics model on the previous section, and as in the scalar field theory above, we now take the potential energy to dominate in the Schrodinger equation near the singularity. As a result the wave function has a phase given by

$$\psi[A_i(x), t] = e^{-i\{G(t)V[A_i]\}}\psi_0[A_i(x)]. \tag{71}$$

The phase factor is identical to that found in Eq. (43) and diverges, as $t \to 0$, if $p > 1$, resulting in the wave function being singular at $t \to 0$. The wave function above can be regarded as being dependent on only the transverse components of the vector potential. Alternatively, we can take the wave function to be dependent on a general gauge potential (with $A_0 = 0$), and then impose Gauss's law,

$$\partial_i \frac{\delta \psi}{\delta A_i} = 0, \tag{72}$$

on it.

Similarly one can calculate the expectation value of the energy near $t = 0$. It has the same form as in Eq. (54), with the expectation value of the potential energy now being given by

$$\langle V \rangle = \int DA_i |\psi_0[A_i(x)]|^2 V[A_i(x)], \tag{73}$$

where $V[A_i(x)]$ is given in Eq. (70). We see that the energy diverges in the gauge theory as well, as $(-t)^{(-p)}$, for all p, unless the state is such that $\langle V \rangle$ vanishes.

We conclude this section with two important comments. First, naively one would have thought that as the dilaton becomes small perturbation theory should become a good approximation. However we have seen in our analysis of the toy model in the last section that this is in fact not true. In the X description for the toy model, the time dependent dilaton drives the system to large values of X resulting in the quartic term being non-negligible. A similar argument also holds in the gauge theory. The cubic and quartic interactions terms are not small near $t = 0$, and as a result perturbation theory is not a good approximation. Second, it should be emphasized that our analysis is valid for finite N and finite $g_{\rm YM}^2 N$. In fact from Eq. (71) we see that the behavior of the wave functional near $t = 0$ is essentially determined by $G(t)$ and is independent of the details of the potential $V[A_i(x)]$. In the dual closed string theory, this means that this conclusion is valid in the presence of string and quantum corrections.

A more critical look at the field theory analysis

The central assumption in the field theory discussion above was that in the Schrodinger equation the potential energy which scales like $e^{-\Phi}$ dominates over the kinetic energy, which has a prefactor e^{Φ} in front of it. This assumption was motivated by our earlier analysis in quantum mechanics. However, field theory differs from quantum mechanics in having an infinite number of degrees of freedom, and one might worry that this introduces additional subtleties and complications.[11] We turn to an examination of these issues below.

The wave function for the ground state of the harmonic oscillator with action Eq. (14) in the $\tilde{x}$ description is given by Eq. (35). This is an exact result. If the potential energy is dominant the wave function has the form Eq. (43). From the exact result we can ask how close to $t = 0$ must one come for this approximation to become a good one. The phase factor in the wave function Eq. (35) contains the factor $(\frac{\dot{f}}{f})^* + \frac{\dot{\Phi}}{2}$. As discussed in Eq. (B23) of Appendix B, this phase factor has a power series expansion in $t\omega_0$, near $t = 0$, of form

$$\left(\frac{\dot{f}}{f}\right)^* + \frac{\dot{\Phi}}{2} = \frac{1}{t}\left(\frac{1 - 2\nu + p}{2}\right) + 2c_2\omega_0^2 t + \cdots. \tag{74}$$

The ellipses stand for terms which are suppressed when

$$t\omega_0 \ll 1. \tag{75}$$

Now the first term in Eq. (74) vanishes due to Eq. (17). The second term is therefore the leading one and it is easy to see that this gives agreement with Eq. (43).

The conclusion is that the potential energy dominates over the kinetic energy term when t is small enough and satisfies the condition in Eq. (75). We can also understand this on the basis of the general arguments given in Sec. III for the self-consistency of this approximation. From Eq. (47) we see that the kinetic energy term goes like

$$-\frac{e^{\Phi}}{2}\partial_{\tilde{x}}^2 \psi \sim t^2 e^{-\Phi} V'(\tilde{x})^2 \psi \sim t^2 e^{-\Phi}(\omega_0^2 \tilde{x})^2 \psi, \tag{76}$$

while the potential energy term is

$$e^{-\Phi} V(\tilde{x})\psi \sim e^{-\Phi}(\omega_0 \tilde{x})^2 \psi. \tag{77}$$

Thus for the latter to dominate, Eq. (75) must be true.

Consider now a field theory in the quadratic approximation. Since there are an infinite number of modes, for any nonzero and arbitrarily small time t, there will always be some modes with high enough frequency for which the condition Eq. (75) is not met and thus for which the wave functional will not be well approximated by Eq. (61) or Eq. (71). Including interactions will couple these modes to low-momentum frequency potentially making the wave functional for all modes to be different from Eqs. (62) and (71).

It is of course well known that in dealing with the infinite numbers of degrees of freedom in a field theory it is first useful to introduce a cutoff or regulator, which makes the number of degrees of freedom finite, and then ask what happens as the cutoff is removed. For ease of discussion consider a momentum space cutoff Λ (in the gauge theory one needs a more sophisticated regulator to preserve gauge invariance, but this will not change the essential points in our discussion). The above analysis suggests that if we take the time t to vanish while keeping Λ fixed and finite, then the potential energy should dominate for times t meeting the condition

$$t\Lambda \ll 1 \tag{78}$$

[11] We are grateful to David Gross, in particular, for emphasizing this point to us.

and our conclusions in the previous section will be correct near $t = 0$. More generally one might expect that these conclusions are valid as long as we take $t \to 0$, before we take $\Lambda \to \infty$.

Unfortunately, it is not easy to make this argument precise. Additional complications can arise in field theory due to the effects of renormalization.[12] Usually renormalization in field theory is discussed in terms of an effective Lagrangian which changes under renormalization group flow. For our purposes in the discussion above the Schrodinger picture has been more useful. Including the effects of renormalization in the Schrodinger picture though is a complicated issue that we have not fully sorted out. Presumably the Hamiltonian which governs the time evolution of the wave functional needs to be well defined by appropriate operator ordering and this introduces the effects of renormalization.

One would expect that at least some of the consequences of renormalization can be incorporated by first constructing an effective Lagrangian by integrating out high-frequency modes and then using this effective Lagrangian to construct a Hamiltonian that governs the time evolution of the surviving low-frequency modes. Since the $\mathcal{N} = 4$ theory is conformally invariant any renormalization of the effective Lagrangian must be due to the time dependence of the dilaton and thus operators which are induced by this renormalization must have coefficients proportional to derivatives of the dilaton. In turn such operators could then also change the Hamiltonian resulting in extra operators in it with coefficients proportional to time derivatives of the dilaton.

The essential reason why in our discussion the potential energy dominates is that it scales like

$$V \sim e^{-\Phi},$$

whereas the kinetic energy term has a prefactor e^{Φ} in front of it that suppresses it. However suppose as an example of the consequences of renormalization the potential energy acquires an extra term which arises at one loop so that it now has the form

$$V = e^{-\Phi}\left[\mathcal{O}_1 + e^{\Phi}\frac{\dot{\Phi}^2}{\Lambda^2}\mathcal{O}_2\right].$$

Here $\mathcal{O}_1$ is the operator corresponding to the magnetic field energy, Λ is the cutoff scale, and $\mathcal{O}_2$ is the additional operator which arises at one loop. If the condition

$$p - 2 < 0 \tag{79}$$

is met, this second term could get important close to $t = 0$. This will mean that one has to include additional loop effects that arise beyond one loop as well. Resumming these effects could well lead to a much smaller potential energy. For example, if these corrections take the form of a geometric series, we would get schematically

$$V \sim e^{-\Phi}\left[1 + e^{\Phi}\frac{(\dot{\Phi})^2}{\Lambda^2} + (\frac{e^{\Phi}(\dot{\Phi})^2}{\Lambda^2})^2) + \cdots\right];$$

we would get after resumming

$$V \sim e^{-\Phi}\left[\frac{\Lambda^2}{\Lambda^2 - e^{\Phi}(\dot{\Phi})^2}\right] \sim \frac{\Lambda^2 e^{-2\Phi}}{(\dot{\Phi})^2}, \tag{80}$$

sufficiently close to $t = 0$, if the condition Eq. (79) is met. We see that the effects of renormalization can therefore suppress the potential energy term. In particular, if

$$e^{-2\Phi}\frac{\Lambda^2}{\dot{\Phi}^2} \sim e^{\Phi} \tag{81}$$

this suppression would make the kinetic energy term comparable. It is easy to see that Eq. (81) will be met for small enough time,t, if

$$p < 2/3. \tag{82}$$

The summary is that our analysis in the field theory is not complete and our conclusions about the gauge theory being singular should be taken as being suggestive but not conclusive. To analyze the gauge theory in a well-defined manner one must introduce a regulator. Once this is done a very rich set of counterterms are allowed in the process of renormalization, and such counterterms can potentially invalidate our conclusions. This would happen if they make the potential energy comparable to the kinetic energy or even smaller than it near $t = 0$ thereby significantly changing the form of the wave function and potentially making the gauge theory nonsingular. Whether this happens or not requires a detailed understanding of renormalization in the gauge theory in these time dependent backgrounds. This is a fairly complicated subject and we leave it for the future.

We end with some comments. Introducing a UV regulator in the gauge theory is dual to introducing a boundary in the bulk that regulates the IR behavior. Some consequences of renormalization, in the null-dependent case, have been worked out in [5], where indeed terms in the Wilsonian action with coefficients proportional to $\frac{(\Phi')^2}{\Lambda^2}$ were found.

V. COSMOLOGICAL SOLUTIONS

The motivation for this investigation came from trying to understand some cosmological solutions [2–4]. These solutions can be thought of as deformations of $AdS_5 \times S^5$ and have a dual description in terms of the $\mathcal{N} = 4$ theory with a time dependent dilaton. As the dilaton becomes small on the boundary the bulk curvature becomes larger and larger, eventually becoming singular at $t = 0$. In this section we ask what the above analysis in the gauge theory teaches us about these cosmological solutions.

[12] We thank the referee for emphasizing this point to us.

We begin with a brief review of these solutions and then return to the gauge theory later.

