PHYSICAL REVIEW D **79,** 045014 (2009)

Trace anomaly and massless scalar degrees of freedom in gravity

Maurizio Giannotti* and Emil Mottola†

Theoretical Division, T-2, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA

(Received 26 November 2008; published 18 February 2009)

The trace anomaly of quantum fields in electromagnetic or gravitational backgrounds implies the existence of massless scalar poles in physical amplitudes involving the stress-energy tensor. Considering first the axial anomaly and using QED as an example, we compute the full one-loop triangle amplitude of the fermionic stress tensor with two current vertices, $\langle T^{\mu\nu}J^{\alpha}J^{\beta}\rangle$, and exhibit the scalar pole in this amplitude associated with the trace anomaly, in the limit of zero electron mass $m \to 0$. To emphasize the infrared aspect of the anomaly, we use a dispersive approach and show that this amplitude and the existence of the massless scalar pole is determined completely by its ultraviolet finite terms, together with the requirements of Poincaré invariance of the vacuum, Bose symmetry under interchange of J^{α} and J^{β}, and vector current and stress-tensor conservation. We derive a sum rule for the appropriate positive spectral function corresponding to the discontinuity of the triangle amplitude, showing that it becomes proportional to $\delta(k^2)$ and therefore contains a massless scalar intermediate state in the conformal limit of zero electron mass. The effective action corresponding to the trace of the triangle amplitude can be expressed in local form by the introduction of two scalar auxiliary fields which satisfy massless wave equations. These massless scalar degrees of freedom couple to classical sources, contribute to gravitational scattering processes, and can have long range gravitational effects.

DOI: 10.1103/PhysRevD.79.045014 PACS numbers: 03.70.+k, 04.60.−m, 04.62.+v, 12.20.−m

I. INTRODUCTION

Quantum effects are most commonly associated with short distance physics. The basic reason for this is that the fluctuations of most fields have a finite correlation length, and hence their correlations fall off rapidly at large distances. For relativistic fields which are massive, the correlation length is m^{-1}, and the fall off is exponential, e^{-mr}. This is an elementary example of decoupling. In the limit of very large mass, $m \to \infty$, the quantum effects of a heavy field become negligible at any finite distance scale.

In the opposite limit of massless fields $m \to 0$, the correlation length becomes infinite. Decoupling no longer holds, and it becomes possible for quantum correlations to extend over very great distances on even macroscopic scales. Such infrared effects are more pronounced the lower the spacetime dimensionality. In $d = 2$ the two-point propagator function of a free massless scalar field, rather than falling off, grows logarithmically in the separation of the points. Massless conformal field theories in $d = 2$ have been studied extensively by a variety of methods, and it is clear that their fluctuations are non-negligible and have important physical consequences in the infrared [1].

In massless field theories in two dimensions, the conformal group algebra and its central extension play important roles. The central term, a Schwinger term in the commutation algebra of stress tensors in flat space, may also be recognized as the trace anomaly of the stress-energy tensor defined in curved space [1,2]. The anomaly in the trace of the stress-energy tensor, corresponds to a well-defined additional term in the effective action which has long range effects [3]. As one illustrative example of these infrared effects, one can show that the curved space anomaly modifies the critical scaling exponents of the two-dimensional Ising model at its second order phase transition point [4]. This modification of the critical exponents is associated with the fluctuations of the spacetime metric at large distance scales. The gravitational metric fluctuations may be described by an additional massless scalar field, the conformal or Liouville mode, whose dynamics is generated and required by the conformal anomaly.

In two dimensions the central term related to the anomaly in curved space can be seen already in the two-point correlation of stress tensors, $\langle T^{\mu\nu}(x)T^{\alpha\beta}(y)\rangle$. The meaning of this term is most clearly deduced from the momentum space representation, where the corresponding amplitude exhibits a massless pole in the conformal limit [5]. This massless pole corresponds to a $\delta(k^2)$ in the corresponding imaginary part, describing a propagating massless scalar degree of freedom in the two-particle intermediate state of the cut diagram. The same kinematics applies to the pole and massless scalar state in the correlator of electromagnetic currents $\langle J^{\mu}(x)J^{\nu}(y)\rangle$ in the Schwinger model of two-dimensional massless electrodynamics [6,7]. The infrared effects of the anomaly may be understood as the result of the fluctuations of this additional massless degree of freedom.

In dimensions greater than two, infrared effects due to anomalies are both more subtle and less well studied. Since the correlation functions of canonical free theories fall off as power laws at large distances for $d > 2$, at first sight

*maurizio@lanl.gov
†emil@lanl.gov

there would seem to be little possibility of enhanced infrared effects in higher dimensions. QCD is a notable counterexample, where the growth of the effective coupling at larger distances leads to large quantum fluctuations and infrared confinement. The renormalization group flow $\beta(g^2)$ of the coupling arises from the same breaking of scale invariance by quantum fluctuations which gives rise to the conformal anomaly [8].

In the most familiar case of the axial anomaly, a massless pseudoscalar pole does appear in the triangle amplitude $\langle J_5^\mu J^\alpha J^\beta \rangle$ [9,10], in the chiral limit of vanishing fermion mass, a feature we review in the next section. This example of the axial anomaly in massless quantum electrodynamics (QED) shows that infrared relevant fluctuations due to anomalies can occur even in $d = 4$, and that triangle amplitudes are the simplest ones to reveal these effects. In QCD, the lightest pseudoscalar state is the pion, whose mass vanishes in the chiral limit of zero quark mass. By identifying this state with the massless pole appearing in the perturbative anomaly in the chiral limit, the low energy rate of neutral pion decay, $\pi^0 \to 2\gamma$ is determined by the short distance colored quark degrees of freedom in the one-loop $\langle J_5^\mu J^\alpha J^\beta \rangle$ amplitude [11,12]. The agreement of the measured rate with the coefficient obtained with $N_C = 3$ quarks is a striking confirmation of both QCD and the infrared effects of the anomaly. This well-known example of anomaly matching [13] shows that anomalies can provide a mechanism for short distance quantum degrees of freedom to have long distance or low energy consequences.

Although the special role of the triangle diagram in $d = 4$ has been emphasized in [14] in the context of the chiral anomaly some time ago, to date there has been no clear indication of a massless pole or infrared degrees of freedom in flat space amplitudes involving the energy-momentum tensor. It is known that in $d = 4$ the trace anomaly in curved space involves geometric invariants that are quadratic in the Riemnann curvature tensor [2,15]. This has the immediate consequence that the simplest amplitude in four dimensional flat spacetime that can show any direct evidence of the full curved space anomaly is the three-point function of stress tensors, $\langle T^{\mu\nu}(x) T^{\alpha\beta}(y) T^{\gamma\delta}(z) \rangle$, indicating again the importance of triangle amplitudes in $d = 4$.

In this paper we address the possibility for low energy quantum effects in gravity, analogous to those in gauge theories, due to the corresponding trace anomaly, and, in particular, for additional massless scalar degrees of freedom with long range effects which can modify the predictions of classical general relativity on macroscopic and even cosmological scales. We present a complete calculation of the one-loop triangle amplitude $\langle T^{\mu\nu} J^\alpha J^\beta \rangle$ in QED, for all values of the kinematical invariants. This amplitude contains the same basic kinematics as both the more familiar chiral triangle $\langle J_5^\mu J^\alpha J^\beta \rangle$, and the more complicated amplitude $\langle TTT \rangle$ involving three stress tensors. The $\langle TJJ \rangle$ amplitude is sensitive to the trace anomaly of the one-loop stress-tensor expectation value in a background electromagnetic potential A_μ (rather than a gravitational background curvature). By calculating this amplitude for arbitrary electron mass m, both the decoupling limit $m \to \infty$, and the conformal limit $m \to 0$, where the massless pole in the amplitude appears can be studied. It is the latter limit that reveals the consequences for low energy gravity.

Following methods that have been used previously for the chiral anomaly [16,17], we show that the $\langle TJJ \rangle$ amplitude can be determined completely from general principles of Poincaré invariance of the vacuum, Bose symmetry under interchange of J^α and J^β, and the Ward identities of vector current and stress-energy conservation, once its finite tensor components are given. These finite components can be determined unambiguously from the imaginary part of the cut triangle amplitude, with the real part obtained by dispersion relations which require no subtractions (other than charge renormalization in one particular component). This dispersive approach based upon the finite parts of the amplitude emphasizes the infrared aspect of the anomaly, making it clear that the anomaly is finite, well defined and uniquely determined, independent of UV regularization scheme, provided only that the amplitude is defined in a way consistent with the nonanomalous low energy symmetries of the theory.

In the conformal limit of massless QED, the two-particle intermediate state of the cut triangle diagram has a delta-function contribution at $k^2 = 0$. Because this state couples to the stress tensor, it contributes to gravitational scattering amplitudes at arbitrarily low energies. We demonstrate that the trace part of the $\langle TJJ \rangle$ amplitude containing this massless intermediate scalar state and its gravitational couplings may be described by the introduction of local massless scalar degrees of freedom, which render the trace part of the one-loop effective action local. The auxiliary field description introduced recently in the context of curved space [18] reproduces the trace part of the amplitude exactly in flat space, and the massless pole in the trace part of the flat space $\langle TJJ \rangle$ amplitude is precisely the propagator of these scalar fields.

In QED, the scalar state may also be understood as a two-particle correlation of $e^+ e^-$ which in the massless limit moves collinearly at the speed of light in a total spin-0 configuration. When the electron mass is nonzero, the singularity at $k^2 = 0$ is replaced by a resonance with a width of order m^2. However, the corresponding spectral function obeys a sum rule, which shows that although broadened, and eventually decoupled for larger m, the scalar state survives deformations away from the conformal limit. In this sense it behaves analogously to the pion in QCD.

The paper is organized as follows. In the next section we review the axial anomaly in QED in four dimensions, using

the spectral representation and dispersion relations to emphasize its infrared character, exhibiting the massless 0^- intermediate state, and the finite spectral sum rule in this case. In Sec. III we give the auxiliary field description of the chiral amplitude, showing that the massless pseudoscalar state can be described by a local effective field theory. In Sec. IV we turn to the main task of evaluating the $\langle TJJ\rangle$ amplitude in QED. Imposing the Ward identities, we show that the full amplitude is determined by its finite terms and imaginary parts for any m and its three kinematic invariants, independent of any specific UV regularization method. In Sec. V we evaluate its trace, isolate the anomaly, and discuss its relation to the β function and scaling violation. In Sec. VI we give the spectral representation of the $\langle TJJ\rangle$ amplitude, derive the corresponding finite sum rule, and show that a $\delta(k^2)$ appears in the appropriate spectral function in the conformal limit of massless electrons. In Sec. VI the foregoing results are compared with the auxiliary field representation of the anomaly given in [18] and are shown to coincide exactly in the trace sector. In Sec. VIII we show how the anomalous amplitude contributes to gravitational scattering of photons by a source, prove that the anomaly pole induces a massless scalar interaction and propagating intermediate state in this scattering process, and provide the effective action description of the scattering by scalar exchange. Finally, Sec. IX contains a concise summary of our results. Technical details of extracting the finite parts of the $\langle TJJ\rangle$ amplitude are given in Appendix A, while the proofs of some identities needed in the text are given in Appendix B.

II. THE AXIAL ANOMALY IN QED

In order to exhibit the relationship between anomalies and massless degrees of freedom, we review first the familiar case of the axial anomaly in QED in this section [11,19,20]. Although the triangle anomaly has been known for quite some time, the general behavior of the amplitude off the photon mass shell, its spectral representation, the appearance of a massless pseudoscalar pole, and its infrared aspects generally have received only limited attention [9,21]. It is this generally less emphasized infrared character of the axial anomaly upon which we focus here.

The vector and axial currents in QED are defined by[1]

$$J^\mu(x) = \bar\psi(x)\gamma^\mu\psi(x), \tag{1a}$$

$$J_5^\mu(x) = \bar\psi(x)\gamma^\mu\gamma^5\psi(x). \tag{1b}$$

The Dirac equation,

$$-i\gamma^\mu(\partial_\mu - ieA_\mu)\psi + m\psi = 0, \tag{2}$$

implies that the vector current is conserved,

$$\partial_\mu J^\mu = 0, \tag{3}$$

while the axial current apparently obeys

$$\partial_\mu J_5^\mu = 2im\bar\psi\gamma^5\psi \qquad \text{(classically)}. \tag{4}$$

In the limit of vanishing fermion mass $m \to 0$, the classical Lagrangian has a $U_{\rm ch}(1)$ global symmetry under $\psi \to e^{i\alpha\gamma^5}\psi$, in addition to $U(1)$ local gauge invariance, and J_5^μ is the Noether current corresponding to this chiral symmetry. As is well known, both symmetries cannot be maintained simultaneously at the quantum level. Let us denote by $\langle J_5^\mu(z)\rangle_A$ the expectation value of the chiral current in the presence of a background electromagnetic potential A_μ. Enforcing $U(1)$ gauge invariance (3) on the full quantum theory leads necessarily to a finite axial current anomaly,

$$\partial_\mu\langle J_5^\mu\rangle_A|_{m=0} = \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = \frac{e^2}{2\pi^2}\mathbf{E}\cdot\mathbf{B}, \tag{5}$$

in a background electromagnetic field.

The second variation of $\langle J_5^\mu(z)\rangle_A$ with A_μ then set to zero,

$$\Gamma^{\mu\alpha\beta}(z;x,y) \equiv -i\frac{\delta^2\langle J_5^\mu(z)\rangle_A}{\delta A_\alpha(x)\delta A_\beta(y)}\bigg|_{A=0} = -i(ie)^2\langle\mathcal{T}J_5^\mu(z)J^\alpha(x)J^\beta(y)\rangle|_{A=0} \tag{6}$$

is thus the primary quantity of interest. By translational invariance of the Minkowski vacuum at $A = 0$, this amplitude depends only upon the coordinate differences $x - z$ and $y - z$. Hence with no loss of generality we may fix $z = 0$. Taking the Fourier transform of (6) and removing the factor of total momentum conservation, $(2\pi)^4\delta^4(k - p - q)$, we obtain

$$\Gamma^{\mu\alpha\beta}(p,q) \equiv -i\int d^4x\int d^4y e^{ip\cdot x+iq\cdot y}\frac{\delta^2\langle J_5^\mu(0)\rangle_A}{\delta A_\alpha(x)\delta A_\beta(y)}\bigg|_{A=0} = ie^2\int d^4x\int d^4y e^{ip\cdot x+iq\cdot y}\langle\mathcal{T}J_5^\mu(0)J^\alpha(x)J^\beta(y)\rangle|_{A=0}. \tag{7}$$

At the lowest one-loop order it is given by the triangle diagram of Fig. 1, plus the Bose symmetrized diagram with the photon legs interchanged. The chiral current expectation value in position space can be reconstructed from this momentum space amplitude by

$$\langle J_5^\mu(z)\rangle_A = \frac{i}{2}\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\int d^4x\int d^4y e^{-ip\cdot(x-z)} \times e^{-iq\cdot(y-z)}\Gamma^{\mu\alpha\beta}(p,q)A_\alpha(x)A_\beta(y) + \ldots \tag{8}$$

up to second order in the gauge field background A_μ.

Elementary power counting indicates that the triangle diagram of Fig. 1 is superficially linearly divergent. The

[1]We use the conventions that $\{\gamma^\mu,\gamma^\nu\} = -2g^{\mu\nu} = 2\text{diag}(+---)$, so that $\gamma^0 = (\gamma^0)^\dagger$, and $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = (\gamma^5)^\dagger$ are Hermitian, and $\text{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\epsilon^{\mu\nu\rho\sigma}$, where $\epsilon^{\mu\nu\rho\sigma} = -\epsilon_{\mu\nu\rho\sigma}$ is the fully antisymmetric Levi-Civita tensor, with $\epsilon_{0123} = +1$.

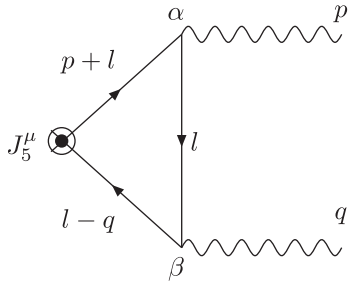

FIG. 1. The axial anomaly triangle diagram.

formal reason why (3) and (4) cannot both be maintained at the quantum level is that verifying them requires the ability to shift the loop momentum integration variable l in the triangle amplitude. Because the diagram is linearly divergent, such shifts are inherently ambiguous, and can generate finite extra terms. It turns out that there is no choice for removing the ambiguity which satisfies both the vector and chiral Ward identities simultaneously, and one is forced to choose between them. Thus although the ambiguity results in a well-defined finite term, the axial anomaly has most often been presented as inherently a problem of regularization of an apparently ultraviolet linearly divergent loop integral [11,19,20].

