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Dynamics of strings between walls

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Configurations of vortex strings stretched between or ending on domain walls were previously found to be $1/4$ Bogomol'nyi-Prasad-Sommerfield (BPS) states in $\mathcal{N} = 2$ supersymmetric gauge theories in $3 + 1$ dimensions. Among zero modes of string positions, the center of mass of strings in each region between two adjacent domain walls is shown to be non-normalizable whereas the rests are normalizable. We study dynamics of vortex strings stretched between separated domain walls by using two methods, the moduli space (geodesic) approximation of full $1/4$ BPS states and the charged particle approximation for string end points in the wall effective action. In the first method we explicitly obtain the effective Lagrangian in the strong coupling limit, which is written in terms of hypergeometric functions, and find the 90° scattering for head-on collision. In the second method the domain wall effective action is assumed to be $U(1)^N$ gauge theory, and we find a good agreement between two methods for well-separated strings.

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I. INTRODUCTION

Dirichlet (D-)branes [1] have been necessary ingredients to study nonperturbative dynamics of string theory since their discovery. They are defined as end points of open strings. The low-energy effective theory on a D-brane is described by the Dirac-Born-Infeld (DBI) action. Strings ending on a D-brane can be realized as solitons or solutions with a source term in the DBI action. These solitons are called BIons [2]. Usually these solitons are constructed as deformations of the D-brane surface such as a spike. It is not easy to construct a string stretched between D-branes as a soliton of the DBI theory. One reason for difficulty is that no DBI action for multiple D-branes is known so far.

Solitons resembling strings ending on D-branes have been found in a field theory framework [3]. They have given an exact solution of vortex strings ending on a domain wall in a $\mathbf{C}P^1$ nonlinear sigma model. This theory or the $\mathbf{C}P^N$ extension was known to admit single or multiple domain wall solutions [4–6]. Assuming the DBI action on the effective action on a single domain wall, they have further shown that the soliton can be identified with a BIon, a soliton on a D2-brane [3], and so have called it a "D-brane soliton." It has been extended to a solution in $U(1)$ gauge theory coupled to two charged Higgs fields [7]. Exact solutions of multiple domain walls have been constructed in $U(N)$ gauge theory in the strong coupling limit by introducing the "moduli matrix" [8]. By extending this method, the most general solutions of D-brane solitons have been constructed [9] which offer *exact* (*analytic*) solutions of multiple domain walls with an arbitrary number of vortex strings stretched between (ending on) domain walls in the strong coupling limit, see Fig. 1. Some aspects of these solitons have been studied. We have found an object with a monopole charge which contributes negatively to the total energy of composite solitons in $U(1)$ gauge theory [9]. It has later been called a "boojum" and studied extensively [10]. In the case of $U(N)$ gauge theory a monopole confined by vortices is also admitted [11–14], which can be understood as a kink in non-Abelian vortices found earlier [15]. The moduli space has been studied [16] for composite solitons consisting of domain walls, vortex

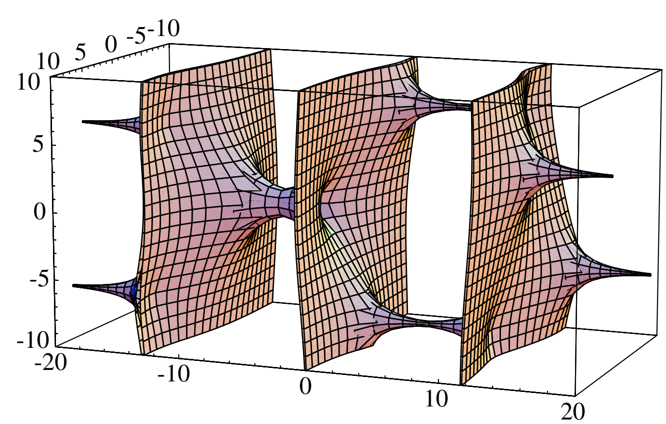

FIG. 1 (color online). An example of the *exact* solution of the D-brane soliton in the strong coupling limit. A same energy surface is plotted. A figure is taken from [9].

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strings, and monopoles. See review papers [16–19] for recent developments of Bogomol'nyi-Prasad-Sommerfield (BPS) composite solitons. It has been proposed that domain walls actually can be regarded as D-branes after taking into account quantum corrections by loop effect of vortex end points [20] (see also [21]). It has been proposed that this provides some field theoretical model of the open-closed string duality.

In this paper we study classical dynamics of D-brane solitons in $3 + 1$ dimensions by using two methods and compare their results. One is to use the moduli space (geodesic) approximation found by Manton in studying the monopole dynamics [22,23]. In this approximation, geodesics on the moduli space of solitons correspond to dynamics or a scattering of solitons. So far the moduli approximation has been used to describe the classical scattering of particlelike solitons such as monopoles in three space dimensions, vortices in two space dimensions [24,25], and kinks in one space dimension [6,26], which are $1/2$ BPS states. As the first example of composite solitons, it has been recently applied to dynamics of domain wall networks [27,28], which are $1/4$ BPS states [29]. Here we apply it to dynamics of D-brane solitons, vortex strings stretched between domain walls.

The other is to use a charged particle approximation of solitons and a domain wall effective action. It was suggested by Manton that monopoles can be regarded as particles with magnetic and scalar charges [23,30]. It was used to derive an asymptotic metric on the moduli space of well-separated BPS monopoles [31]. On the other hand, the effective action on a single domain wall is a free Lagrangian of the $U(1)$ Nambu-Goldstone zero mode and the translational zero mode. This $U(1)$ zero mode can be dualized to a $U(1)$ gauge field in the $2 + 1$ dimensional world volume of the wall [3], then the effective Lagrangian becomes a dual $U(1)$ gauge theory plus one neutral scalar field. It has been found by Shifman and Yung [7] that string end points can be regarded as electrically charged particles in a dual $U(1)$ gauge theory of the domain wall effective action.

We generalize this discussion to N parallel domain walls. As the effective theory of well-separated N domain walls, we propose $U(1)^N$ gauge theory and N scalar fields corresponding to wall positions. Then we use the particle approximation for end points of strings on the domain walls. By comparing the moduli metric derived by the moduli approximation for the full $1/4$ BPS configurations, we find a good agreement in the asymptotic metric.

This is instructive for clarifying the similarity or difference between D-branes and field theory solitons. The BPS monopoles can be realized as a D1–D3 bound state, D1-branes stretched between separated D3-branes. The end points of the D1-branes at the D3-branes can be regarded as BPS monopoles in the D3-brane effective action [32]. The monopole (or D1-brane) dynamics by the particle approximation in the D3-brane effective theory is parallel to our second derivation of the vortex-string dynamics as the charged particles in the domain wall effective theory. The only difference is the number of codimensions of string end points, which is three for D1–D3 and two for vortex strings on walls. In fact our asymptotic metric is similar to that of monopoles [30,31] by replacing $1/r$ by $\log r$, where r is the distance between solitons. However, there exists a crucial difference when N host branes (D3-branes or walls) coincide. The effective theory of D3-branes is in fact $U(N)$ gauge theory with several adjoint Higgs fields, reducing to the $U(1)^N$ gauge group only when eigenvalues of the adjoint Higgs field (positions of N D3-branes) are different from each other. In contrast to this, our effective theory on domain walls does not become the $U(N)$ gauge theory even when domain walls coincide.[1]

This paper is organized as follows. In Sec. II A we briefly explain the $1/4$ BPS equations in the $U(N_C)$ gauge theory with N_F hypermultiplets. In Sec. II B we review $1/4$ BPS wall-vortex systems in the $U(1)$ gauge theory. In Sec. III we first construct a general form of the effective Lagrangian of $1/4$ BPS solitons in the $U(N_C)$ gauge theory by applying the method to obtain a manifestly supersymmetric effective action on BPS solitons [36]. Next we use it to examine normalizability of zero modes of $1/4$ BPS wall-vortex systems in the $U(1)$ gauge theory. Here we assume that each vacuum region between two adjacent domain walls has the same number of vortices. It is easy to see that zero modes related to domain walls or vortices with infinite lengths are non-normalizable. We find the center of mass of vortex strings in each vacuum region is also non-normalizable, and the other zero modes are normalizable. In Sec. IV we give examples of (1, 1, 1), (2, 2, 2), (0, 2, 0), and $(n, 0, n)$ in the $U(1)$ gauge theory with three flavors admitting two domain walls. Here (n_1, n_2, n_3) represent configurations in which n_1 and n_3 strings end on the left (right) domain wall from outside and n_2 strings are stretched between the domain walls, see Fig. 2 below. We obtain the effective Lagrangian explicitly by using the exact solutions in the strong coupling limit. The effective Lagrangian is given as a nonlinear sigma model on the moduli space. In the case of (1, 1, 1), we find the position of the vortex living in the middle vacuum is non-normalizable. We give the physical explanation of the divergence in the effective Lagrangian. In the cases of (2, 2, 2) and (0, 2, 0), the relative positions of vortices in the middle region gives normalizable modes and we find the 90° scattering for the head-on collision of those vortices. Metrics of both configurations can be expressed in terms of

[1]If we consider domain walls with degenerate masses for Higgs scalar fields, we have $U(N)$ Nambu-Goldstone modes for coincident domain walls [33–35]. Taking a duality has been achieved only for $3 + 1$-dimensional wall world volume, where dual fields are non-Abelian two-form fields rather than Yang-Mills fields [35].

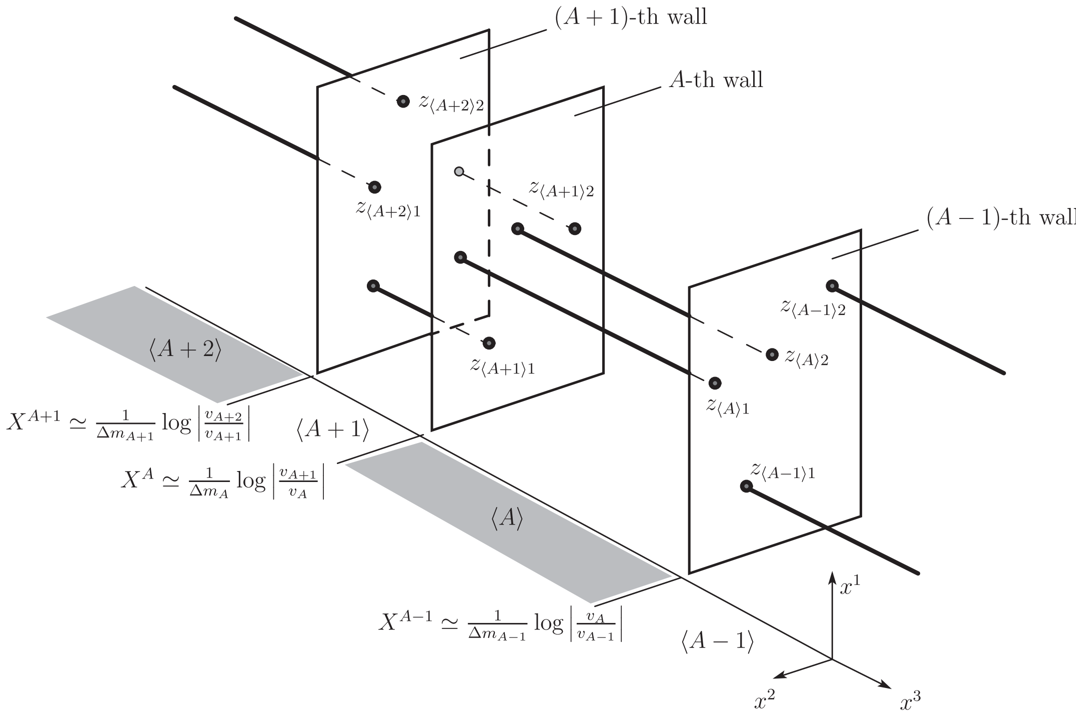

FIG. 2. An example of the D-brane soliton $(\cdots, n_{A-1}, n_A, n_{A+1}, n_{A+2}, \cdots) = (\cdots, 2, 2, 2, 2, \cdots)$ in the $U(1)$ gauge theory.

hypergeometric functions. The $(n, 0, n)$ example is a bit strange in the sense that no vortices can move. Since the region of the middle vacuum is finite in this case, there exists a normalizable moduli parameter for the size of the middle region. We obtain the effective Lagrangian for the modulus. In Sec. V we obtain vortex-string dynamics from a dual effective theory on domain walls. A dual effective theory on N well-separated domain walls is the $U(1)^N$ gauge theory with N real scalar fields parametrizing the wall positions, and end points of the vortex strings can be viewed as particles with scalar charges and electric charges. We obtain a general effective Lagrangian which describes dynamics of charged particles. We find a good agreement to the results obtained by the moduli space approximation of full 1/4 BPS configurations. Section VI is devoted to a conclusion and discussion. In Appendix A, we evaluate the Kähler metrics for (2, 2, 2) and (0, 2, 0) configurations. In Appendix B, the asymptotic Kähler metrics are examined. In Appendix C, we show that the asymptotic metric obtained in Sec. V is Kähler by writing down the Kähler potential explicitly. In Appendix D, we discuss the dual effective theory on multiple domain walls.

II. COMPOSITE SOLITONS OF WALLS, VORTICES, AND MONOPOLES

A. BPS equations and their solutions

Let us here briefly present our model admitting the 1/4 BPS composite solitons of domain walls, vortices, and monopoles (see [16] for a review). Our model is a 3 + 1 dimensional $\mathcal{N} = 2$ supersymmetric $U(N_C)$ gauge theory with $N_F(>N_C)$ massive hypermultiplets in the fundamental representation. The bosonic components in the vector multiplet are gauge fields $W_M (M = 0, 1, 2, 3)$, the two real adjoint scalar fields $\Sigma_\alpha (\alpha = 1, 2)$, and those in the hypermultiplet are the $SU(2)_R$ doublets of the complex scalar fields $H^i (i = 1, 2)$, which we express as $N_C \times N_F$ matrices. The bosonic part of the Lagrangian is given by

$$\mathcal{L} = \mathrm{Tr}\Big[-\frac{1}{2g^2} F_{MN}F^{MN} + \frac{1}{g^2}\mathcal{D}_M\Sigma_\alpha \mathcal{D}^M\Sigma^\alpha + \mathcal{D}_M H^i(\mathcal{D}^M H^i)^\dagger\Big] - V, \tag{2.1}$$

$$V = \mathrm{Tr}\Big[\frac{1}{g^2}\sum_{a=1}^{3}(Y^a)^2 + (H^iM - \Sigma_1 H^i)(H^iM - \Sigma_1 H^i)^\dagger + \Sigma_2 H^i(\Sigma_2 H^i)^\dagger - \frac{1}{g^2}[\Sigma_1, \Sigma_2]^2\Big], \tag{2.2}$$

where g is a $U(N)$ gauge coupling constant, and we have defined $Y^a \equiv \frac{g^2}{2}(c^a \mathbf{1}_{N_C} - (\sigma^a)^j_i H^i(H^j)^\dagger)$ with c^a an $SU(2)_R$ triplet of the Fayet-Iliopoulos (FI) parameters. In the following, we choose the FI parameters as $c^a = (0, 0, c > 0)$ by using $SU(2)_R$ rotation without loss of generality. We use the space-time metric $\eta_{MN} = \mathrm{diag}(+1, -1, -1, -1)$, the covariant derivatives are defined as $\mathcal{D}_M\Sigma_\alpha = \partial_M\Sigma_\alpha + i[W_M, \Sigma_\alpha]$, $\mathcal{D}_M H^i = (\partial_M + iW_M)H^i$, and the field strength is defined as $F_{MN} = -i[\mathcal{D}_M, \mathcal{D}_N] = \partial_M W_N - \partial_N W_M + i[W_M, W_N]$. M is a real $N_F \times N_F$ diagonal mass matrix, $M = \mathrm{diag}(m_1, m_2, \cdots, m_{N_F})$. In this paper we consider nondegenerate real masses, chosen as $m_1 > m_2 > \cdots > m_{N_F}$.

If we turn off all the mass parameters, the moduli space of vacua is the cotangent bundle over the complex Grassmannian $T^*Gr_{N_F,N_C}$ [37]. Once the mass parameters $m_A(A = 1, \cdots, N_F)$ are turned on and chosen to be fully nondegenerate ($m_A \neq m_B$ for $A \neq B$), almost all points of the vacuum manifold are lifted and only $N_F!/[N_C!(N_F - N_C)!]$ discrete points on the base manifold Gr_{N_F,N_C} are left to be the supersymmetric vacua [38]. Each vacuum is characterized by a set of N_C different indices $\langle A_1 \cdots A_{N_C}\rangle$ such that $1 \leq A_1 < \cdots < A_{N_C} \leq N_F$. In these discrete vacua, the vacuum expectation values are determined as

$$\langle H^{1rA}\rangle = \sqrt{c}\delta^{A_r}{}_A, \qquad \langle H^{2rA}\rangle = 0,$$
$$\langle \Sigma_1\rangle = \mathrm{diag}(m_{A_1}, \cdots, m_{A_{N_C}}), \qquad \langle \Sigma_2\rangle = 0, \tag{2.3}$$

where the color index r runs from 1 to N_C, and the flavor index A runs from 1 to N_F.