A. The gravity solutions

The solutions arise in IIB theory and are deformations of $AdS_5 \times S^5$. The S^5 factor is unchanged in the deformations, and accordingly we will omit it below and only discuss the solution in the remaining five dimensions.

The first solution we consider has the five-dimensional metric

$$ds^2 = \frac{1}{z^2}[dz^2 + |\sinh(2t)|\Big[-dt^2 + \frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)]\Big], \tag{83}$$

and dilaton

$$e^{\phi(t)} = g_s|\tanh t|^{\sqrt{3}}. \tag{84}$$

This solution was discussed in [4].

In the far past, as $t \to -\infty$, the dilaton goes to a constant, and the metric becomes AdS_5. One can see this by going to coordinates

$$r = \frac{R}{\sqrt{\eta^2 - r^2}}, \qquad e^{-t} = \sqrt{\eta^2 - R^2}, \tag{85}$$

in which the metric, Eq. (83), and dilaton, Eq. (84), take the form

$$ds^2 = \frac{1}{z^2}\Big[dz^2 + \Big|1 - \frac{1}{(\eta^2 - R^2)^2}\Big|[-d\eta^2 + dR^2 + R^2 d\Omega_2^2]\Big]. \tag{86}$$

The far past, $t \to -\infty$, corresponds to $(\eta^2 - R^2) \to \infty$; it is clear from Eq. (86) that the metric asymptotes to AdS_5 in this limit. The dilaton in these coordinates is

$$e^{\phi} = \Big|\frac{\eta^2 - R^2 - 1}{\eta^2 - R^2 + 1}\Big|^{\sqrt{3}}. \tag{87}$$

At $t = 0$ the solution Eq. (83) has a singularity. The curvature scalar diverges like $R \sim \frac{1}{t^3}$ as $t \to 0$.

The dilaton vanishes as $t \to 0$, Eq. (84). Thus the singularity occurs at weak string coupling. This singularity is the main focus of our analysis.

The region $t < 0$, which is the region of space-time before the singularity, maps to $\eta^2 - R^2 > 1$, in the (η, R) coordinates, while the singularity, which is at $t = 0$, maps to the locus $\eta^2 - R^2 = 1$.

Another 5D solution is given by

$$ds^2 = \frac{1}{z^2}[dz^2 + |2t|[-dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2], \tag{88}$$

with dilaton

$$e^{\Phi(t)} = g_s|t|^{\sqrt{3}}. \tag{89}$$

This solution does not asymptote to AdS_5 in the far past, as $t \to -\infty$. However its behavior at the singularity, as $t \to 0$, is very similar to the solution discussed above, Eqs. (83) and (84). The $z =$ const hypersurfaces in both metrics are of the FRW form. The difference between the two metrics is that this 4D FRW cosmology has constant negative curvature in the first case, Eq. (83) while it is flat in the second case, Eq. (88). This difference is increasingly unimportant near the singularity, where the dominant source of stress energy is provided by the diverging time derivative of the dilaton, rather than the spatial curvature. Since the dilaton is essentially identical near the singularity, at $t \to 0$, in both cases, the resulting space-times also are essentially the same.

In Sec. VI we will explore some additional cosmological solutions and comment on their gauge theory duals. Some of these solutions differ from the two solutions discussed above at early times but their behavior near $t = 0$ becomes the same as in the solutions above.

B. The gauge theory duals

For purposes of studying the field theory dual, we start with the first bulk solution considered above, Eqs. (83) and (84), or equivalently Eqs. (86) and (87). This solution asymptotes in the far past to AdS_5 with a constant dilaton. As discussed in [2,3], this corresponds to starting in the far past with the $N = 4$ super Yang-Mills theory in the vacuum state. The space-time, Eq. (83), has a boundary at $z \to 0$. We see that the metric on the boundary is conformal to flat space. As was discussed in [4], we can then take the metric of the space-time in which the dual gauge theory lives to be the 4D Minkowski-space metric.

Since this is an important point let us pause to briefly comment on it further. In general a Weyl transformation in the boundary theory corresponds to a Penrose-Brown-Henneaux (PBH) transformation in the bulk. The explicit PBH transformation which gives rise to a flat boundary metric for Eq. (83) was found in [4]. The metric in Eq. (83) has a second order pole as $z \to 0$. The metric on the boundary defined by

$$ds_4^2 \equiv \lim_{z\to 0} z^2 g_{\mu\nu}dx^\mu dx^\nu, \tag{90}$$

where x^μ denotes coordinates on the boundary, is

$$ds_4^2 = |\sinh(t)|\Big[-dt^2 + \frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\Big]. \tag{91}$$

After the PBH transformation the resulting four-dimensional metric is given by

$$ds_4^2 = e^{-2t}\left[-dt^2 + \frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{92}$$

This is in fact flat space in Milne coordinates. It is easy to see this. The coordinate transformation, Eq. (85), turns this metric into the familiar Minkowski metric

$$ds_4^2 = -d\eta^2 + dR^2 + R^2 d\Omega^2. \tag{93}$$

The region to the past of the singularity is given by $-\infty < t < 0$, in Eq. (92), and maps to the region $\eta^2 - R^2 > 1$, which is part of one of the Milne wedges. The rest of this Milne wedge is given by $0 < t < \infty$, to which the metric Eq. (92) automatically extends. The boundary of the Milne wedge lies at $t \to \infty$. Starting from $t \to -\infty$ one arrives at the singularity, at $t = 0$, before reaching the boundary of the Milne wedge.

The dual gauge theory knows about the time dependence of the bulk through the varying dilaton. The exponential of the dilaton is equal to the coupling constant of the Yang-Mills theory, Eq. (2). Thus the varying dilaton gives rise to a varying Yang-Mills coupling. The dilaton depends on the Milne time coordinate t, Eq. (92), and takes the form Eq. (84).

In summary we see that the dual gauge theory to this cosmological solution lives in flat space with a varying dilaton which vanishes at the singularity. The analysis in the preceding sections can now be used to determine the nature of this singularity. We turn to this in the next subsection.

Before that, it is also useful to discuss the dual to the second cosmological solution introduced in the previous subsection, with metric and dilaton given by Eqs. (88) and (89). In this case the dual gauge theory also lives in flat space, but the dilaton depends on Minkowski time instead of Milne time. To see this note that from Eq. (88) it follows that the boundary metric as defined by Eq. (90) is conformally flat. After a suitable PBH transformation the boundary metric becomes that of flat Minkowski space,

$$ds^2 = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \tag{94}$$

The dilaton dependence is given in Eq. (89); we see that it is only a function of Minkowski time.

The first solution, Eq. (84), has a dilaton which goes to a constant in the far past, and the dual gauge theory starts in the vacuum state of the $N = 4$ theory as $t \to -\infty$. In contrast in the second solution, Eq. (89), the dilaton blows up in the far past, this makes the dual map to the boundary theory less clear. It will turn out that the behavior at the singularity of the two theories is similar[13] near the singularity, and a little easier to analyze in the second case, where the variation is with respect to Minkowski time. For this reason, and for the limited purpose of asking questions near the singularity, we will focus some of the following discussion on the second hologram.

[13]This similarity is parallel with the fact, mentioned in the previous section, that the two corresponding bulk solutions also behave similarly near the singularity.

C. Gauge theory and gravity

In this section we relate what was learned in the analysis of the gauge theory above to the specific cosmological solutions of interest.

We see from Eqs. (84) and (89) that the solutions correspond to

$$p = \sqrt{3}. \tag{95}$$

In particular this means that for these solutions $p > 1$. We now see that, with the provisos discussed above, the analysis in the gauge theory suggests that the system becomes genuinely sick as $t \to 0^-$. In particular the wave function in the Schrodinger picture acquires a wildly oscillating phase, Eq. (43), and thus does not have a good limit, as $t \to 0$. This also means the state cannot be sensibly continued past $t = 0$. We remind the reader that these conclusions are not definitive, in particular, various caveats discussed in Sec. apply here, and there are additional issues having to do with renormalization that we have not adequately discussed in this work.

If true, the conclusions should hold regardless of the state of the system. In the far past, for the solution, Eqs. (83) and (84), the state is known to be the vacuum of the $\mathcal{N} = 4$ theory. However, as time progresses and the dilaton becomes smaller the state evolves. The analysis in the gauge theory leading to the wave function of form, Eq. (71), is only valid very close to $t = 0$, where the state would be different in general from the vacuum.

It also follows from the gauge theory, Eq. (54), that the energy diverges like $(-t)^{-\sqrt{3}}$, as $t \to 0$. If our approximations hold, the only way to avoid this conclusion would be if the system, which starts in the vacuum state in the far past, evolves to a nongeneric state near $t = 0$ for which $\langle V \rangle$, Eq. (52), vanishes. While this seems unlikely, since such states are nongeneric, one cannot rule out this possibility. Note however that even in this case the discussion of the previous two paragraphs would continue to hold and the system would become sick as $t \to 0^-$. Also, as was mentioned in Sec. III, even in such a nongeneric state the fluctuations in the energy, which are suppressed at large N would still diverge and the gauge theory at finite N would still be singular.

The cosmological solution in Eqs. (83) and (84) maps to a boundary theory where the dilaton depends on Milne time, rather than Minkowski time. This does not make an essential difference to the analysis in the gauge theory which in the above sections was carried out for the dilaton being a function of Minkowski time. This issue is analyzed further in Appendix E. The Milne case corresponds to having a nontrivial metric in the boundary theory. The dilaton vanishes at the point $t = 0$ which is perfectly smooth in Milne coordinates, thus the nontrivial metric does not make an essential difference to the discussion of the singularity.

More generally one can consider other cosmological solutions, which are different in the far past, but which also behave like the two examples discussed above, near $t = 0$ where the dilaton vanishes. In all these cases, as long as the metric on the boundary is well behaved at $t = 0$ the above discussion should apply. In Sec. VI, we give examples of BKL cosmologies, with the dilaton, which at late times asymptote to the example, Eqs. (88) and (89). It follows from the discussion above that, subject to the caveats mentioned above, the singularity in all these solutions then is a genuine sickness of the theory.

D. Concluding comments

We conclude this section with some more comments on the relation between the gravity solutions and their gauge theory description.