There is an alternative derivation of the axial anomaly that emphasizes instead its infrared character. The idea of this approach is to use the tensor structure of the triangle amplitude to extract its well-defined ultraviolet *finite* parts, which are homogeneous of degree three in the external momenta p and q. Then the remaining parts of the full amplitude may be determined by the joint requirements of Lorentz covariance, Bose symmetry under interchange of the two-photon legs, and electromagnetic current conservation,

$$p_\alpha \Gamma^{\mu\alpha\beta}(p,q) = 0 = q_\beta \Gamma^{\mu\alpha\beta}(p,q), \tag{9}$$

at the two vector vertices. By this method the full one-loop triangle contribution to $\Gamma^{\mu\alpha\beta}(p,q)$, becomes completely determined in terms of well-defined ultraviolet finite integrals which require no further regularization [16,17]. The divergence of the axial current may then be computed unambiguously, and one obtains (5) in the limit of vanishing fermion mass [11]. There is of course no contradiction between these two points of view, since it is the same Ward identities which are imposed in either method, and in the conformal limit of vanishing fermion mass the infrared and ultraviolet behavior of the triangle amplitude are one and the same.

Since we will apply this method to the trace anomaly amplitude in the next section, let us first review the calculation in the axial current case. One first uses the Poincaré invariance of the vacuum to assert that $\Gamma^{\mu\alpha\beta}(p,q)$ can be expanded in the set of all three-index pseudotensors constructible from the p and q, with the correct Lorentz transformation properties. There are exactly eight such pseudotensors, which are listed in Table I, the first two of which are linear in p or q, while the remaining six are homogeneous of degree three in the external momenta. Since the amplitude $\Gamma^{\mu\alpha\beta}(p,q)$ has mass dimension one, any regularization ambiguity can appear only in the coefficients of the tensors which are linear in momenta, i.e. $\varepsilon^{\mu\alpha\beta\lambda}p_\lambda$ and $\varepsilon^{\mu\alpha\beta\lambda}q_\lambda$. The coefficients of these tensors have mass dimension zero and are therefore potentially logarithmically divergent. On the other hand, the remaining six tensors in Table I, homogeneous of degree three in p and q can appear in $\Gamma^{\mu\alpha\beta}(p,q)$ multiplied only by scalar loop integrals with *negative* mass dimension, -2, which are completely convergent in the ultraviolet. If these scalar coefficient functions can be extracted unambiguously, then vector current conservation can be used to determine the coefficients of the remaining two tensors of dimension one. Indeed the general amplitude satisfying (9) must be a linear combination of only the six linear combinations defined below and listed in Table II. Since the tensors $\varepsilon^{\mu\alpha\beta\lambda}p_\lambda$ and $\varepsilon^{\mu\alpha\beta\lambda}q_\lambda$ appear only in those linear combinations which satisfy (9), their coefficients are determined unambiguously by the finite coefficients multiplying the tensors of degree three.

TABLE I. The 8 three-index pseudotensor monomials into which $\Gamma^{\mu\alpha\beta}(p,q)$ can be expanded.

$\epsilon^{\mu\alpha\beta\lambda}p_\lambda$	$\epsilon^{\mu\alpha\rho\sigma}p^\beta p_\rho q_\sigma$	$\epsilon^{\mu\beta\rho\sigma}p^\alpha p_\rho q_\sigma$	$\epsilon^{\alpha\beta\rho\sigma}p^\mu p_\rho q_\sigma$
$\epsilon^{\mu\alpha\beta\lambda}q_\lambda$	$\epsilon^{\mu\alpha\rho\sigma}q^\beta p_\rho q_\sigma$	$\epsilon^{\mu\beta\rho\sigma}q^\alpha p_\rho q_\sigma$	$\epsilon^{\alpha\beta\rho\sigma}q^\mu p_\rho q_\sigma$

To make the procedure of extraction of finite terms of the amplitude completely unambiguous, one may first calculate the imaginary part of the cut triangle amplitude in Fig. 2 at timelike k^2, which is finite, and then construct the real part by a dispersion relation. For the mass dimension -2 terms, the dispersion relations constructing the real parts of the amplitude from its imaginary parts are finite and require no subtractions [9,21].

To construct the tensors satisfying (9), let us define first the two-index pseudotensor,

$$\upsilon^{\alpha\beta}(p,q) \equiv \epsilon^{\alpha\beta\rho\sigma}p_\rho q_\sigma, \tag{10}$$

which satisfies

$$\upsilon^{\alpha\beta}(p,q) = \upsilon^{\beta\alpha}(q,p), \tag{11a}$$

$$p_\alpha \upsilon^{\alpha\beta}(p,q) = 0 = q_\beta \upsilon^{\alpha\beta}(p,q). \tag{11b}$$

TABLE II. The 6 third rank pseudotensors obeying (12).

i	$\tau_i^{\mu\alpha\beta}(p,q)$
1	$-p\cdot q\epsilon^{\mu\alpha\beta\lambda}p_\lambda - p^\beta \upsilon^{\mu\alpha}(p,q)$
2	$p^2\epsilon^{\mu\alpha\beta\lambda}q_\lambda + p^\alpha \upsilon^{\mu\beta}(p,q)$
3	$p^\mu \upsilon^{\alpha\beta}(p,q)$
4	$p\cdot q\epsilon^{\mu\alpha\beta\lambda}q_\lambda + q^\alpha \upsilon^{\mu\beta}(p,q)$
5	$-q^2\epsilon^{\mu\alpha\beta\lambda}p_\lambda - q^\beta \upsilon^{\mu\alpha}(p,q)$
6	$q^\mu \upsilon^{\alpha\beta}(p,q)$

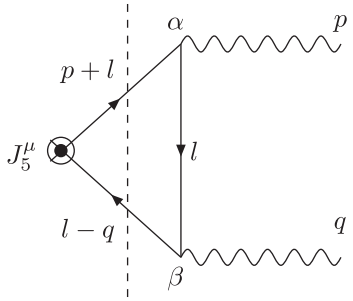

FIG. 2. Discontinuity or imaginary part of the triangle diagram obtained by cutting two lines.

By taking general linear combinations of the eight pseudotensors in Table I, we find then that there are exactly six third rank pseudotensors, $\tau_i^{\mu\alpha\beta}(p,q)$, $i=1,\ldots,6$ which can be constructed from them to satisfy the conditions (9),

$$p_\alpha \tau_i^{\mu\alpha\beta}(p,q) = 0 = \tau_i^{\mu\alpha\beta}(p,q) q_\beta = 0, \qquad i = 1,\ldots,6, \tag{12}$$

given in Table II. Hence we may express the amplitude (7) satisfying (9) in the form,

$$\Gamma^{\mu\alpha\beta}(p,q) = \sum_{i=1}^{6} f_i \tau_i^{\mu\alpha\beta}(p,q), \tag{13}$$

where $f_i = f_i(k^2; p^2, q^2)$ are dimension -2 scalar functions of the three invariants, p^2, q^2, and k^2.

We note also that the full amplitude (7) must be Bose symmetric,

$$\Gamma^{\mu\alpha\beta}(p,q) = \Gamma^{\mu\beta\alpha}(q,p). \tag{14}$$

Since $\tau_{i+3}^{\mu\alpha\beta}(p,q) = \tau_i^{\mu\beta\alpha}(q,p)$ for $i=1,2,3$, it follows that the six scalar coefficient functions f_i also fall into three Bose conjugate pairs, i.e.

$$f_1(k^2; p^2, q^2) = f_4(k^2; q^2, p^2), \tag{15a}$$
$$f_2(k^2; p^2, q^2) = f_5(k^2; q^2, p^2), \tag{15b}$$
$$f_3(k^2; p^2, q^2) = f_6(k^2; q^2, p^2), \tag{15c}$$

related by interchange of p^2 and q^2.

Actually, owing to the algebraic identity obeyed by the ϵ symbol,

$$g^{\alpha\beta}\epsilon^{\mu\nu\rho\sigma} + g^{\alpha\mu}\epsilon^{\nu\rho\sigma\beta} + g^{\alpha\nu}\epsilon^{\rho\sigma\beta\mu} + g^{\alpha\rho}\epsilon^{\sigma\beta\mu\nu} + g^{\alpha\sigma}\epsilon^{\beta\mu\nu\rho} = 0, \tag{16}$$

in four dimensions, the six tensors τ_i are not linearly independent, and form an overcomplete basis. The identity (16) leads to the relations,

$$\tau_3^{\mu\alpha\beta}(p,q) = \tau_1^{\mu\alpha\beta}(p,q) + \tau_2^{\mu\alpha\beta}(p,q), \tag{17a}$$
$$\tau_6^{\mu\alpha\beta}(p,q) = \tau_4^{\mu\alpha\beta}(p,q) + \tau_5^{\mu\alpha\beta}(p,q). \tag{17b}$$

Thus the tensors τ_3 and τ_6 could be eliminated completely by means of (17), and the full amplitude expressed entirely in terms of the linearly independent and complete basis set of only the four tensors, τ_1, τ_2, τ_4, and τ_5. Indeed this has been the general practice in the literature on the axial anomaly [11,16,19].

Eliminating these or any of the other two tensors is not necessary for our purposes, and we choose instead to work with the overcomplete set of six tensors listed in Table II. This will have the consequence that the coefficient functions f_i are determined only up to the freedom to choose arbitrary coefficients of the linear combinations (17), i.e. to shift each of the coefficients f_i by an arbitrary scalar function h via the rule,

$$f_1(k^2; p^2, q^2) \to f_1(k^2; p^2, q^2) + h(k^2; p^2, q^2), \tag{18a}$$
$$f_2(k^2; p^2, q^2) \to f_2(k^2; p^2, q^2) + h(k^2; p^2, q^2), \tag{18b}$$
$$f_3(k^2; p^2, q^2) \to f_3(k^2; p^2, q^2) - h(k^2; p^2, q^2), \tag{18c}$$

with the shift in f_4, f_5, f_6 determined by (15) by the interchange of p^2 and q^2. The arbitrary function h drops out of the final amplitude by use of (17).

The computation of the finite coefficients given in the literature [11,16] amounts to a specific choice of the arbitrary function h (by the order in which the γ matrix traces are performed), and yields

$$f_1 = f_4 = \frac{e^2}{\pi^2}\int_0^1 dx \int_0^{1-x} dy \frac{xy}{D}, \tag{19a}$$
$$f_2 = \frac{e^2}{\pi^2}\int_0^1 dx \int_0^{1-x} dy \frac{x(1-x)}{D}, \tag{19b}$$
$$f_5 = \frac{e^2}{\pi^2}\int_0^1 dx \int_0^{1-x} dy \frac{y(1-y)}{D}, \tag{19c}$$
$$f_3 = f_6 = 0, \tag{19d}$$

where the denominator of the Feynman parameter integral is given by

$$D \equiv p^2 x(1-x) + q^2 y(1-y) + 2p\cdot q xy + m^2 = (p^2 x + q^2 y)(1-x-y) + xyk^2 + m^2, \tag{20}$$

strictly positive for $m^2 > 0$, and spacelike momenta, k^2, p^2, $q^2 > 0$. Thus each of the dimension -2 scalar coefficient functions f_i in (19) are finite and free of any UV regularization ambiguities, and the full amplitude $\Gamma^{\mu\alpha\beta}(p,q)$ satisfying

(i) Lorentz invariance of the vacuum,
(ii) Bose symmetry (14),
(iii) vector current conservation (9),
(iv) unsubtracted dispersion relation of real and imaginary parts,

with the finite imaginary parts determined by the cut triangle diagram of Fig. 2, is given by (13) and (19), without any need of regularization of ultraviolet divergent loop integrals at any step.

Contraction of the finite amplitude $\Gamma^{\mu\alpha\beta}(p,q)$ with the momentum $k_\mu = (p+q)_\mu$ entering at the axial vector vertex can now be computed unambiguously, and we obtain

$$k_\mu \Gamma^{\mu\alpha\beta}(p,q) = \mathcal{A} v^{\alpha\beta}(p,q), \tag{21}$$

with

$$\begin{aligned}\mathcal{A} &= p\cdot q f_1 + p^2 f_2 + (p^2 + p\cdot q) f_3 + p\cdot q f_4 \\ &\quad + q^2 f_5 + (p\cdot q + q^2) f_6 \\ &= 2p\cdot q f_1 + p^2 f_2 + q^2 f_5,\end{aligned} \tag{22}$$

by (19). Substituting the explicit Feynman parameter integrals of (19) in (22), and using (20), (22) becomes

$$\begin{aligned}\mathcal{A}(k^2; p^2, q^2) &= \frac{e^2}{\pi^2}\int_0^1 dx \int_0^{1-x} dy \frac{D - m^2}{D} \\ &= \frac{e^2}{2\pi^2} - \frac{e^2}{\pi^2} m^2 \int_0^1 dx \int_0^{1-x} dy \frac{1}{D}.\end{aligned} \tag{23}$$

The second term proportional to m^2 is what would be expected from the naive axial vector divergence (4) [17]. The first term in (23) in which the denominator D is cancelled in the numerator is

$$\frac{e^2}{\pi^2}\int_0^1 dx \int_0^{1-x} dy = \frac{e^2}{2\pi^2}, \tag{24}$$

and which remains finite and nonzero in the limit $m \to 0$ is the axial anomaly.

Thus the finite anomalous term is unambiguously determined by our four requirements above, and may be clearly identified even for finite m, when the chiral symmetry is broken. This construction of the amplitude from only symmetry principles and its finite parts may be regarded as a proof that the same finite axial anomaly must arise in *any* regularization of the original triangle amplitude which respects these symmetries and leaves the finite parts unchanged. Explicit calculations in dimensional regularization and Pauli-Villars regularization schemes, which respect these symmetries confirm this [22].

The spectral representations for the triangle amplitude functions,

$$f_i(k^2; p^2, q^2) = \int_0^\infty ds \frac{\rho_i(s; p^2, q^2)}{k^2 + s}, \tag{25}$$

used to compute the finite parts also aid in the physical interpretation of the infrared aspect of the anomaly. If one defines the function,

$$\begin{aligned}S(x, y; p^2, q^2) &\equiv \frac{(p^2 x + q^2 y)(1 - x - y) + m^2}{xy} \\ &= \frac{D}{xy} - k^2,\end{aligned} \tag{26}$$

and substitutes the identity,

$$\frac{1}{D} = \int_0^\infty \frac{ds}{xy(k^2 + s)} \delta(s - S) \tag{27}$$

valid for $p^2, q^2, m^2 \geq 0$, into the expressions (19), interchanging the order of the s and x, y integrations, the spectral representation (25) of the amplitude is obtained, with

$$\rho_1(s; p^2, q^2) = \frac{e^2}{\pi^2}\int_0^1 dx \int_0^{1-x} dy \delta(s - S), \tag{28a}$$

$$\rho_2(s; p^2, q^2) = \frac{e^2}{\pi^2}\int_0^1 dx \int_0^{1-x} dy \frac{1-x}{y} \delta(s - S), \tag{28b}$$

$$\begin{aligned}\rho_5(s; p^2, q^2) &= \frac{e^2}{\pi^2}\int_0^1 dx \int_0^{1-x} dy \frac{1-y}{x} \delta(s - S) \\ &= \rho_2(s; q^2, p^2).\end{aligned} \tag{28c}$$

From the definition of the function $S(x, y; p^2, q^2)$ in (26) it follows that

$$2p\cdot q + p^2 \frac{1-x}{y} + q^2 \frac{1-y}{x} = \frac{D - m^2}{xy} = k^2 + S - \frac{m^2}{xy}, \tag{29}$$

and therefore, from (28),

$$2p\cdot q \rho_1 + p^2 \rho_2 + q^2 \rho_5 = (k^2 + s)\rho_{\mathcal{A}} - m^2 \rho_0, \tag{30}$$

where

$$\rho_{\mathcal{A}}(s; p^2, q^2) \equiv \rho_1(s; p^2, q^2), \quad \text{and} \tag{31a}$$

$$\rho_0(s; p^2, q^2) \equiv \frac{e^2}{\pi^2}\int_0^1 dx \int_0^{1-x} dy \frac{1}{xy} \delta(s - S). \tag{31b}$$

The relation (30) with (31) for the imaginary part of the cut triangle amplitude can be compared to (22) and (23) for the corresponding real part. Defined for Euclidean spacelike four-momenta, $k^2, p^2, q^2 > 0$, they are continued to timelike four-momenta $k^2 < 0$, by means of an $-i\epsilon$ prescription in the denominators of (25). Then the imaginary part of the chiral amplitude (7), corresponding to the cut diagram illustrated in Fig. 2 is given by (30) evaluated at $s = -k^2 > 0$, i.e.,

$$(2p\cdot q \rho_1 + p^2 \rho_2 + q^2 \rho_5)|_{s=-k^2} = -m^2 \rho_0|_{s=-k^2}, \tag{32}$$

which shows that $\rho_{\mathcal{A}} = \rho_1$ drops out of (30) for $s = -k^2$ on shell, and the finite imaginary part of the amplitude is completely nonanomalous for timelike k^2.