The 1/4 BPS equations for composite solitons of walls, vortices, and monopoles can be obtained by the usual Bogomol'ny completion of the energy density [9,16,17,19] as

$$\mathcal{D}_2\Sigma - F_{31} = 0, \qquad \mathcal{D}_1\Sigma - F_{23} = 0, \tag{2.4}$$

$$\mathcal{D}_3\Sigma - F_{12} - \frac{g^2}{2}(c\mathbf{1}_{N_C} - HH^\dagger) = 0, \tag{2.5}$$

$$\mathcal{D}_1 H + i\mathcal{D}_2 H = 0, \qquad \mathcal{D}_3 H + \Sigma H - HM = 0, \tag{2.6}$$

where $H \equiv H^1$, $\Sigma \equiv \Sigma_1$, and H^2, Σ_2 have been suppressed since they do not contribute to soliton solutions for $c > 0$. These equations describe composite solitons consisting of monopoles, vortices with codimensions in the $z \equiv x^1 + ix^2$ plane, and walls perpendicular to the x_3 direction.[2] The Bogomol'ny bound for the energy density $\mathcal{E}$ is given as

$$\mathcal{E} \geq t_w + t_v + t_m + \partial_m J_m. \tag{2.7}$$

Here t_w, t_v, t_m are the energy densities for walls, vortices, and monopoles, respectively, given by

$$t_w = c\partial_3 \mathrm{Tr}\,\Sigma, \qquad t_v = -c\,\mathrm{Tr}\,B_3,$$
$$t_m = \frac{2}{g^2}\partial_m \mathrm{Tr}(\Sigma B_m), \tag{2.8}$$

where $B_m = \frac{1}{2}\epsilon_{mnl}F_{nl}(m, n, l = 1, 2, 3)$. The monopole charge t_m can be either positive or negative, corresponding to monopoles and boojums, respectively. The last term in Eq. (2.7) containing $J_m(m = 1, 2, 3)$, which are defined by

$$J_1 \equiv \mathrm{Re}(-i\,\mathrm{Tr}(H^\dagger \mathcal{D}_2 H)),$$
$$J_2 \equiv \mathrm{Re}(i\,\mathrm{Tr}(H^\dagger \mathcal{D}_1 H)), \tag{2.9}$$
$$J_3 \equiv -\mathrm{Tr}(H^\dagger(\Sigma - M)H),$$

is a correction term which does not contribute to the total energy.

Since Eqs. (2.4), which are equivalent to $[\mathcal{D}_1 + i\mathcal{D}_2, \mathcal{D}_3 + \Sigma] = 0$, provide the integrability condition for the operators $\mathcal{D}_1 + i\mathcal{D}_2$ and $\mathcal{D}_3 + \Sigma$, we can introduce an $N_C \times N_C$ invertible complex matrix function $S(z, \bar{z}, x_3) \in GL(N_C, \mathbf{C})$ defined by [9]

$$\Sigma + iW_3 \equiv S^{-1}\partial_3 S, \qquad [(\mathcal{D}_3 + \Sigma)S^{-1} = 0], \tag{2.10}$$

$$W_1 + iW_2 \equiv -2iS^{-1}\bar{\partial}S, \qquad [(\mathcal{D}_1 + i\mathcal{D}_2)S^{-1} = 0], \tag{2.11}$$

with $\bar{\partial} \equiv \partial/\partial\bar{z}$. With the form of Eqs. (2.10) and (2.11), Eq. (2.4) is satisfied and Eq. (2.6) is solved by

$$H = S^{-1}(z, \bar{z}, x_3)H_0(z)e^{Mx_3}. \tag{2.12}$$

Here $H_0(z)$ is an $N_C \times N_F$ matrix whose elements are arbitrary holomorphic functions of z. We call it the "moduli matrix" since it contains all the moduli parameters of solutions as we will see shortly. Let us define an $N_C \times N_C$ Hermitian matrix

$$\Omega \equiv SS^\dagger, \tag{2.13}$$

invariant under the $U(N_C)$ gauge transformations. The remaining BPS equation (2.5) can be rewritten in terms of Ω as [9]

$$\frac{1}{g^2 c}[4\partial_z(\Omega^{-1}\bar{\partial}_z\Omega) + \partial_3(\Omega^{-1}\partial_3\Omega)] = \mathbf{1}_{N_C} - \Omega^{-1}\Omega_0, \tag{2.14}$$

$$\Omega_0 \equiv \frac{1}{c}H_0 e^{2Mx_3}H_0^\dagger. \tag{2.15}$$

This equation is called the master equation for the wall-vortex-monopole system. This reduces to the master equation for the 1/2 BPS domain walls if we omit the z dependence ($\partial_z = \partial_{\bar{z}} = 0$) and for the 1/2 BPS vortices if we omit the x_3 dependence ($\partial_3 = 0$) and set $M = 0$. It determines S for a given moduli matrix H_0 up to the gauge symmetry $S \to SU^\dagger$, $U \in U(N_C)$ and then the physical fields can be obtained through Eqs. (2.10), (2.11), and (2.12). The master equation (2.14) has a symmetry which we call "V transformations"

$$H_0(z) \to V(z)H_0(z), \qquad S(z, \bar{z}, x_3) \to V(z)S(z, \bar{z}, x_3), \tag{2.16}$$

where $V(z) \in GL(N_C, \mathbf{C})$ has components holomorphic

[2]When there exists a flux on a domain wall world volume, the domain wall is tilted [9] resembling noncommutative monopoles.

with respect to z. The moduli matrices related by this V transformation are physically equivalent $H_0(z) \sim V(z)H_0(z)$ since they do not change the physical fields. Therefore the total moduli space of this system, defined by all topological sectors patched together, is given by a set of the whole holomorphic matrix $H_0(z)$ divided by the equivalence relation $H_0(z) \sim V(z)H_0(z)$. Therefore the parameters contained in the moduli matrix $H_0(z)$ after fixing the redundancy of the V transformation can be interpreted as the moduli parameters, namely, the coordinates of the moduli space of the BPS configurations.

B. Composite solitons of vortices and domain walls

The moduli matrix offers a powerful tool to study the moduli space of the $1/4$ BPS composite solitons, because it exhausts all possible BPS configurations. In this paper we study the dynamics in the Abelian-Higgs model with $N_{\rm F}(\geq 2)$ flavors, in which the moduli matrix is an $N_{\rm F}$ vector. To this end we summarize here how the moduli matrix represents (1) the SUSY vacua, (2) $1/2$ BPS domain walls, (3) $1/2$ BPS vortices, and (4) $1/4$ BPS composite states.

(1) $N_{\rm F}$ discrete SUSY vacua. In the Ath vacuum $\langle A\rangle$ $(A = 1, 2, \cdots, N_{\rm F})$, only the Ath element is nonzero with the rest being zero,

$$\langle A\rangle\text{: } H_0 = (0, \cdots, 0, 1, 0, \cdots, 0). \tag{2.17}$$

(2) $N_{\rm F} - 1$ multiple $1/2$ BPS domain walls. When the Ath and the Bth elements $(A > B)$ and the elements between them are nonzero constants in H_0, it represents $A - B$ multiple domain walls interpolating between two vacua $\langle A\rangle$ and $\langle B\rangle$. The most general configurations are obtained when all the elements are nonzero constants:

$$H_0 = (h_1, h_2, \cdots, h_{N_{\rm F}}), \qquad h_A \in \mathbf{C}. \tag{2.18}$$

If some of the $\{h_A\}$ vanish, the corresponding vacua disappear. Namely, the domain walls adjacent to the vacua collapse. In order to estimate the position of the domain wall interpolating between $\langle A\rangle$ and $\langle A + 1\rangle$, let us define the weight of the vacuum $\langle A\rangle$ by

$$\exp(\mathcal{W}^{\langle A\rangle}) \equiv h_A e^{m_A x_3}. \tag{2.19}$$

In the region where only one of the weights $\exp(\mathcal{W}^{\langle A\rangle})$ is large, the solution of the master equation (2.14) is approximately given by $\Omega \approx \exp(\mathcal{W}^{\langle A\rangle})$. In such regions, the energy density of the domain wall $t_{\rm w}$ vanishes since it is given by $t_{\rm w} = \frac{c}{2}\partial_3^2 \log\Omega$. The Ath domain wall exists where the weights $\exp(\mathcal{W}^{\langle A\rangle})$ and $\exp(\mathcal{W}^{\langle A+1\rangle})$ of the two vacua $\langle A\rangle$ and $\langle A + 1\rangle$ are balanced. Its position $x_3 = X^A$ can be estimated by

$$\Delta m_A X^A + i\sigma^A \simeq \log\left(\frac{h_{A+1}}{h_A}\right), \tag{2.20}$$

with $\Delta m_A \equiv m_A - m_{A+1} > 0$. Here the imaginary part σ^A represents an associated phase modulus of the wall. The tension of this domain wall is $c\Delta m_A$.

(3) $1/2$ BPS vortices in the vacuum $\langle A\rangle$. When only the Ath element $h_A(z)$ of H_0 is a polynomial function of z with the rest zero,

$$H_0 = (0, \cdots, 0, h_A(z), 0, \cdots, 0), \tag{2.21}$$

$$h_A(z) = v_A(z - z_{\langle A\rangle 1})(z - z_{\langle A\rangle 2}) \cdots (z - z_{\langle A\rangle k_A}), \tag{2.22}$$

it represents multiple vortices in the vacuum $\langle A\rangle$ extending to infinity $(x_3 \to \pm\infty)$. The degree of the polynomials $n_A = \deg[h_A(z)]$ is identical to the number of the vortices in the vacuum $\langle A\rangle$, and the zeros $z_{\langle A\rangle 1}, \cdots, z_{\langle A\rangle k_A}$ of $h_A(z)$ represent the vortex positions. We call the infinitely long straight vortices, generated by the above moduli matrix, the ANO vortices. The tension of each vortex is $2\pi c$, and its transverse size is of the order $1/(g\sqrt{c})$. The ANO vortex becomes singular in the strong gauge coupling limit $g \to \infty$.

(4) $1/4$ BPS states (D-brane solitons). The most general composite states of vortex strings ending on (or stretched between) domain walls are given by the moduli matrix

$$H_0 = (h_1(z), h_2(z), \cdots, h_{N_{\rm F}}(z)), \tag{2.23}$$

where the Ath element $h_A(z)$ is of the form of Eq. (2.22) with the degree n_A. Here n_A vortices exist in the vacuum $\langle A\rangle$ and are suspended between $(A - 1)$th and the Ath domain walls for $A \neq 1, N_{\rm F}$, or ending on the first or the $(N_{\rm F} - 1)$th domain wall for $A = 1, N_{\rm F}$. We denote such a D-brane soliton by $(n_1, n_2, \cdots, n_{N_{\rm F}})$; see Fig. 2.

Let us define a z-dependent generalization of the weight of the vacuum $\langle A\rangle$ by

$$\exp(\mathcal{W}^{\langle A\rangle}(z)) \equiv h_A(z)e^{m_A x_3}. \tag{2.24}$$

Domain walls are curved and their positions in the x_3 direction depend on z in general. The z-dependent position $X^A(z, \bar{z})$ of the Ath domain wall and its associated (z-dependent) phase $\sigma(z, \bar{z})$ can be estimated by equating two weights (2.24), to yield

$$\Delta m_A X^A(z, \bar{z}) + i\sigma^A(z, \bar{z}) \simeq \log\left(\frac{h_{A+1}(z)}{h_A(z)}\right). \tag{2.25}$$

One can quickly see an approximate configuration from this rough estimation. Exact solutions and the approximations (2.25) are compared in Fig. 3. If a vortex ends on a domain wall, it pulls the domain wall towards its direction to give the logarithmic bending of the domain wall (2.25).

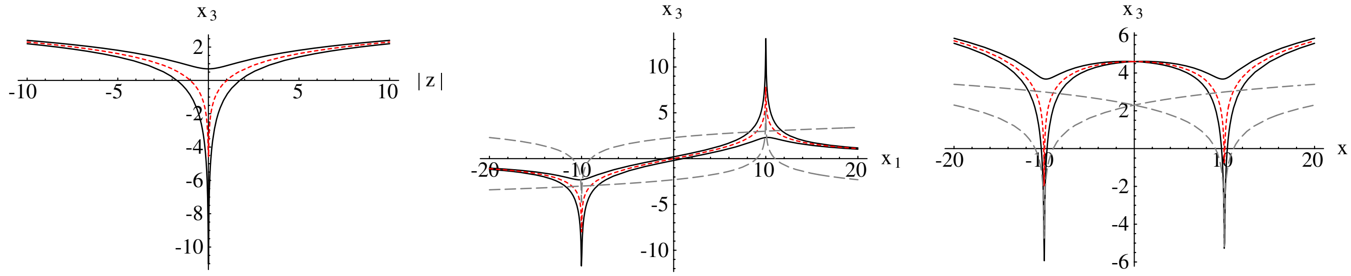

FIG. 3 (color online). The {left, middle, right} panel shows {(0, 1), (1, 1), (0, 2)} configuration, respectively. The solid line is the exact solution for the equal energy density contour of $\mathcal{E} = 1/2$ while the broken line is an approximate curve given by Eq. (2.25). Gray long-dashed lines correspond to the configurations that one of two vortices is removed away to infinity. The parameters are chosen to be $c = 1$ and $M = \mathrm{diag}(1/2, -1/2)$.

When the same number of vortices end on the domain wall from both sides, that is, $n_A = n_{A+1}$, then the domain wall is asymptotically flat

$$\Delta m_A X^A(z, \bar{z}) + i\sigma^A(z, \bar{z}) \to \log\left(\frac{v_{A+1}}{v_A}\right), \quad \text{as } |z| \to \infty. \tag{2.26}$$

The correction terms of order $\log z$ correspond to deformation by the vortices ending on the domain walls. We can read the deformation near the ith vortex at $z = z_{\langle A\rangle i}$ in the vacuum $\langle A\rangle$, which ends on the Ath domain wall from the right

$$\begin{aligned}\Delta m_A X^A(z_{\langle A\rangle i}, \bar{z}_{\langle A\rangle i}) \simeq & -\log\epsilon + \log\left|\frac{v_{A+1}}{v_A}\right| \\ & + \sum_j \log|z_{\langle A\rangle i} - z_{\langle A+1\rangle j}| \\ & - \sum_{j\neq i} \log|z_{\langle A\rangle i} - z_{\langle A\rangle j}|.\end{aligned} \tag{2.27}$$

Here the first term with $0 < \epsilon \ll 1$ (UV cutoff) comes from the vortex at $z = z_{\langle A\rangle i}$ and the second term represents the host Ath domain wall. The third and the fourth terms correspond to the deformation by the rest of the vortices. The deformation by the rest of the vortices ending from the same side make the vortex longer while that by the other vortices ending from the opposite side shorten the vortex; see Fig. 3.

Furthermore, one can estimate the transverse size of vortices as follows. If we look at the region sufficiently away from the domain walls, we can ignore x_3 dependence of the configurations. In such a region one can take a slice with fixed x_3 and can regard the configuration as semilocal vortices in 2 + 1 dimensions. The configuration is determined by Ω_0 typically taking the form of

$$\Omega_0 = |z - z_0(x_3)|^2 + |a(x_3)|^2, \tag{2.28}$$

up to an overall factor independent of z. Here z_0 stands for the position and $|a|$ for the transverse size of the semilocal vortex. Let us show two concrete examples of the D-brane soliton $(n_1, \cdots, n_{N_F})$.

(i) $(n_1, n_2) = (0, 1)$ case: For instance, we consider the mass matrix $M = \mathrm{diag}(m/2, -m/2)$ with the moduli matrix $H_0 = (1, z - z_0)$ and obtain

$$\begin{aligned}\Omega_0 &= e^{mx_3} + |z - z_0|^2 e^{-mx_3} \\ &= e^{-mx_3}(|z - z_0|^2 + e^{2mx_3}).\end{aligned} \tag{2.29}$$

The (z-dependent) transverse size can be read as $|a(x_3)| = e^{mx_3}$; see Fig. 3 (left-most).