For the supergravity solutions, Eqs. (83), (84), (88), and (89), the stress energy tensor was calculated in Sec. 5 of [4]. For small t the stress tensor diverges like

$$T_{\mu\nu} \sim \frac{N^2}{t^4}. \tag{96}$$

This calculation was made in the supergravity approximation which breaks down when the dilaton becomes small enough. At very small values of the dilaton the gauge theory analysis carried out in Sec. IV becomes valid, subject to the various caveats discussed in Sec. . We have seen in Sec. IV that the energy density according to this analysis goes like[14]

$$\langle\rho\rangle \sim N^2/|t|^p. \tag{97}$$

(The factor of N^2 was not explicitly displayed in the discussion in Sec. IV. It is easy to see that in the quadratic approximation, where the gauge theory is free, the N^2 independent color degrees of freedom, in the presence of the time dependent dilaton, give rise to an energy density that scales in this way with N. In the presence of interactions, the energy density should continue to scale like N^2, to leading order in N.) Since $p = \sqrt{3} < 4$, we see that the growth of energy in the regime where the dilaton is very small is much slower than that in supergravity.

In the cosmological solutions we are studying here, the energy being pumped in by the dilaton does not lead to the formation of a black hole in the regime where supergravity is valid, for $t < 0$. This means that in the dual gauge theory the energy being pumped in is not thermalizing. To understand this we note that in the strongly coupled gauge theory, which is dual to supergravity, the thermalization time scale τ, is expected to be of order

$$\tau \sim \frac{1}{T} \sim \left(\frac{N^2}{\rho}\right)^{1/4}, \tag{98}$$

where ρ is the energy density, and T is the temperature. From stress energy tensor, Eq. (96), we see that[15]

$$\tau \sim t. \tag{99}$$

Thermalization would occur if the external source is pumping in energy on a time scale much slower than τ. In the situation at hand, this time scale is of order

$$t_{\rm pump} \sim \frac{\rho}{\dot{\rho}}. \tag{100}$$

Using Eq. (96) we see that $t_{\rm pump}$ is also of order t. Thus the rate at which the dilaton is pumping in energy equals the thermalization or relaxation rate of the system. This explains why thermalization does not happen and in the dual a black hole does not form. Another way to see this is that from Eqs. (98) and (99) it follows that

$$\frac{\dot{T}}{T^2} \sim O(1), \tag{101}$$

so that the temperature changes too rapidly for thermalization.

We have seen that within our approximations, a dilaton which decreases all the way to zero results in a singular gauge theory. Effects of renormalization could, in principle, tame the singularity. In any case, if the dilaton profile is modified such that e^{Φ} never vanishes, but can become small at $t = 0$, a smooth time evolution beyond this time is possible. Let us consider a situation where the dilaton varies in a smooth fashion reaching a minimum value and then increasing again, approaching a constant in the far future which corresponds to a large value for the 't Hooft coupling. The time dependent dilaton will typically lead to the system having some nonzero energy density in the far future, and one would expect that given enough time this energy would thermalize. This suggests that in the dual closed string description supergravity will eventually become a good approximation in the far future, and the dual geometry in the far future will be that of a black hole in AdS_5.

An interesting question to ask is how the formation of the black hole depends on the dilaton time variation. In particular whether the black hole is always a good description of the geometry by the time the supergravity approxi-

[14]The initial state in the far past is the vacuum which is translationally invariant. Also, the dilaton only depends on time and does not break this translational symmetry. Thus the potential energy in Eq. (70) scales like the volume, leading in turn from Eq. (51) to an energy density which is finite and given by Eq. (97).

[15]Note that Eq. (96) is valid when $|t| \ll 1$ for the solution Eq. (83). This is consistent with the supergravity approximation being valid, as follows from Eq. (104).

mation becomes good in the future, or whether for a suitable dilaton time profile the formation of the black hole can be delayed, until much after the supergravity description becomes valid. This question is difficult to answer with our current level of understanding. In particular, in the course of the time evolution of the dilaton it goes through a region where the 't Hooft coupling is of order unity. This region is very difficult to analyze, since neither the gravity nor the gauge theory descriptions are tractable then.

Finally, we have not been very precise in the discussion above about exactly when the supergravity approximation breaks down. The metric for the solutions we are considering is in Eqs. (83) and (88). For small t the curvature goes like

$$R_{\rm curv} = \frac{z^2}{(-t)^3}. \qquad (102)$$

In this expression we have set

$$R_{\rm AdS} = (4\pi g_s N)^{1/4} \qquad (103)$$

to unity, and we are also working in a 5D Einstein frame. After taking this into account one finds that the string frame curvature is given by

$$R_{\rm curv}^{\rm string} \sim \frac{1}{(-t)^{(3-(\sqrt{3}/2))}} \frac{z^2}{R_{\rm AdS}^2}. \qquad (104)$$

As $t \to 0^-$ we see that the curvature blows up, resulting in a singularity as we have discussed above. The supergravity approximation breaks down when the curvature gets to be of order string scale. We see that when this condition becomes valid depends on the radial coordinate z. In particular, for all finite t, $z \to \infty$ has diverging curvature. This point corresponds to the locus where the past and future horizons of AdS_5 (in the Poincaré coordinates we are using) intersect. The significance of this curvature singularity is unclear to us.[16] As time increases, the region with high curvature (of order string scale or more) grows, moving to smaller z, eventually leading to a singularity for all z as $t \to 0$.

The calculation of curvature, Eq. (104), makes it clear that the singularity (in string frame) is due to two effects which are tied together in these cosmological solutions. First, the dilaton vanishes causing the string frame curvature to blow up. Second, an infinite amount of energy is dumped into the system due to the vanishing dilaton source. If the dilaton profile is altered so that it attains a nonvanishing minimum value at $t = 0$, the total energy put into the system would be finite. If the minimum value of the dilaton is small enough one expects that the curvature in string units at $t = 0$ still becomes much bigger than unity, resulting in the breakdown of supergravity.

[16]Note that the past Poincaré horizon at $z \to \infty$, $t \to -\infty$, with $|z/t|$ held fixed is nonsingular, as was discussed in [4].

VI. BEHAVIOR NEAR SINGULARITIES AND BKL COSMOLOGIES

We saw in Sec. V that the two solutions which were considered had very similar behavior near the $t = 0$ singularity. This was because the stress energy near the singularity was dominated by the dilaton which had essentially the same behavior near $t = 0$ in these two cases.

The behavior of the dilaton near the singularity is in fact shared by a much larger class of cosmological solutions. The key point is that the Einstein frame metric for the class of solutions we have considered may be written in suitable coordinates as [2]

$$ds^2 = \frac{1}{z^2}[dz^2 + \tilde{g}_{\mu\nu}(x)dx^\mu dx^\nu] + d\Omega_5^2. \qquad (105)$$

This solves the ten-dimensional supergravity equations provided

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\nabla_\mu\Phi\nabla_\nu\Phi = 0, \qquad \frac{1}{\sqrt{-\tilde{g}}}\partial_\mu(\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\partial_\nu\Phi) = 0, \qquad (106)$$

and the 5-form field strength is standard

$$F = \omega_5 + \star\omega_5. \qquad (107)$$

In other words, any solution of 3 + 1-dimensional dilaton gravity may be lifted to a solution of ten-dimensional supergravity. This means that we can use the well-known analysis of Belinski, Lifshitz, and Khlalatnikov and subsequent work [14–19] to make useful statements about AdS cosmologies.

For instance, we can consider a cosmological solution where the spatial 3-metric is one of the general homogenous spaces in the Bianchi classification (see, e.g., [14] for a lucid treatment in the 4D context), with vanishing $g_{0,\alpha}$ components:

$$ds_4^2 = -dt^2 + \eta_{ab}(t)(e^a_\alpha dx^\alpha)(e^b_\beta dx^\beta), \qquad (108)$$

where $(e^a_\alpha dx^\alpha)$ are a triad of 1-forms defining symmetry directions.[17] $\eta_{ab}(t)$ are general time dependent coefficients which can be solved for from the Einstein equations, with components decomposed along the frame. Assuming a spatially homogenous dilaton gives $\partial_a\Phi = e^\alpha_a\partial_\alpha\Phi = 0$, with $\partial_0\Phi$ nonvanishing, so that $R^a_{(a)}$ vanish, with $R^0_0 = \frac{1}{2}\times(\partial_0\Phi)^2$. More details on the Bianchi-IX solution can be found in Appendix F.

The main point of BKL is that close to a spacelike singularity, physics becomes ultralocal. For dilaton-driven

[17]Starting with the 1-form triad $(e^a_\alpha dx^\alpha)$, a labeling the vectors in the triad, we can obtain the dual vectors e^α_a, satisfying $e^a.e_b = \delta^a_b$. Then the symmetry algebra acting on the homogenous space (i.e., the spatial metric) in question is obtained as the algebra of the differential operators $X_a = e^\alpha_a\partial_\alpha$.

cosmologies, this results in a Kasner-like solution in which the time dependent part of the dilaton is precisely of the form we have been analyzing.

A simple illustration is provided by a general conformally flat boundary metric and dilaton

$$ds^2 = F(\bar{x}, t)(-dt^2 + d\bar{x}^2)\Phi = \Phi(\bar{x}, t). \tag{109}$$

Near a singularity (which may be chosen to be at $t = 0$ without loss of generality) we will assume that space derivatives may be ignored compared to time derivatives (which would typically blow up). However terms which contain mixed derivatives need to be retained [14]. This results in the following system of equations for Φ and $f \equiv \log F$:

$$\begin{aligned} \partial_t^2 f(\bar{x}, t) + [\partial_t f(\bar{x}, t)]^2 = 0, \\ \partial_t f(\bar{x}, t)\partial_t \Phi(\bar{x}, t) + \partial_t^2 \Phi(\bar{x}, t) = 0, \\ 3\partial_t^2 f(\bar{x}, t) + [\partial_t \Phi(\bar{x}, t)]^2 = 0, \\ \partial_t f(\bar{x}, t)\partial_i f(\bar{x}, t) - 2\partial_t \partial_i f(\bar{x}, t) - \partial_t \Phi(\bar{x}, t)\partial_i \Phi(\bar{x}, t) = 0. \end{aligned} \tag{110}$$

The general solution of (110) is given by

$$\begin{aligned} \Phi(\bar{x}, t) = \sqrt{3}\ln(t + D(\bar{x})) + C_1(\bar{x}), \\ f(\bar{x}, t) = \ln(t + D(\bar{x})) + C_2(\bar{x}). \end{aligned} \tag{111}$$

The last equation in (110) imposes the following relation on C_2 and C_1:

$$C_2(\bar{x}) = \sqrt{3}C_1(\bar{x}). \tag{112}$$

By choosing $D = 0$, one can see that the behavior of the fields is the same as in Eqs. (88) and (89) for the symmetric Kasner case.