The anomaly in Re $\mathcal{A}$ comes about because of the cancellation of the $k^2 + s$ in the denominator of the unsubtracted dispersion integrals (25) and the same factor in the spectral function sum, (30), resulting in the finite integral,

$$\begin{aligned}\int_0^\infty ds \rho_{\mathcal{A}}(s; p^2, q^2) &= \frac{e^2}{\pi^2}\int_0^1 dx \int_0^{1-x} dy \int_0^\infty ds \delta(s - S) \\ &= \frac{e^2}{2\pi^2},\end{aligned} \tag{33}$$

independent of $p^2, q^2, m^2 \geq 0$. Thus the anomalous divergence of the axial vector current is tied to an ultraviolet finite *sum rule* (33), for the associated spectral density $\rho_{\mathcal{A}}(s) = \rho_1(s)$.

The finite sum rule (33) and relationship (30) between the spectral functions is critical to the infrared aspect of the axial anomaly. Using $2p \cdot q = k^2 - p^2 - q^2$, and rearranging (30) we find

$$\rho_{\mathcal{A}} = \frac{p^2}{s}(\rho_2 - \rho_1) + \frac{q^2}{s}(\rho_5 - \rho_1) + \frac{m^2}{s}\rho_0. \tag{34}$$

It is easy to see from the Feynman parameter representations (28) that the differences, $\rho_2 - \rho_1$ and $\rho_5 - \rho_1$, are positive for spacelike p^2, q^2. Hence the function obeying the sum rule (33) is expressed in (34) as a sum of nonnegative contributions for spacelike or null p^2 and q^2. If the limits $p^2, q^2, m^2 \to 0^+$ are taken (in any order), some of the spectral functions $\rho_2 - \rho_1$, $\rho_2 - \rho_1$, and ρ_0 develop logarithmic singularities, but each term on the right side of (34) multiplied by p^2, q^2, or m^2 approaches zero. Hence the spectral function $\rho_{\mathcal{A}}$ *vanishes* pointwise for all $s > 0$ in this combined limit. In order for this to be consistent with the sum rule (33), $\rho_{\mathcal{A}}(s)$ must develop a $\delta(s)$ singularity at $s = 0$ in this limit. It is straightforward either to calculate the function $\rho_{\mathcal{A}}(s; q^2, p^2)$ from the relations above for any p^2, q^2, m^2 and verify the appearance of a more and more sharply peaked spectral density in the limits, $p^2, q^2, m^2 \to 0^+$, or alternatively, to evaluate $\rho_{\mathcal{A}} = \rho_1$ directly from (28a) in this limit, where from (26) the function $S(x, y; 0, 0)$ vanishes identically. Then by interchanging the limits and integrations over x, y, from (28a) and (31a) we obtain

$$\begin{aligned}\lim_{p^2,q^2,m^2\to 0^+} \rho_{\mathcal{A}}(s; p^2, q^2) &= \frac{e^2}{\pi^2}\int_0^1 dx \int_0^{1-x} dy\, \delta(s) \\ &= \frac{e^2}{2\pi^2}\delta(s).\end{aligned} \tag{35}$$

Hence the appearance of the $\delta(s)$ in this limit is explicit in this representation. This delta function shows that a massless pseudoscalar appears in the intermediate state of the cut triangle amplitude when $p^2 = q^2 = m^2 = 0$.

To examine this infrared behavior in more detail, it is instructive to consider the case of $p^2 = q^2 = 0$, while still retaining m as an infrared regulator. In this case from (26), $S = m^2/xy$, and we easily find[2]

$$\begin{aligned}&\rho_{\mathcal{A}}(s; p^2, q^2)|_{p^2=q^2=0} \\ &= \frac{e^2}{\pi^2}\int_0^1 dx \int_0^{1-x} dy\, \delta\left(s - \frac{m^2}{xy}\right) \\ &= \frac{m^2}{s}\rho_0(s; 0, 0) \\ &= \frac{e^2}{\pi^2}\frac{m^2}{s^2}\ln\left\{\frac{1 + \sqrt{1 - \frac{4m^2}{s}}}{1 - \sqrt{1 - \frac{4m^2}{s}}}\right\}\theta(s - 4m^2).\end{aligned} \tag{36}$$

As expected for $m^2 \to 0$ the spectral function (36) vanishes pointwise for all $s > 0$. However by making the change of variables $s = 4m^2/(1 - u^2)$ it is straightforward to verify that the integral over s of the spectral function (36) is independent of m and given by (33). Because of (22) and (30) with $2p \cdot q = k^2$, the spectral integral (25) for the full amplitude f_1 is

$$\begin{aligned}f_1(k^2)|_{p^2=q^2=0} &= \frac{1}{k^2}\left[\frac{e^2}{2\pi^2} - m^2\int_0^\infty ds\, \frac{\rho_0(s; 0, 0)}{k^2 + s}\right] \\ &= \frac{\mathcal{A}(k^2; 0, 0; m^2)}{k^2},\end{aligned} \tag{37}$$

due to the sum rule (33). This shows that the amplitude f_1 develops a pole at $k^2 = 0$ when $p^2 = q^2 = m^2 = 0$, corresponding to the $\delta(s)$, (35) in the imaginary part in the same limit [9].

When the fermion mass is nonzero the amplitude (37) can also be written in the form,

$$f_1(k^2)|_{p^2=q^2=0} = m^2\int_{4m^2}^\infty \frac{ds}{s}\frac{\rho_0(s; 0, 0)}{k^2 + s}, \tag{38}$$

which shows that the amplitude has no pole divergence as $k^2 \to 0$ with $m^2 > 0$ fixed [23]. Because of the sum rule and relations (36), the residue $\mathcal{A}(k^2; 0, 0; m^2)$ of the pole vanishes in this limit. This may be understood as a consequence of decoupling, for with no other scales remaining, the limit $k^2/m^2 \to 0$ is equivalent to the limit $m^2 \to \infty$ with k^2 fixed, in which case the entire fluctuation represented by the triangle diagram should vanish on physical grounds. The spectral function representation and determination of the anomaly by its finite parts builds in this decoupling limit $m^2 \to \infty$ automatically. This decoupling behavior is also inherent in the Pauli-Villars regularization of the triangle amplitude, since the first anomalous first term in (23) is exactly the negative of the second term in the limit of infinite mass. Conversely, if $m = 0$ then the amplitude (37) behaves like k^{-2} for all k^2, in both the infrared and ultraviolet, as would be expected for a conformal theory with no intrinsic mass or momentum scale.

For comparison we may consider $\rho_2(s)$, given by

$$\rho_2(s; 0, 0) = \rho_5(s; 0, 0) = \frac{e^2}{2\pi^2}\frac{1}{s}\sqrt{1 - \frac{4m^2}{s}}\theta(s - 4m^2). \tag{39}$$

[2]The conjectured form of the spectral function of Ref. [10], Eq. (11.50), disagrees with the exact result, (36), although the qualitative conclusions are unchanged.

Because of its slower fall off with s, the integral of $\rho_2(s)$ over s does not converge, and does not obey a finite sum rule. Its corresponding amplitude,

$$f_2|_{p^2=q^2=0} = \frac{e^2}{2\pi^2 k^2}\left\{-2+\sqrt{1+\frac{4m^2}{k^2}}\ln\left[\frac{\sqrt{1+\frac{4m^2}{k^2}}+1}{\sqrt{1+\frac{4m^2}{k^2}}-1}\right]\right\} \to \frac{e^2}{2\pi^2 k^2}\ln\left(\frac{k^2}{m^2}\right), \tag{40}$$

does not possess a pole or even a finite limit as $m^2 \to 0$ or $k^2/m^2 \to \infty$. Thus this limit, equivalent to m^2 fixed and $k^2 \to \infty$ is purely ultraviolet in character, and cannot be interpreted in terms of an infrared massless state with a finite spectral weight.

The full amplitude (13) for $p^2 = q^2 = 0$ becomes

$$\Gamma^{\mu\alpha\beta}(p,q)|_{p^2=q^2=0} = f_1(k^2;0,0)[\tau_1^{\mu\alpha\beta}(p,q)+\tau_4^{\mu\alpha\beta}(p,q)] + f_2(k^2;0,0)[\tau_2^{\mu\alpha\beta}(p,q)+\tau_5^{\mu\alpha\beta}(p,q)] = f_1(k^2;0,0)k^\mu v^{\alpha\beta}(p,q) + [f_2(k^2;0,0)-f_1(k^2;0,0)][\tau_2^{\mu\alpha\beta}(p,q)+\tau_5^{\mu\alpha\beta}(p,q)], \tag{41}$$

by use of the identities, (17). For vanishing $p^2 = q^2 = 0$, the tensors,

$$\tau_2^{\mu\alpha\beta}(p,q)|_{p^2=q^2=0} = p^\alpha v^{\mu\beta}(p,q), \tag{42a}$$

$$\tau_5^{\mu\alpha\beta}(p,q)|_{p^2=q^2=0} = -q^\beta v^{\mu\alpha}(p,q), \tag{42b}$$

have zero contraction with photon wave amplitudes obeying the transversality condition, $p^\alpha \tilde{A}_\alpha(p) = q^\beta \tilde{A}_\beta(q) = 0$. Hence the term involving the $\tau_2 + \tau_5$ in (41) drops out entirely in the full matrix element of $J_5^\mu(0)$ between the vacuum and a physical two-photon state $|p,q\rangle$, giving simply

$$\langle 0|J_5^\mu(0)|p,q\rangle = i\Gamma^{\mu\alpha\beta}(p,q)\tilde{A}_\alpha(p)\tilde{A}_\beta(q)|_{p^2=q^2=0} = if_1(k^2;0,0)k^\mu v^{\alpha\beta}(p,q)\tilde{A}_\alpha(p)\tilde{A}_\beta(q) \xrightarrow[m\to 0]{} \frac{ie^2}{2\pi^2 k^2}k^\mu v^{\alpha\beta}(p,q)\tilde{A}_\alpha(p)\tilde{A}_\beta(q), \tag{43}$$

where the last line follows from (37) for $m = 0$. This exhibits the pole at $k^2 = (p+q)^2 = 0$. Thus the singular infrared behavior required by the anomaly survives in the full on shell matrix element to physical transverse photons. The residue of the pole is determined by the anomalous divergence,

$$\langle 0|\partial_\mu J_5^\mu(0)|p,q\rangle = ik_\mu\langle 0|J_5^\mu(0)|p,q\rangle = -\frac{e^2}{2\pi^2}v^{\alpha\beta}(p,q)\tilde{A}_\alpha(p)\tilde{A}_\beta(q) \tag{44}$$

to be nonvanishing when $m = 0$.

By examining the expressions above one can see that the full amplitude exhibits propagating polelike behavior for $k^2 \gg p^2, q^2, m^2$, while for finite p^2, q^2, m^2, the pole appears to soften into a resonance and there is no singularity when $k^2 \leq \min(|p^2|, |q^2|, m^2)$. Thus a strict infrared pole at $k^2 = 0$ exists in the triangle amplitude only for zero mass fermions, and it couples to the physical amplitude only when $p^2 = q^2 = m^2 = 0$, giving the full answer for the on shell matrix element (43) of the chiral current to physical transverse photons only in this case [23]. However, because of the sum rule (33), the pseudoscalar state implied by the anomaly is present at any momentum or mass scale, while from (37) its coupling to $\langle J_5^\mu\rangle$ and photons simply becomes weaker for $k^2 \leq \min(|p^2|, |q^2|, m^2)$, and decouples entirely as $k^2 \to 0$ for any of $(|p^2|, |q^2|, m^2)$ finite.

This appearance of a massless pseudoscalar in the chiral amplitude (7) in the two-fermion intermediate state in the limit of massless fermions is reminiscent of the Schwinger model, i.e. massless electrodynamics in $1+1$ dimensions, where it is also related to the anomaly [6]. In each case one can use the fermion mass as an infrared regulator to examine the appearance of the anomaly pole in the amplitude or delta function in its imaginary part as the limit $m^2 \to 0$, for $k^2 < 0$ timelike. In each case when one finally arrives at the limit of null four-momenta, the intermediate state which gives rise to the pole is a massless electron-positron pair moving exactly collinearly at the speed of light [24]. Thus even the $3+1$ dimensional case becomes effectively $1+1$ dimensional in this limit, which accounts for the infrared enhancement. The only essential difference between $d = 2$ and $d = 4$ dimensions appears to be the necessity of going to a more complicated three-point amplitude in the $d = 4$ case to reveal the anomaly pole. The special role of the kinematics of the triangle diagram for this infrared enhancement in gauge theories in $3+1$ dimensions has been emphasized previously in [14].

III. THE AUXILIARY FIELD DESCRIPTION OF THE AXIAL ANOMALY AND ANOMALOUS CURRENT COMMUTATORS

The appearance of a massless pseudoscalar pole in the triangle anomaly amplitude suggests that this can be described as the propagator of a pseudoscalar field which couples to the axial current. Indeed it is not difficult to find the field description of the pole. To do so let us note first that the axial current expectation value $\langle J_5^\mu\rangle_A$ can be obtained from an extended action principle in which we introduce an axial vector field, $\mathcal{B}_\mu$ into the Dirac

Lagrangian,

$$i\bar{\psi}\gamma^{\mu}(\overrightarrow{\partial}_{\mu} - ieA_{\mu})\psi - m\bar{\psi}$$
$$\rightarrow i\bar{\psi}\gamma^{\mu}(\overrightarrow{\partial}_{\mu} - ieA_{\mu} - ig\gamma^{5}\mathcal{B}_{\mu})\psi - m\bar{\psi}\psi \quad (45)$$

so that the variation of the corresponding action with respect to $\mathcal{B}_{\mu}$ gives

$$\frac{\delta\mathcal{S}}{\delta\mathcal{B}_{\mu}} = g\langle J_5^{\mu}\rangle_A. \quad (46)$$

Henceforth we shall set the axial vector coupling $g = 1$. Next let us decompose the axial vector $\mathcal{B}_{\mu}$ into its transverse and longitudinal parts,

$$\mathcal{B}_{\mu} = \mathcal{B}_{\mu}^{\perp} + \partial_{\mu}\mathcal{B} \quad (47)$$

with $\partial^{\mu}\mathcal{B}_{\mu}^{\perp} = 0$ and $\mathcal{B}$ a pseudoscalar. Then, by an integration by parts in the action corresponding to (45), we have

$$\partial_{\mu}\langle J_5^{\mu}\rangle_A = -\frac{\delta\mathcal{S}}{\delta\mathcal{B}}. \quad (48)$$

Thus the axial anomaly (5) implies that there is a term in the one-loop effective action in a background A_{μ} and $\mathcal{B}_{\mu}$ field, linear in $\mathcal{B}$ of the form,

$$\mathcal{S}_{eff} = -\frac{e^2}{16\pi^2}\int d^4x\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\mathcal{B}, \quad (49)$$

or since $\partial^{\lambda}\mathcal{B}_{\lambda} = \Box\mathcal{B}$,

$$\mathcal{S}_{\text{eff}} = -\frac{e^2}{16\pi^2} \times \int d^4x\int d^4y[\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}]_x\Box_{xy}^{-1}[\partial^{\lambda}\mathcal{B}_{\lambda}]_y, \quad (50)$$

where $\Box_{xy}^{-1}$ is the Green's function for the massless scalar wave operator $\Box = \partial_{\mu}\partial^{\mu}$. Thus from (46), this nonlocal action gives [25]

$$\langle J_5^{\mu}\rangle_A = \frac{e^2}{16\pi^2}\partial^{\mu}\Box^{-1}\epsilon^{\alpha\beta\rho\sigma}F_{\alpha\beta}F_{\rho\sigma}, \quad (51)$$

which explicitly exhibits the massless scalar pole in the massless limit of (54), and which agrees with the explicit calculation of the physical $\langle 0|J_5^{\mu}|p, q\rangle$ triangle amplitude to two photons (43) in the previous section for $p^2 = q^2 = m^2 = 0$.