(ii) $(n_1, n_2, n_3) = (1, 1, 1)$ case: For instance, we consider the mass matrix $M = \mathrm{diag}(m/2, 0, -m/2)$ with the moduli matrix $H_0 = (z - z_1, e^{ml/4}(z + z_1), z - z_1)$ for the coincident outer vortices

$$\begin{aligned}\Omega_0 &= |z - z_1|^2 e^{mx_3} + e^{ml/2}|z + z_1|^2 \\ &\quad + |z - z_1|^2 e^{-mx_3} \\ &= 2|z - z_1|^2 \cosh mx_3 + e^{ml/2}|z + z_1|^2 \\ &= (A + B)\left[\left|z - \frac{A - B}{A + B} z_1\right|^2 \right. \\ &\quad \left. + \left(1 - \left(\frac{A - B}{A + B}\right)^2\right)|z_1|^2\right],\end{aligned} \tag{2.30}$$

with $A \equiv 2\cosh mx_3$ and $B \equiv e^{ml/2}$. The physical meaning of l is the distance between two domain walls. The (z-dependent) position of the vortex is given by $z_0(x_3) = \frac{A-B}{A+B} z_1$ and their sizes are by $|a(x_3)| = |z_1|\sqrt{1 - (\frac{A-B}{A+B})^2}$. Therefore, one can see that $z_0(x_3 \to \pm\infty) \to z_1$ and the position of the middle vortex is $z_0(x_3 = 0) = -z_1$ if $l \gg 1$. The size of the outer vortices reduces $2|z_1|\sqrt{\frac{B}{A}} \sim 2|z_1|e^{(m/2)((l/2)-|x_3|)} \to 0$ as $|x_3| \to \infty$. On the other hand, if the separation of the two walls are sufficiently large so that $B \gg A$, the size is estimated by $2|z_1|\sqrt{\frac{A}{B}} \sim 2|z_1|e^{-(m/2)((l/2)-|x_3|)}$. Thus the center of the middle vortex ($x_3 = 0$) has the smallest size $|a(x_3 = 0)| = 2|z_1|e^{-ml/2}$. It is exponentially small with respect to the wall distance l, but still finite.

However, the size becomes zero when all the vortices are coincident ($z_1 = 0$).

The vortices ending on the domain walls are not the usual ANO vortices. They are also deformed by domain walls and their transverse sizes are not constant along x_3 anymore. The sizes depend on the positions of the vortices. The ANO vortices appear at $z = z_1, z_2, \cdots, z_{k'}$ only when all elements of the moduli matrix have common zeros as

$$H_0(z) = (z - z_1)(z - z_2)\cdots(z - z_{k'}) \times H_0'(z), \quad (2.31)$$

without poles in $H_0'(z)$. We call such a moduli matrix factorizable. In the strong gauge coupling limit, these ANO vortices become singular. On the other hand, the other general vortices ending on (stretched between) domain walls remain regular with finite transverse sizes in the strong coupling limit.

III. EFFECTIVE LAGRANGIAN OF 1/4 BPS WALL-VORTEX SYSTEMS

A. General form of the effective Lagrangian

Now let us construct the effective Lagrangian of the full 1/4 BPS composite solitons.[3] Zero modes on the background BPS solutions will play a main role in the effective theory while all the massive modes will be ignored in the following. As we will see shortly, the composite solitons have normalizable zero modes and non-normalizable zero modes. Only the normalizable zero modes can be promoted to dynamical degrees of freedom in the effective theory. In this subsection we construct a formal form of the effective Lagrangian without identifying which zero modes are normalizable. We will discuss the problem of the normalizability in the next subsection by extending our method to obtain a manifestly supersymmetric effective action on BPS solitons [36].

If there are normalizable zero modes ϕ^i, we can give them weak dependence on time (slow-move approximation *à la* Manton [22,23])

$$H_0(\phi^i) \to H_0(\phi^i(t)). \quad (3.1)$$

We introduce "the slow-movement order parameter" λ, which is assumed to be much smaller than the other typical mass scales in the problem. There are two characteristic mass scales: one is the mass difference $|\Delta m|$ of hypermultiplets, and the other is $g\sqrt{c}$ in front of the master equation. Therefore, we assume that

$$\lambda \ll \min(|\Delta m|, g\sqrt{c}). \quad (3.2)$$

The nonvanishing fields in the 1/4 BPS background have contributions independent of λ, namely, we assume that

$$H^1 = \mathcal{O}(1), \qquad W_m = \mathcal{O}(1), \qquad \Sigma_1 = \mathcal{O}(1). \quad (3.3)$$

The derivatives of these fields with respect to time are assumed to be of order λ expressing the weak dependence on time. The vanishing fields in the background can now have nonvanishing values, induced by the fluctuations of the moduli parameters of order λ. Therefore, we assume that

$$\partial_0 = \mathcal{O}(\lambda), \qquad W_0 = \mathcal{O}(\lambda),$$
$$H^2 = \mathcal{O}(\lambda), \qquad \Sigma_2 = \mathcal{O}(\lambda). \quad (3.4)$$

Then the covariant derivative $\mathcal{D}_0 = \mathcal{O}(\lambda)$ has consistent λ dependence.

If we expand the full equations of motion of the Lagrangian (2.1) in powers of λ, we find that the $\mathcal{O}(1)$ equations are automatically satisfied due to the BPS equations (2.4), (2.5), and (2.6). The next leading $\mathcal{O}(\lambda)$ equation is the equation for W_0, which is called the Gauss law

$$0 = -\frac{2}{g^2}\mathcal{D}_m F_{m0} + \frac{2i}{g^2}[\Sigma_1, \mathcal{D}_0\Sigma_1] + i(H^1\mathcal{D}_0 H^{1\dagger} - \mathcal{D}_0 H^1 H^{1\dagger}), \quad (3.5)$$

with $m = 1, 2, 3$. In order to obtain the effective Lagrangian of order λ^2 of the composite solitons, we have to solve this equation and determine the configuration of W_0.

As a consequence of Eq. (3.1), the moduli matrix $H_0(\phi^i)$ depends on time through the time-dependent moduli parameter $\phi^i(t)$. Note that for the fields which depend on time only through $\phi^i(t)$, the derivatives with respect to time satisfy

$$\partial_0 = \delta_0 + \delta_0^\dagger, \quad (3.6)$$

where we have defined the differential operators δ_0 and $\delta_0^\dagger$ by

$$\delta_0 \equiv \sum_i \partial_0\phi^i \frac{\partial}{\partial\phi^i}, \qquad \delta_0^\dagger \equiv \sum_i \partial_0\bar{\phi}^i \frac{\partial}{\partial\bar{\phi}^i}, \quad (3.7)$$

in order to distinguish chiral ϕ^i and antichiral $\bar{\phi}^i$ multiplets of preserved supersymmetry. Using these operators, the Gauss law (3.5) can be solved to yield [36]

$$W_0 = i(\delta_0 S^\dagger S^{\dagger-1} - S^{-1}\delta_0^\dagger S). \quad (3.8)$$

The effective Lagrangian is obtained by substituting these solutions into the fundamental Lagrangian (2.1) and integrating over the codimensional coordinates x^1, x^2, and x_3. We retain the terms up to $\mathcal{O}(\lambda^2)$ since we are interested in the leading nontrivial part in powers of λ, and we ignore total derivative terms which do not contribute to the effective Lagrangian. Then the effective Lagrangian for the composite solitons can be obtained as

$$L_{\rm eff} = K_{i\bar{j}}\frac{d\phi^i}{dt}\frac{d\bar{\phi}^j}{dt}, \quad (3.9)$$

[3] Our main interest in this paper is the dynamics of vortices between domain walls in the Abelian gauge theory. However, the general formula obtained in this section can be applied to other composite solitons in non-Abelian gauge theory.

$$K_{i\bar{j}} = \int d^3x \,\mathrm{Tr}\Big[\partial_i\partial_{\bar{j}}\Big(c\log\Omega + \frac{1}{2g^2}(\Omega^{-1}\partial_3\Omega)^2\Big) + \frac{4}{g^2}\{\partial_{\bar{z}}(\Omega\partial_i\Omega^{-1})\partial_{\bar{j}}(\Omega\partial_z\Omega^{-1}) - \partial_{\bar{z}}(\Omega\partial_z\Omega^{-1})\partial_{\bar{j}}(\Omega\partial_i\Omega^{-1})\}\Big]. \tag{3.10}$$

This is a nonlinear sigma model whose target space is the moduli space of the $1/4$ BPS configurations. The metric on the moduli space is a Kähler metric, which can be obtained from the following Kähler potential[4]

$$K = \int d^3x \,\mathrm{Tr}\Big[c\psi + ce^{-\psi}\Omega_0 + \frac{1}{2g^2}(e^{-\psi}\partial_3 e^{\psi})^2 + \frac{4}{g^2}\int_0^1 dt\int_0^t ds\bar{\partial}\psi e^{-sL_\psi}\partial\psi\Big], \tag{3.11}$$

where $\psi \equiv \log\Omega$ and the operation L_ψ is defined by

$$L_\psi \times X = [\psi, X]. \tag{3.12}$$

This general form reduces to the effective Lagrangian for either $1/2$ BPS domain walls or $1/2$ BPS vortices if one considers the moduli matrix of the corresponding $1/2$ BPS states.

B. Normalizability of zero modes

The moduli matrix contains both normalizable and non-normalizable zero modes because it exhausts all possible configurations. There exist two kinds of zero modes appearing in the moduli matrix as we saw in the previous section. One is related to positions and phases of domain walls which form complex numbers. The other represents the positions of vortices. In general, zero modes changing the boundary conditions at infinities are non-normalizable. For example, zero modes related to domain walls or vortices with infinite lengths are apparently non-normalizable because the infinite extent of the solitons brings divergence in the integration. However, the opposite is not true; zero modes fixing the boundary conditions are not always normalizable but sometimes are non-normalizable. The purpose of this section is to examine if zero modes for vortices with finite lengths stretched between domain walls are normalizable or not.

Now let us analyze the divergences of the Kähler potential (3.11) in order to find out which modes are normalizable and which are not. Since the solutions of the master equation (2.14) have been assumed to be smooth, the divergences of the Kähler potential can appear only from the integration around the boundaries at infinity. The composite solitons have two kinds of boundaries: one is along $|z| \to \infty$ where we see no vortices, and the other is along $x_3 \to \pm\infty$ where we see no domain walls. We will discuss the behaviors of solutions near these two boundaries separately.

From now on, we consider Abelian gauge theories ($N_C = 1$) for simplicity. First let us consider the boundary along $|z| \to \infty$. The master equation (2.14) can be rewritten in terms of $\psi = \log\Omega$ as

$$(4\partial_z\partial_{\bar{z}} + \partial_3^2)\psi = g^2c(1 - e^{-\psi}\Omega_0). \tag{3.13}$$

For simplicity, we will assume all domain walls are asymptotically flat, that is, each vacuum has the same number of vortices. Let us denote the number of vortices as k. Then Ω_0 is given by

$$\Omega_0 \equiv \frac{1}{c}H_0 e^{2Mx_3}H_0^\dagger = \sum_{A=1}^{N_F}|h_A(z)|^2 e^{2m_A x_3} \tag{3.14}$$

$$h_A(z) = v_A(z - z_{\langle A\rangle 1})(z - z_{\langle A\rangle 2})\cdots(z - z_{\langle A\rangle k}) = v_A(z^k - a_A z^{k-1} + \cdots). \tag{3.15}$$

The moduli parameter v_A controls the weight of the vacuum $\langle A\rangle$ in Eq. (2.24), and thus is related to positions of the domain walls separating the vacuum $\langle A\rangle$ from the adjacent vacua. The moduli parameters a_A are related to the center of mass Z_A^c of vortices in the vacuum $\langle A\rangle$ as

$$Z_A^c \equiv \frac{z_{\langle A\rangle 1} + \cdots + z_{\langle A\rangle k}}{k} = \frac{a_A}{k}. \tag{3.16}$$

Let us introduce new functions defined as

$$\tilde{h}_A(z) \equiv \frac{h_A(z)}{z^k}, \tag{3.17}$$

$$\tilde{\Omega}_0(z, \bar{z}, x_3) \equiv \frac{\Omega_0(z, \bar{z}, x_3)}{|z|^{2k}}, \tag{3.18}$$

$$\tilde{\psi}(z, \bar{z}, x_3) \equiv \psi(z, \bar{z}, x_3) - \log|z|^{2k}. \tag{3.19}$$

The master equation (3.13) does not change in terms of these functions except for the appearance of the delta function[5] in z. Since we are interested in the boundary along $|z| \to \infty$, let us ignore the delta function in the following discussion:

$$(4\partial_z\partial_{\bar{z}} + \partial_3^2)\tilde{\psi} = g^2c(1 - e^{-\tilde{\psi}}\tilde{\Omega}_0). \tag{3.20}$$

If we take the limit $|z| \to \infty$, $\tilde{\Omega}_0$ becomes

$$\tilde{\Omega}_0 \to \Omega_{0\mathrm{w}} \equiv \sum_{A=1}^{N_F}|v_A|^2 e^{2m_A x_3}, \tag{3.21}$$

[4]Although this Kähler potential is divergent, it can be made finite without changing the Kähler metric by the Kähler transformation, namely, by adding terms $f(x, \phi)$ and $\overline{f(x, \phi)}$ which are (anti)holomorphic with respect to the normalizable moduli parameters to the integrand of Eq. (3.11).

[5]This redefinition transforms our master equation to the so-called Taubes's equation [39] in the case of vortices without domain walls.

which is nothing but the source for domain walls without vortices. Therefore, the solution $\tilde{\psi}(z, \bar{z}, x_3)$ approaches the domain wall solution in the large $|z|$ region. We denote it as $\psi_{\rm w}(x_3)$

$$\tilde{\psi}(z, \bar{z}, x_3) \to \psi_{\rm w}(x_3) \quad \text{as } |z| \to \infty. \tag{3.22}$$

Now let us analyze the effects of vortices on the asymptotic behavior. Note that $\tilde{\Omega}_0$ can be expanded as

$$\tilde{\Omega}_0(z, \bar{z}, x_3) = \Omega_{0\rm w}(x_3) - \sum_{A=1}^{N_{\rm F}} |v_A|^2 \left(\frac{a_A}{z} + \frac{\bar{a}_A}{\bar{z}}\right) e^{2m_A x_3} + \mathcal{O}\left(\frac{1}{z^2}\right). \tag{3.23}$$

We will assume that $\tilde{\psi}$ can be also expanded as

$$\tilde{\psi}(z, \bar{z}, x_3) = \psi_{\rm w}(x_3) + \frac{\varphi(x_3)}{z} + \frac{\bar{\varphi}(x_3)}{\bar{z}} + \mathcal{O}\left(\frac{1}{z^2}\right). \tag{3.24}$$

If we substitute the asymptotic forms (3.23) and (3.24) into the master equation (3.20) and expand it in terms of z, we find that the $\mathcal{O}(1)$ equation gives the master equation for domain walls

$$\partial_3^2 \psi_{\rm w} = g^2 c(1 - e^{-\psi_{\rm w}} \Omega_{0\rm w}), \tag{3.25}$$

and the $\mathcal{O}(z^{-1})$ equation gives

$$\partial_3^2 \varphi = g^2 c\left(\Omega_{0\rm w} \varphi + \sum_{A}^{N_{\rm F}} |v_A|^2 a_A e^{2m_A x_3}\right) e^{-\psi_{\rm w}}. \tag{3.26}$$

Using Eq. (3.25), we can find the solution of Eq. (3.26) as

$$\varphi(x_3) = -\sum_{A=1}^{N_{\rm F}} v_A a_A \frac{\partial \psi_{\rm w}(x_3)}{\partial v_A}. \tag{3.27}$$