A similar result holds for a general class of diagonal metrics with nonvanishing Weyl tensor [16].

In fact, in dilaton-driven cosmology for any space-time dimension greater than 3, the approach to a singularity is characterized by a *finite* number of oscillations between Kasner-like solutions. Consider, for example, homogeneous cosmologies of type Bianchi-IX

$$\begin{aligned} ds^2 = -dt^2 + (a_1^2(t)l_\alpha l_\beta + a_2^2(t)m_\alpha m_\beta \\ + a_3^2(t)n_\alpha n_\beta)dx^\alpha dx^\beta, \end{aligned} \tag{113}$$

where l, m, n are the three frame vectors e^1, e^2, e^3. The Kasner-like solutions are obtained when the spatial curvatures can be ignored,

$$a_i(t) \sim t^{p_i}, \qquad \Phi \sim \alpha \log(t), \tag{114}$$

where the Kasner exponents satisfy

$$\sum_i p_i = 1, \qquad \sum_i p_i^2 = 1 - \frac{\alpha^2}{2}. \tag{115}$$

The effects of the spatial curvature results in oscillations between different sets of p_i's until all the p_i's are positive.

The transition between different Kasner regimes lead to an interesting attractor behavior. Consider the case of $3 + 1$ dimensions. Let p_- denote a negative Kasner exponent and $p_+ > 0$ being either of the other two positive exponents. These transitions can be expressed as the iterative map

$$\begin{aligned} p_i^{(n+1)} = \frac{-p_-^{(n)}}{1 + 2p_-^{(n)}}, \qquad p_j^{(n+1)} = \frac{p_+^{(n)} + 2p_-^{(n)}}{1 + 2p_-^{(n)}}, \\ \alpha_{(n+1)} = \frac{\alpha_n}{1 + 2p_-^{(n)}}, \end{aligned} \tag{116}$$

for the bounce from the (n)-th to the $(n + 1)$-th Kasner regime with exponents p_i, p_j. With each bounce, we see that α increases, shrinking the allowed space of $\{p_i\}$, as detailed in Appendix F. For $\alpha \geq 1$, the allowed window of $\{p_i\}$ pinches sufficiently forcing all $p_i > 0$, at which point oscillations cease and the system settles in the $p_i > 0$ attractor region. The rate of increase of α is small for small α, since $\alpha_{n+1} - \alpha_n = \alpha_n(\frac{-2p_-}{1+2p_-})$, so that a system with near constant dilaton ($\alpha \sim 0$) takes a long iteration time to "flow" towards the $p_i > 0$ attractor region.

Furthermore many distinct initial Kasner regimes can flow to the same attractor point characterized by a set of positive p_i's. The details of the derivation are given in Appendix F. Note that in the absence of a dilaton the number of such oscillations is infinite even though the proper time to the singularity is finite [14,16,17], simply because in this case all the p_i's cannot be positive.

Finally, these attractor flows exhibit some degree of chaotic behavior, in the sense that small changes to the initial conditions give rise to drastic changes in the final endpoints, as elaborated in Appendix F. A quick glimpse at this is obtained by considering exponents $\{p_1, p_2, p_3, \alpha = 0\}$ corresponding to a nondilatonic asymmetric Kasner cosmology which oscillates indefinitely, and perturbing infinitesimally. Now $\alpha^2 = 2(1 - \sum p_i'^2)$ is generically nonzero (although small). This latter set $\{p_i', \alpha \neq 0\}$ thus flows to the attractor region, while the former oscillates indefinitely, showing that a small change in the former gives a drastically different endpoint.

Note that in our setting we have frozen the 5-form field strength and all the other supergravity fields. This is because we want to embed BKL-type cosmologies in the AdS setup used in this paper. In general supergravity theories, a BKL-type analysis shows that a general solution (which excites all fields) in the supergravities which follow from string theories and 11D supergravity exhibit an infinite number of oscillations between different Kasner regimes [15,18], similar to pure gravity [14,16,17].

The fact that the general BKL analysis for gravity-dilaton system can be carried over to a discussion of a class of AdS cosmologies is interesting. However, the symmetric Kasner is the only solution whose Weyl curvature vanishes.[18] In the AdS context this means that it is only for this case that we can have PBH transformations to choose a flat boundary. In other cases, the metric on which the dual conformal field theories live is generically singular and one would expect that the gauge theory would be singular as well.

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APPENDIX A: DILATON COUPLINGS IN THE YANG-MILLS THEORY

In this appendix we discuss the Lagrangian which appears in Eq. (3), especially the dependence of the dilaton in it.

In the standard AdS/CFT dictionary the operator dual to the dilaton is determined by the superconformal symmetry to be an appropriate descendant of the chiral primary obtained by symmetrizing two scalars. Of particular importance to this paper is the fact that in Eq. (3) the dilaton only couples to the kinetic energy term for the gauge fields and does not couple to the kinetic energy terms of the scalars and the fermions. To see this it is enough to consider the $U(1)$ theory. We follow the notation in [13]. The supercharges are Q^I_α, $\bar{Q}_{I\dot{\alpha}}$. Here, the index I upstairs (downstairs) denotes a $4(\bar{4})$ of $SU(4)$, and α, $\dot{\alpha}$ are indices for the two different spinor representations of $SO(3,1)$.

The scalars transform like a real six-dimensional representation of $SU(4)$. For the limited purpose of carrying out the supersymmetry analysis it is useful to denote the scalar fields by $\Phi_{[IJ]}$, where the square brackets indicated antisymmetrization. The scalars satisfy the condition

$$(\Phi_{[IJ]})^* = \frac{1}{2}\epsilon^{IJKL}\Phi_{KL}. \tag{A1}$$

The traceless symmetric product of two scalars gives rise to a field which transforms in the 20-dimensional representation of $SU(4)$. This is a chiral primary of the full superconformal algebra. We denote it by

$$T_{IJKL} = \Phi_{[IJ]}\Phi_{[KL]} - \frac{1}{4!}\epsilon_{IJKL}\Phi_{[PQ]}\Phi_{[RS]}\epsilon^{PQRS}. \tag{A2}$$

Note this field satisfies the tracelessness condition, $\epsilon^{IJKL}T_{IJKL} = 0$.

The operator which the complexified dilaton-axion couples to is [13]

$$\hat{O} = \epsilon^{\alpha\beta}\epsilon^{\gamma\delta}Q^I_\alpha Q^J_\beta Q^K_\gamma Q^L_\delta T_{IJKL}. \tag{A3}$$

The supersymmetry transformations in the $U(1)$ theory are (up to possible numerical factors)

$$[Q^I_\alpha, \Phi_{JK}] = \delta^I_J\Psi_{k\alpha} - \delta^I_K\Psi_{J\alpha}, \tag{A4}$$

$$\{Q^I_\beta, \Psi_{I\alpha}\} = \delta^I_J F_{\alpha\beta}, \tag{A5}$$

and,

$$[Q^I_\alpha, F_{\beta\gamma}] = 0. \tag{A6}$$

Here $\Psi_{I\alpha}$, $F_{\alpha,\beta}$ denote the fermionic partners and the gauge fields. It is then easy to see that this gives

$$\hat{O} \sim F^2. \tag{A7}$$

In particular, $\hat{O}$ does not contain any coupling to the scalar or fermion kinetic terms.

This result is consistent with Eq. (3). It is also consistent with the statement that the dilaton couples to the on shell Lagrangian once we allow for a total derivative term involving the scalars.

A further check on Eq. (3) may be obtained as follows. Consider deformed $AdS_5 \times S^5$ with a ten-dimensional string frame metric $g_{\alpha\beta}(x)$ and a dilaton $\Phi(x)$. The indices (α, β) are ten-dimensional indices, e.g., $\alpha = (\mu, a)$ where $\mu = 0, \cdots 3$ and $a = 1 \cdots 6$. We will only consider backgrounds such that $g_{a\mu} = 0$. Then the Lagrangian for the dual theory for small deformations is

$$\begin{aligned}\mathcal{L} = \sqrt{-g_4}e^{-\Phi}\,\mathrm{Tr}\Big[&-\frac{1}{4}g^{\mu\mu'}g^{\nu\nu'}F_{\mu\nu}F_{\mu'\nu'} \\ &-\frac{1}{2}g^{\mu\nu}D_\mu X^a D_\nu X^b g_{ab} + \frac{1}{4}[X^a, X^b][X^c, X^d]g_{ac}g_{bd} \\ &+\frac{1}{2}\bar{\Psi}\Gamma^A e^\mu_A[-iD_\mu, \Psi] + \frac{1}{2}\bar{\Psi}\Gamma^A e^a_A[X^b, \Psi]g_{ab}\Big].\end{aligned} \tag{A8}$$

Here g_4 denotes the determinant $\det(g_{\mu\nu})$ and A denotes a frame index and e^a_A is the string frame vierbein. One way to see this is to consider the Yang-Mills theory in the Coulomb branch with $SU(N) \to SU(N-1) \times U(1)$. Then the effective action for the $U(1)$ part should be given by the Dirac-Born-Infeld (DBI) action for a 3-brane in this geometry. In this action, the dilaton factor e^Φ appears as an

[18] Analyzing the Weyl tensor components shows that the Weyl tensor vanishes identically only for flat space and the symmetric Kasner space-time. For a generic asymmetric Kasner space-time with exponents (p_1, p_2, p_3), some of the nonvanishing Weyl tensor components diverge as $t^{2p_m - 2}$, where p_m is one of the exponents p_i.

overall factor, provided everything is written in terms of the string frame metric. The leading order (two derivative) terms of this action can be obtained by simply replacing the $SU(N)$ fields in the original Yang-Mills action by the $U(1)$ part. It is then easy to see that if we make this replacement in (A8), we get the correct leading terms of the DBI action.