The nonlocal action (50) can be recast into a local form by the introduction of two pseudoscalar auxiliary fields η and χ satisfying the second order linear equations of motion,

$$\Box\eta = -\partial^{\lambda}\mathcal{B}_{\lambda}, \quad (52a)$$

$$\Box\chi = \frac{e^2}{8\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \quad (52b)$$

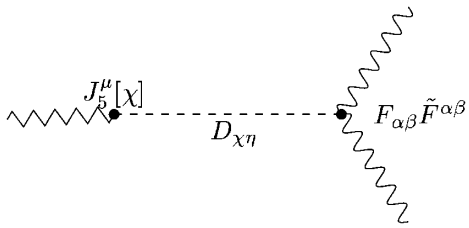

FIG. 3. Tree diagram of the effective action (53) showing the massless propagator $D_{\chi\eta}$ representing the massless 0^- state in the triangle amplitude (43) to physical photons when the fermion mass $m = 0$.

Then one can verify that the local quadratic action functional,

$$\mathcal{S}_{\text{eff}}[\eta, \chi; A, \mathcal{B}] = \int d^4x\Big\{(\partial^{\mu}\eta)(\partial_{\mu}\chi) - \chi\partial^{\mu}\mathcal{B}_{\mu} + \frac{e^2}{8\pi^2}\eta F_{\mu\nu}\tilde{F}^{\mu\nu}\Big\} \quad (53)$$

yields back the equations of motion (52) when freely varied with respect to χ and η, respectively, while evaluating to (50) upon using these equations of motion to solve for and eliminate the auxiliary fields. In the auxiliary field form of the effective action, (53) the quantum expectation value of the chiral current (46) is given by

$$J_5^{\mu}[\chi] = \frac{\delta\mathcal{S}_{\text{eff}}}{\delta\mathcal{B}_{\mu}} = \partial^{\mu}\chi = \frac{e^2}{16\pi^2}\partial^{\mu}\Box^{-1}\epsilon^{\alpha\beta\rho\sigma}F_{\alpha\beta}F_{\rho\sigma} \quad (54)$$

at least insofar as its anomalous divergence is concerned. The effective action (53) reproduces the anomalous divergence of $\langle J_5^{\mu}\rangle$, but not necessarily the nonanomalous parts of the tensor amplitude $\Gamma^{\mu\alpha\beta}(p, q)$. The η and χ fields and their propagator are a local field representation of a massless 0^- state propagating in the physical matrix element (43) with $p^2 = q^2 = m^2 = 0$, which may be represented by the effective tree diagram with source $F_{\mu\nu}\tilde{F}^{\mu\nu}$ on one end and $\partial^{\lambda}\mathcal{B}_{\lambda}$ on the other, as in Fig. 3.

The same diagram also represents the *vector* current expectation value $\langle J^{\mu}\rangle_B$ in the presence of a background axial field $\mathcal{B}_{\lambda}$ and gauge field A_{μ}, also implied by the original triangle diagram (Fig. 1) upon reversing the roles of the axial vertex and one of the vector vertices, i.e.

$$J^{\mu}[\eta] = \frac{\delta\mathcal{S}_{\text{eff}}}{\delta A_{\mu}} = -\frac{e^2}{4\pi^2}\tilde{F}^{\nu\mu}\partial_{\nu}\eta = \frac{e^2}{4\pi^2}\tilde{F}^{\nu\mu}\partial_{\nu}\Box^{-1}\partial^{\lambda}\mathcal{B}_{\lambda}. \quad (55)$$

This crossing symmetry or equivalently, the fact that the nonlocal action (49) involves a mixed term involving both $F_{\mu\nu}\tilde{F}^{\mu\nu}$ and $\partial^{\lambda}\mathcal{B}_{\lambda}$ is the reason why two massless pseudoscalar auxiliary fields rather than just one are required to describe the amplitude correctly through a local effective action. A single auxiliary field would necessarily produce

unwanted direct $F_{\mu\nu}\tilde{F}^{\mu\nu}\Box^{-1}F_{\mu\nu}\tilde{F}^{\mu\nu}$ and $\partial^\lambda\mathcal{B}_\lambda\Box^{-1}\partial^\lambda\mathcal{B}_\lambda$ terms in the effective action, not present in QED. Note that because of the mixed field kinetic term in (53), the only propagator function that can appear is the mixed one, $D_{\chi\eta} = i\langle\mathcal{T}\chi\eta\rangle$.

Several additional remarks concerning the effective action (53) are in order. First, the axial anomaly, and hence the fields η and χ and their propagator $D_{\chi\eta}$ are present for any p^2, q^2, m^2, although they decouple from the physical amplitude $\Gamma^{\mu\alpha\beta}(p, q)$ as $k^2 \to 0$ if any one of p^2, q^2, m^2 are greater than zero [10,26]. The massless η and χ fields decouple from all physical processes involving electrons in that case, and the amplitude has a resonant peak at $s \sim (p^2, q^2, m^2)$ as in (36), rather than a sharp $\delta(s)$ behavior. Because of the sum rule (33) the resonance has the same total probability when integrated over s, but the massless propagator $D_{\chi\eta} = i\langle\mathcal{T}\chi\eta\rangle$ saturates the physical on shell amplitude (43), and may be substituted in its place only when $p^2 = q^2 = m^2 = 0$.

Second, since it contains kinetic terms for the auxiliary fields η and χ, the effective action (53) describes two massless pseudoscalar degrees of freedom. These degrees of freedom are two-particle 0^- correlated e^+e^- states, and η and χ are pseudoscalar composite fields of bilinears of $\bar{\psi}$ and ψ. In fact, η and χ may be defined by their relations to the vector and axial currents J^μ and J_5^μ of the underlying Dirac theory (1) by (55) and (54), respectively. Hence varying η and χ and treating them as true degrees of freedom is equivalent to varying the bilinear current densities J^μ and J_5^μ, according to (55) and (54).

Third, these pseudoscalar degrees of freedom are implied also by a canonical operator treatment. Taking the effective action (53) as defining canonical momenta conjugate to the η and χ fields via

$$\Pi_\eta \equiv \frac{\delta\mathcal{S}_{\text{eff}}}{\delta\dot{\eta}} = -\dot{\chi}, \tag{56a}$$

$$\Pi_\chi \equiv \frac{\delta\mathcal{S}_{\text{eff}}}{\delta\dot{\chi}} = -\dot{\eta}, \tag{56b}$$

and imposing the equal time canonical commutation relations,

$$[\eta(t, \vec{x}), \Pi_\eta(t, \vec{x}')] = i\delta^3(\vec{x} - \vec{x}') = [\chi(t, \vec{x}), \Pi_\chi(t, \vec{x}')], \tag{57}$$

we find that the currents defined by (54) and (55) satisfy the commutation relations,

$$[J^0(t, \vec{x}), J_5^0(t, \vec{x}')] = -\frac{e^2}{2\pi^2}\tilde{F}^{0j}\partial_j[\eta(t, \vec{x}), \dot{\chi}(t, \vec{x}')] = -\frac{ie^2}{2\pi^2}\mathbf{B}\cdot\nabla\delta^3(\vec{x} - \vec{x}'), \tag{58a}$$

$$[J^i(t, \vec{x}), J_5^0(t, \vec{x}')] = -\frac{e^2}{2\pi^2}\tilde{F}^{ij}\partial_j[\eta(t, \vec{x}), \dot{\chi}(t, \vec{x}')] = -\frac{ie^2}{2\pi^2}(\mathbf{E}\times\nabla)^i\delta^3(\vec{x} - \vec{x}'), \tag{58b}$$

which are the anomalous commutation relations deducible from the covariant $\mathcal{T}^*$ time ordering of the currents required by the axial anomaly [20,27]. In other words, the canonical commutation relations of the auxiliary fields with the kinetic terms in the effective action (53) are *required* by the anomalous equal time commutators of the currents J^μ and J_5^ν. This suggests that η and χ should be treated as *bona fide* quantum degrees of freedom in their own right. Because of the unique kinematic status of the triangle diagram [14], and the nonrenormalization of the axial anomaly [28], the structure of the effective action (53) and commutation relations (58) are not modified by any higher order processes.

Fourth, we observe that the energy corresponding to (53) is not positive definite. This in itself should not be surprising, since the action (53) is a finite effective action in which the formally infinite energy of the Dirac sea has been effectively subtracted (by the counterterms needed to impose gauge invariance, not encountered explicitly in our approach). Under some boundary conditions, this finite subtracted energy can be negative, as in the Casimir effect. The conditions under which this is true requires a careful analysis of the surface terms which we have neglected so far in our discussion. In fact, because both $\partial_\mu\mathcal{B}^\mu$ and $F_{\mu\nu}\tilde{F}^{\mu\nu} = \partial_\mu(\epsilon^{\mu\nu\alpha\beta}A_\nu F_{\alpha\beta})$ are total derivatives, the action (53) changes only by a surface term under constant shifts of either η or χ, and there are two Noether currents,

$$K_A^\mu \equiv \frac{e^2}{8\pi^2}\epsilon^{\mu\nu\alpha\beta}A_\nu F_{\alpha\beta} - \partial^\mu\chi, \tag{59a}$$

$$K_{\mathcal{B}}^\mu \equiv \mathcal{B}^\mu + \partial^\mu\eta, \tag{59b}$$

with corresponding Noether charges which are conserved by the equations of motion (52). The dynamics of the η and χ fields is partly constrained by these conservation laws, and should be considered together with the dynamics of vector and axial vector sources A_μ and $\mathcal{B}_\mu$.

Finally, since the effective action (53) explicitly exhibiting these two pseudoscalar fields is nothing but a rewriting of the nonlocal form of the effective action for massless QED in the presence of an axial vector source, the massless degrees of freedom they represent have not been added in to the theory in an *ad hoc* manner. They are contained in QED as soon as it is extended by an arbitrary axial vector coupling as in (45), and are a necessary consequence of the axial anomaly, which in turn is required by imposition of all the other symmetries.

In condensed matter physics, or electrodynamics at finite temperature or in polarizable media, where Lorentz invariance is broken, it is a familiar circumstance that there are low energy collective modes of the many-body theory, which are not part of the single particle constituent spectrum. This occurs also *in vacuo* in the two-dimensional massless Schwinger model, whose anomaly and longitudinal "photon" can be described by the introduction of an effective scalar field composed of an e^+e^- pair [7]. In

3 + 1 dimensions, relativistic kinematics and symmetries severely limit the possibilities for the appearance of such composite massless scalars, with the triangle anomaly the only known example [14]. The fact that the e^+e^- pair becomes collinear in the massless limit shows that this effectively reduces the dimensionality back to 1 + 1. In the well studied 1 + 1 dimensional case, the commutation relations of fermion bilinear currents J^μ and J^ν_5, which create the composite e^+e^- massless state are due to the anomaly [29]. Evidently from (57) and (58), a similar phenomenon occurs in the triangle amplitude in 3 + 1 dimensions.

To conclude this section, one may ask: if there are massless pseudoscalar states in a weakly coupled theory like QED, which has been subjected to such exquisitely accurate tests, how could they have escaped detection? The answer to this is twofold. First, as we have seen these massless pseudoscalars do not couple to real QED with a finite electron mass, except at $k^2 \gg m^2$, so in massive QED they have no effects on low energy or long range electromagnetic interactions. Second, and more importantly, they require an axial vector source $\mathcal{B}_\mu$ (as well as a nonzero $F\tilde{F}$). In pure QED there is no axial vector coupling, i.e. $g = 0$ in (45). Indeed it is impossible to introduce such a coupling into a $U(1)$ gauge theory with a dynamical axial vector $\mathcal{B}_\mu$ field without the breakdown of Ward identities necessary to the ultraviolet renormalizability of the theory. While it is theoretically possible to introduce a nondynamical axial vector source, except for π^0 decay where indeed the axial anomaly with a quark triangle amplitude dominates [12], it seems to be difficult to realize such a source in nature, at least on macroscopic scales. In this situation the appearance of a massless pseudoscalar pole in the QED triangle amplitude, and its description by massless auxiliary fields is an interesting curiosity, illustrating the logical and kinematical possibility that anomalies may lead to unexpected consequences for the long distance physics in higher dimensions as well as lower ones, but which does not affect any predictions of QED in four dimensions with $g\mathcal{B}_\mu \equiv 0$.

IV. THE $\langle TJJ\rangle$ TRIANGLE AMPLITUDE IN QED

In this section we consider the amplitude for the trace anomaly in flat space that most closely corresponds to the triangle amplitude for the axial current anomaly reviewed in the previous section, and give a complete calculation of the full $\langle T^{\mu\nu}J^\alpha J^\beta\rangle$ amplitude for all values of the mass and the off shell kinematic invariants. Although the tensor structure of this amplitude is more involved than the axial vector case, the kinematics is essentially the same, and the appearance of the massless pole is very much analogous to the axial case.

The fundamental quantity of interest is the expectation value of the energy-momentum tensor bilinear in the fermion fields in an external electromagnetic potential A_μ,

$$\langle T^{\mu\nu}\rangle_A = \langle T^{\mu\nu}_{\text{free}}\rangle_A + \langle T^{\mu\nu}_{\text{int}}\rangle_A \tag{60}$$

where

$$T^{\mu\nu}_{\text{free}} = -i\bar\psi\gamma^{(\mu}\overleftrightarrow{\partial}^{\nu)}\psi + g^{\mu\nu}(i\bar\psi\gamma^\lambda\overleftrightarrow{\partial}_\lambda\psi - m\bar\psi\psi) \tag{61a}$$

$$T^{\mu\nu}_{\text{int}} = -eJ^{(\mu}A^{\nu)} + eg^{\mu\nu}J^\lambda A_\lambda, \tag{61b}$$

are the contributions to the stress tensor of the free and interaction terms of the Dirac Lagrangian (45). The notations, $t^{(\mu\nu)} \equiv (t^{\mu\nu} + t^{\nu\mu})/2$ and $\overleftrightarrow{\partial}_\mu \equiv (\vec\partial_\mu - \overleftarrow\partial_\mu)/2$, for symmetrization and antisymmetrization have been used. The expectation value $\langle T^{\mu\nu}\rangle_A$ satisfies the partial conservation equation,

$$\partial_\nu\langle T^{\mu\nu}\rangle_A = eF^{\mu\nu}\langle J_\nu\rangle_A, \tag{62}$$

upon formal use of the Dirac equation of motion (2). Just as in the chiral case, the relation is formal because of the *a priori* ill-defined nature of the bilinear product of Dirac field operators at the same spacetime point in (61). Energy-momentum conservation in full QED (i.e. when the electromagnetic field A^μ is also quantized) requires adding to the fermionic $T^{\mu\nu}$ of (61) the electromagnetic Maxwell stress tensor,

$$T^{\mu\nu}{}_{\text{max}} = F^{\mu\lambda}F^\nu{}_\lambda - \tfrac{1}{4}g^{\mu\nu}F^{\lambda\rho}F_{\lambda\rho} \tag{63}$$

which satisfies $\partial_\nu T^{\mu\nu}{}_{\text{max}} = -F^{\mu\nu}J_\nu$. This cancels (62) at the operator level, so that the full stress tensor of QED is conserved upon using Maxwell's equations, $\partial_\nu F^{\mu\nu} = J^\mu$. Since in our present treatment A_μ is an arbitrary external potential, rather than a dynamical field, we consider only the fermionic parts of the stress tensor (61) whose expectation value satisfies (62) instead.

At the classical level, i.e. again formally, upon use of (2), the trace of the fermionic stress tensor obeys

$$T^{\mu(\text{cl})}{}_\mu \equiv g_{\mu\nu}T^{\mu\nu(\text{cl})} = -m\bar\psi\psi \qquad \text{(classically)}, \tag{64}$$

analogous to the classical relation for the axial current (4). From this it would appear that $\langle T^{\mu\nu}\rangle_A$ will become traceless in the massless limit $m \to 0$, corresponding to the global dilation symmetry of the classical theory with zero mass. However, as in the case of the classical chiral symmetry, this symmetry under global scale transformations cannot be maintained at the quantum level, without violating the conservation law satisfied by a related current, in this case the partial conservation law (62), implied by general coordinate invariance. Requiring that (62) *is* preserved at the quantum level necessarily leads to a well-defined anomaly in the trace, [8,30,31], namely,

$$\langle T^\mu{}_\mu\rangle_A|_{m=0} = -\frac{e^2}{24\pi^2}F_{\mu\nu}F^{\mu\nu}, \tag{65}$$

analogous to (5). It is the infrared consequences of this modified, anomalous trace identity and the appearance of massless scalar degrees of freedom for vanishing electron mass $m = 0$ that we wish to study.