Let us now substitute these asymptotic behaviors (3.24) and (3.27) into the Kähler potential (3.11). Using the fact that the solution of the master equation is the extremum of the Kähler potential, we obtain

$$\begin{aligned} K &= \int d^2 z \left(K_{\rm w} + \frac{1}{|z|^2} \sum_{A,B=1}^{N_{\rm F}} (a_A v_A)(\bar{a}_B \bar{v}_B) \frac{\partial^2 K_{\rm w}}{\partial v_A \partial \bar{v}_B} + \mathcal{O}\left(\frac{1}{z^4}\right)\right) \\ &\simeq \pi L^2 K_{\rm w} + 2\pi \log L \sum_{A,B=1}^{N_{\rm F}} (a_A v_A)(\bar{a}_B \bar{v}_B) \frac{\partial^2 K_{\rm w}}{\partial v_A \partial \bar{v}_B} + \text{const} + \mathcal{O}(L^{-2}), \end{aligned} \tag{3.28}$$

where L is the infrared cutoff $|z| < L$, and $K_{\rm w}$ is the Kähler potential for domain walls

$$\begin{aligned} K_{\rm w}(v_A, \bar{v}_A) &= \int dx_3 \, \mathrm{Tr}\left[c\psi_{\rm w} + c e^{-\psi_{\rm w}} \Omega_{0\rm w} + \frac{1}{2g^2}(e^{-\psi_{\rm w}} \partial_3 e^{\psi_{\rm w}})^2\right] \\ &\approx \sum_{A=1}^{N_{\rm F}-1} \frac{c}{\Delta m_A} \left(\log \left|\frac{v_{A+1}}{v_A}\right|\right)^2, \end{aligned} \tag{3.29}$$

where the last line is valid for well-separated walls. From the Kähler potential equation (3.28) we find that the leading terms in the Kähler metric for the moduli parameters v_A are proportional to L^2 and diverge in the limit $L \to \infty$. Therefore moduli parameters v_A, which are contained in $\Omega_{0\rm w}$, correspond to non-normalizable zero modes. This is because the infinitely extended domain walls move with infinite kinetic energy when the parameters v_A vary. However, the above result says that the center of mass positions of vortices, a_A, in each vacuum are also non-normalizable even if the vortices have finite lengths. The divergent part of the effective Lagrangian associated with the motion of the parameters a_A is

$$\begin{aligned} &2\pi \log L \sum_{A,B=1}^{N_{\rm F}} \frac{da_A}{dt} \frac{d\bar{a}_B}{dt} v_A \bar{v}_B \frac{\partial^2 K_{\rm w}}{\partial v_A \partial \bar{v}_B} \\ &\approx \pi c \log L \sum_{A=1}^{N_{\rm F}-1} \frac{1}{\Delta m_A} \left|\frac{da_A}{dt} - \frac{da_{A+1}}{dt}\right|^2. \end{aligned} \tag{3.30}$$

The intuitive explanation is given as follows. In the presence of vortices, the positions of domain walls actually depend on the positions of vortices. For example, the position and the corresponding phase of the Ath domain wall interpolating the two vacua $\langle A \rangle$ and $\langle A+1 \rangle$ is given by

$$\Delta m_A X^A(z, \bar{z}) + i\sigma^A(z, \bar{z}) = \log\left(\frac{v_{A+1}}{v_A}\right) + \log\left(\frac{z^k - a_{A+1} z^{k-1} + \cdots}{z^k - a_A z^{k-1} + \cdots}\right). \tag{3.31}$$

Let us perturb the center of mass positions of vortices a_A and a_{A+1} with v_A and v_{A+1} fixed, to yield

$$\delta(\Delta m_A X^A + i\sigma^A) = \delta a_A \frac{z^{k-1}}{z^k - a_A z^{k-1} + \cdots} - \delta a_{A+1} \frac{z^{k-1}}{z^k - a_{A+1} z^{k-1} + \cdots}. \tag{3.32}$$

Therefore, if a_A is promoted to a dynamical degree of freedom to have the weak dependence on time, there appears kinetic energy of the domain wall with the tension $T_A \equiv c\Delta m_A$

$$\int d^2z \frac{T_A}{2}\left(\frac{dX^A}{dt}\right)^2 \approx \frac{\pi c}{2\Delta m_A} \log L \left|\frac{da_A}{dt} - \frac{da_{A+1}}{dt}\right|^2. \tag{3.33}$$

The same amount of kinetic energy appears from the phase σ^A of the domain wall. Thus the non-normalizability of a_A given in Eq. (3.30) can be understood as the divergent kinetic energy of domain walls.

Although the vortex dynamics is studied in this paper using the moduli space approximation, it is interesting to consider here what happens in the full field theory dynamics. As explained above, if we perturb the center of mass of the vortex system, it has to be accompanied by changes of positions of domain walls in both sides. This means that the zero mode of the center of mass of the vortex system is coupled to zero modes of domain walls. Consequently, if we give a finite energy to the zero mode of the center of mass of the vortex system, the energy is radiated away through zero modes of domain walls, and the center mass of the vortex system eventually settles down on an equilibrium position. Such a radiation of zero modes of domain walls cannot be dealt within the moduli space approximation. In the following section, we will assume using moduli space approximation that the zero mode of the center of mass of the vortex system is frozen.

Now let us consider the boundaries along $x_3 \to \pm\infty$ directions. As in the previous case, we define the following new functions for the limit $x_3 \to +\infty$:

$$\check{\Omega}_0(z, \bar{z}, x_3) \equiv \Omega_0(z, \bar{z}, x_3) e^{-2m_1 x_3} = \sum_{A=1}^{N_F} |h_A(z)|^2 e^{-2(m_1 - m_A)x_3}, \tag{3.34}$$

$$\check{\psi}(z, \bar{z}, x_3) \equiv \psi(z, \bar{z}, x_3) - 2m_1 x_3. \tag{3.35}$$

Note that we have chosen the mass parameters such that $m_1 > m_2 > \cdots > m_{N_F}$. The master equation (3.13) does not change in terms of these functions as before. If we take the limit $x_3 \to \infty$, $\check{\Omega}_0$ becomes

$$\check{\Omega}_0 \to \Omega_0^{\mathrm{v}} \equiv |h_1(z)|^2, \tag{3.36}$$

which is nothing but the source for vortices in vacuum $\langle 1 \rangle$. Therefore, the solution $\check{\psi}(z, \bar{z}, x_3)$ approaches the vortex solution in the large x_3 region, which we denote as $\psi_{\mathrm{v}}(z, \bar{z})$,

$$\check{\psi}(z, \bar{z}, x_3) \longrightarrow \psi_{\mathrm{v}}(z, \bar{z}) \quad \text{as } x_3 \to \infty, \tag{3.37}$$

$$4\partial_z \partial_{\bar{z}} \psi_{\mathrm{v}} = g^2 c(1 - e^{-\psi_{\mathrm{v}}} \Omega_0^{\mathrm{v}}). \tag{3.38}$$

We are interested in the effects of domain walls on the asymptotic behavior. Note that $\check{\Omega}_0$ behaves as

$$\check{\Omega}_0 = |h_1(z)|^2 + |h_2(z)|^2 e^{-2(m_1 - m_2)x_3} + \cdots \tag{3.39}$$

in the large x_3 region. The second term is strongly suppressed by the exponential factor in contrast to the previous case. The solution $\check{\psi}(z, \bar{z}, x_3)$ should also behave as

$$\check{\psi}(z, \bar{z}, x_3) = \psi_{\mathrm{v}}(z, \bar{z}) + \phi(z, \bar{z}) e^{-m_v x_3} + \mathcal{O}(e^{-2m_v x_3}), \tag{3.40}$$

where m_v stands for the lowest mass scale of the bulk modes in the right-most vacuum $\langle 1 \rangle$. Since the second term is exponentially suppressed and does not give a divergence, the parameters contained in the function $\phi(z, \bar{z})$ correspond to normalizable zero modes. Therefore, only the function $\psi_{\mathrm{v}}(z, \bar{z})$ has non-normalizable zero modes, which are positions of vortices living in vacuum $\langle 1 \rangle$. The same argument holds for the $x_3 \to -\infty$ direction.

In summary, in the case of flat domain walls, non-normalizable zero modes are positions of domain walls v_A, the center of mass of vortices in each vacuum $Z_A^c = a_A/n_A$, and positions of infinitely long vortices $z_{\langle 1 \rangle i}$ and $z_{\langle N_F \rangle i}$ in vacuum $\langle 1 \rangle$ and vacuum $\langle N_F \rangle$, respectively.

IV. DYNAMICS OF 1/4 BPS WALL-VORTEX SYSTEMS

Now let us construct the effective Lagrangian of vortices between domain walls. We will discuss the Abelian gauge theory with three flavors. The mass parameters are taken as $M = \mathrm{diag}(\frac{m}{2}, 0, -\frac{m}{2})$, and we denote the numbers of vortices in three vacua by (n_1, n_2, n_3). In what follows, we will take the strong gauge coupling limit ($g \to \infty$) in order to calculate the effective action analytically. In the strong coupling limit, the master equation (2.14) becomes an algebraic equation and is analytically solved as

$$\Omega = \Omega_0 \equiv \frac{1}{c} H_0 e^{2Mx_3} H_0^\dagger. \tag{4.1}$$

The Kähler metric equation (3.10) also takes a simple form in the strong coupling limit

$$K_{i\bar{j}} = c \int d^3x \partial_i \partial_{\bar{j}} \log \det \Omega_0. \tag{4.2}$$

Although the ANO vortices linearly extending to infinity like Eq. (2.31) shrink to singular configurations since their sizes $1/(g\sqrt{c})$ tend to zero in this limit, vortex strings with finite length between domain walls do not (its size behaves as e^{-mL} where L is separation between walls). Therefore we can construct low energy effective theories for vortex strings between domain walls in the strong coupling limit.

A. Numbers of vortices: (1, 1, 1)

First let us consider the case in which each vacuum has a single vortex. This configuration admits no normalizable zero modes. However, it will give us an explicit example of the non-normalizable modes which we have discussed in Sec. III B. The general form of the moduli matrix is given by

$$H_0 = \sqrt{c}(v_1(z - z_{\langle 1\rangle 1}), v_2(z - z_{\langle 2\rangle 1}), v_3(z - z_{\langle 3\rangle 1})). \tag{4.3}$$

Since we are interested in the vortex in vacuum $\langle 2\rangle$, we set $z_{\langle 1\rangle 1} = z_{\langle 3\rangle 1} = 0$ and $v_1 = v_3 = 1$, and define $v_2 \equiv v$ and $z_{\langle 2\rangle 1} \equiv z_0$,

$$H_0 = \sqrt{c}(z, v(z - z_0), z). \tag{4.4}$$

The positions of domain walls can be estimated by weights of vacua (2.24). Both domain walls are asymptotically flat, and the asymptotic distance between these domain walls, that is, the length of the vortex in vacuum $\langle 2\rangle$, is given by

$$l_{\langle 2\rangle} = \frac{4}{m}\log|v|. \tag{4.5}$$

Energy densities of configurations in a plane containing vortices are shown for several values of moduli parameters in Fig. 4.

Now we give the weak time dependence to z_0 and investigate the dynamics of the middle vortex. The explicit solution of the master equation (2.14) can be obtained in the strong coupling limit $g \to \infty$ as

$$\Omega \to \Omega_0 = |z|^2 e^{mx_3} + |v|^2|z - z_0|^2 + |z|^2 e^{-mx_3}. \tag{4.6}$$

Let us substitute this solution into the Kähler metric (4.2). Leaving the integration along the x_3-coordinate, the Kähler metric can be calculated as

$$\begin{aligned} K_{z_0\bar{z}_0} &\equiv c\int d^3x \frac{2|vz|^2\cosh(mx_3)}{(|v|^2|z - z_0|^2 + 2|z|^2\cosh mx_3)^2} \\ &= \pi c\int dx_3\bigg[\frac{2|v|^2\cosh mx_3}{(2\cosh mx_3 + |v|^2)^2}\log\frac{L^2}{|z_0|^2} \\ &\quad + \frac{|v|^2(|v|^2 - 2\cosh mx_3)}{(2\cosh mx_3 + |v|^2)^2} - \frac{2|v|^2\cosh mx_3}{(2\cosh mx_3 + |v|^2)^2} \\ &\quad \times \log\bigg(\frac{2|v|^2\cosh mx_3}{(2\cosh mx_3 + |v|^2)^2}\bigg)\bigg], \end{aligned} \tag{4.7}$$

where L is the infrared cutoff in the z-plane $|z| < L$. As we saw in the previous section, the Kähler metric contains the logarithmic divergence. The complicated metric (4.7) reduces to a simple form when we take the limit of $|v| \to \infty$

$$K_{z_0\bar{z}_0} \approx \pi c\bigg(\frac{4}{m}\log|v| - \frac{4}{m}\log|z_0| + \frac{4}{m}\log L + \text{const}\bigg). \tag{4.8}$$

The physical meaning of the metric is clear in this form. Since the tension of the vortex is $2\pi c$ and its length is given in Eq. (4.5), the first term in Eq. (4.8) corresponds to kinetic energy of the vortex. According to Eq. (3.33) and the following comments, the kinetic energy of two domain walls can be calculated as

$$T_{\text{wall}} = \frac{4\pi c}{m}\log L\frac{dz_0}{dt}\frac{d\bar{z}_0}{dt}, \tag{4.9}$$

where we have identified $z_0 = a_2$ and $\Delta m_A = m/2$ in Eq. (3.33). This coincides with the third term in Eq. (4.8). This is the origin of the non-normalizability of z_0.

Note that the moduli space has the singularity at $z_0 = 0$. This is because we have fixed the vortices in vacuum $\langle 1\rangle$ and vacuum $\langle 3\rangle$ at the same position. When the vortex in $\langle 2\rangle$ also comes to the same position, it results in a single ANO vortex which is infinitely long and becomes singular in the strong coupling limit with its shrinking size $1/(g\sqrt{c}) \to 0$. The singularity can be removed by dislocating the outer vortices. We will discuss this issue in the next example.

B. Numbers of vortices: (2, 2, 2)

Let us next consider the case in which each vacuum has a pair of vortices. The general form of the moduli matrix is given by

$$\begin{aligned} H_0 = \sqrt{c}(&v_1(z - z_{\langle 1\rangle 1})(z - z_{\langle 1\rangle 2}), v_2(z - z_{\langle 2\rangle 1})(z - z_{\langle 2\rangle 2}), \\ &v_3(z - z_{\langle 3\rangle 1})(z - z_{\langle 3\rangle 2})). \end{aligned} \tag{4.10}$$

Since we are interested in the relative motion of vortices in vacuum $\langle 2\rangle$, we set $z_{\langle 1\rangle i} = 0 = z_{\langle 3\rangle j}$ and $v_1 = v_3 = 1$, and define $v_2 \equiv v$ and $z_{\langle 2\rangle 1} = -z_{\langle 2\rangle 2} \equiv z_0$,

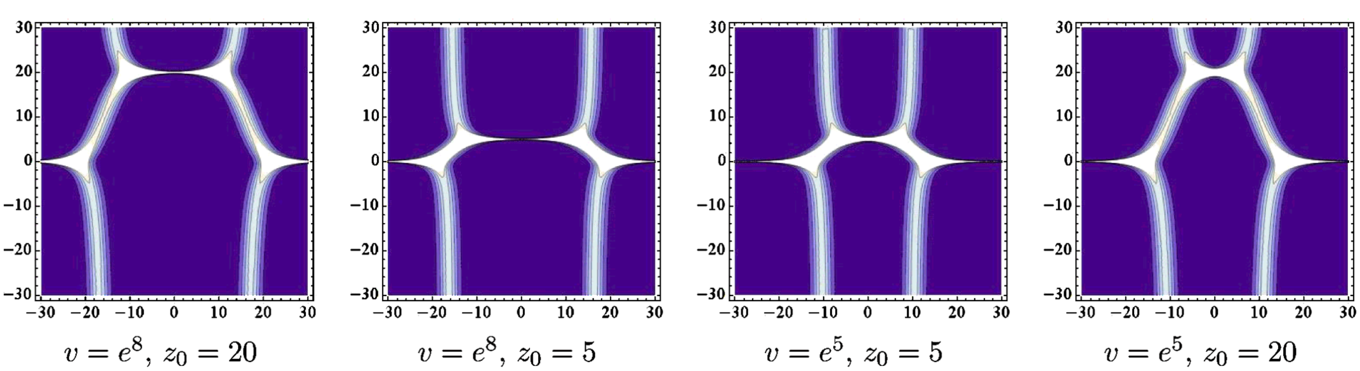

FIG. 4 (color online). The energy densities in a plane containing vortices in the strong coupling limit $g \to \infty$ with $c = 1$ and $m = 1$. Vertical lines are walls and horizontal lines are vortices.

$$H_0 = \sqrt{c}(z^2, v(z^2 - z_0^2), z^2). \tag{4.11}$$

The distance between two domain walls is also given by Eq. (4.5) in the present case. Energy densities of configurations in a plane containing vortices are shown for several values of moduli parameters in Fig. 5.