We now need to express the Lagrangian (A8) in terms of the ten-dimensional Einstein frame metric $G_{\alpha\beta}$,

$$G_{\alpha\beta} = e^{-\Phi/2} g_{\alpha\beta}. \tag{A9}$$

This leads to the Lagrangian

$$\begin{aligned}\mathcal{L} = \sqrt{-G}\,\mathrm{Tr}\Big[&-\frac{1}{4} e^{-\Phi} G^{\mu\mu'} G^{\nu\nu'} F_{\mu\nu} F_{\mu'\nu'} \\ &- \frac{1}{2} G^{\mu\nu} D_\mu X^a D_\nu X^b G_{ab} + \frac{1}{4} e^{\Phi} [X^a, X^b] \\ &\times [X^c, X^d] G_{ac} G_{bd} + \frac{1}{2} e^{\Phi/4} \bar{\Psi} \Gamma^A (e_E)^\mu_A [-iD_\mu, \Psi] \\ &+ \frac{1}{2} e^{3\Phi/4} \bar{\Psi} \Gamma^A (e_E)^a_A [X^b, \Psi] G_{ab} \Big],\end{aligned} \tag{A10}$$

where $(e_E)^a_A$ denotes the Einstein frame vierbein. Now consider the field redefinition for the fermion fields

$$\Psi \to e^{-\Phi/8} \Psi. \tag{A11}$$

This will absorb the dilaton factor in front of the quadratic term, but will give rise to an additional term of the form,[19] $\bar{\Psi}\Gamma^A \Psi (\partial_\mu \phi)(e_E)^\mu_A$. However, $\bar{\Psi}\Gamma^A\Psi = 0$ by virtue of the Majorana condition.

In our setup, it is the Einstein metric which is flat. This gives rise to Eq. (3).

Finally, let us mention that Eq. (3) is invariant under the conformal transformation, $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$, with the scalars and fermions transforming in the standard manner, once the $\frac{1}{6} R X^2$ term is also included.

APPENDIX B: REDUCTION TO SINGLE HARMONIC OSCILLATOR AND THE SCHRODINGER PICTURE WAVE FUNCTION

Consider the 3 + 1-dimensional quadratic theory

$$S = \int dt d^3 x \frac{1}{2} [\dot{X}^2 - (\partial_i X)^2 - m^2(t) X^2]. \tag{B1}$$

We work in a box of volume $V \equiv L^3$, with periodic boundary conditions. Define the modes $X_{\mathbf{n}}$ to satisfy the equation

$$X = \sum_{\mathbf{n}} X_{\mathbf{n}} e^{i[2\pi(\mathbf{n}\cdot\mathbf{x}/\mathbf{L})]}. \tag{B2}$$

The action, Eq. (B1), becomes

$$S = \int dt \frac{V}{2} \Big[\dot{X}_{\mathbf{n}} \dot{X}_{-\mathbf{n}} + \Big(2\pi \frac{\mathbf{n}}{\mathbf{L}}\Big)^2 + m^2(t) |X_{\mathbf{n}}|^2 \Big]. \tag{B3}$$

Taking

$$\sqrt{V} X_{\mathbf{n}} \to X_{\mathbf{n}} \tag{B4}$$

gives the action for a single mode

$$S = \int dt [|\dot{X}_{\mathbf{n}}|^2 - \omega^2(t) |X_{\mathbf{n}}|^2], \tag{B5}$$

with a time dependent frequency

$$\omega^2(t) = w^2(t) = \Big(2\pi \frac{\mathbf{n}}{\mathbf{L}}\Big)^2 + m^2(t). \tag{B6}$$

Next, we calculate the wave function for the ground state, Eq. (26), in the X and $\tilde{X}$ descriptions.

In the position space representation the ground state wave function is given by

$$\psi(x, t) = \langle x, t | 0 \rangle, \tag{B7}$$

where $|x, t\rangle$ is an eigenstate of the operator $\hat{X}(t)$, Eq. (21). By definition, $|x, t\rangle$ satisfies the condition

$$\hat{X} |x, t\rangle = x |x, t\rangle. \tag{B8}$$

In this representation, $\hat{P}$, the canonically conjugate variable to $\hat{X}$, is the operator

$$\hat{P} = -i\partial_x. \tag{B9}$$

Now from the definition of the ground state, Eq. (26), and the expression for $\hat{X}$, $\hat{P}$, in terms of the creation and annihilation operators, Eqs. (21) and (22), it follows that the ground state satisfies the condition

$$\hat{P} |0\rangle = \Big(\frac{\dot{f}}{f}\Big)^* \hat{X} |0\rangle. \tag{B10}$$

From the properties discussed above it then follows that the wave function $\psi(x, t)$, Eq. (B7), satisfies the equation

$$-i\partial_x \psi(x, t) = \Big(\frac{\dot{f}}{f}\Big)^* x \psi(x, t). \tag{B11}$$

This can be easily integrated to give

$$\psi(x, t) = C(t) e^{i[(\dot{f}/f)^*(x^2/2)]}. \tag{B12}$$

The time dependent function $C(t)$ is determined by requiring that Schrodinger's equation,

$$-\frac{1}{2} \partial_x^2 \psi(x, t) + \frac{1}{2} \omega^2(t) \psi(x, t) = i\partial_t \psi(x, t), \tag{B13}$$

is met. It is straightforward to see that this gives Eq. (27).

The function $f(t)$ is defined in Eq. (16). At small t,

$$H_\nu^{(1)}(-\omega_0 t) \simeq i N_\nu(-\omega_0 t) \sim c_1 (\omega_0 t)^{-\nu}, \tag{B14}$$

where c_1 is a constant. This leads to Eq. (29).

The probability density to find the system between x and $x + dx$ is given by $|\psi(x, t)|^2$. From Eq. (27) this takes the form

[19] This includes the transformation of the spin connection.

$$|\psi(x,t)|^2 = \frac{|A|^2}{|f|} e^{-[(\omega_0 x^2)/|f|^2]}. \tag{B15}$$

We have used the fact that $f(t)$ solves Eq. (19), and has the asymptotic value, Eq. (20). This means that the Wronskian, which is time independent, is given by

$$f(t)\dot{f}^*(t) - f^*(t)\dot{f}(t) = 2i\omega_0. \tag{B16}$$

We see from Eq. (B15) that the probability density is a Gaussian with a width $\sqrt{(\Delta x)^2} \sim |f|$. This diverges as $t \to 0$, since $|f|$ blows up, Eq. (29). Thus the probability density gets more and more uniformly spread out as one approaches the singularity.

The phase factor, Eq. (30), arises from the limiting form of $f(t)$ given in Eq. (29).

Next we turn to the wave function for the ground state in the $\tilde{X}$ variable. The steps are analogous to those above. The $\tilde{X}$ and X variables are related by Eq. (8). From the Lagrangian, Eq. (14), it follows that the conjugate momentum, $\hat{\tilde{P}}$, is given by

$$\hat{\tilde{P}} = e^{-\Phi}\dot{\hat{\tilde{X}}} = e^{-\Phi/2}\Big(\hat{P} + \frac{\dot{\Phi}}{2}\hat{X}\Big), \tag{B17}$$

where we have used the relation $\dot{\hat{X}} = \hat{P}$.

The relation, Eq. (B10), then leads to

$$\hat{\tilde{P}}|0\rangle = e^{-\Phi}\Big(\Big(\frac{\dot{f}}{f}\Big)^* + \frac{\dot{\Phi}}{2}\Big)\hat{\tilde{X}}|0\rangle. \tag{B18}$$

Let $|\tilde{x}, t\rangle$ be eigenstates of $\tilde{X}$, satisfying the condition[20]

$$\hat{\tilde{X}}|\tilde{x}, t\rangle = \tilde{x}|\tilde{x}, t\rangle. \tag{B19}$$

The wave function in the $|\tilde{x}\rangle$ representation, $\tilde{\psi}(\tilde{x}, t)$, then satisfies the condition

$$-i\partial_{\tilde{x}}\tilde{\psi} = e^{-\Phi}\Big[\Big(\frac{\dot{f}}{f}\Big)^* + \frac{\dot{\Phi}}{2}\Big]\tilde{x}\,\tilde{\psi}\,. \tag{B20}$$

In addition the Schrodinger equation which takes the form

$$-\frac{e^{\Phi}}{2}\frac{\partial^2\tilde{\psi}}{\partial\tilde{x}^2} + \frac{e^{-\Phi}}{2}\omega_0^2\tilde{\psi} = i\partial_t\tilde{\psi} \tag{B21}$$

must be satisfied. This leads to the solution, Eq. (35). The probability density $|\tilde{\psi}|^2$, Eq. (36), is then obtained as in the discussion above leading up to Eq. (B15). To understand the behavior of the phase factor discussed in Eq. (37), we note that $f(t)$ is defined in Eq. (16). At small t it then follows from the behavior of the Neumann function N_ν that

$$f(t) \simeq c_1(-\omega_0 t)^{(1/2)-\nu}(1 + c_2(-\omega_0 t)^2). \tag{B22}$$

Thus,

$$\Big(\frac{\dot{f}}{f}\Big)^* + \frac{\dot{\Phi}}{2} = \frac{1}{t}\Big(\frac{1-2\nu+p}{2}\Big) + 2c_2\omega_0^2 t. \tag{B23}$$

Here we have used Eq. (11). This leads to Eq. (37), after noting Eq. (17).

The expectation value for $\hat{\tilde{P}}^2$ can be calculated from the wave function Eq. (35). Alternatively, for this purpose, we can directly work in the Heisenberg picture. From Eq. (B17) and the expression for $\hat{X}$, $\hat{P}$, in terms of the creation and annihilation operators, Eqs. (21) and (22), we get that in the vacuum state

$$\langle\hat{\tilde{P}}^2\rangle = \frac{e^{-\Phi}}{2\omega_0}\Big|\dot{f} + \frac{\dot{\Phi}}{2}f\Big|^2. \tag{B24}$$

Using Eq. (B23) and related discussion above, this leads to Eq. (39).