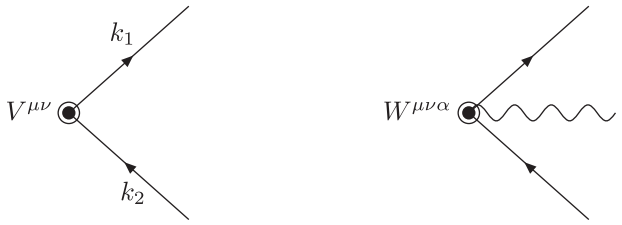

FIG. 4. The two kinds of vertices contributing to the stress-tensor amplitude (68).

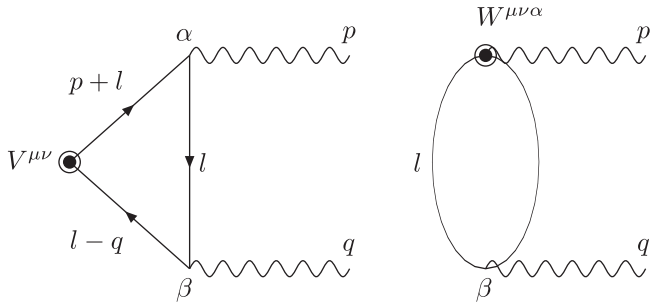

FIG. 5. The two kinds of amplitudes contributing to the stress-tensor amplitude (68).

Our first task is to evaluate the full amplitude at one-loop order obtained by taking two functional variations of the expectation value (60) with respect to the external potential, and then evaluating at vanishing external field, $A = 0$. In position space this is

$$\begin{aligned}\Gamma^{\mu\nu\alpha\beta}(z;x,y) &\equiv \frac{\delta^2\langle T^{\mu\nu}(z)\rangle_A}{\delta A_\alpha(x)\delta A_\beta(y)}\bigg|_{A=0} \\ &= (ie)^2\langle T^{\mu\nu}_{\text{free}}(z)J^\alpha(x)J^\beta(y)\rangle_{A=0} \\ &\quad + \delta^4(x-z)g^{\alpha(\mu}\Pi^{\nu)\beta}(z,y) \\ &\quad + \delta^4(y-z)g^{\beta(\mu}\Pi^{\nu)\alpha}(z,x) \\ &\quad - g^{\mu\nu}[\delta^4(x-z)-\delta^4(y-z)]\Pi^{\alpha\beta}(x,y),\end{aligned} \tag{66}$$

where

$$\Pi^{\alpha\beta}(x,y) \equiv -e\frac{\delta\langle J^\alpha(x)\rangle_A}{\delta A_\beta(y)}\bigg|_{A=0} = -ie^2\langle J^\alpha(x)J^\beta(y)\rangle_{A=0} \tag{67}$$

is the electromagnetic polarization tensor in zero external field. Going over to momentum space and factoring out the resulting factor of momentum conservation, $(2\pi)^4\delta^4(k-p-q)$, we obtain

$$\Gamma^{\mu\nu\alpha\beta}(p,q) = \int d^4x\int d^4y e^{ip\cdot x+iq\cdot y}\Gamma^{\mu\nu\alpha\beta}(z=0;x,y), \tag{68}$$

which receives contributions from the two kinds of vertices derived from the two terms in (61), namely,

$$\begin{aligned}V^{\mu\nu}(k_1,k_2) &= \tfrac{1}{4}[\gamma^\mu(k_1+k_2)^\nu + \gamma^\nu(k_1+k_2)^\mu] \\ &\quad - \tfrac{1}{2}g^{\mu\nu}[\gamma^\lambda(k_1+k_2)_\lambda + 2m], \qquad (69a)\\ W^{\mu\nu\alpha} &= -\tfrac{1}{2}(\gamma^\mu g^{\nu\alpha} + \gamma^\nu g^{\mu\alpha}) + g^{\mu\nu}\gamma^\alpha, \qquad (69b)\end{aligned}$$

respectively, represented in Fig. 4.

At the one-loop level the amplitude (68) is represented by the diagrams in Fig. 5, together with the two others with the photon lines interchanged. The first of these diagrams with the vertex $V^{\mu\nu}$ gives the contribution,

$$\begin{aligned}{}^V\Gamma^{\mu\nu\alpha\beta}(p,q) &= (-1)(ie)^2\int\frac{id^4l}{(2\pi)^4}\,\text{tr}\bigg\{V^{\mu\nu}(l+p,l-q)\frac{-i}{\not l+\not p+m}\gamma^\alpha\frac{-i}{\not l+m}\gamma^\beta\frac{-i}{\not l-\not q+m}\bigg\} \\ &= -e^2\int\frac{d^4l}{(2\pi)^4}\frac{\text{tr}\{V^{\mu\nu}(l+p,l-q)(-\not l-\not p+m)\gamma^\alpha(-\not l+m)\gamma^\beta(-\not l+\not q+m)\}}{[(l+p)^2+m^2][(l-q)^2+m^2][l^2+m^2]},\end{aligned} \tag{70}$$

while the second diagram in Fig. 5 with the vertex $W^{\mu\nu\alpha}$ gives

$$\begin{aligned}{}^W\Gamma^{\mu\nu\alpha\beta}(p,q) &= (-1)i(ie)^2\int\frac{id^4l}{(2\pi)^4}\,\text{tr}\bigg\{W^{\mu\nu\alpha}\frac{-i}{\not l+m}\gamma^\beta\frac{-i}{\not l-\not q+m}\bigg\} = e^2\int\frac{d^4l}{(2\pi)^4}\frac{\text{tr}\{W^{\mu\nu\alpha}(\not l+m)\gamma^\beta(\not l-\not q+m)\}}{[l^2+m^2][(l-q)^2+m^2]} \\ &= -\frac{1}{2}g^{\nu\alpha}\Pi^{\mu\alpha}(q) - \frac{1}{2}g^{\mu\alpha}\Pi^{\nu\alpha}(q) + g^{\mu\nu}\Pi^{\alpha\beta}(q),\end{aligned} \tag{71}$$

where

$$\Pi^{\alpha\beta}(p) = \int d^4x e^{ip\cdot(x-y)}\Pi^{\alpha\beta}(x,y) = e^2\int\frac{d^4l}{(2\pi)^4}\frac{\text{tr}\{\gamma^\alpha(\not l+m)\gamma^\beta(\not l-\not p+m)\}}{[l^2+m^2][(l-p)^2+m^2]} \tag{72}$$

is the Fourier transform of (67). The additional factor of i in the loop integration measure of (70) and (71) comes from the continuation to Euclidean momenta: $l_0 \to il_4$. As usual these loop integrals are formal and divergent, since the one-loop polarization requires regularization and renormalization, which we postpone for the moment. The

second set of diagrams with the W interaction vertex give rise to the second set of terms in (66) explicitly proportional to the polarization (72). As we shall see, we actually require only the finite parts of the first diagram in Fig. 5, together with the equivalent diagram obtained by interchanging p and q and α and β. The contribution of the contact terms of the second kind of diagram in Fig. 5 with the $W^{\mu\nu\alpha}$ vertex will be determined from the Ward identities.

The full one-loop contribution to the amplitude (68) is the Bose symmetric sum,

$$\begin{aligned}\Gamma^{\mu\nu\alpha\beta}(p,q) &= {}^{V}\Gamma^{\mu\nu\alpha\beta}(p,q) + {}^{V}\Gamma^{\mu\nu\beta\alpha}(q,p) \\ &\quad + {}^{W}\Gamma^{\mu\nu\alpha\beta}(p,q) + {}^{W}\Gamma^{\mu\nu\beta\alpha}(q,p) \\ &= \Gamma^{\mu\nu\beta\alpha}(q,p). \end{aligned} \tag{73}$$

Vector current conservation $\partial_\mu J^\mu = 0$ implies that both the polarization, and this amplitude should satisfy the Ward identities,

$$p_\alpha \Pi^{\alpha\beta}(p) = q_\beta \Pi^{\alpha\beta}(q) = 0; \tag{74a}$$

$$p_\alpha \Gamma^{\mu\nu\alpha\beta}(p,q) = q_\beta \Gamma^{\mu\nu\alpha\beta}(p,q) = 0. \tag{74b}$$

The first of these relations implies that $\Pi^{\alpha\beta}(p)$ is transverse, i.e.

$$\Pi^{\alpha\beta}(p) = (p^2 g^{\alpha\beta} - p^\alpha p^\beta)\Pi(p^2). \tag{75}$$

In addition, the partial conservation law (62) implies that the amplitude in (66) should also satisfy the Ward identity,

$$\begin{aligned}\frac{\partial}{\partial z^\nu}\Gamma^{\mu\nu\alpha\beta}(z;x,y) &= \Pi^{\beta\nu}(y,z)(g^{\alpha\mu}\delta^\lambda_\nu - \delta^\alpha_\nu g^{\mu\lambda}) \\ &\quad \times \frac{\partial}{\partial x^\lambda}\delta^4(x-z) + \Pi^{\alpha\nu}(x,z) \\ &\quad \times (g^{\beta\mu}\delta^\lambda_\nu - \delta^\beta_\nu g^{\mu\lambda})\frac{\partial}{\partial y^\lambda}\delta^4(y-z). \end{aligned} \tag{76}$$

In momentum space this becomes

$$\begin{aligned}k_\nu \Gamma^{\mu\nu\alpha\beta}(p,q) &= (g^{\mu\alpha}p_\nu - \delta^\alpha_\nu p^\mu)\Pi^{\beta\nu}(q) \\ &\quad + (g^{\mu\beta}q_\nu - \delta^\beta_\nu q^\mu)\Pi^{\alpha\nu}(p), \end{aligned} \tag{77}$$

or combining with (75), we obtain

$$\begin{aligned}&k_\nu \Gamma^{\mu\nu\alpha\beta}(p,q) \\ &= (q^\mu p^\alpha p^\beta - q^\mu g^{\alpha\beta}p^2 + g^{\mu\beta}q^\alpha p^2 - g^{\mu\beta}p^\alpha p\cdot q)\Pi(p^2) \\ &\quad + (p^\mu q^\alpha q^\beta - p^\mu g^{\alpha\beta}q^2 + g^{\mu\alpha}p^\beta q^2 - g^{\mu\alpha}q^\beta p\cdot q)\Pi(q^2). \end{aligned} \tag{78}$$

As already remarked, all of these relations are formal since both $\Gamma^{\mu\nu\alpha\beta}(p,q)$ and $\Pi^{\alpha\beta}(p)$ are ill defined *a priori* and require some procedure to extract the finite terms.

Formally one could use (64) to obtain an identity for the trace on the amplitude $g_{\mu\nu}\Gamma^{\mu\nu\alpha\beta}$. Like (74) and (78) this trace relation can be proven if and only if shifting the loop integration variable l in the integrals (70) and (71) is allowed, and terms which are odd in the loop integration variable are dropped. Since the loop integrals in (70)–(72) are formally quadratically divergent they are not well defined as they stand, so that formal manipulations of this kind can yield ambiguous or incorrect conclusions.

The standard method of dealing with such ill-defined expressions is regularization. Any regularization method that preserves the Ward identities of gauge invariance (74) and coordinate covariance (78) may be used, such as dimensional regularization, Pauli-Villars regularization, or the Schwinger-DeWitt heat kernel method. It is important to recognize that regularization amounts to supplying additional information which defines an ill-defined expression, by requiring that certain symmetries of the classical theory be strictly maintained at the quantum level. Here we shall follow the alternative approach, exactly parallel to the previous treatment of the axial triangle anomaly in Sec. II, which does not require any explicit choice of covariant regularization scheme. Instead we *define* the Lorentz covariant tensor amplitude $\Gamma^{\mu\nu\alpha\beta}(p,q)$ by its *finite* terms, together with the requirement that the full amplitude satisfies the Ward identities (74) and (78). Then the joint requirements of:

(i) Lorentz invariance of the vacuum,
(ii) Bose symmetry (73),
(iii) vector current conservation (74),
(iv) unsubtracted dispersion relation of real and imaginary parts, and
(v) energy-momentum tensor conservation (78),

are sufficient to determine the full amplitude $\Gamma^{\mu\nu\alpha\beta}(p,q)$ in terms of its explicitly finite pieces, and yield a well-defined finite trace anomaly. As in the axial anomaly case considered previously, this method of constructing the full $\Gamma^{\mu\nu\alpha\beta}(p,q)$ may be regarded as a proof that the same finite trace anomaly must be obtained in any regularization scheme that respects (i–v) above.

Lorentz invariance of the vacuum is assumed first by expanding the amplitude in terms of all the possible tensors with four indices depending on p^α, q^β and the flat spacetime metric $g^{\mu\nu}$. There are $2^4 = 16$ tensors with all four Lorentz indices (μ,ν,α,β) are carried by either p or q, $2^2 \times 6 = 24$ tensors in which two of the four indices are carried by the symmetric metric tensor $g^{\mu\nu}$ and the other two by either p^μ or q^μ, and just 3 tensors in which the four indices are distributed over a product of two metric tensors with no factors of p or q. The complete set of these 43 tensor monomials is given in Table III. Lorentz covariance requires that the amplitude $\Gamma^{\mu\nu\alpha\beta}(p,q)$ must be expandable in this complete set of 43 tensors with scalar coefficient functions of the three invariants p^2, q^2, and $p\cdot q$, or equivalently p^2, q^2, and $k^2 = (p+q)^2$.

TABLE III. The 43 fourth rank tensor monomials into which $\Gamma^{\mu\nu\alpha\beta}(p,q)$ can be expanded.

$p^\mu p^\nu p^\alpha p^\beta$	$p^\mu p^\nu p^\alpha q^\beta$	$p^\mu p^\nu q^\alpha q^\beta$	$p^\mu q^\nu q^\alpha p^\beta$	$p^\mu q^\nu q^\alpha q^\beta$	$g^{\mu\nu} g^{\alpha\beta}$
$q^\mu q^\nu q^\alpha q^\beta$	$p^\mu p^\nu q^\alpha p^\beta$	$p^\mu q^\nu p^\alpha q^\beta$	$q^\mu p^\nu q^\alpha p^\beta$	$q^\mu p^\nu q^\alpha q^\beta$	$g^{\alpha\mu} g^{\beta\nu}$
	$p^\mu q^\nu p^\alpha p^\beta$	$q^\mu p^\nu p^\alpha q^\beta$	$q^\mu q^\nu p^\alpha p^\beta$	$q^\mu q^\nu p^\alpha q^\beta$	$g^{\alpha\nu} g^{\beta\mu}$
	$q^\mu p^\nu p^\alpha p^\beta$			$q^\mu q^\nu q^\alpha p^\beta$	
$p^\mu p^\nu g^{\alpha\beta}$	$p^\beta p^\nu g^{\alpha\mu}$	$p^\beta p^\mu g^{\alpha\nu}$	$p^\alpha p^\nu g^{\beta\mu}$	$p^\mu p^\alpha g^{\beta\nu}$	$p^\alpha p^\beta g^{\mu\nu}$
$p^\mu q^\nu g^{\alpha\beta}$	$p^\beta q^\nu g^{\alpha\mu}$	$p^\beta q^\mu g^{\alpha\nu}$	$p^\alpha q^\nu g^{\beta\mu}$	$p^\mu q^\alpha g^{\beta\nu}$	$p^\alpha q^\beta g^{\mu\nu}$
$q^\mu p^\nu g^{\alpha\beta}$	$q^\beta p^\nu g^{\alpha\mu}$	$q^\beta p^\mu g^{\alpha\nu}$	$q^\alpha p^\nu g^{\beta\mu}$	$q^\mu q^\alpha g^{\beta\nu}$	$q^\alpha q^\beta g^{\mu\nu}$
$q^\mu q^\nu g^{\alpha\beta}$	$q^\beta q^\nu g^{\alpha\mu}$	$q^\beta q^\mu g^{\alpha\nu}$	$q^\alpha q^\nu g^{\beta\mu}$		