Now let us give the weak time dependence to z_0 and investigate the dynamics of the middle vortices. The explicit solution of the master equation (2.14) can be obtained in the strong coupling limit $g \to \infty$

$$\Omega \to \Omega_0 = |z|^4 e^{mx_3} + |v|^2 |z^2 - z_0^2|^2 + |z|^4 e^{-mx_3}. \tag{4.12}$$

Let us substitute this solution into the Kähler metric (4.2). After integrating the z-coordinates, we obtain the Kähler metric

$$K_{z_0\bar{z}_0} = 2\pi c \int dx_3 k E(k), \tag{4.13}$$

where $E(k)$ is the complete elliptic integral of the second kind

$$E(k) \equiv \int_0^{\pi/2} d\theta \sqrt{1 - k^2 \sin^2\theta}, \tag{4.14}$$

with the x_3-dependent parameter k

$$k = \left(\frac{|v|^2}{2\cosh mx_3 + |v|^2}\right)^{1/2}. \tag{4.15}$$

The metric does not depend on z_0 and its value can be written as a sum of the hypergeometric functions (see Appendix A). The asymptotic value of this metric for large $|v|$ is given by (see Appendix B)

$$K_{z_0\bar{z}_0} \approx \frac{8\pi c}{m} \log|4v|. \tag{4.16}$$

The leading term in the effective Lagrangian coincides with the kinetic energy of two vortices with length $l_{\langle 2\rangle} = \frac{2}{m}\log|v|^2$ and tension $T_v = 2\pi c$

$$L_{\rm eff} \approx \left(\frac{8\pi c}{m}\log|v| + \frac{16\pi c}{m}\log 2\right)|\dot{z}_0|^2 = \left(T_v l_{\langle 2\rangle} + \frac{16\pi c}{m}\log 2\right)|\dot{z}_0|^2. \tag{4.17}$$

The independence of $L_{\rm eff}$ on the IR cutoff L shows the normalizability of the moduli z_0. Therefore it makes sense to consider its dynamics. Since we cannot distinguish two vortices, the geometry of the moduli space is a cone, $\mathbf{C}/\mathbf{Z}_2$. Here $\mathbf{Z}_2$ denotes the exchange of the vortices and acts on the coordinate as $z_0 \to -z_0$. In fact, a good coordinate of the moduli space is not z_0 but z_0^2, which appears naturally in the moduli matrix (4.11). The moduli space has the singularity at $z_0 = 0$. As we explained in the previous section, this is because we have fixed the outer vortices at the same position. When the vortices in vacuum $\langle 2\rangle$ also come to the same position, they result in two ANO vortices which are singular in the strong coupling limit $g \to \infty$. The singularity can be removed by dislocating the outer vortices. For instance, let us consider the moduli matrix given in the form

$$H_0 = \sqrt{c}((z - z_1)^2, v(z^2 - z_0^2), (z + z_1)^2). \tag{4.18}$$

The vortices in vacuum $\langle 1\rangle$ are located at $z = z_1$, and the vortices in vacuum $\langle 3\rangle$ are at $z = -z_1$. The Kähler metric in the strong coupling limit is given as

$$K_{z_0\bar{z}_0} = c\int d^3x \frac{4|v|^2|z_0|^2(|z - z_1|^4 e^{mx_3} + |z + z_1|^4 e^{-mx_3})}{(|z - z_1|^4 e^{mx_3} + |v|^2|z^2 - z_0^2|^2 + |z + z_1|^4 e^{-mx_3})^2}. \tag{4.19}$$

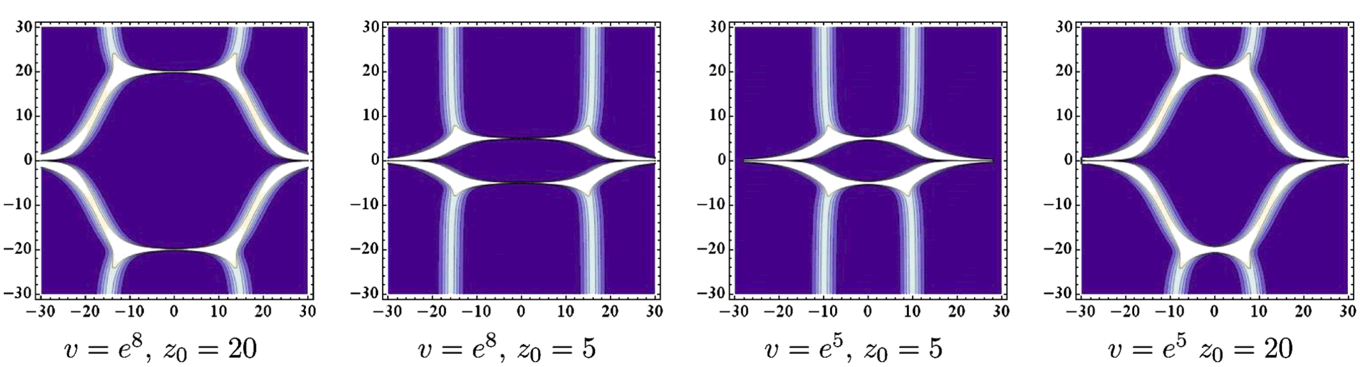

FIG. 5 (color online). The energy densities in a plane containing vortices in the strong coupling limit $g \to \infty$ with $c = 1$ and $m = 1$. Vertical lines are walls and horizontal lines are vortices.

The metric starts at $\mathcal{O}(|z_0|^2)$ and can be expanded around $z_0 = 0$ as

$$\begin{aligned} ds^2 &\simeq |z_0|^2(A + Bz_0^2 + \bar{B}\bar{z}_0^2 + \mathcal{O}(|z_0|^4))dz_0 d\bar{z}_0 \\ &= (A + BZ + \bar{B}Z + \mathcal{O}(|Z|^2))dZd\bar{Z}, \end{aligned} \quad (4.20)$$

where $Z \equiv z_0^2$ is a good complex coordinate on the moduli space. Since the constant $A \equiv (K_{z_0\bar{z}_0}/2|z_0|^2)|_{z_0=0}$ is nonzero, the scalar curvature does not diverge at $Z = 0$. Therefore, the moduli space is nonsingular at the origin, and the vortices scatter with a right angle in head-on collisions. On the other hand, the asymptotic metric for $|v|^2 \gg 1 \gg |z_1/z_0|$ is given by (see Appendix B)

$$K_{z_0\bar{z}_0} \approx \frac{8\pi c}{m} \log\left|\frac{4vz_0^2}{z_0^2 - z_1^2}\right|. \quad (4.21)$$

This coincides with Eq. (4.17) when $z_1 = 0$. The leading term is again identified with the kinetic term of the vortex of length $l_{\langle 2\rangle} = \frac{2}{m}\log|v|^2$.

C. Numbers of vortices: (0, 2, 0)

Let us next consider the case in which only the middle vacuum has a pair of vortices. This is the case where walls are not asymptotically flat, but may be useful as a building block for more complicated configurations. The general form of the moduli matrix is given by

$$H_0 = \sqrt{c}(v_1, v_2(z - z_{\langle 2\rangle 1})(z - z_{\langle 2\rangle 2}), v_3). \quad (4.22)$$

We are interested in the relative motion of vortices in vacuum $\langle 2\rangle$. Although we have not discussed the cases in which domain walls are logarithmically bending in Sec. III B, it turns out that the relative motion of the two vortices is the normalizable zero mode even in such cases. Let us set $v_1 = v_3 = 1$, $z_{\langle 2\rangle 1} = -z_{\langle 2\rangle 2} \equiv z_0$, and $v_2 \equiv v$,

$$H_0 = \sqrt{c}(1, v(z - z_0)(z + z_0), 1). \quad (4.23)$$

Energy densities of configurations in a plane containing vortices are shown for several values of moduli parameters in Fig. 6.

Now let us give the weak time dependence to z_0 and investigate the dynamics of the middle vortices. The explicit solution of the master equation (2.14) can be obtained in the strong coupling limit $g \to \infty$

$$\Omega \to \Omega_0 = e^{mx_3} + |v|^2|z^2 - z_0^2|^2 + e^{-mx_3}. \quad (4.24)$$

Let us substitute this solution into the Kähler metric (4.2). After integrating the z-coordinates similarly to the case of the number of vortices (2, 2, 2) in the previous subsection, we obtain the Kähler metric as an integral over the complete elliptic integral of the second kind $E(k)$ defined in Eq. (4.14)

$$K_{z_0\bar{z}_0} = 2\pi c\int dx_3 kE(k), \quad \text{with} \quad k = \left(\frac{|vz_0^2|^2}{2\cosh mx_3 + |vz_0^2|^2}\right)^{1/2}. \quad (4.25)$$

The metric has the same form as that of the previous example (4.13). However, the variable k in $E(k)$ is now defined differently from the case of (2, 2, 2), v is now replaced by vz_0^2. Integrating over x_3, we obtain the Kähler metric as a sum of the hypergeometric functions (see Appendix A). If we expand the Kähler metric around $|vz_0^2|^2 = 0$, we obtain

$$\begin{aligned} dx^2 &= 2K_{z_0\bar{z}_0}dz_0d\bar{z}_0 \to \frac{\pi^{3/2}c}{m}\left(\Gamma(1/4)^2 - \frac{3}{2}\Gamma(3/4)^2|vz_0^2|^2 \right. \\ &\quad \left. + \mathcal{O}(|vz_0^2|^4)\right)|vz_0^2|dz_0d\bar{z}_0 \\ &= \frac{\pi^{3/2}|v|c}{4m}\left(\Gamma(1/4)^2 - \frac{3}{2}\Gamma(3/4)^2|vZ|^2 \right. \\ &\quad \left. + \mathcal{O}(|vZ|^4)\right)dZd\bar{Z}. \end{aligned} \quad (4.26)$$

Since the coordinate $Z \equiv z_0^2$ is a good coordinate even at the origin, it shows that the moduli space is nonsingular at the origin and the vortices scatter with a right angle in head-on collisions. If we take the opposite limit $|vz_0^2|^2 \to \infty$, the metric can be calculated as (see Appendix B)

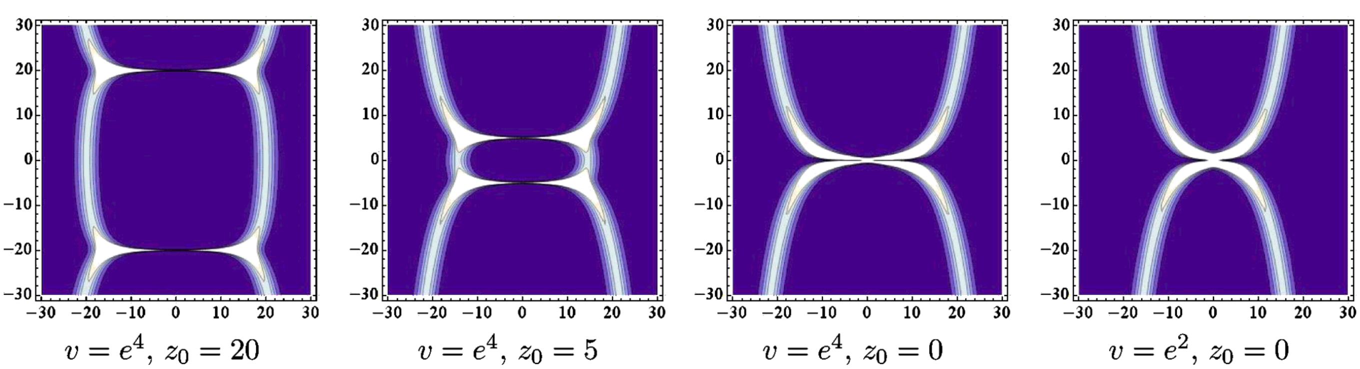

FIG. 6 (color online). The energy densities in a plane containing vortices in the strong coupling limit $g \to \infty$ with $c = 1$ and $m = 1$. Vertical lines are walls and horizontal lines are vortices.

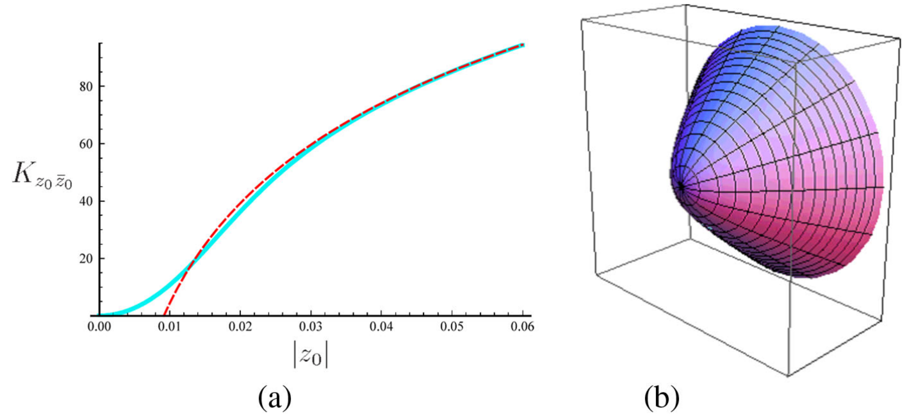

FIG. 7 (color online). A metric of the moduli space for $c = 1$, $m = 1$, $v = e^8$. (a) The numerically calculated metric (solid line) and the asymptotic metric $K_{z_0\bar{z}_0} \approx \frac{8\pi c}{m} \log|4vz_0^2|(|vz_0^2| \to \infty)$ (dashed line). (b) The moduli space isometrically embedded into three-dimensional Euclidean space $\mathbf{R}^3$.

$$K_{z_0\bar{z}_0} \approx \frac{8\pi c}{m} \log|4vz_0^2|. \tag{4.27}$$

Since the domain walls are logarithmically bending in the present case, the definition of the distance between domain walls is not clear. However, at the center of mass of two vortices, the distance between domain walls is given by

$$l_{\langle 2\rangle} = \frac{4}{m} \log|vz_0^2|. \tag{4.28}$$

It can be considered as the typical lengths of the vortices (see Fig. 6). Therefore, the above asymptotic metric (4.27) can be understood as the kinetic energy of two vortices. Figure 7 shows a numerically calculated metric and the moduli space embedded into $\mathbf{R}^3$.

D. Numbers of vortices: $(n, 0, n)$

In Sec. III B we have seen that the moduli parameters $v_A (A = 1, \cdots, N_F)$ correspond to non-normalizable zero modes if there exist the same number of vortices in each vacuum region. In fact, this is not necessarily the case if there exist different numbers of vortices in each vacuum region. The simplest such example is the configuration described by the following moduli matrix:

$$H_0 = (z^n, v, z^n). \tag{4.29}$$

The Kähler metric for the moduli parameter v is finite for $n \geq 2$. The relative distance of two walls is determined from Eq. (2.25) as $\Delta X(z, \bar{z}) = \frac{4}{m} \log|v/z^n|$. Therefore, in the region $|z| > |v|^{1/n}$, two walls are compressed into one wall located at $x_3 = 0$ and its position is unchanged under the variation of the moduli parameter $v \to v + \delta v$. Several plots of the energy densities are shown in Fig. 8. The Kähler metric for the moduli parameter v is given in the strong coupling limit by

$$\begin{aligned} K_{v\bar{v}} &= c \int d^3x \frac{2|z|^{2n} \cosh(mx_3)}{(2|z|^{2n} \cosh(mx_3) + |v|^2)^2} \\ &= \frac{\pi^2 c}{2n^2 m} \frac{|v|^{(2/n)-2}}{\sin(\pi/n)} \frac{\Gamma(\frac{1}{2n})^2}{\Gamma(\frac{1}{n})}. \end{aligned} \tag{4.30}$$

In terms of the coordinate $u \equiv v^{1/n}$, the metric can be written as

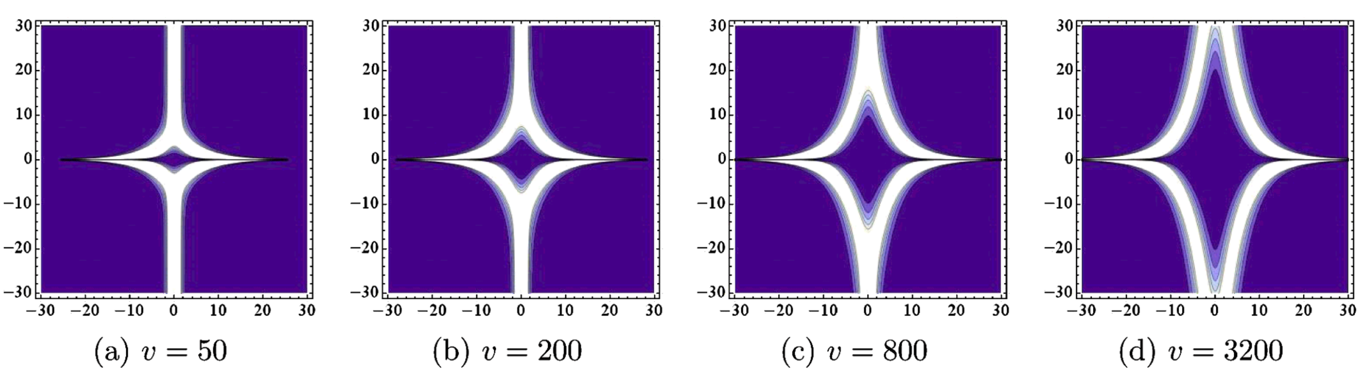

FIG. 8 (color online). The energy densities in a plane containing vortices in the strong coupling limit $g \to \infty$ with $m = 1$, $c = 1$, $n = 2$. Vertical lines are walls and horizontal lines are vortices.

$$ds^2 = 2K_{v\bar{v}}dvd\bar{v} = \frac{\pi^2 c}{m\sin(\pi/n)}\frac{\Gamma(\frac{1}{2n})^2}{\Gamma(\frac{1}{n})}dud\bar{u}. \qquad (4.31)$$

The moduli space is a cone $\mathbf{C}/\mathbf{Z}_n$ and has a singularity at $v = 0$. In the limit $v \to 0$, the moduli matrix can be factorized as

$$H_0 = (z^{2n}, v, z^{2n}) \to z^{2n}(1, 0, 1). \qquad (4.32)$$

This indicates the appearance of ANO vortices in the limit $v \to 0$. The existence of the singularity in the moduli space reflects the fact that the vortices become ANO vortices which are singular in the strong coupling limit $g \to \infty$.