A similar analysis can be carried out for a coherent state, defined by

$$a|s\rangle = \alpha|s\rangle. \tag{B25}$$

This leads to a wave function

$$\tilde{\psi}(\tilde{x}, t) = \frac{A}{\sqrt{f^* e^{\Phi/2}}} e^{[ie^{-\Phi}(\tilde{x}^2/2)[(\dot{f}/f)^* + (\dot{\Phi}/2)]} \times e^{[(\alpha\sqrt{2\omega_0}\tilde{x})/(f^* e^{\Phi/2})]} e^{[i\omega_0\alpha^2\int(dt/(f^*)^2)]}. \tag{B26}$$

The extra terms, compared to the ground state wave function, which are dependent on α, are both well defined in the limit $t \to 0$. Thus this wave function has the same type of singularity as the ground state wave function.

APPENDIX C: SUBLEADING CONTRIBUTIONS TO ENERGY

In this appendix we calculate the subleading contributions to the energy. These contributions would be the dominant ones if $\langle V\rangle$ vanishes, as discussed in Sec. III D, and need to be calculated to understand when the energy remains finite, as $t \to 0$.

A subleading contribution arises from the kinetic energy term. From the wave function, Eq. (43), we find that for $p > 1$ this is given by

$$\langle KE\rangle \simeq \frac{e^{\Phi}}{2}G^2\langle(V')^2\rangle \sim (-t)^{(2-p)}, \tag{C1}$$

and diverges if $p > 2$. For $p < 1$, since $G(t)$ is small near $t = 0$,

[20] We also require that the completeness relation,

$$\int d\tilde{x}|\tilde{x}, t\rangle\langle\tilde{x}, t| = \mathbf{I},$$

is satisfied. A similar relation, with $\tilde{x}$ replaced by x, is also satisfied by the states $|x, t\rangle$. This tells us that the states $|\tilde{x}, t\rangle$ are related to the states $|x, t\rangle$ introduced above, by the relation

$$|\tilde{x}, t\rangle = e^{-\Phi/4}|x\rangle = e^{-\Phi/2}|\tilde{x}, t\rangle.$$

$$\langle KE\rangle \simeq \frac{e^{\Phi}}{2}\int dx|\psi_0'|^2 \sim (-t)^p. \tag{C2}$$

This does not diverge as $t \to 0$.

Another subleading correction arises due to a correction in the absolute magnitude of ψ which in turn leads to a correction in $\langle V\rangle$. We write

$$\psi(x,t) = e^{-iG(t)V(\tilde{x})}\psi_0(\tilde{x})[1 + S_1(\tilde{x},t)]. \tag{C3}$$

Since we are interested in the corrections to the absolute value of ψ we take S_1 to be real. From the Schrodinger equation we get

$$\frac{e^{\Phi}}{2}\left[-\mathrm{Im}\left(\frac{\psi_0''}{\psi_0}\right) + 2G(t)V'\,\mathrm{Re}\left(\frac{\psi_0'}{\psi_0}\right) + G(t)V''\right] = \frac{\partial S_1}{\partial t}. \tag{C4}$$

For $p > 1$ the second and third terms within the square brackets on the left hand side dominate, leading to

$$S_1 = \left[\int dt e^{\Phi}G(t)\right]\left[V'\,\mathrm{Re}\left(\frac{\psi_0'}{\psi_0}\right) + \frac{V''}{2}\right], \tag{C5}$$

$$= \frac{1}{2(1-p)}t^2\left[V'\,\mathrm{Re}\left(\frac{\psi_0'}{\psi_0}\right) + \frac{V''}{2}\right]. \tag{C6}$$

We see that this goes like t^2, as $t \to 0$, and does not diverge.

For $p < 1$ the first term on the left-hand side of Eq. (C4) dominates, giving

$$S_1 = \frac{1}{2}\,\mathrm{Im}\left(\frac{\psi_0''}{\psi_0}\right)\left[\int dt e^{\Phi}\right], \tag{C7}$$

$$= -\frac{1}{2(1+p)}\,\mathrm{Im}\left(\frac{\psi_0''}{\psi_0}\right)(-t)^{(1+p)}. \tag{C8}$$

This term goes like $(-t)^{(1+p)}$ and also does not diverge. Since S_1 vanishes as $t \to 0$, the resulting correction to $\langle V\rangle$ and therefore to the energy also vanishes.

Thus the conclusion is that except for the case where $p > 2$, in which case the kinetic energy itself gives a divergent contribution, it is enough to have $\langle V\rangle$ as defined in Eq. (52) to vanish, to ensure that the expectation value of the energy stays finite.

APPENDIX D: PARTICLE PRODUCTION

In this appendix we detail the calculation of particle production at the quadratic level in the case where e^{Φ} does not become zero at any point, but can become small. For this purpose we choose a dilaton profile of the following form:

$$e^{\Phi(t)} = g_s|t|^p, \quad |t| > \epsilon, \quad e^{\Phi(t)} = g_s|\epsilon|^p, \quad |t| < \epsilon. \tag{D1}$$

We will first perform the analysis for each individual momentum mode, $\tilde{X}_k$.

As explained above, one should work in the variables $\tilde{X}$ since these are the variables which have a finite limit as $t \to 0$. The equation of motion for $\tilde{X}_k$ is

$$\left[\frac{d}{dt}\left(e^{-\Phi(t)}\frac{d}{dt}\right) + \omega_0^2 e^{-\Phi(t)}\right]\tilde{X}_k = 0. \tag{D2}$$

Clearly, it is convenient to work with a time variable τ defined by

$$e^{-\Phi(t)}\frac{d}{dt} = \frac{d}{d\tau}. \tag{D3}$$

As is standard, we will solve (D2) separately in the regions $t < -\epsilon$, $-\epsilon < t < \epsilon$, and $t > \epsilon$ and then match $\tilde{X}_k$ and $\partial_\tau \tilde{X}_k$ across $t = \pm\epsilon$. With the profile given in (D1), a solution which is purely positive frequency at $t \to -\infty$ is given by

$$\begin{aligned}
&\tilde{X}_k(t) = (-\omega_0 t)^\nu H_\nu^{(1)}(-\omega_0 t), \qquad t \le -\epsilon\\
&\tilde{X}_k(t) = A\exp\left[i\frac{\omega_0 t^{p+1}}{\epsilon^p(p+1)}\right] + B\exp\left[-i\frac{\omega_0 t^{p+1}}{\epsilon^p(p+1)}\right],\\
&\qquad -\epsilon \le t \le \epsilon,\\
&\tilde{X}_k(t) = (\omega_0 t)^\nu[CH_\nu^{(1)}(\omega_0 t) + DH_\nu^{(2)}(\omega_0 t)], \qquad t \ge \epsilon,
\end{aligned} \tag{D4}$$

where ν has been defined in (17). The Bogoliubov coefficients C and D will be determined by the matching conditions. After a standard calculation we get the following expressions for C and D:

$$\begin{aligned}
C &= \frac{i\pi}{4}(\omega_0\epsilon)\bigg\{\cos\left(\frac{2\omega_0\epsilon}{p+1}\right)[H_\nu^{(1)}(\omega_0\epsilon)H_{\nu-1}^{(2)}(\omega_0\epsilon)\\
&\quad + H_\nu^{(2)}(\omega_0\epsilon)H_{\nu-1}^{(1)}(\omega_0\epsilon)] - \sin\left(\frac{2\omega_0\epsilon}{p+1}\right)\\
&\quad \times[H_{\nu-1}^{(1)}(\omega_0\epsilon)H_{\nu-1}^{(2)}(\omega_0\epsilon) - H_\nu^{(2)}(\omega_0\epsilon)H_\nu^{(1)}(\omega_0\epsilon)]\bigg\},\\
D &= -\frac{i\pi}{4}(\omega_0\epsilon)\bigg\{2\cos\left(\frac{2\omega_0\epsilon}{p+1}\right)H_\nu^{(1)}(\omega_0\epsilon)H_{\nu-1}^{(1)}(\omega_0\epsilon)\\
&\quad - \sin\left(\frac{2\omega_0\epsilon}{p+1}\right)[(H_{\nu-1}^{(1)}(\omega_0\epsilon))^2 - (H_\nu^{(1)}(\omega_0\epsilon))^2]\bigg\}.
\end{aligned} \tag{D5}$$

In deriving these we have used the following property for any Bessel function $Z_\nu(x)$:

$$\frac{d}{dx}[x^\nu Z_\nu(x)] = x^\nu Z_{\nu-1}(x). \tag{D6}$$

A straightforward calculation verifies the unitarity relation

$$|C|^2 - |D|^2 = -1. \tag{D7}$$

Let us first consider the limit $\omega_0\epsilon \ll 1$. Using the standard expansions for the Hankel functions,

$$H_\nu^{(1)}(x) = \frac{ix^\nu}{2^\nu \sin(\pi\nu)\Gamma(1+\nu)}\bigg\{e^{-i\pi\nu}\bigg(1 - \frac{x^2}{4(\nu+1)} + O(x^4)\bigg) - \bigg(\frac{x}{2}\bigg)^{2\nu}\frac{\Gamma(1+\nu)}{\Gamma(1-\nu)} \times \bigg(1 - \frac{x^2}{4(1-\nu)} + O(x^4)\bigg)\bigg\}, \tag{D8}$$

we find

$$C = \frac{i\pi}{4}\frac{1}{\sin^2(\pi\nu)2^{2\nu}[\Gamma(1-\nu)]^2}\bigg[-2^{2p+2}\frac{(\omega_0\epsilon)^{1-p}}{1-\nu} - 2^{p+2}\nu\cos(\pi\nu)\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} + \frac{2(\omega_0\epsilon)^{1-p}}{p+1}2^{2p+2}\bigg]. \tag{D9}$$

Thus there is a qualitatively different behavior of the Bogoliubov coefficient for $p > 1$ and for $p < 1$ as $(\omega_0\epsilon) \to 0$. When $p > 1$ the coefficients C and D both diverge in this limit (of course maintaining the unitarity relation). When $p < 1$ they both tend to finite limits

$$\lim_{\omega_0\epsilon\to 0} C = -i\cot(\pi\nu), \qquad \lim_{\omega_0\epsilon\to 0} D = -ie^{-i\pi\nu}\mathrm{cosec}(\pi\nu). \tag{D10}$$

This difference between the cases $p < 1$ and $p > 1$ is the Heisenberg picture manifestation of the behavior of the Schrodinger picture wave functional.