Since the amplitude (68) has total mass dimension 2, the scalar coefficient functions multiplying the tensors in Table III which are homogeneous of degree 4 in p and q have mass dimension -2. These coefficients can be extracted in terms of loop integrals which are UV quadratically convergent and finite. Then the coefficients of the remaining tensors are determined by the Ward identities of vector current and stress-tensor conservation. Let us define the two-index tensors,

$$u^{\alpha\beta}(p,q) \equiv (p\cdot q)g^{\alpha\beta} - q^\alpha p^\beta, \tag{79a}$$

$$w^{\alpha\beta}(p,q) \equiv p^2q^2g^{\alpha\beta} + (p\cdot q)p^\alpha q^\beta - q^2p^\alpha p^\beta - p^2q^\alpha q^\beta, \tag{79b}$$

each of which satisfies the conditions of Bose symmetry,

$$u^{\alpha\beta}(p,q) = u^{\beta\alpha}(q,p), \tag{80a}$$

$$w^{\alpha\beta}(p,q) = w^{\beta\alpha}(q,p), \tag{80b}$$

and vector current conservation,

$$p_\alpha u^{\alpha\beta}(p,q) = 0 = q_\beta u^{\alpha\beta}(p,q), \tag{81a}$$

$$p_\alpha w^{\alpha\beta}(p,q) = 0 = q_\beta w^{\alpha\beta}(p,q). \tag{81b}$$

These tensors may be obtained from the variation of local gauge invariant quantities $F_{\mu\nu}F^{\mu\nu}$ and $(\partial_\mu F^\mu{}_\lambda)(\partial_\nu F^{\nu\lambda})$ respectively, via

$$u^{\alpha\beta}(p,q) = -\frac{1}{4}\int d^4x \int d^4y e^{ip\cdot x+iq\cdot y}\frac{\delta^2\{F_{\mu\nu}F^{\mu\nu}(0)\}}{\delta A_\alpha(x)A_\beta(y)}, \tag{82a}$$

$$w^{\alpha\beta}(p,q) = \frac{1}{2}\int d^4x \int d^4y e^{ip\cdot x+iq\cdot y}\frac{\delta^2\{\partial_\mu F^\mu{}_\lambda \partial_\nu F^{\nu\lambda}(0)\}}{\delta A_\alpha(x)A_\beta(y)}. \tag{82b}$$

Making use of $u^{\alpha\beta}(p,q)$ and $w^{\alpha\beta}(p,q)$, one finds that of the 43 tensors in Table III, there are exactly 13 linearly independent four-tensors $t_i^{\mu\nu\alpha\beta}(p,q)$, $i = 1,\ldots,13$, which satisfy

$$p_\alpha t_i^{\mu\nu\alpha\beta}(p,q) = 0 = q_\beta t_i^{\mu\nu\alpha\beta}(p,q), \qquad i = 1,\ldots,13. \tag{83}$$

These 13 tensors are cataloged in Table IV.

This set of 13 tensors is linearly independent for generic k^2, p^2, q^2 different from zero. Five of the 13 are Bose symmetric, namely,

$$t_i^{\mu\nu\alpha\beta}(p,q) = t_i^{\mu\nu\beta\alpha}(q,p), \qquad i = 1,2,7,8,13, \tag{84}$$

while the remaining eight tensors form four pairs related by Bose symmetry:

$$t_3^{\mu\nu\alpha\beta}(p,q) = t_5^{\mu\nu\beta\alpha}(q,p), \tag{85a}$$

$$t_4^{\mu\nu\alpha\beta}(p,q) = t_6^{\mu\nu\beta\alpha}(q,p), \tag{85b}$$

$$t_9^{\mu\nu\alpha\beta}(p,q) = t_{10}^{\mu\nu\beta\alpha}(q,p), \tag{85c}$$

$$t_{11}^{\mu\nu\alpha\beta}(p,q) = t_{12}^{\mu\nu\beta\alpha}(q,p). \tag{85d}$$

Expanding the amplitude (70) in this basis,

$$\Gamma^{\mu\nu\alpha\beta}(p,q) = \sum_{i=1}^{13} F_i(k^2;p^2,q^2)t_i^{\mu\nu\alpha\beta}(p,q). \tag{86}$$

Bose symmetry implies that the scalar functions F_1, F_2, F_7, F_8, and F_{13} are symmetric under the interchange of p^2 and q^2, while the remaining eight functions form four pairs related by the interchange of p^2 and q^2:

TABLE IV. The 13 fourth rank tensors satisfying (83).

i	$t_i^{\mu\nu\alpha\beta}(p,q)$
1	$(k^2g^{\mu\nu} - k^\mu k^\nu)u^{\alpha\beta}(p,q)$
2	$(k^2g^{\mu\nu} - k^\mu k^\nu)w^{\alpha\beta}(p,q)$
3	$(p^2g^{\mu\nu} - 4p^\mu p^\nu)u^{\alpha\beta}(p,q)$
4	$(p^2g^{\mu\nu} - 4p^\mu p^\nu)w^{\alpha\beta}(p,q)$
5	$(q^2g^{\mu\nu} - 4q^\mu q^\nu)u^{\alpha\beta}(p,q)$
6	$(q^2g^{\mu\nu} - 4q^\mu q^\nu)w^{\alpha\beta}(p,q)$
7	$[p\cdot qg^{\mu\nu} - 2(q^\mu p^\nu + p^\mu q^\nu)]u^{\alpha\beta}(p,q)$
8	$[p\cdot qg^{\mu\nu} - 2(q^\mu p^\nu + p^\mu q^\nu)]w^{\alpha\beta}(p,q)$
9	$(p\cdot qp^\alpha - p^2q^\alpha)[p^\beta(q^\mu p^\nu + p^\mu q^\nu) - p\cdot q(g^{\beta\nu}p^\mu + g^{\beta\mu}p^\nu)]$
10	$(p\cdot qq^\beta - q^2p^\beta)[q^\alpha(q^\mu p^\nu + p^\mu q^\nu) - p\cdot q(g^{\alpha\nu}q^\mu + g^{\alpha\mu}q^\nu)]$
11	$(p\cdot qp^\alpha - p^2q^\alpha)[2q^\beta q^\mu q^\nu - q^2(g^{\beta\nu}q^\mu + g^{\beta\mu}q^\nu)]$
12	$(p\cdot qq^\beta - q^2p^\beta)[2p^\alpha p^\mu p^\nu - p^2(g^{\alpha\nu}p^\mu + g^{\alpha\mu}p^\nu)]$
13	$(p^\mu q^\nu + p^\nu q^\mu)g^{\alpha\beta} + p\cdot q(g^{\alpha\nu}g^{\beta\mu} + g^{\alpha\mu}g^{\beta\nu}) - g^{\mu\nu}u^{\alpha\beta} - (g^{\beta\nu}p^\mu + g^{\beta\mu}p^\nu)q^\alpha - (g^{\alpha\nu}q^\mu + g^{\alpha\mu}q^\nu)p^\beta$

$$F_3(k^2; p^2, q^2) = F_5(k^2; q^2, p^2), \quad (87a)$$
$$F_4(k^2; p^2, q^2) = F_6(k^2; q^2, p^2), \quad (87b)$$
$$F_9(k^2; p^2, q^2) = F_{10}(k^2; q^2, p^2), \quad (87c)$$
$$F_{11}(k^2; p^2, q^2) = F_{12}(k^2; q^2, p^2), \quad (87d)$$

corresponding to (85). Thus there are nine independent scalar functions in the amplitude (86), five of them completely symmetric, and four of them possessing both symmetric and antisymmetric terms under the $p^2 \leftrightarrow q^2$ interchange, for $5 + (4 \times 2) = 13$ scalar amplitudes in all. From Table IV, we observe that all but t_{13} contain terms which are homogeneous of degree four in the external momenta, whose coefficients we may constrain from the finite parts of the amplitude.

We have chosen this basis so that only the first two of the 13 tensors possess a nonzero trace,

$$g_{\mu\nu} t_1^{\mu\nu\alpha\beta}(p, q) = 3k^2 u^{\alpha\beta}(p, q), \quad (88a)$$
$$g_{\mu\nu} t_2^{\mu\nu\alpha\beta}(p, q) = 3k^2 w^{\alpha\beta}(p, q), \quad (88b)$$

while the remaining 11 tensors are traceless,

$$g_{\mu\nu} t_i^{\mu\nu\alpha\beta}(p, q) = 0, \qquad i = 3, \ldots, 13. \quad (89)$$

Moreover because of Eqs. (65) and (82), we have chosen the basis t_i in anticipation of the result that in the limit of zero fermion mass, the entire trace anomaly will reside only in the first amplitude function, $F_1(k^2; p^2, q^2)$.

To proceed, we now fix as many of the 13 scalar functions F_i as possible by examining the finite terms in the formal expressions (70) and (71). To this end we perform the indicated Dirac algebra and introduce the Feynman parametrization (A2) of the product of propagator denominators. Then we make any necessary shifts in the loop integration variable l in (70) and (71) in order to extract only those terms for which the four indices $(\mu, \nu, \alpha, \beta)$ are carried by the external momentum vectors p and q in various combinations, which therefore can be removed from the loop integration. The remaining loop integration is then finite for these terms and can be extracted unambiguously. The details of this computation are given in Appendix A.

To make this procedure completely rigorous, one can calculate first only the discontinuities of the amplitude, continued to timelike four-momenta, using the Cutkosky rule replacement,

$$\frac{1}{(l + p)^2 + m^2} \to 2\pi i \theta(l^0 + p^0) \delta((l + p)^2 + m^2) \quad (90)$$

for the lines cut as in Fig. 2 in the chiral case. Owing to these delta functions in the cut diagram, the discontinuity of (70) is completely finite. Then for those terms (and only those terms) multiplying tensors of degree 4 in p and q, the real parts may be constructed from the discontinuous imaginary parts by *unsubtracted* dispersion relations, and are completely finite as well. Since there are 12 such tensors of degree 4 in the external momenta, listed in Table V, there are 12 finite scalar coefficient functions $C_j(k^2; p^2, q^2)$ multiplying them which are defined in this way. The explicit form of their corresponding imaginary parts ρ_j is given by (132) in Sec. VI. It is not difficult to show from the linear independence of the tensors in Table V, and general analyticity properties of the amplitude, that the finite coefficient functions C_j of mass dimension -2 obtained in this way from their imaginary parts are identical to those obtained by the recipe of shifting the loop integration variable in the original full amplitude, in Feynman parametrized form, and identifying the terms multiplying each tensor listed in Table V. This is of course also the same result for the finite terms that is obtained if the loop integration were regularized in a covariant way, such as in the dimensional regularization or Pauli-Villars schemes, in which the shift of the loop integration variable is permitted. It is also noteworthy that this procedure of

TABLE V. The 12 tensors with four free indices $(\mu\nu\alpha\beta)$ on p, q which appear in the amplitude (86), with finite scalar coefficient functions $C_j(k^2; p^2, q^2)$ and corresponding polynomials in the Feynman parametrized form, (92).

j	C_j = coefficient of	$c_j(x, y)$
1	$p^\mu p^\nu p^\alpha p^\beta$	$-4x^2(1 - x)(1 - 2x)$
2	$(p^\mu q^\nu + q^\mu p^\nu) p^\alpha p^\beta$	$-x(1 - x)(1 - 4x + 8xy) + xy$
3	$q^\mu q^\nu p^\alpha p^\beta$	$2x(1 - 2y)(1 - x - y + 2xy)$
4	$p^\mu p^\nu p^\alpha q^\beta$	$-2x(1 - x)(1 - 2x)(1 - 2y)$
5	$(p^\mu q^\nu + q^\mu p^\nu) p^\alpha q^\beta$	$x(1 - x)(1 - 2y)^2 + y(1 - y)(1 - 2x)^2$
6	$q^\mu q^\nu p^\alpha q^\beta$	$-2y(1 - y)(1 - 2x)(1 - 2y)$
7	$p^\mu p^\nu q^\alpha p^\beta$	$2xy(1 - 2x)^2$
8	$(p^\mu q^\nu + q^\mu p^\nu) q^\alpha p^\beta$	$-2xy(1 - 2x)(1 - 2y)$
9	$q^\mu q^\nu q^\alpha p^\beta$	$2xy(1 - 2y)^2$
10	$p^\mu p^\nu q^\alpha q^\beta$	$2y(1 - 2x)(1 - x - y + 2xy)$
11	$(p^\mu q^\nu + q^\mu p^\nu) q^\alpha q^\beta$	$-y(1 - y)(1 - 4y + 8xy) + xy$
12	$q^\mu q^\nu q^\alpha q^\beta$	$-4y^2(1 - 2y)(1 - y)$

extracting the finite parts of (70) relies only upon the terms involving the $V^{\mu\nu}$ vertex in the triangle diagram of Fig. 5, and not the contributions of the $W^{\mu\nu\alpha}$ vertex which are proportional to the polarization tensor (72) and divergent.

The 12 scalar coefficient functions listed in Table V are not all independent. Owing to the tensor structure in the table imposed by Bose symmetry and vector current conservation, two pairs of the coefficients are trivially dependent upon one other, namely,

$$-(p\cdot q)C_1 = q^2 C_4, \tag{91a}$$
$$-(p\cdot q)C_{12} = p^2 C_6, \tag{91b}$$

so that only 10 independent coefficient functions can be determined from the finite parts of (86).

Inspection of Table V shows also that the coefficients C_5 and C_8 are automatically Bose symmetric, while the remaining 10 coefficients occur in five Bose conjugate pairs, viz. (C_1, C_{12}), (C_2, C_{11}), (C_3, C_{10}), (C_4, C_6), and (C_7, C_9), so that, for example, $C_1(k^2; p^2, q^2) = C_{12}(k^2; q^2, p^2)$. Explicit formulas for all 12 finite coefficient functions may be given in the Feynman parametrized form,

$$C_j(k^2; p^2, q^2) = \frac{e^2}{4\pi^2}\int_0^1 dx \int_0^{1-x} dy \frac{c_j(x,y)}{p^2 x(1-x) + q^2 y(1-y) + 2xy p\cdot q + m^2}, \tag{92}$$

where the polynomials $c_i(x, y)$ for $i = 1, \ldots, 12$ are listed in Table V. From (92) with the help of Table V it is straightforward to verify relations (91) and identify the Bose conjugate pairs of coefficient functions by the interchange of x and y. These relations are verified in Appendix B.

Identifying the coefficients of the finite amplitudes multiplying the 12 tensors of Table V with those in (86) and Table IV gives 10 relations, which we group into the following three sets. First we have the three relations,

$$F_1 + 4F_3 = C_7, \tag{93a}$$
$$F_1 + 4F_5 = C_9, \tag{93b}$$
$$F_1 + 2F_7 = \frac{p^2 C_2 + q^2 C_{11}}{p\cdot q} + \frac{2p^2 q^2}{(p\cdot q)^2} C_5 + C_8, \tag{93c}$$

which multiply only the $u^{\alpha\beta}$ tensor. Next we have the three relations,

$$F_2 + 4F_4 = \frac{C_{10}}{p^2}, \tag{94a}$$
$$F_2 + 4F_6 = \frac{C_3}{q^2}, \tag{94b}$$
$$F_2 + 2F_8 = -\frac{C_5}{p\cdot q}, \tag{94c}$$

multiplying only the $w^{\alpha\beta}$ tensor. Finally, we have the four relations,

$$F_9 = \frac{C_2}{p\cdot q} + \frac{q^2 C_5}{(p\cdot q)^2}, \tag{95a}$$
$$F_{10} = \frac{p^2 C_5}{(p\cdot q)^2} + \frac{C_{11}}{p\cdot q}, \tag{95b}$$
$$F_{11} = \frac{C_3}{2q^2} - \frac{C_{12}}{2p^2}, \tag{95c}$$
$$F_{12} = \frac{C_{10}}{2p^2} - \frac{C_1}{2q^2}, \tag{95d}$$

multiplying tensors that do not appear in either $u^{\alpha\beta}$ or $w^{\alpha\beta}$. The first three of the relations (93) determine three linear combinations of the four functions F_1, F_3, F_5, and F_7 in terms of the finite coefficient functions C_i, leaving only one of these four functions to be determined. Likewise the second three relations (94) determine three linear combinations of the four functions, F_2, F_4, F_6, and F_8 in terms of the C_i, leaving only one of these four functions to be determined. Finally the last four relations (95) determine the four functions F_9, F_{10}, F_{11}, and F_{12} completely in terms of the C_i, and leave only F_{13} to be determined.