V. VORTEX DYNAMICS IN A DUAL EFFECTIVE THEORY ON WALLS

So far, we have calculated the metric on the 1/4 BPS moduli space and investigated the dynamics of vortices suspended between the domain walls, using the moduli space approximation. Now let us obtain the vortex dynamics from another point of view.

A. General formalism

Let us first consider the single vortex ending on the single domain wall in the minimal Abelian-Higgs model with $N_F = 2$ and see how the vortex ending on the wall appears in the effective theory on the domain wall world volume. The 1/2 BPS domain wall is described by a single complex parameter $\phi = e^{\Delta m X + i\sigma} \in \mathbf{C}^* \simeq \mathbf{C} - \{0\} \simeq \mathbf{R} \times S^1$ in the moduli matrix $H_0 = (h_1, h_2) \simeq (1, \phi)$; see Eq. (2.18). The real part X corresponds to the position of the domain wall [see Eq. (2.20)], and σ is its phase. The effective theory on the wall turned out to be a free theory

$$\mathcal{L}_{\rm w} = \frac{c\Delta m}{2}\left[(\partial_\mu X)^2 + \frac{1}{\Delta m^2}(\partial_\mu \sigma)^2\right] \qquad (5.1)$$

via the generic expression Eq. (3.11) [35,36]

$$K_{\rm w} = \frac{c}{4\Delta m}(\log|\phi|^2)^2. \qquad (5.2)$$

The moduli matrix given in Eq. (2.23) tells us that we should identify the vortex ending on the wall at $z = z_0$ as the following configuration:

$$\phi(z, \bar{z}) = e^{\Delta m X(z,\bar{z}) + i\sigma(z,\bar{z})} = e^{\Delta m X_0 + i\sigma_0}\frac{(z - z_0)}{L}, \qquad (5.3)$$

where we have introduced a "boundary" at $|z| = L \gg |z_0|$ in the z-plane for later convenience. The parameter L plays the role of the cutoff for the IR divergence of the non-normalizable modes. The constants X_0 and σ_0 respectively represent the position and the phase of the background domain wall at $z = L + z_0$. Notice that under the identification we have added two points $\phi = 0, \infty (X = \pm\infty)$ to $\mathbf{C}^*$ resulting in the target space $\mathbf{C}P^1$. In this sense, the above realization of the vortex is thought of as the 1/2 BPS lump on the domain wall effective action.[6] Let us set $X_0 = \frac{1}{\Delta m}\log L$, $\sigma_0 = 0$, and $z_0 = 0$ in the following for simplicity. The vortex causes the logarithmic bending of the domain wall

$$X = \frac{1}{\Delta m}\log|\phi| = \frac{1}{\Delta m}\log|z|. \qquad (5.4)$$

This is consistent with the bulk point of view. We also find that if we walk around the vortex in the z-plane, the phase of the domain wall also winds once

$$\sigma = \theta, \qquad (z = x_1 + ix_2 = re^{i\theta}). \qquad (5.5)$$

Equation (5.2) is the free theory of the real scalar field X and the periodic field $\sigma \in S^1$ in $2 + 1$ dimensions. The phase degree of freedom of the domain wall $\sigma(x^\mu) \in S^1$ in $2 + 1$ dimensional world volume can be dualized into an Abelian gauge field as [7]

$$F_{\mu\nu} = \frac{e^2}{2\pi}\epsilon_{\mu\nu\rho}\partial^\rho\sigma, \qquad e^2 \equiv \frac{4\pi^2 c}{\Delta m}. \qquad (5.6)$$

If we also rescale the scalar field $X(x^\mu)$ as

$$\log\phi = \Delta m X + i\sigma = \frac{\Delta m}{2\pi c}\Phi + i\sigma = \frac{2\pi}{e^2}\Phi + i\sigma, \qquad (5.7)$$

the effective Lagrangian has the simple form

$$\mathcal{L}_{\rm w} = \left(\frac{1}{2e^2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}\right). \qquad (5.8)$$

In terms of the dual gauge field, the phase winding (5.5) corresponds to the electric field for a static source with unit electric charge

$$F_{0r} = E_r = \frac{e^2}{2\pi}\frac{1}{r}, \qquad (5.9)$$

and the electrostatic potential is given by

$$A_0 = -\frac{e^2}{2\pi}\log|z|. \qquad (5.10)$$

Furthermore, the logarithmic bending (5.4) yields the scalar potential

$$\Phi = \frac{e^2}{2\pi}\log|z|. \qquad (5.11)$$

Therefore, the vortex at rest can be viewed as a charged particle in the effective theory, which gives the scalar field (5.11) and the electric field (5.10). When the electric charge

[6] The BPS equation is $\bar{\partial}_z\phi = 0$. The solution of this BPS equation satisfies the equation of motion with a source term $\partial_{\bar{z}}\partial_z(X + i\sigma/\Delta m) = \pm\frac{\pi}{2\Delta m}\delta^2(z - z_0)$ corresponding to the addition of the points $X = \pm\infty$.

(vortex) moves at a constant velocity $u = v^1 + iv^2$, ($u \equiv \dot{z}_0$), the potentials are Lorentz boosted as

$$\Phi = \frac{e^2}{2\pi} \log|L_u(z - z_0)|, \tag{5.12}$$

$$A_\mu = -\frac{e^2}{2\pi} \frac{v_\mu}{\sqrt{1-|u|^2}} \log|L_u(z - z_0)|, \tag{5.13}$$

with $v^\mu = (1, v^1, v^2)$ and

$$L_u(z - z_0) \equiv \frac{1}{2}\left[\left(\frac{1}{\sqrt{1-|u|^2}} + 1\right)(z - z_0) + \frac{u}{\bar{u}}\left(\frac{1}{\sqrt{1-|u|^2}} - 1\right)(\bar{z} - \bar{z}_0)\right]. \tag{5.14}$$

We can confirm that these configurations satisfy equations of motion with the moving charged particle,

$$\begin{aligned} \partial_\mu \partial^\mu \Phi &= -e^2 \delta^2(z - z_0)\sqrt{1-|u|^2} \\ \partial_\nu F^{\mu\nu} &= -e^2 \delta^2(z - z_0) v^\mu. \end{aligned} \tag{5.15}$$

Notice that the vortex ending on the wall from the other side corresponds to the moduli matrix $H_0 = (z, 1) \sim (1, 1/z)$, namely $X = -\frac{1}{\Delta m}\log|z|$, $\sigma = -\theta$. This implies that it generates the potentials with the sign opposite to that in Eqs. (5.12) and (5.13). Furthermore, if the BPS vortex is replaced by an anti-BPS vortex, we find that only the sign of the phase σ flips without any change to X. We consider only BPS vortices in the following discussion.

We can extend this analysis to the case of multiple domain walls. When all the domain walls are well separated, we can assume that the dual theory[7] is a $U(1)^N$ gauge theory with N neutral Higgs fields Φ^A ($A = 1, \cdots, N$)

$$\mathcal{L}_{\rm w} = \sum_{A=1}^{N}\left[\frac{1}{2e_A^2}\partial_\mu \Phi^A \partial^\mu \Phi^A - \frac{1}{4e_A^2}F^A_{\mu\nu}F^{A\mu\nu}\right]. \tag{5.16}$$

Here the scalar fields Φ^A ($A = 1, \cdots, N$) are identified with the position of the domain wall between the vacua $\langle A \rangle$ and $\langle A+1 \rangle$ as $X^A = \frac{1}{2\pi c}\Phi^A$. The constants $e_A (A = 1, \cdots, N)$ are the gauge coupling constants on the world volume of the domain wall between vacua $\langle A \rangle$ and $\langle A+1 \rangle$,

$$e_A^2 = \frac{4\pi^2 c}{\Delta m_A}. \tag{5.17}$$

The ith vortex living in vacuum $\langle B \rangle$ positioned at $z_{\langle B \rangle i}$ with a velocity $u_{\langle B \rangle i} \equiv \dot{z}_{\langle B \rangle i}$ yields the scalar field and the electric field on the world volume of neighboring domain walls[8]

$$(\Phi^A)_{(B,i)} = (\delta_{A+1,B} - \delta_{AB})\frac{e_A^2}{2\pi} G(u_{\langle B \rangle i}; z - z_{\langle B \rangle i}), \tag{5.18}$$

$$(A^A_\mu)_{(B,i)} = -(\delta_{A+1,B} - \delta_{AB})\frac{e_A^2}{2\pi}\frac{v_{\mu\langle B \rangle i}}{\sqrt{1-|u_{\langle B \rangle i}|^2}} \times G(u_{\langle B \rangle i}; z - z_{\langle B \rangle i}), \tag{5.19}$$

where $v^\mu_{\langle B \rangle i} = (1, {\rm Re}[u_{\langle B \rangle i}], {\rm Im}[u_{\langle B \rangle i}])$ and G is the Green's function given by

$$\begin{aligned} G(u_{\langle B \rangle i}; z - z_{\langle B \rangle i}) &= \log|L_{u_{\langle B \rangle i}}(z - z_{\langle B \rangle i})| - \log L + f(u_{\langle B \rangle i}), \\ f(u) &\approx \mathcal{O}(u^2). \end{aligned} \tag{5.20}$$

Here we have added the last two terms so that the Green's function vanishes at the boundary $|z| = L$.

Now let us assume that the dynamics of the ith vortex living in vacuum $\langle A \rangle$ can be regarded as that of an electric charge moving in the background potential produced by the other vortices. We shall suppose that the effect of the Lorentz scalar potential Φ^B ($B \neq A$) is to change the rest mass of the vortex ending on the domain wall. This is consistent with the fact that the vortices cause the logarithmic bending of the domain wall and it leads to the change of the length of the other vortices ending on the domain wall, and thus the masses of the vortices. With these assumptions, the Lagrangian for the ith vortex in vacuum $\langle A \rangle$ is given by that for the charged particle [23,30]

$$\begin{aligned} L_i^{\langle A \rangle} &= \sum_B (\delta_{A,B} - \delta_{A-1,B})(-\tilde{\Phi}^B\sqrt{1-|u_{\langle A \rangle i}|^2} - \tilde{A}^B_\mu v^\mu_{\langle A \rangle i}) \\ &\approx \sum_B (\delta_{A,B} - \delta_{A-1,B})\left(-\tilde{\Phi}^B + \frac{\tilde{\Phi}^B}{2}|u_{\langle A \rangle i}|^2 - \tilde{A}^B_\mu v^\mu_{\langle A \rangle i}\right), \end{aligned} \tag{5.21}$$

where $\tilde{\Phi}^B$, $\tilde{A}^B_0$, $\tilde{\mathbf{A}}^B$ are the values of the fields produced by the other particles at the location of the particle $z = z_{\langle A \rangle i}$,

$$\begin{aligned} \tilde{\Phi}^B &\equiv \langle \Phi^B \rangle + \sum_{(C,j)} (\Phi^B)_{(C,j)}|_{z = z_{\langle A \rangle i}}, \\ \tilde{A}^B_\mu &= \sum_{(C,j)} (A^B_\mu)_{(C,j)}|_{z = z_{\langle A \rangle i}}. \end{aligned} \tag{5.22}$$

Here we implicitly assume that the fields due to the particle in problem (ith vortex in vacuum $\langle A \rangle$) are excluded in the sum and $\langle \Phi \rangle$ is the vacuum expectation value (VEV) of the scalar field at the boundary $|z| = L$. Let us note that Eq. (5.21) gives the Lagrangian for the particle A under the background potential produced by all the other particles. To obtain the total Lagrangian for all particles including mutual interactions, we need to sum over the interaction terms only once for each pair of particles. Substituting Eqs. (5.18) and (5.19) into Eq. (5.21) and summing up the kinetic terms and the interaction terms from all pairs of particles, we obtain the effective

[7] In fact, we can obtain the dual $U(1)^N$ gauge theory by dualizing N compact scalar fields σ^A; see Appendix D.

[8] Note that the N domain walls divide the three-dimensional space into $N+1$ different vacuum regions. We use indices A and B to label both the domain walls and vacuum regions: the indices A and B run from 1 to N for domain walls, and the labels $\langle A \rangle$ and $\langle B \rangle$ run from 1 to $N+1$ for vacuum regions; see Fig. 2.

Lagrangian as

$$L_{\rm eff} = \sum_{(A,i)} \frac{\langle\Phi^{A-1}\rangle - \langle\Phi^{A}\rangle}{2}|u_{\langle A\rangle i}|^2 + \sum_{(A,i),(B,j)}{}' \left(\frac{C_{AB}}{2}\log\frac{|z_{\langle A\rangle i} - z_{\langle B\rangle j}|}{L}\right)|u_{\langle A\rangle i} - u_{\langle B\rangle j}|^2, \tag{5.23}$$

$$C_{AB} \equiv 2\pi c\left[\left(\frac{1}{\Delta m_A} + \frac{1}{\Delta m_{A-1}}\right)\delta_{AB} - \frac{1}{\Delta m_A}\delta_{A,B-1} - \frac{1}{\Delta m_B}\delta_{A-1,B}\right], \tag{5.24}$$

where $\sum'$ means that the sum is taken only once for each pair of the index sets (A, i) and (B, j) such that $(A, i) \neq (B, j)$.

The general Lagrangian equation (5.23) can be interpreted as an asymptotic effective Lagrangian for the vortices between the domain walls. The dynamics of the vortices are well described by the Lagrangian equation (5.23) when the domain walls are well separated in the x_3-direction and the vortices are well separated in the z-plane. We will compare it with the results obtained in Sec. IV by taking several examples in the following. The general form itself also has some good properties. One is that the sigma model metric is Kähler as shown in Appendix C. The other is that the IR divergences $\log L$ in Eq. (5.23) are completely canceled out when $\sum_i u_{\langle A\rangle i} = 0$ and $u_{\langle 1\rangle i} = u_{\langle N+1\rangle i} = 0$, that is, the center of mass of vortices in each vacuum and semi-infinite vortices in vacuum $\langle 1\rangle$ and $\langle N+1\rangle$ do not move. This is consistent with the argument of normalizability in Sec. III B. Furthermore, it correctly reproduces the IR divergence in Eq. (3.30) for the center of mass of vortices in each vacuum.

Before concluding this subsection, let us comment on the effect of the bulk coupling constant g which we have ignored in the discussion above. If the coupling constant is finite, we should take into account the boojum charges, which have negative contributions to the energy corresponding to the binding energy between vortices and domain walls. Since vortices become lighter by the amount of boojum charges, the kinetic terms in the effective Lagrangian (5.23) should be replaced as

$$\frac{M_{{\rm v}\langle A\rangle}}{2}|u_{\langle A\rangle i}|^2 = \frac{\langle\Phi^{A-1}\rangle - \langle\Phi^{A}\rangle}{2}|u_{\langle A\rangle i}|^2 \to \frac{\langle\Phi^{A-1}\rangle - \langle\Phi^{A}\rangle + B_g^{A-1} + B_g^{A}}{2}|u_{\langle A\rangle i}|^2. \tag{5.25}$$

Here B_g^A is the boojum charge between the Ath domain wall and a vortex living in vacuum $\langle A\rangle$

$$B_g^A = -\frac{2\pi\Delta m_A}{g^2} < 0. \tag{5.26}$$

Another interpretation of the shifts of the vortex masses Eq. (5.25) is given as follows. If the gauge coupling constant g is finite, the Ath domain wall has its typical width [7,16]

$$d_A \equiv \frac{2\Delta m_A}{g^2 c} = -\frac{B_g^A}{\pi c}. \tag{5.27}$$

Since the length of the vortices $l_{\langle A\rangle}$ is measured as the distance between the surfaces of the Ath and $(A-1)$th domain walls, the mass of the vortices $M_{{\rm v}\langle A\rangle} \equiv 2\pi c l_{\langle A\rangle}$ is given by

$$M_{{\rm v}\langle A\rangle} = 2\pi c l_{\langle A\rangle} = 2\pi c\left(\frac{\langle\Phi^{A-1}\rangle - \langle\Phi^{A}\rangle}{2\pi c} - \frac{d_{A-1} + d_A}{2}\right) = \langle\Phi^{A-1}\rangle - \langle\Phi^{A}\rangle + B_g^{A-1} + B_g^{A}. \tag{5.28}$$

Here $(\langle\Phi^{A-1}\rangle - \langle\Phi^{A}\rangle)/2\pi c$ is the distance between the middle points of the Ath and $(A-1)$th domain walls. For more details, see Appendix D.