The analysis performed above was with an abrupt modification of the dilaton profile. However we expect that the $\omega_0\epsilon \ll 1$ behavior would continue to be similar for a smooth modification.

In the above analysis there is a finite amount of particle production for every momentum mode for $p < 1$, independent of the value of ω_0, in the limit $\omega_0\epsilon \to 0$. It is interesting to estimate the total amount of energy produced. However, for this estimate we need to perform the calculation for a dilaton profile which tends to a constant at early and late times, in keeping with our overall scenario. For a smooth dilaton profile, the ultraviolet behavior ($\omega_0\epsilon \gg 1$) is then expected to be exponentially damped, $|C|^2 \sim e^{-\omega_0\epsilon}$, so that the total energy produced is finite.

It should be emphasized again that all the considerations of this appendix relate to the quadratic approximation. As we have seen this is *not* a good approximation in our problem. An estimate of the total amount of energy produced in the real problem has to take into account the effects of interactions which become stronger at later times. This requires a lot more detailed knowledge of strong coupling physics.

APPENDIX E: THE MILNE BACKGROUND

In this appendix we analyze the behavior when the boundary theory lives in Milne space with the metric Eq. (92), with a dilaton

$$e^\Phi = g_s|\tanh(t)|^{\sqrt{3}}. \tag{E1}$$

The metric Eq. (92), up to the overall conformal factor, e^{-2t}, is

$$ds^2 = -dt^2 + \frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{E2}$$

This is a space of constant negative curvature. Since the gauge theory is conformally invariant it is equivalent to consider it in the background metric, Eq. (E2), and with dilaton, Eq. (E1).

Below, we first analyze a scalar field with Lagrangian[21]

$$S = \int d^4x\sqrt{-g}e^{-\Phi}\bigg\{\frac{1}{2}(\partial\tilde{X})^2 + \frac{1}{12}R\tilde{X}^2\bigg\} \tag{E3}$$

in the background with the metric, Eq. (E2), and the dilaton, Eq. (E1). Thereafter, we turn to the gauge field. The analysis in the scalar field theory, Eq. (E3), is not identical to the gauge theory, but quite analogous.

We can mode decompose the scalar field into modes which are eigenfunctions of the 3D spatial Laplacian. From [20], we see that, for the metric, Eq. (E2), the modes satisfy the equation

$$\nabla^3 y_{\mathbf{k}} = -(k^2+1)y_{\mathbf{k}}. \tag{E4}$$

The functions $y_{\mathbf{k}}$ are normalized to satisfy the condition

$$\int d^3x\sqrt{h}y_{\mathbf{k}}(x)y^*_{\mathbf{k}'}(x') = \delta^3(\mathbf{k}-\mathbf{k}'). \tag{E5}$$

Here h_{ij} is the spatial part of the metric, Eq. (E2). We can expand the field $\tilde{X}$ in these modes,

$$\tilde{X} = \int d^3k\tilde{X}_{\mathbf{k}}(t)y_{\mathbf{k}}. \tag{E6}$$

This gives rise to decoupled oscillators for each mode, with the Lagrangian

$$S = \int dtd^3ke^{-\Phi}\{|\dot{\tilde{X}}_{\mathbf{k}}|^2 - k^2|\tilde{X}_{\mathbf{k}}|^2\}, \tag{E7}$$

where we have used the fact that the Ricci scalar, $R = -6$, for the metric, Eq. (E2). We see that for each mode the Lagrangian, Eq. (E7), is essentially the same as Eq. (31), with k^2 being identified with ω_0^2. Since the dilaton asymptotically goes to a constant here, Eq. (E1), the Lagrangian for each mode reduces to that of a standard harmonic oscillator in the far past or future.

Our discussion in Sec. V then leads to the conclusion that the wave function has a phase factor which is singular as $t \to 0$, in this case as well. The phase factor is given by Eq. (61). The potential energy $V[\tilde{X}]$ for the Lagrangian Eq. (E3) is

[21] The conclusions would be essentially the same without the curvature coupling term $\frac{1}{12}R\tilde{X}^2$.

$$V[\tilde{X}] = \int d^3x\sqrt{h}\left[h^{ij}\frac{1}{2}\partial_i\tilde{X}\partial_j\tilde{X} + \frac{1}{12}R\tilde{X}^2\right]. \quad \text{(E8)}$$

For the case of the gauge field, in the background metric, Eq. (E2), with the dilaton, Eq. (E1), an analysis similar to that carried out here and in the previous appendix can be done. Once again choosing the Coulomb gauge is convenient. In this gauge one finds that the wave function has a phase factor which near the singularity takes the form Eq. (72), with

$$V[A_i(x)] = \frac{1}{4}\int d^3x\sqrt{-h}F_{ij}F^{ij}. \quad \text{(E9)}$$

This phase factor diverges, leading to a singular wave function.

APPENDIX F: UNIVERSAL BEHAVIOR NEAR SINGULARITIES

In this appendix we discuss some aspects of the universality of Kasner-like behavior near spacelike singularities in the class of models we consider. We will consider ten-dimensional metrics of the form (105) so that the $3+1$-dimensional Ricci tensor $R_{\mu\nu}$ and the dilaton $\Phi(x)$ satisfies the Eqs. (106). It is therefore sufficient to discuss $3+1$-dimensional dilaton cosmologies.

Consider an AdS cosmology where the 4-metric is a Bianchi-IX space-time. Using a BKL-type argument [14,16,17], we show the space-time near the singularity has Kasner-like behavior. Furthermore, the dilaton drives the system towards an attractor region, where all the exponents are positive $p_i > 0$, through a finite number of Kasner oscillations. This analysis can be directly extended to all homogeneous spaces with the results that we either have no oscillations at all or the number of oscillation is finite. This suggests that the symmetric Kasner singularity is generic and independent of the spatial 3-geometry, being either flat or any of Bianchi homogeneous spaces. It is worth mentioning that with no dilaton, symmetric Kasner solutions do not exist and the canonical BKL analysis gives an oscillatory approach to the singularity, with transitions between distinct asymmetric Kasner regimes.

Let us take the $3+1$-dimensional boundary metric $\tilde{g}_{\mu\nu}(x)$ in (105) to be of Bianchi-IX type, which has the following form:

$$ds_4^2 = -dt^2 + (a^2(t)l_\alpha l_\beta + b^2(t)m_\alpha m_\beta + c^2(t)n_\alpha n_\beta)dx^\alpha dx^\beta, \quad \text{(F1)}$$

where l, m, n are the three frame vectors e^1, e^2, e^3 (for explicit form of the metric see, for example, [14] page 390). The spatial symmetry algebra here is $SU(2)$. a, b, and c are three independent scale factors.[22]

[22]If we take equal scale factors $a=b=c$, the spatial metric becomes $d\sigma^2 = (dx_1^2 + dx_2^2 + dx_3^2 + \cos x^1 dx^2 dx^3)$, with constant curvature, $R_{ij} = \frac{1}{2}\gamma_{ij}$, $R = \frac{3}{2}$.

If we assume that the dilaton is spatially homogeneous, then $\partial_a\Phi = e_a^\alpha\partial_\alpha\Phi = 0$, with $\partial_0\Phi$ nonvanishing. Decomposing the Ricci tensor along the frame, we then have

$$\begin{aligned} R^1_{(1)} &= \frac{(\dot{a}bc)^{\cdot}}{abc} - \frac{1}{2(abc)^2}[(b^2-c^2)^2 - a^4] = 0,\\ R^2_{(2)} &= \frac{(a\dot{b}c)^{\cdot}}{abc} - \frac{1}{2(abc)^2}[(a^2-c^2)^2 - b^4] = 0, \quad \text{(F2)}\\ R^3_{(3)} &= \frac{(ab\dot{c})^{\cdot}}{abc} - \frac{1}{2(abc)^2}[(a^2-b^2)^2 - c^4] = 0, \end{aligned}$$

$$R^0_0 = \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} = -\frac{1}{2}(\dot{\Phi})^2, \quad \text{(F3)}$$

and the dilaton field equation is given by

$$\ddot{\Phi} + \dot{\Phi}\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}\right) = 0. \quad \text{(F4)}$$

The above system of ordinary differential equations is very difficult to solve analytically but if we ignore curvature terms (i.e., terms in (F2) with no time derivatives) one has $a = t^{p_1}$, $b = t^{p_2}$, $c = t^{p_3}$, $e^\Phi = t^\alpha$,

$$\sum p_i = 1, \qquad \sum p_i^2 = 1 - \frac{\alpha^2}{2}, \quad \text{(F5)}$$

as an approximate solution near the singularity at $t=0$.

In the usual BKL analysis, $\alpha = 0$, which forces one of the p_i to be negative. The negative p_i means that time evolution towards the singularity necessarily makes one of the curvature terms (treated as a perturbation to the time-derivative terms) dominate the others at some point, for instance $a^4 \sim t^{-|p_1|}$ (if $p_1^{(0)} < 0$). This forces the metric to evolve and transit from one Kasner regime to another according to the following law:

$$p_1^{(1)} = \frac{-p_1^{(0)}}{1+2p_1^{(0)}}, \quad p_2^{(1)} = \frac{p_2^{(0)}+2p_1^{(0)}}{1+2p_1^{(0)}}, \quad p_3^{(1)} = \frac{p_3^{(0)}+2p_1^{(0)}}{1+2p_1^{(0)}}. \quad \text{(F6)}$$

However, with a nontrivial dilaton, we can have one of the following situations: All $p_i > 0$, in which case no transitions take place since curvature terms (perturbations) die off as we approach the singularity, e.g.,

$$[(b^2-c^2)^2 - a^4] \sim -a^4 \sim t^{4/3} \to 0$$

near the singularity $t \to 0$. Other Ricci components have similar behavior. This means that the symmetric Kasner case with all $p_i = \frac{1}{3}$ is stable against these perturbations as we approach this dilaton-driven symmetric Kasner singularity and there is no forced transition to a distinct Kasner regime. The other possibility is that one of the p_i's is negative, in this case, we can have a finite number of oscillatory transitions between different Kasner regimes. This occurs since with every transition α increases and as it reaches a specific value ($\alpha = 1$) all the p_i become positive [see (F5)], and then no further transitions occur. If we consider any of the other types of Bianchi spaces, we should replace the curvature terms in (F2) with that of this Bianchi space. But these terms have no time derivatives so they will not change the leading behavior of the solution. Furthermore, these curvature terms either die off as we approach the singularity, in which case we have no oscillation at all, or one of them ($p_1 < 0$) gets larger as we approach the singularity. This leads to a finite number of oscillations as in Bianchi-IX. Again the dilaton will drive the system to an attractor region where the oscillation stops.