The information needed to fix the remaining three functions comes from our fifth and final requirement on (86), namely, the Ward identity (78). The contraction, $k_\mu \Gamma^{\mu\nu\alpha\beta}(p, q)$ gives six independent three-tensors obeying vector current conservations, and therefore six conditions on the amplitude, $\Gamma^{\mu\nu\alpha\beta}(p, q)$,

$$-p^2 F_3 + (3q^2 + 4p\cdot q)F_5 + (2p^2 + p\cdot q)F_7 - p^2 q^2 F_{10} - p^2(p^2 + p\cdot q)F_9 + p^2 q^2 F_{11} = 0, \tag{96a}$$
$$p^2 F_4 - (3q^2 + 4p\cdot q)F_6 - (2p^2 + p\cdot q)F_8 - p\cdot q F_{10} + (q^2 + 2p\cdot q)F_{11} = 0, \tag{96b}$$
$$-p\cdot q(p^2 + p\cdot q)F_9 - q^2(q^2 + p\cdot q)F_{11} + F_{13} + \Pi(p^2) = 0, \tag{96c}$$

and their Bose symmetry conjugates under interchange of p and q,

$$(3p^2 + 4p\cdot q)F_3 - q^2F_5 + (2q^2 + p\cdot q)F_7 - p^2q^2F_9 - q^2(q^2 + p\cdot q)F_{10} + p^2q^2F_{12} = 0, \tag{97a}$$

$$-(3p^2 + 4p\cdot q)F_4 + q^2F_6 - (2q^2 + p\cdot q)F_8 - p\cdot qF_9 + (p^2 + 2p\cdot q)F_{12} = 0, \tag{97b}$$

$$-p\cdot q(q^2 + p\cdot q)F_{10} - p^2(p^2 + p\cdot q)F_{12} + F_{13} + \Pi(q^2) = 0. \tag{97c}$$

It is evident that the symmetrized sum of (96a) and (97a) provides one new relation between F_3, F_5, and F_7, needed together with the first three relations of (93) to determine F_1, F_3, F_5, and F_7 completely in terms of the C_i. In this way we find

$$F_1 = \frac{C_7 + C_8 + C_9}{3} + \frac{p^2}{3k^2}(-C_1 + C_3 + C_8 - C_9) + \frac{q^2}{3k^2}(-C_7 + C_8 + C_{10} - C_{12}), \tag{98a}$$

$$F_3 = \frac{2C_7 - C_8 - C_9}{12} + \frac{p^2}{12k^2}(C_1 - C_3 - C_8 + C_9) + \frac{q^2}{12k^2}(C_7 - C_8 - C_{10} + C_{12}), \tag{98b}$$

$$F_5 = \frac{-C_7 - C_8 + 2C_9}{12} + \frac{p^2}{12k^2}(C_1 - C_3 - C_8 + C_9) + \frac{q^2}{12k^2}(C_7 - C_8 - C_{10} + C_{12}), \tag{98c}$$

$$F_7 = \frac{-C_7 + 2C_8 - C_9}{6} + \frac{p^2}{6k^2}(C_1 - C_3 - C_8 + C_9) + \frac{q^2}{6k^2}(C_7 - C_8 - C_{10} + C_{12}) + \frac{p^2q^2}{(p\cdot q)^2}C_5 + \frac{p^2C_2 + q^2C_{11}}{2(p\cdot q)}. \tag{98d}$$

Likewise the symmetrized sum of (96b) and (97b) provides one new relation between F_4, F_6, and F_8, needed together with the second three relations of (94) to determine F_2, F_4, F_6, and F_8 completely in terms of the C_i. This gives

$$F_2 = \frac{C_1}{3q^2} + \frac{C_{12}}{3p^2} + \frac{-C_1 + 2C_2 - 2C_5 + 2C_{11} - C_{12}}{3k^2}, \tag{99a}$$

$$F_4 = -\frac{C_1}{12q^2} + \frac{3C_{10} - C_{12}}{12p^2} + \frac{C_1 - 2C_2 + 2C_5 - 2C_{11} + C_{12}}{12k^2}, \tag{99b}$$

$$F_6 = \frac{-C_1 + 3C_3}{12q^2} - \frac{C_{12}}{12p^2} + \frac{C_1 - 2C_2 + 2C_5 - 2C_{11} + C_{12}}{12k^2}, \tag{99c}$$

$$F_8 = -\frac{C_5}{2p\cdot q} - \frac{C_1}{6q^2} - \frac{C_{12}}{6p^2} + \frac{C_1 - 2C_2 + 2C_5 - 2C_{11} + C_{12}}{6k^2}. \tag{99d}$$

With F_9, F_{10}, F_{11}, F_{12} given previously by (95) in terms of the finite coefficient functions C_i, the final function F_{13} is determined from the symmetrized sum of (96c) and (97c) to be

$$F_{13} = -\frac{\Pi(p^2) + \Pi(q^2)}{2} + \frac{p^2q^2}{p\cdot q}C_5 + \frac{p^4C_4 + q^4C_6}{4p\cdot q} + \frac{p\cdot q}{4}(2C_2 + C_3 + C_{10} + 2C_{11}) + \frac{p^2}{4}(2C_2 + C_4 + 2C_5 + C_{10}) + \frac{q^2}{4}(C_3 + 2C_5 + C_6 + 2C_{11}). \tag{100}$$

In this way all the coefficient functions F_i and hence the entire amplitude $\Gamma^{\mu\nu\alpha\beta}(p, q)$ in (86) is determined from its finite parts C_i, and the one-loop polarization tensor Π, by enforcing the conservation Ward identities on the amplitude. In particular the trace terms involving F_1 and F_2 in (98a) and (99a) are determined unambiguously by this procedure, and we shall show in the next section that F_1 contains the anomaly. Clearly if we had not enforced the conservation Ward identity relations (96) or (97), the trace is not determined, and could be required to satisfy the corresponding identities of conformal invariance in the massless limit removing any trace anomaly, at the price of violating the conservation identities (96) and (97).

Because (96) and (97) potentially overdetermine the coefficients F_i, we note they give three additional conditions on the finite coefficients C_i from their antisymmetric parts,

$$p^2(C_1 - 2C_2 + C_3 - 2C_7 + 2C_8) - 2p\cdot q(C_7 - C_9) - q^2(2C_8 - 2C_9 + C_{10} - 2C_{11} + C_{12}) = 0, \tag{101a}$$

$$-p^4C_1 + q^4C_{12} - 2p\cdot q(p^2C_1 - q^2C_{12}) + p^2q^2(-2C_2 + C_3 - C_{10} + 2C_{11}) = 0, \tag{101b}$$

$$p^2(-2C_2 + C_4 + 2C_5 + C_{10}) + p\cdot q(-2C_2 - C_3 + C_{10} + 2C_{11}) + q^2(-C_3 - 2C_5 - C_6 + 2C_{11}) - \frac{p^4}{q^2}C_1 + \frac{q^4}{p^2}C_{12} + 2[\Pi(p^2) - \Pi(q^2)] = 0, \tag{101c}$$

which must be satisfied identically, for a consistent solution to exist. The method for verifying that these three conditions are indeed satisfied by the C_i is given in Appendix B.

We note also that the difference,

$$\Pi(p^2) - \Pi(q^2) = (q^2 - p^2) \int_0^\infty \frac{ds}{q^2 + s} \frac{\rho_\Pi(s)}{p^2 + s} \quad (102)$$

in the last of the relations (101) is finite, as required by the fact that all the C_i are finite. Here

$$\rho_\Pi(s) \equiv \frac{1}{\pi} \mathrm{Im}\Pi|_{p^2=-s} = \frac{e^2}{12\pi^2}\left(1 + \frac{2m^2}{s}\right)\sqrt{1 - \frac{4m^2}{s}}\theta(s - 4m^2), \quad (103)$$

is the familiar spectral function of the one-loop photon polarization in QED, which tends to a constant for $s \gg 4m^2$. Using the spectral representation of Π, the renormalization of Eq. (100) may be accomplished by defining

$$\Pi_R(p^2) \equiv \Pi(p^2) - \Pi(0) = -p^2 \int_0^\infty \frac{ds}{s} \frac{\rho_\Pi(s)}{p^2 + s} = -\frac{e^2}{2\pi^2} \int_0^1 dx x(1 - x) \ln\left[1 + x(1 - x)\frac{p^2}{m^2}\right], \quad (104)$$

so that

$$F_{13} = (F_{13})_R + \Pi(0), \quad (105)$$

with the logarithmically divergent $\Pi(0)$ removed by charge renormalization,

$$\frac{1}{e_R^2} = \frac{1}{e^2}[1 + \Pi(0)]. \quad (106)$$

Indeed the tensor multiplying the logarithmically divergent $\Pi(0)$ is

$$t_{13}^{\mu\nu\alpha\beta}(p, q) = -\int d^4x \int d^4y e^{ip\cdot x + iq\cdot y} \frac{\delta^2 T_{\max}^{\mu\nu}(0)}{\delta A_\alpha(x)\delta A_\beta(y)}, \quad (107)$$

which must be added to $\Gamma^{\mu\nu\alpha\beta}(p, q)$ in full QED. Thus unlike the chiral $\langle J_5^\mu J^\alpha J^\beta\rangle$ amplitude considered previously, one renormalization of the amplitude $\langle T^{\mu\nu} J^\alpha J^\beta\rangle$ involving the stress tensor is necessary. However since charge renormalization enters only through the tensor t_{13} which is proportional to the stress tensor of the classical electromagnetic field, which is traceless, the trace of $\Gamma^{\mu\nu\alpha\beta}(p, q)$ which resides in the t_1 and t_2 tensors is finite, and unaffected by renormalization.

V. THE TRACE ANOMALY AND SCALING VIOLATION

Having determined completely the amplitude $\Gamma^{\mu\nu\alpha\beta}(p, q)$ from its finite parts by the principles of Lorentz covariance, gauge invariance, and general coordinate invariance, we come now to the relations that would be expected from the classical conformal invariance of the theory in the massless limit, $m \to 0$.

The $\langle TJJ\rangle$ triangle diagram with the first vertex replaced by the naive classical trace (64) is

$$\Lambda^{\alpha\beta}(p, q) \equiv -m(ie)^2 \int d^4x \int d^4y e^{ip\cdot x + iq\cdot y}\langle\bar\psi\psi J^\alpha(x) J^\beta(y)\rangle, = -e^2 m \int \frac{id^4l}{(2\pi)^4} \mathrm{tr}\left\{\frac{-i}{\not l + \not p + m}\gamma^\alpha \frac{-i}{\not l + m}\gamma^\beta \frac{-i}{\not l - \not q + m} + \frac{-i}{\not l + \not q + m}\gamma^\beta \frac{-i}{\not l + m}\gamma^\alpha \frac{-i}{\not l - \not p + m}\right\}. \quad (108)$$

This amplitude formally satisfies the conditions of vector current conservation,

$$p_\alpha \Lambda^{\alpha\beta}(p, q) = 0 = q_\beta \Lambda^{\alpha\beta}(p, q), \quad (109)$$

if one is free to shift the loop momentum integration variable l in (108). Although the integral is superficially linearly divergent, it is in fact at worst only logarithmically divergent because of the Dirac gamma matrix trace, and one factor of l is replaced by m. We may extract the factors of external momenta p and q, which multiply quadratically convergent integrals in $d = 4$ dimensions, and determine the finite parts in the same manner as the full amplitude. Then as before, we can determine (108) by Lorentz covariance, and vector current conservation, (109), in complete analogy to the case of the full amplitude (68) considered in the previous section. The evaluation is given in Appendix A.

Since there are only 2 two-index tensors composed of p, q, and the metric tensor, which satisfy the conservation conditions (109), namely, the tensors $u^{\alpha\beta}$ and $w^{\alpha\beta}$ defined by (79), we must have

$$\Lambda^{\alpha\beta}(p, q) = G_1 u^{\alpha\beta}(p, q) + G_2 w^{\alpha\beta}(p, q), \quad (110)$$

with G_1 and G_2 scalar functions of the three invariants k^2, p^2, q^2, and m^2. By identifying the coefficients of the finite terms proportional to the four-tensors homogeneous of degree two, viz. $p^\alpha p^\beta$, $q^\alpha q^\beta$, $p^\alpha q^\beta$, and $q^\alpha p^\beta$, we obtain in Appendix A,

$$G_1 = -\frac{e^2}{2\pi^2} m^2 \int_0^1 dx \int_0^{1-x} dy \frac{(1-4xy)}{D}, \tag{111a}$$

$$G_2 = -\frac{e^2}{2\pi^2} \frac{m^2}{p\cdot q} \int_0^1 dx \int_0^{1-x} dy \frac{(1-2x)(1-2y)}{D}, \tag{111b}$$

with D given by (20). Both of these vanish in the massless limit, and would be the expected values of the trace of the full amplitude (68), absent any anomalies.

We now compare $\Lambda^{\alpha\beta}(p,q)$ of (110) and (111) to the exact trace of the full amplitude (86)

$$g_{\mu\nu}\Gamma^{\mu\nu\alpha\beta}(p,q) = 3k^2 F_1 u^{\alpha\beta}(p,q) + 3k^2 F_2 w^{\alpha\beta}(p,q), \tag{112}$$

computed in the previous section by requiring the Ward identities (96) and (97) of stress-tensor conservation. The functions F_1 and F_2, given by (98a) and (99a), respectively, are completely determined by that procedure. Let us consider the second term in (112) first. Since the tensor $w^{\alpha\beta}$ corresponds to a local dimension six term in the effective action, c.f. (82b), we do not expect it to contain any anomaly, i.e. we expect $3k^2F_2 = G_2$, which vanishes in the limit $m \to 0$. To see this explicitly requires the following simple algebraic identity satisfied by the C_i coefficients, viz.,

$$C_2 - C_4 - C_5 - C_6 + C_8 + C_{11} = 0, \tag{113}$$

as is easily verified by direct substitution of Table V into (92). Subtracting twice this identity from $3k^2F_2$ given by (99a), and using also Eq. (91), we find

$$3k^2F_2 = -\frac{1}{p\cdot q}(p^2C_4 + q^2C_6 + 2p\cdot qC_8). \tag{114}$$

In this form one may substitute the Feynman parametrization of the coefficients (92), using Table V to find

$$\begin{aligned} 3k^2F_2 &= \frac{e^2}{2\pi^2 p\cdot q}\int_0^1 dx\int_0^{1-x} dy(1-2x)(1-2y)\frac{(D-m^2)}{D} \\ &= -\frac{e^2}{2\pi^2}\frac{m^2}{p\cdot q}\int_0^1 dx\int_0^{1-x} dy\frac{(1-2x)(1-2y)}{D}, \end{aligned} \tag{115}$$

since the first integral independent of m^2 in which the denominator D is cancelled,

$$\int_0^1 dx\int_0^{1-x} dy(1-2x)(1-2y) = 0 \tag{116}$$

in fact vanishes identically. Comparing (115) with (111b) we verify that indeed

$$3k^2F_2 = G_2 \tag{117}$$

is nonanomalous, and vanishes in the $m \to 0$ limit.

Turning next to the F_1 term in the full trace, (112), we need the following identity satisfied by the C_i coefficients,

$$\begin{aligned} p^2(C_1 - 2C_2 + C_3 + 2C_4 - 4C_8) + q^2(2C_6 - 4C_8 + C_{10} \\ - 2C_{11} + C_{12}) + 2p\cdot q(C_3 - 2C_5 - C_7 - C_9 + C_{10}) \\ = 0. \end{aligned} \tag{118}$$

which is verified in Appendix B. Adding this quantity to $3k^2F_1$ given by (98a) yields

$$\begin{aligned} 3k^2F_1 &= p^2(-2C_2 + 2C_3 + 2C_4 + C_7 - 2C_8) \\ &\quad + q^2(2C_6 - 2C_8 + C_9 + 2C_{10} - 2C_{11}) \\ &\quad + 2p\cdot q(C_3 - 2C_5 + C_8 + C_{10}). \end{aligned} \tag{119}$$

By substituting the Feynman parametrization integrals (92) and using again Table V, we obtain in this case,

$$\begin{aligned} 3k^2F_1 &= \frac{e^2}{2\pi^2}\int_0^1 dx\int_0^{1-x} dy(1-4xy)\frac{(D-m^2)}{D} \\ &= \frac{e^2}{6\pi^2} - \frac{e^2m^2}{2\pi^2}\int_0^1 dx\int_0^{1-x} dy\frac{(1-4xy)}{D}, \end{aligned} \tag{120}$$

since

$$\int_0^1 dx\int_0^{1-x} dy(1-4xy) = \frac{1}{3}, \tag{121}$$

which unlike (116) does not vanish.