B. Numbers of vortices: (1, 1, 1)

Let us consider the Abelian gauge theory with three flavors and assume that each vacuum has a single vortex. We have already obtained the effective Lagrangian for the middle vortex in Sec. IV A. We use the same mass parameters given as $M = {\rm diag}(\frac{m}{2}, 0, -\frac{m}{2})$, and set the outer vortices at $z_{\langle 1\rangle 1} = z_{\langle 3\rangle 1} = 0$ as before. The gauge couplings in the effective theory on the domain walls are given by

$$e^2 \equiv e_1^2 = e_2^2 = \frac{8\pi^2 c}{m}. \tag{5.29}$$

Since the first domain wall is positioned at $X^1 = \frac{2}{m}\log|v|$ and the second domain wall is at $X^2 = -\frac{2}{m}\log|v|$, the vacuum expectation value of the adjoint scalar field is

$$\langle\Phi^1\rangle = \frac{e^2}{2\pi}\log|v|, \qquad \langle\Phi^2\rangle = -\frac{e^2}{2\pi}\log|v|. \tag{5.30}$$

If we substitute these to Eq. (5.23), we obtain the effective Lagrangian for the middle vortex

$$L_{\rm eff} = \pi c\left(\frac{4}{m}\log|v| - \frac{4}{m}\log\frac{|z_0|}{L}\right)|\dot{z}_0|^2. \tag{5.31}$$

This result coincides with the asymptotic metric in Eq. (4.8).

C. Numbers of vortices: (2, 2, 2)

Next let us consider the case in which each vacuum has a pair of vortices. We have already obtained the effective Lagrangian for the relative motion of the middle vortices in Sec. IV B. The gauge couplings and the vacuum expectation value of the scalar field are the same as in the previous example. Let us set the vortices in vacuum $\langle 1\rangle$ at $z_{\langle 1\rangle 1} = z_{\langle 1\rangle 2} = z_1$ and the vortices in vacuum $\langle 3\rangle$ at $z_{\langle 3\rangle 1} = z_{\langle 3\rangle 2} = -z_1$ as in Sec. IV B. Since we are interested in the relative motion of the vortices in vacuum $\langle 2\rangle$, we take $z_{\langle 2\rangle 1} = -z_{\langle 2\rangle 2} = z_0$. The effective Lagrangian is given by

$$L_{\rm eff} = \pi c\biggl(\frac{8}{m}\log|v| - \frac{8}{m}\log|z_0 - z_1| - \frac{8}{m}\log|z_0 + z_1| + \frac{16}{m}\log|2z_0|\biggr)|\dot{z}_0|^2, \tag{5.32}$$

Note that the divergence terms are exactly canceled out. The second term comes from the interactions with the vortices in vacuum $\langle 1\rangle$ and the third from the vortices in vacuum $\langle 3\rangle$. The last term represents the interactions of two vortices in vacuum $\langle 2\rangle$. The effective Lagrangian for the relative motion of two vortices can be obtained as

$$L_{\rm eff} = \biggl[\frac{8\pi c}{m}\log|v| + \frac{16\pi c}{m}\log 2 + \mathcal{O}\biggl(\biggl(\frac{z_1}{z_0}\biggr)^2, \biggl(\frac{\bar{z}_1}{\bar{z}_0}\biggr)^2\biggr)\biggr]|\dot{z}_0|^2. \tag{5.33}$$

This coincides with the previous result in Eq. (4.17).

D. Numbers of vortices: (0, 2, 0)

Next let us consider the case in which only the middle vacuum has a pair of vortices. We have already obtained the effective Lagrangian for the relative motion of the vortices in Sec. IV C. The gauge couplings are the same as in the previous examples. In this case, walls logarithmically bend even at the boundary and the VEV of the scalar fields depend on the cutoff L as

$$\langle\Phi^1\rangle = \frac{e^2}{2\pi}\log(|v|L^2), \qquad \langle\Phi^2\rangle = -\frac{e^2}{2\pi}\log(|v|L^2). \tag{5.34}$$

We will find that this vacuum expectation value gives the correct answer in the following. Since we are interested in the relative motion of the vortices, we take $z_{\langle 2\rangle 1} = -z_{\langle 2\rangle 2} = z_0$. The effective Lagrangian for the first vortex in vacuum $\langle 2\rangle$ is given by

$$L_{\rm eff} = \pi c\biggl(\frac{8}{m}\log(|v|L^2) + \frac{16}{m}\log\frac{|2z_0|}{L}\biggr)|\dot{z}_0|^2. \tag{5.35}$$

The second term comes from the interaction of two vortices and the cutoff dependence vanishes again. The effective Lagrangian for the relative motion of two vortices can be obtained as

$$L_{\rm eff} = \biggl(\frac{8\pi c}{m}\log|vz_0^2| + \frac{16\pi c}{m}\log 2\biggr)|\dot{z}_0|^2. \tag{5.36}$$

This coincides with the previous result in Eq. (4.27).

In summary, this method correctly reproduces the asymptotic metric on the moduli space. If the domain walls are well separated in the x_3-direction and the vortices are well separated from other vortices in the z-plane, we can trust the Lagrangian in Eq. (5.23).

VI. CONCLUSION AND DISCUSSION

We have investigated dynamics of the 1/4 BPS solitons in $\mathcal{N} = 2$ supersymmetric $U(N_{\rm C})$ gauge theory with $N_{\rm F}$ hypermultiplets in $3 + 1$ dimensions. The 1/4 BPS solitons are composite of different solitons: monopoles, boojums, vortex strings, and parallel domain walls. Neither the vortex strings of infinite length nor the domain walls can move because of their infinite masses. On the other hand, the monopoles pieced by the vortices and the vortices of finite length suspended between the domain walls may move. We have considered two different methods to study this interesting dynamics of solitons; the one is the so-called moduli approximation *à la* Manton and the other is the charged particle approximation for string end points in the wall effective action. After reviewing the moduli space of the 1/4 BPS states in Sec. II, we have derived the formal low energy effective action which describes the slow-move soliton dynamics and have specified which moduli parameters are normalizable and which are not in Sec. III. Since we are primarily interested in the 1/4 BPS dynamics in the $U(1)$ gauge theory, we have no monopoles. Clearly only the vortices with finite length can have finite masses and have a chance to give a normalizable mode. In spite of the finite length and mass, we have found that the center of masses of the vortices in each vacuum are actually non-normalizable. In Sec. IV we have dealt with several examples of (1, 1, 1), (2, 2, 2), (0, 2, 0), and $(n, 0, n)$. In order to study it analytically, we have taken the strong gauge coupling limit where the gauge theory reduces to the massive $\mathbf{C}P^{N_{\rm F}-1}$ nonlinear sigma model. With the first example, we have seen that the low energy effective action can be intuitively understood as the normal kinetic energies of domain walls and the vortices. We have also understood the origin of the non-normalizability of the middle vortex in spite of the finiteness of its mass. The (2, 2, 2) provides us with a simple example of the vortex dynamics. The dynamical degree of freedom is only the relative position of the vortices in the middle vacuum. We have studied two situations. The first setup is tuned in such a way that all the outer semi-infinite vortices are positioned at the origin of the z-plane and the center of mass of the middle vortices is put on the origin as well. It turned out that the moduli space is $\mathbf{C}/\mathbf{Z}_2$ and we fall into its conical singularity as the middle vortices go to the origin. The next setup is taken so that the outer vortices are dislocated from the origin and are put separately. This removes the singularity and we have found the 90° scattering for head-on collisions. The (0, 2, 0) is the example where the domain walls are not asymptotically flat. We have seen the 90° scattering for head-on collision also here. The $(n, 0, n)$ is completely different from the others. There are no dynamical vortices but there exists one complex parameter associated with the middle vacuum which is enclosed by the adjacent walls. The metric of the moduli space has been found $\mathbf{C}/\mathbf{Z}_n$, and the conical singularity reflects that n ANO vortices appear when the middle vacuum disappears. Our last attempt to reveal the dynamics of the vortices ending on the domain walls has been done from the viewpoint of the effective action on the host domain walls. As first shown in Ref. [7], the effective action on the single domain wall can be dualized to the $U(1)$ gauge theory with a free real scalar in $2 + 1$ dimensions. The vortex ending on the wall can be, then, identified with an electrically charged particle. We have applied the idea for the well-separated $N_{\rm F} - 1$ domain wall system and the vortices suspended between them. To this end, we have considered the $U(1)^{N_{\rm F}-1}$ gauge theory with $N_{\rm F} - 1$ real scalar fields as the dual theory. Vortices ending on a domain wall from the right-hand side have the opposite $U(1)$ charges to those ending from the left-hand side. Our effective action is given in Eq. (5.23). It is worth while to mention that our Lagrangian is the $2 + 1$-dimensional analogue of the Lagrangian given by Manton who calculated the velocity dependent interactions between well-separated BPS monopoles in $3 + 1$ dimensions [30]. We have tested our second approach to the case of (1, 1, 1), (2, 2, 2), and (0, 2, 0). It is gratifying that two different methods give us the same asymptotic interactions.

It is not easy to construct a string stretched between D-branes as a soliton of the Born-Infeld theory. On the contrary, our method of the moduli matrix allows us to construct easily the configurations of vortex string stretched between walls. It is worth emphasizing that the dynamics of these composite solitons can be analyzed without any logical or practical difficulty in our method of the moduli matrix.

It is an interesting future work to generalize our analysis to $U(N_{\rm C})$ gauge theory. For instance, a characteristic feature of the non-Abelian gauge theories is that we can have monopoles (with positive energy contribution) rather than boojums (with negative energy contribution). It has been found in the case of webs of domain walls that the non-Abelian and Abelian junction can be interchanged during the course of geodesic motion [29]. A similar dynamical metamorphosis may also be expected in the case of the wall-vortex-monopole system. It is also interesting to further generalize to arbitrary gauge groups [40] such as the SO gauge group [41].

In this paper we have assumed that the masses of the Higgs fields (hypermultiplets) are nondegenerate and real. When some masses are degenerate the model enjoys non-Abelian flavor symmetry and a part of them broken by the wall configurations appear in the domain wall effective action [33–35], where some modes (called non-Abelian clouds) appear between domain walls [35] whereas the rest are localized around each wall as usual. Accordingly non-Abelian vortex strings [15] can end on these non-Abelian domain walls [33]. Classification of possible configurations is still lacking in this case. In particular non-Abelian semilocal vortex strings [42,43] have not been studied in the presence of domain walls. For instance (non-)normalizability of orientational zero modes is quite nontrivial even in the absence of domain walls; they are non-normalizable for a single vortex with nonzero size moduli [42] but become normalizable with vanishing size moduli [43]. Moreover relative orientational moduli of multiple vortices are normalizable [43]. Classification of possible configurations and dynamics of these configurations should be done as a natural extension of this paper.

Stationary time-dependent configurations carry a conserved Noether charge. Such configurations are called dyonic (Q-)solitons. Dyonic instantons were found as an extension of dyons and have been studied by many authors [44]. Dyonic domain walls [45,46] and a dyonic network of domain walls [28] have been studied so far. Dyonic non-Abelian vortices are also studied recently [47]. The dyonic extension of the wall-vortex system is still $1/4$ BPS in four space-time dimensions [46], which can be realized if we introduce imaginary masses for the Higgs fields (hypermultiplets) and a linear time dependence on corresponding phases.

Our considerations of dynamics are classical so far. The quantization of solitons is one of interesting future direction. First, (semiclassical) the first quantization of monopoles was done by using the moduli space, to obtain quantum mechanics on the moduli space [48]. One should be able to generalize this even to a composite system. For instance, by quantizing the sector (0, 2, 0) of two strings, we will obtain a quantum scattering of strings. It is interesting to compare this with a scattering of W-bosons since our configurations resemble fundamental strings between D2-branes.[9] Second, the second quantization of solitons is more interesting. It has been suggested that it is crucial to take account of the quantum dynamics of solitons in order to see the precise parallel of our field theory solitons with the D-branes [20,21]. The second quantization of solitons is an intriguing delicate problem which is worth examining.

We have studied the moduli space and dynamics of $1/4$ BPS composite systems such as domain wall webs (networks) [29] and vortex strings between domain walls as in this paper. There exists another interesting $1/4$ BPS composite system. In $4 + 1$ dimensions instantons are particle-like solitons, and they can lie inside vortex sheets in the Higgs phase. So far instantons inside a straight vortex plane as a host soliton were studied [12,14]. Their dynamics is identical to that of sigma model lumps, because the instantons can be regarded as lumps in the effective theory on the vortex plane which is typically the $\mathbf{C}P^1$ model. Recently more interesting configurations of instantons liv-

[9] We thank Koji Hashimoto for suggesting this problem.

ing inside a vortex network as a host soliton have been found [49]. In this case the host soliton has a geometry of the Riemann surface so the instanton dynamics is more ample and interesting to explore. Solitons in different dimensions are connected by duality such as T duality between domain walls and vortices [50,51]. In [51] it has been used to study statistical mechanics of vortices. This method should be extendible to the present case of vortex strings between domain walls.

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APPENDIX A: EVALUATION OF KÄHLER METRIC

In Eqs. (4.13) and (4.25), we have seen that the Kähler metrics take the form

$$K_{z_0\bar{z}_0} = 2\pi c \int dx_3 kE(k), \qquad k \equiv \sqrt{\frac{|a|^2}{|a|^2 + 2\cosh(mx_3)}}, \tag{A1}$$

with $a = v$ for Eq. (4.13) and $a = vz_0^2$ for Eq. (4.25). This integral can be evaluated by expanding the integrand in terms of a as

$$kE(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+1}\left(\frac{(2n+1)!!}{(2n)!!}\right)^2\left(\frac{|a|}{\sqrt{2\cosh(mx_3)}}\right)^{2n+1}. \tag{A2}$$

Then integrating term by term, we obtain the Kähler metric as

$$\begin{aligned} K_{z_0\bar{z}_0} &= \pi^2 c\int_{-\infty}^{\infty} dx_3 \sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}\left(\frac{(2n+1)!!}{(2n)!!}\right)^2\left(\frac{|a|}{\sqrt{2\cosh(mx_3)}}\right)^{2n+1} \\ &= \frac{\pi^{5/2}c}{\sqrt{2}m}\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n\frac{1}{2n+1}\left(\frac{(2n+1)!!}{(2n)!!}\right)^2\frac{\Gamma(\frac{1}{4}+\frac{n}{2})}{\Gamma(\frac{3}{4}+\frac{n}{2})}|a|^{2n+1} \\ &= \frac{\pi^{3/2}c}{2m}|a|\left[\Gamma(1/4)^2{}_4F_3\left(\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{5}{4};\frac{1}{2},\frac{1}{2},1;\frac{|a|^4}{4}\right) - \frac{3}{2}|a|^2\Gamma(3/4)^2{}_4F_3\left(\frac{3}{4},\frac{3}{4},\frac{5}{4},\frac{7}{4};1,\frac{3}{2},\frac{3}{2};\frac{|a|^4}{4}\right)\right], \end{aligned} \tag{A3}$$

where ${}_4F_3(a_1, a_2, a_3, a_4; b_1, b_2, b_3; z)$ is the generalized hypergeometric function defined by

$${}_4F_3(a_1, a_2, a_3, a_4; b_1, b_2, b_3; z) = \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n(a_3)_n(a_4)_n}{(b_1)_n(b_2)_n(b_3)_n}\frac{z^n}{n!}, \tag{A4}$$

with $(a)_n \equiv a(a+1)(a+2)\cdots(a+n-1)$.