We will now see that the dilaton in fact drives the system towards an "attractor" region given by $p_1,\ p_2,\ p_3 > 0$, through a finite number of oscillations. Once the system reaches this region, there is no further oscillation.

The equations

$$\sum_i p_i = 1, \qquad \sum_i p_i^2 = 1 - \frac{\alpha^2}{2}, \tag{F7}$$

for a Kasner-like cosmology with dilaton $e^{\Phi} = t^{\alpha}$ can be described by the following parametrization:

$$p_1 = x, \qquad p_2 = \frac{1-x}{2} + \frac{\sqrt{1-\alpha^2+2x-3x^2}}{2}, \qquad p_3 = \frac{1-x}{2} - \frac{\sqrt{1-\alpha^2+2x-3x^2}}{2}, \tag{F8}$$

in terms of p_1, α. For a solution to exist, the radical being positive forces

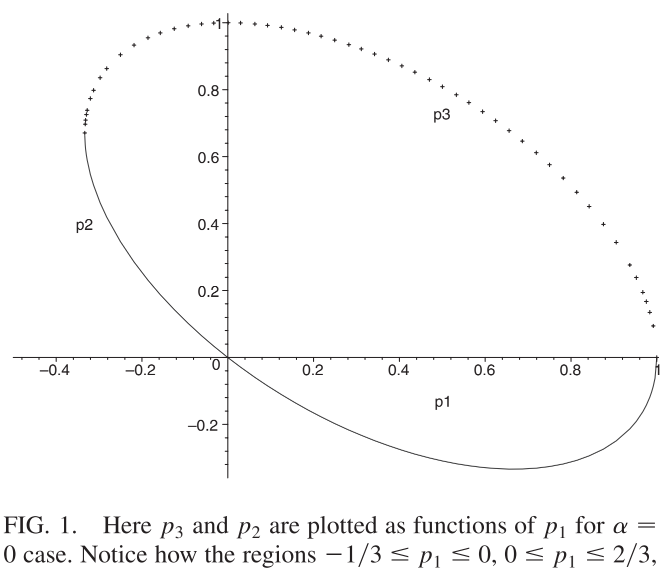

FIG. 1. Here p_3 and p_2 are plotted as functions of p_1 for $\alpha = 0$ case. Notice how the regions $-1/3 \le p_1 \le 0$, $0 \le p_1 \le 2/3$, $2/3 \le p_1 \le 1$ have the same values for p_i's.

$$\frac{1-\sqrt{4-3\alpha^2}}{3} \le p_1 \le \frac{1+\sqrt{4-3\alpha^2}}{3}. \tag{F9}$$

The above range for p_1 can be divided into three regions. In these regions the values of the p_i's are permuted among each other, as in Fig. 1. To avoid redundancy one should constrain p_1 to one region which we choose to be

$$\frac{1-\sqrt{4-3\alpha^2}}{3} \le p_1 \le \frac{2-\sqrt{4-3\alpha^2}}{6}. \tag{F10}$$

The square root here being positive implies $\alpha^2 \le \frac{4}{3}$, i.e., $|\alpha| \le \frac{2}{\sqrt{3}} \sim 1.1547$. The lower limit on p_1 becomes positive if $4 - 3\alpha^2 \le 1$, i.e., $\alpha^2 \ge 1$. At this point, $x = 0$, $\alpha^2 = 1$, all $p_i > 0$. This shrinking of the allowed space of $\{p_i\}$ is a key difference from the case $\alpha = 0$ without dilaton.

Now let us say the system starts with say $p_1^{(0)} = x^{(0)} < 0$. Then there is a transition to a new Kasner regime with $p_i^{(1)}$ and $\alpha^{(1)}$ given by

$$p_1^{(1)} = \frac{-p_1^{(0)}}{1+2p_1^{(0)}}, \qquad p_2^{(1)} = \frac{p_2^{(0)}+2p_1^{(0)}}{1+2p_1^{(0)}}, \qquad p_3^{(1)} = \frac{p_3^{(0)}+2p_1^{(0)}}{1+2p_1^{(0)}}, \qquad \alpha^{(1)} = \frac{\alpha^{(0)}}{1+2p_1^{(0)}}. \tag{F11}$$

Now, $p_1^{(0)} < 0$ means that $\alpha^{(1)} > \alpha^{(0)}$, i.e., α increases under the Kasner transition. More generally, for $p_- < 0$ and $p_+ > 0$ being either of the other two positive exponents, this can be expressed as the iterative map

$$p_i^{(n+1)} = \frac{-p_-^{(n)}}{1+2p_-^{(n)}}, \qquad p_j^{(n+1)} = \frac{p_+^{(n)}+2p_-^{(n)}}{1+2p_-^{(n)}}, \qquad \alpha_{(n+1)} = \frac{\alpha_n}{1+2p_-^{(n)}}, \tag{F12}$$

for the bounce from the (n)-th to the ($n+1$)-th Kasner regime with exponents p_i, p_j. The fixed point of this transformation is $\alpha = 0$, and it is unstable for $p_- < 0$ [an iterative map $x_{n+1} = f(x_n)$ has an unstable fixed point $x_* = f(x_*)$ if $f'(x_*) > 1$]. Furthermore, the rate of increase of α is small for small α, since

$$\alpha_{n+1} - \alpha_n = \alpha_n\left(\frac{-2p_-}{1+2p_-}\right). \tag{F13}$$

Thus a system with near constant dilaton ($\alpha \sim 0$) takes a long iteration time to flow towards the $p_i > 0$ attractor region (although the flow is ensured due to the unstable fixed point). For instance, with $p_1^0 = x_0 = 0.3$, $\alpha_0 = 0.001$, the system flows (initially slowly) to $p_i > 0$ after 15 oscillations, with $\alpha_{15} = 1.0896$.

This flow towards the $p_i > 0$ attractor region in $\{p_i\}$ space can be seen geometrically: the intersection of the sphere $\sum p_i^2 = 1 - \frac{\alpha^2}{2}$ with the plane $\sum p_i = 1$ is a circle on the plane. Under the bounce iterations, the sphere radius shrinks and so the circle radius also shrinks until the circle lies entirely in the $p_i > 0$ quadrant.

The finiteness of the number of oscillations means that the bulk cosmology flows towards the $p_i > 0$ attractor region, driven by the dilaton.

Before we close this section we would like to comment on the nature of the flow towards the attractor region. Inverting (F12), we get $p_- = \frac{p_i^{(n+1)}}{1+2p_i^{(n+1)}}$ and so on.[23] Thus we can trace back from the symmetric Kasner, giving (up to five iterations) the flow

$$\left(-\frac{1}{5}, \frac{9}{35}, \frac{33}{35}\right) \to \left(-\frac{5}{21}, \frac{7}{21}, \frac{19}{21}\right) \to \left(-\frac{3}{11}, \frac{5}{11}, \frac{9}{11}\right) \to \left(-\frac{1}{5}, \frac{3}{5}, \frac{3}{5}\right) \to \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right). \tag{F14}$$

An alternative distinct flow to the same symmetric Kasner endpoint begins at $(\frac{-9}{29}, \frac{15}{29}, \frac{23}{29})$, merging with the above flow at $(-\frac{3}{11}, \frac{5}{11}, \frac{9}{11})$. With each step backwards, α decreases. This shows that there are multiple trajectories that get attracted to any of the points in the $\{p_i > 0\}$ attractor region, perhaps as for any attractorlike behavior.

Furthermore we suspect that the flow exhibit chaotic behavior, i.e., small changes in the initial conditions give rise to drastic changes in the final endpoints. For example, consider changing the starting point for the flow (F14) above by a small perturbation (by $\frac{1}{70} \sim 0.014$, i.e., a 7% change to the smallest exponent, $-\frac{1}{5}$). This gives

$$\left(-\frac{13}{70}, \frac{9}{35}, \frac{65}{70}\right) \to \left(-\frac{2}{11}, \frac{13}{44}, \frac{39}{44}\right) \to \left(-\frac{3}{28}, \frac{2}{7}, \frac{23}{28}\right) \to \left(\frac{1}{11}, \frac{3}{22}, \frac{17}{22}\right), \tag{F15}$$

the flow endpoint being distinct from the symmetric Kasner.

We have used rational Kasner exponents above for simplicity in illustration: more generally, one expects that there exist "nearby" (not necessarily rational) Kasner exponents p_i with nonconstant dilaton ($\alpha \neq 0$) in the neighborhood of exponents with constant dilaton ($\alpha = 0$). In this case, a small change in the exponents in the set $\{p_i, \alpha \neq 0\}$ would give exponents in the set $\{p_i, \alpha = 0\}$, which latter set belong to the canonical BKL analysis and oscillate forever, thus exhibiting no attractor behavior. More explicitly consider exponents $\{p_1, p_2, p_3, \alpha = 0\}$ corresponding to a nondilatonic asymmetric Kasner cosmology which oscillates indefinitely, and perturb infinitesimally as $\{p_1' = p_1 + \epsilon, p_2' = p_2, p_3' = p_3 - \epsilon\}$. Now, $\alpha^2 = 2(1 - \sum p_i'^2) \sim 4(p_3 - p_1)\epsilon \neq 0$, if $p_1 \neq p_3$. This latter set $\{p_i', \alpha \neq 0\}$ thus flows to the attractor region, while the former does not. These examples and arguments suggest that small perturbations to initial conditions apparently give rise to large departures from the endpoints, in other words, chaotic behavior.

[23] Since $|p_-^{(n)}| < \frac{1}{2}$ is always true, therefore $p_i^{(n+1)} = \frac{-p_-^{(n)}}{1+2p_-^{(n)}} > 0$. This means that for each of the other two distinct $p_j^{(n+1)} > 0$, we can potentially trace back to a distinct $p_-^{(n)} < 0$. This gives a tree with two flows starting with a given point $\{p_i^{(n+1)}\}$. Similarly at every previous point, the tree forks into two.

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