APPENDIX B: ASYMPTOTIC KÄHLER METRIC

In this section we derive the asymptotic Kähler metric equations (4.21) and (4.27). In both cases the moduli matrix takes the form

$$H_0 = (\varphi_1(z), v\varphi_2(z, z_0), \varphi_3(z)), \tag{B1}$$

with

$$\begin{aligned} &\varphi_1 = (z-z_1)^2, \qquad \varphi_2 = (z^2 - z_0^2), \qquad \varphi_3 = (z+z_1)^2 \\ &\quad \text{for } (k_1, k_2, k_3) = (2, 2, 2), \\ &\varphi_1 = 1, \qquad \varphi_2 = (z^2 - z_0^2), \qquad \varphi_3 = 1 \\ &\quad \text{for } (k_1, k_2, k_3) = (0, 2, 0). \end{aligned} \tag{B2}$$

The Kähler potential equation (3.11) in the strong coupling limit is given by

$$\begin{aligned} K &= c\int d^3x\mathcal{K} \\ &= c\int d^3x \log(|\varphi_1|^2 e^{mx_3} + |v|^2|\varphi_2|^2 + |\varphi_3|^2 e^{-mx_3}). \end{aligned} \tag{B3}$$

For $x_3 > x_0 \equiv \frac{1}{2m}\log|\varphi_3/\varphi_1|^2$, the integrand can be expanded as

$$\mathcal{K} = \log(|\varphi_1|^2 e^{mx_3} + |v|^2|\varphi_2|^2) - \sum_{n=1}^{\infty} \frac{1}{n}\left(-\frac{|\varphi_3|^2 e^{-mx_3}}{|\varphi_1|^2 e^{mx_3} + |v|^2|\varphi_2|^2}\right)^n, \qquad \text{(B4)}$$

and for $x_3 < x_0$ it can be expanded as

$$\mathcal{K} = \log(|v|^2|\varphi_2|^2 + |\varphi_3|^2 e^{-mx_3}) - \sum_{n=1}^{\infty} \frac{1}{n}\left(-\frac{|\varphi_1|^2 e^{mx_3}}{|v|^2|\varphi_2|^2 + |\varphi_3|^2 e^{-mx_3}}\right)^n. \qquad \text{(B5)}$$

We can show that the contributions to the metric from the terms in the infinite series vanish in the limit $v \to \infty$. Therefore, the asymptotic Kähler metric is given by

$$K \approx c \int d^2z \left[\int_{-\infty}^{x_0} dx_3 \log(|v|^2|\varphi_2|^2 + |\varphi_3|^2 e^{-mx_3}) + \int_{x_0}^{\infty} dx_3 \log(|\varphi_1|^2 e^{mx_3} + |v|^2|\varphi_2|^2)\right], \qquad \text{(B6)}$$

and the asymptotic Kähler metric can be written as

$$K_{z_0\bar{z}_0} = \partial_{z_0}\partial_{\bar{z}_0} K \approx c \int d^2z |v|^2 |\partial_{z_0}\varphi_2|^2 \left[\int_{-\infty}^{x_0} dx_3 \frac{|\varphi_3|^2 e^{mx_3}}{(|\varphi_3|^2 + |v|^2|\varphi_2|^2 e^{mx_3})^2} + \int_{x_0}^{\infty} dx_3 \frac{|\varphi_1|^2 e^{-mx_3}}{(|\varphi_1|^2 + |v|^2|\varphi_2|^2 e^{-mx_3})^2}\right]$$
$$= \frac{2c}{m} \int d^2z \frac{|v|^2 |\partial_{z_0}\varphi_2|^2}{|v|^2|\varphi_2|^2 + |\varphi_3||\varphi_1|}. \qquad \text{(B7)}$$

First, let us consider the case of $(k_1, k_2, k_3) = (2, 2, 2)$. By using Eq. (B7) the asymptotic metric can be calculated as

$$K_{z_0\bar{z}_0} = \frac{2c}{m} \int d^2z \frac{4|v|^2|z_0|^2}{|v|^2|z^2 - z_0^2|^2 + |z^2 - z_1^2|^2} = \frac{2\pi c}{m} \left|\frac{vz_0^2}{z_0^2 - z_1^2}\right| \int_0^{2\pi} d\theta \frac{1}{\sqrt{1 + \frac{||v|^2 z_0^2 + z_1^2|^2}{|v|^2|z_0^2 - z_1^2|^2}\sin^2(2\theta)}}. \qquad \text{(B8)}$$

If we assume that $|v| \gg |z_1|/|z_0|$, the Kähler metric becomes

$$K_{z_0\bar{z}_0} \approx \frac{2\pi c}{m} \left|\frac{vz_0^2}{z_0^2 - z_1^2}\right| \int_0^{2\pi} d\theta \frac{1}{\sqrt{1 + |\frac{vz_0^2}{z_0^2 - z_1^2}|^2 \sin^2(2\theta)}} = \frac{8\pi c}{m} \left|\frac{vz_0^2}{z_0^2 - z_1^2}\right| K\left(i\left|\frac{vz_0^2}{z_0^2 - z_1^2}\right|\right), \qquad \text{(B9)}$$

where the complete elliptic integral of the first kind $K(k)$ is defined by

$$K(k) \equiv \int_0^{\pi/2} d\theta \frac{1}{\sqrt{1 - k^2 \sin^2\theta}}. \qquad \text{(B10)}$$

By using the asymptotic form of the complete elliptic integral

$$kK(ik) \to \log(4k), \qquad k \gg 1, \qquad k \in \mathbf{R}, \qquad \text{(B11)}$$

we obtain the asymptotic Kähler metric as

$$K_{z_0\bar{z}_0} \approx \frac{8\pi c}{m} \log\left|\frac{4vz_0^2}{z_0^2 - z_1^2}\right|. \qquad \text{(B12)}$$

Next, let us consider the case of $(k_1, k_2, k_3) = (0, 2, 0)$. By using Eq. (B7) the asymptotic metric can be calculated as

$$K_{z_0\bar{z}_0} \approx \frac{2c}{m} \int d^2x \frac{4|v|^2|z_0|^2}{|v|^2|z^2 - z_0^2|^2 + 1} = \frac{8\pi c}{m} |vz_0^2| K(i|vz_0^2|) \approx \frac{8\pi c}{m} \log|4vz_0^2|. \qquad \text{(B13)}$$

Here we have used the asymptotic relation equation (B11).

APPENDIX C: KÄHLER POTENTIAL FOR THE ASYMPTOTIC METRIC

In Sec. III we showed that the moduli space of 1/4 BPS configurations is a Kähler manifold. In this section, we check that the Kähler condition is satisfied for the asymptotic metric obtained from the charged particle analysis in Sec. V. From Eq. (5.21), we can read the asymptotic metric as

$$g_{(A,i)\overline{(A,i)}} = \frac{\langle\Phi^{A-1}\rangle - \langle\Phi^A\rangle}{2} + \frac{1}{2} \sum_{(B,j)\neq(A,i)} C_{AB} \log\left|\frac{z_{\langle A\rangle i} - z_{\langle B\rangle j}}{L}\right|, \qquad \text{(C1)}$$

$$g_{(A,i)\overline{(B,j)}} = -\frac{1}{2} C_{AB} \log\left|\frac{z_{\langle A\rangle i} - z_{\langle B\rangle j}}{L}\right|, \qquad (A, i) \neq (B, j). \qquad \text{(C2)}$$

This metric can be obtained from the following Kähler potential:

$$K = \frac{\langle\Phi^{A-1}\rangle - \langle\Phi^A\rangle}{2}|z_{\langle A\rangle i}|^2 + \sum_{(A,i),(B,j)}{}^{\prime} \frac{C_{AB}}{2}\left(\log\left|\frac{z_{\langle A\rangle i} - z_{\langle B\rangle j}}{L}\right| - 1\right)|z_{\langle A\rangle i} - z_{\langle B\rangle j}|^2, \tag{C3}$$

where $\sum'$ means that the sum is taken only once for a pair of the index sets (A, i) and (B, j) such that $(A, i) \neq (B, j)$. The existence of the Kähler potential means that the asymptotic metric equation (C2) obtained in the charged particle analysis is a Kähler metric. The normalizable part of the moduli space, which is free from the divergence in the $L \to \infty$ limit, is a subspace defined by

$$\begin{aligned} z_{\langle 1\rangle i} &= \text{const.} \qquad (i = 1, \cdots, k_1), \\ z_{\langle N\rangle i} &= \text{const.} \qquad (i = 1, \cdots, k_N), \\ \sum_{i=1}^{k_A} z_{\langle A\rangle i} &= \text{const.} \qquad (A = 2, \cdots, N-1). \end{aligned} \tag{C4}$$

The metric on this complex submanifold is given by the induced metric of (C1) and (C2). The pullback of the Kähler form

$$\omega \equiv i\partial\bar{\partial}K = i\sum_{(A,i),(B,j)} g_{(A,i)\overline{(B,j)}} dz_{\langle A\rangle i} \wedge d\bar{z}_{\langle B\rangle j} \tag{C5}$$

onto the subspace equation (C4) gives a closed two-form ω^*. This is because the Kähler form equation (C5) is a closed two-form and the exterior derivative commutes with pullback. Therefore, the submanifold equation (C4) is also a Kähler manifold endowed with the Kähler form ω^*, which is finite in the infinite cutoff limit $L \to \infty$.

APPENDIX D: DUAL EFFECTIVE THEORY ON MULTIPLE DOMAIN WALLS

The effective theory on $N(\geq 2)$ domain walls is described by the positions of N domain walls $X^A = \tilde{X}^A/\Delta m_A$ and the associated phases σ^A $(A = 1, 2, \cdots, N)$ as

$$\mathcal{L}_{\rm w} = \tfrac{1}{2}\mathcal{G}_{AB}(\partial_\mu \tilde{X}^A \partial^\mu \tilde{X}^B + \partial_\mu \sigma^A \partial^\mu \sigma^B). \tag{D1}$$

Here $\mathcal{G}_{AB}$ is the Kähler metric on the domain wall moduli space and depends only on $\tilde{X}^A$

$$\mathcal{G}_{AB}(\tilde{X}) = \frac{1}{2}\frac{\partial^2 K_{\rm w}}{\partial \tilde{X}^A \partial \tilde{X}^B}. \tag{D2}$$

When all the domain walls are well separated $X^1 \gg X^2 \gg \cdots \gg X^N$, the metric of the domain wall moduli space is a flat metric

$$\mathcal{G}_{AB} \simeq \frac{c}{\Delta m_A}\delta_{AB}. \tag{D3}$$

We would like to obtain the dual Lagrangian by dualizing the periodic scalar fields σ^A. First, let us define scalar fields $\Phi^A (A = 1, 2, \cdots, N)$ by

$$\Phi_A \equiv \pi\frac{\partial K_{\rm w}}{\partial \tilde{X}^A}. \tag{D4}$$

The derivative of Φ_A with respect to $\tilde{X}^B$ gives the metric on the domain wall moduli space as

$$\mathcal{G}_{AB} = \frac{1}{2\pi}\frac{\partial \Phi_A}{\partial \tilde{X}^B} = \frac{1}{2\pi}\frac{\partial \Phi_B}{\partial \tilde{X}^A}. \tag{D5}$$

Since we can assume the existence of an inverse of the metric

$$\mathcal{G}^{AC}\mathcal{G}_{CB} = \delta^A_B, \qquad \mathcal{G}^{AB} = 2\pi\frac{\partial \tilde{X}^A}{\partial \Phi_B} = 2\pi\frac{\partial \tilde{X}^B}{\partial \Phi_A}, \tag{D6}$$

the set of $\Phi_A (A = 1, 2, \cdots, N)$ can be interpreted as new coordinates on the domain wall moduli space. Note that the definition of Φ_A is not invariant under the Kähler transformation

$$K_{\rm w}(\phi, \bar{\phi}) \to K_{\rm w}(\phi, \bar{\phi}) + f(\phi) + \overline{f(\phi)}. \tag{D7}$$

However, we can always fix the definition of Φ_A so that the asymptotic values of Φ_A take the form

$$\Phi_A \simeq 2\pi c X^A \tag{D8}$$

when all the domain walls are well separated. Next let us define one-form fields $\tilde{F}^A_\mu (A = 1, 2, \cdots, N)$ by

$$\tilde{F}^A_\mu \equiv \partial_\mu \sigma^A, \tag{D9}$$

and interpret them as new dynamical fields obeying the Bianchi identity $\epsilon^{\mu\nu\rho}\partial_\nu \tilde{F}^A_\rho = 0$. In order to rewrite the Lagrangian in terms of $\tilde{F}^A_\mu$, we have to add a term with Lagrange multipliers $A_{A\mu}$

$$\mathcal{L}_F \propto \epsilon^{\mu\nu\rho} A_{A\mu}\partial_\nu \tilde{F}^A_\rho. \tag{D10}$$

Then, if we eliminate $\tilde{F}^A_\mu$ using the equation of motion, we obtain $U(1)^N$ gauge theory with N neutral scalar fields

$$\tilde{\mathcal{L}}_{\rm w} = \frac{1}{4\pi^2}\mathcal{G}^{AB}\left(\frac{1}{2}\partial_\mu \Phi_A \partial^\mu \Phi_B - \frac{1}{4}F_{A\mu\nu}F_B^{\mu\nu}\right), \tag{D11}$$

where $F_{A\mu\nu} = \partial_\mu A_{A\nu} - \partial_\nu A_{A\mu}$. When all the domain walls are well separated, the effective Lagrangian is simply given by

$$\begin{aligned} \tilde{\mathcal{L}}_{\rm w} &\simeq \sum_{A=1}^{N}\left(\frac{1}{2e_A^2}\partial_\mu \Phi_A \partial^\mu \Phi_A - \frac{1}{4e_A^2}F_{A\mu\nu}F_A^{\mu\nu}\right), \\ e_A^2 &\equiv \frac{4\pi^2 c}{\Delta m_A}. \end{aligned} \tag{D12}$$

The new scalar fields Φ_A have also an interesting physical meaning. We have assumed that scalar fields X^A represent positions of domain walls. However, it is not precisely correct when a domain wall approaches to another domain wall. Let us focus on the $(A-1)$th and Ath domain walls and define their center of mass X_0 and the relative distance R_A by

$$\Delta m_{A-1}X^{A-1} = \Delta m_{A-1}X_0 + \frac{\mu_A}{2}R_A, \qquad \Delta m_A X^A = \Delta m_A X_0 - \frac{\mu_A}{2}R_A, \tag{D13}$$

where μ_A is defined by

$$\mu_A \equiv \frac{2\Delta m_{A-1}\Delta m_A}{\Delta m_{A-1}+\Delta m_A}. \tag{D14}$$

The relative distance R_A can be negative, which does not mean the interchange of domain walls but the compression of two walls, namely, they become a single wall in the limit of $R_A \to -\infty$. Therefore, the parameter R_A loses its meaning as relative distance when the distance between the walls becomes small. An interesting property of the new coordinates Φ_A is that their differences are bounded from below by boojum charges defined in Eq. (5.26),

$$\frac{2\pi}{\mu_A}\frac{\partial K}{\partial R_A} = \Phi_{A-1} - \Phi_A = |B_g^{A-1}| + |B_g^A| + \mathcal{O}(e^{\mu_A R_A}), \qquad R_A \ll -\mu_A. \tag{D15}$$

Since $|B_g^A|/\pi c = 2\Delta m_A/g^2 c$ is equal to the width of the Ath domain wall, $(|B_g^{A-1}| + |B_g^A|)/2\pi c$ can be interpreted as the lower bound of distance between the middle points of the $(A-1)$th and Ath domain walls. Therefore, $\Phi_A/2\pi c$ instead of X^A represents the correct position of the Ath domain wall since it has the correct lower bound equation (D15) and asymptotically coincides with X^A (see Fig. 9).

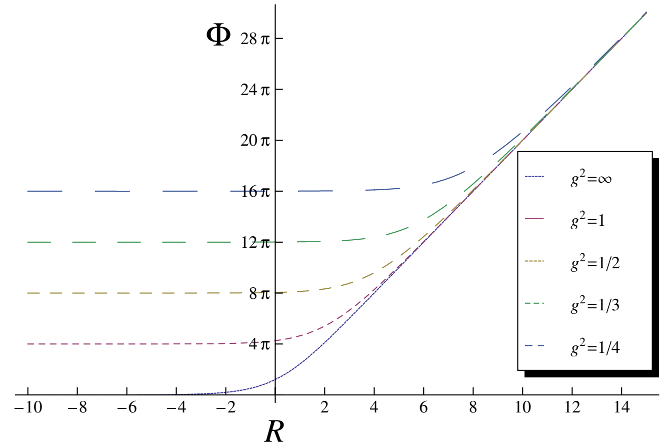

FIG. 9 (color online). The profiles of $\Phi \equiv \Phi_1 - \Phi_2$ as the function of $R \equiv X^1 - X^2$ obtained by numerical calculations for $N_F = 3$, $c = 1$, $m_1 = 1$, $m_2 = 0$, $m_3 = -1$. The function Φ approaches to the constant value $|B_g^1| + |B_g^2| = 4\pi\Delta m/g^2$ as the parameter R becomes small.

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