Since the first term on the right side of (120) in which the factors of D cancel between the numerator and denominator is both nonvanishing and independent of m, the trace is anomalous, and we have

$$3k^2F_1 = \frac{e^2}{6\pi^2} + G_1. \tag{122}$$

Hence the coefficient of the $t_1^{\mu\nu\alpha\beta}(p.q)$ tensor in the full amplitude may be written

$$\begin{aligned} &F_1(k^2; p^2, q^2) \\ &= \frac{e^2}{18\pi^2k^2}\left\{1 - 3m^2\int_0^1 dx\int_0^{1-x} dy\frac{(1-4xy)}{D}\right\}, \end{aligned} \tag{123}$$

giving rise to the nonzero trace,

$$g_{\mu\nu}\Gamma^{\mu\nu\alpha\beta}(p,q)|_{m=0} = \frac{e^2}{6\pi^2}u^{\alpha\beta}(p,q), \tag{124}$$

in the massless limit, which is exactly (65) in momentum space.

It is clear that this nonvanishing trace is completely determined by the finite terms in the amplitude together with the imposition of the Ward identities of stress-tensor

conservation, for if we had not made use of (96), F_1 would be still undetermined, and could be chosen to vanish in the massless limit. Of course, with this choice of F_1 to satisfy the requirement of naive conformal invariance, the conservation identities of (96) or (97) would be violated, there would be an Einstein anomaly, and general coordinate invariance of the theory would be lost. This conflict between symmetries is quite analogous to the chiral case considered previously, where the naive Ward identity of $U_{\rm ch}(1)$ invariance in the massless limit could be maintained by adding an extra term to the amplitude, at the expense of violating the $U(1)$ conservation identities (9).

In the case of the anomalous nonzero trace of the energy-momenum tensor, it is the conformal invariance of the classical theory with massless fermions that cannot be maintained at the quantum level. In $d = 4$ flat spacetime the conformal group is $O(4, 2)$ and its 15 generators consist of global dilations and the four special conformal transformations together with the 10 generators of the Poincaré group. The Noether dilation current,

$$D^\mu = x^\nu T^\mu{}_\nu \tag{125}$$

is divergenceless and the corresponding charge $\int d^3\vec{x} D^0$ is conserved if and only if $T^{\mu\nu}$ is traceless. A nonzero trace implies instead nontrivial scale dependence [8,30].

To see the effect of global scale transformations implied by the trace anomaly, we consider the trace (112) at $k = p + q = 0$. Then, since

$$w^{\alpha\beta}(p, -p) = -p^2 u^{\alpha\beta}(p, -p) = p^2(p^2 g^{\alpha\beta} - p^\alpha p^\beta), \tag{126}$$

upon using (115) and (120), Eq. (112) for the full trace gives

$$\begin{aligned} g_{\mu\nu}\Gamma^{\mu\nu\alpha\beta}(p, -p) &= \left[-\frac{e^2}{6\pi^2} + \frac{e^2 m^2}{2\pi^2}\int_0^1 dx \int_0^{1-x} dy \frac{(1-4xy) + (1-2x)(1-2y)}{p^2(x+y)(1-x-y) + m^2}\right](p^2 g^{\alpha\beta} - p^\alpha p^\beta) \\ &= \left[-\frac{e^2}{6\pi^2} + \frac{e^2 m^2}{\pi^2}\int_0^1 du \frac{u(1-u)}{p^2 u(1-u) + m^2}\right](p^2 g^{\alpha\beta} - p^\alpha p^\beta), \end{aligned} \tag{127}$$

after changing variables to $u = x + y$ and $v = (x - y)/2$, and integrating over v. At the same time we note from (104) that

$$\begin{aligned} 2p^2 \frac{d\Pi_R}{dp^2} &= -\frac{e^2}{6\pi^2} + \frac{e^2 m^2}{\pi^2}\int_0^1 du \frac{x(1-x)}{p^2 x(1-x) + m^2} \\ &= -\frac{\beta(e^2; p^2)}{e^2} \end{aligned} \tag{128}$$

is related to the β function of the electromagnetic coupling e^2. Comparing with (127), we secure [8]

$$\begin{aligned} g_{\mu\nu}\Gamma^{\mu\nu\alpha\beta}(p, -p) &= 2p^2 \frac{d\Pi_R}{dp^2}(p^2 g^{\alpha\beta} - p^\alpha p^\beta) \\ &= -\frac{\beta(e^2; p^2)}{e^2}(p^2 g^{\alpha\beta} - p^\alpha p^\beta), \end{aligned} \tag{129}$$

a result that remains valid to all orders in perturbation theory [30]. Equation (129) may also be derived by differentiating the Ward identity (78) with respect to q^μ (or p^μ), and then setting $q = -p$. Hence the breaking of scale invariance by the trace of the energy-momentum tensor, together with its conservation, may be regarded as responsible for the β function running of the coupling, without any direct reference to ultraviolet renormalization.

We note also that by combining the two terms in (128), the one-loop β function here,

$$\beta(e^2; p^2) = \frac{e^4}{\pi^2} p^2 \int_0^1 dx \frac{x^2(1-x)^2}{p^2 x(1-x) + m^2} \to \frac{e^4}{30\pi^2}\frac{p^2}{m^2}, \tag{130}$$

vanishes as $p^2 \to 0$. This is the correct behavior for the *infrared* running with p^2 of the physical renormalized coupling at momenta small compared to the electron mass, where vacuum polarization is negligible and decoupling of the electron loop must hold [32]. For $p^2 \gg m^2$, $\beta(e^2; p^2) \to e^4/6\pi^2$, which is then identical to the usual ultraviolet β function, calculated e.g. in dimensional regularization where the infrared mass plays no role. We see then the physical necessity of the trace anomaly in a different way, for if the constant first term in (128) determined by the trace anomaly at $m = 0$ were not present, there would be nothing to cancel the second integral as $p^2 \to 0$, and decoupling of heavy degrees of freedom at large distances ($p^2 \ll m^2$) in (130) would not occur.

VI. SPECTRAL REPRESENTATION, SUM RULE, AND THE MASSLESS SCALAR POLE

The physical meaning of the anomaly is further exposed by considering the spectral representation of the amplitude, cut across two of its legs as in Fig. 2. Following the pattern of the chiral case considered previously in (28), the spectral representations for the amplitudes C_i may be introduced by using the definition (26) and the identity (27), to obtain

$$C_j(k^2; p^2, q^2) = \int_0^\infty ds \frac{\rho_j(s; p^2, q^2)}{k^2 + s} \tag{131}$$

with

$$\rho_j(s; p^2, q^2) = \frac{e^2}{4\pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{c_j(x, y)}{xy} \times \delta(s - S(x, y; p^2, q^2)). \tag{132}$$

The ρ_i defined in this way are not necessarily positive, nor are they independent, owing to the relations (101), (113), and (118). Indeed the ρ_j satisfy exactly the same identities as the corresponding C_j, of which they are just the discontinuity or imaginary part as k^2 is analytically continued to $-s$ with $s > 0$. Of interest to us however is only the linear combination which appears in the trace (119). By using (118), and repeating the steps that led from (98a) to (120), we obtain with the help of (26) and Table V,

$$\begin{aligned}
&k^2(\rho_7 + \rho_8 + \rho_9) + p^2(-\rho_1 + \rho_3 + \rho_8 - \rho_9) + q^2(-\rho_7 + \rho_8 + \rho_{10} - \rho_{12}) \\
&= p^2(-2\rho_2 + 2\rho_3 + 2\rho_4 + \rho_7 - 2\rho_8) + q^2(2\rho_6 - 2\rho_8 + \rho_9 + 2\rho_{10} - 2\rho_{11}) + 2p \cdot q(\rho_3 - 2\rho_5 + \rho_8 + \rho_{10}) \\
&= \frac{e^2}{4\pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{\delta(s - S)}{xy} \{p^2(-2c_2 + 2c_3 + 2c_4 + c_7 - 2c_8) + q^2(2c_6 - 2c_8 + c_9 + 2c_{10} - 2c_{11}) \\
&\quad + 2p \cdot q(c_3 - 2c_5 + c_8 + c_{10})\} = \frac{e^2}{2\pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{\delta(s - S)}{xy}(1 - 4xy)(D - m^2) \\
&= \frac{e^2}{2\pi^2} \int_0^1 dx \int_0^{1-x} dy \delta(s - S)(1 - 4xy)\left(k^2 + S - \frac{m^2}{xy}\right) = (k^2 + s)\rho_T - m^2\rho_m,
\end{aligned} \tag{133}$$

where

$$\rho_T(s; p^2, q^2) \equiv \rho_3 - 2\rho_5 + \rho_8 + \rho_{10} = \frac{e^2}{2\pi^2} \int_0^1 dx \int_0^{1-x} dy(1 - 4xy)\delta(s - S(x, y; p^2, q^2)), \quad \text{and} \tag{134a}$$

$$\rho_m(s; p^2, q^2) \equiv \frac{e^2}{2\pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{(1 - 4xy)}{xy} \delta(s - S(x, y; p^2, q^2)), \tag{134b}$$

since $c_3 - 2c_5 + c_8 + c_{10} = 2xy(1 - 4xy)$.

These relations may be compared to their somewhat simpler analogs, (30) and (31), in the chiral case. Since $1 - 4xy \geq 0$ over the indicated range of x, y, both ρ_T and ρ_m are non-negative functions of s for spacelike p^2 and q^2. Notice that at $k^2 = -s$ the quantity ρ_T drops out of (133), so that the discontinuity or imaginary part of (120) vanishes in the conformal limit $m \to 0$, and is nonanomalous.

As in the chiral case (33), we find that the spectral function which determines the anomaly satisfies a sum rule [33],

$$\begin{aligned}
&\int_0^\infty ds \rho_T(s; p^2, q^2) \\
&= \frac{e^2}{2\pi^2} \int_0^1 dx \int_0^{1-x} dy(1 - 4xy) \int_0^\infty ds \delta(s - S) = \frac{e^2}{6\pi^2},
\end{aligned} \tag{135}$$

by (121), which is independent of $p^2 \geq 0$, $q^2 \geq 0$, and $m^2 \geq 0$, since then $S(x, y; p^2, q^2) \geq 0$ and the δ function can be satisfied over the range of $s \geq 0$. On the other hand, using $2p \cdot q = k^2 - p^2 - q^2$, and rearranging the second and last lines of (133) gives

$$\begin{aligned}
\rho_T(s; p^2, q^2) = \frac{p^2}{s}(&-2\rho_2 + \rho_3 + 2\rho_4 + 2\rho_5 + \rho_7 \\
&- 3\rho_8 - \rho_{10}) + \frac{q^2}{s}(-\rho_3 + 2\rho_5 + 2\rho_6 \\
&- 3\rho_8 + \rho_9 + \rho_{10} - 2\rho_{11}) + \frac{m^2}{s}\rho_m.
\end{aligned} \tag{136}$$

Since the ρ_i develop at worst logarithmic singularities in the combined limit $p^2, q^2, m^2 \to 0^+$ (taken in any order), (136) shows that ρ_T vanishes pointwise for all $s > 0$ in this limit. The only way that this can be consistent with the sum rule (135) is if ρ_T develops a δ function singularity at $s = 0$ in this limit. Indeed since the function $S(x, y; p^2, q^2)$ defined by (26) vanishes identically in this limit, we see directly from (134a) that

$$\begin{aligned}
\lim_{p^2, q^2, m^2 \to 0^+} \rho_T(s; p^2, q^2) &= \frac{e^2}{2\pi^2} \int_0^1 dx \int_0^{1-x} dy(1 - 4xy)\delta(s) \\
&= \frac{e^2}{6\pi^2}\delta(s),
\end{aligned} \tag{137}$$

by taking the limits inside the integral. Thus ρ_T exhibits a massless scalar intermediate state in the two-particle cut amplitude.[3]

It is instructive to retain the nonzero fermion mass $m > 0$ as an infrared regulator, in order to study this intermediate state in more detail. Comparing with (36) from the axial anomaly, we find when $p^2 = q^2 = 0$ that

$$\rho_T(s;0,0) = \frac{m^2}{s}\rho_m(s;0,0) = \frac{1}{2}\left(1-\frac{4m^2}{s}\right)\rho_{\mathcal{A}}(s;0,0) = \frac{e^2}{2\pi^2}\frac{m^2}{s^2}\left(1-\frac{4m^2}{s}\right)\ln\left\{\frac{1+\sqrt{1-\frac{4m^2}{s}}}{1-\sqrt{1-\frac{4m^2}{s}}}\right\}\theta(s-4m^2). \tag{138}$$

This function is plotted in Fig. 6.

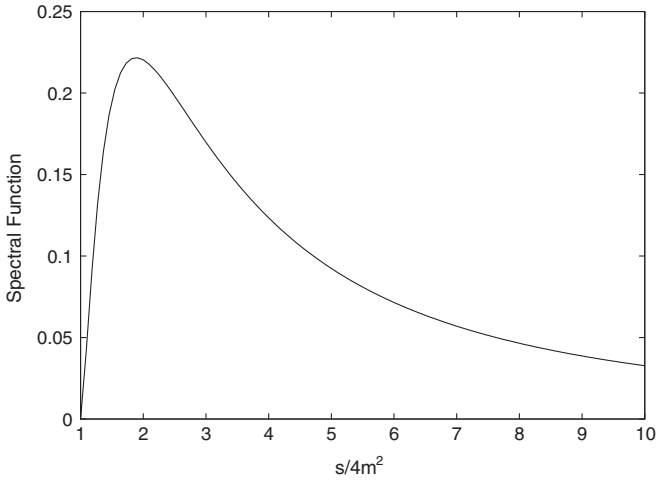

FIG. 6. The spectral function ρ_T of (138) in units of $\frac{e^2}{32\pi^2}m^{-2}$ as a function of $\frac{s}{4m^2}$.

The corresponding amplitude is

$$F_1(k^2;0,0) = \frac{1}{3k^2}\int_0^\infty \frac{ds}{k^2+s}[(k^2+s)\rho_T - m^2\rho_m] = \frac{1}{3k^2}\left[\frac{e^2}{6\pi^2} - m^2\int_0^\infty ds\,\frac{\rho_m(s;0,0)}{k^2+s}\right] \tag{139}$$

which exhibits a pole at $k^2 = 0$ when the fermion mass $m = 0$, i.e.

$$\lim_{p^2,q^2,m^2\to 0^+} F_1(k^2;p^2,q^2) = \frac{e^2}{18\pi^2}\frac{1}{k^2}. \tag{140}$$

When the fermion mass is nonzero the amplitude (139) can also be written in the form,

$$F_1(k^2;0,0) = \frac{m^2}{3}\int_{4m^2}^\infty \frac{ds}{s}\frac{\rho_m(s;0,0)}{k^2+s} \tag{141}$$

which shows that there is no pole divergence as $k^2 \to 0$ with $m^2 > 0$ fixed. This is again the phenomenon of decoupling, as in the cancellation noted previously in (128) and (130), with the place of $p^2 \ll m^2$ being taken here by $k^2 \ll m^2$. Conversely, if $m = 0$ the amplitude (139) behaves like k^{-2} for all k^2, in both the infrared and ultraviolet, as expected in the classical conformal limit of a theory with no intrinsic mass or momentum scale.

It is also instructive to carry out the same steps for the imaginary part of the trace which is nonanomalous, i.e. for the spectral function corresponding to the nonanomalous amplitude F_2. Repeating the steps which led from (99a) and (115), using the identities corresponding to (91) and (113) for their imaginary parts, we obtain

$$\frac{k^2}{q^2}\rho_1 + \frac{k^2}{q^2}\rho_{12} - \rho_1 + 2\rho_2 - 2\rho_5 + 2\rho_{11} - \rho_{12} = -\frac{1}{p\cdot q}(p^2\rho_4 + q^2\rho_6 + 2p\cdot q\rho_8) = \frac{e^2}{2\pi^2}\frac{1}{p\cdot q}\int_0^1 dx\int_0^{1-x} dy\,\frac{\delta(s-S)}{xy}(1-2x)(1-2y)(D-m^2) = \frac{e^2}{2\pi^2}\frac{1}{p\cdot q}\int_0^1 dx\int_0^{1-x} dy\,\delta(s-S)(1-2x)(1-2y)\left(k^2+s-\frac{m^2}{xy}\right). \tag{142}$$

In this case the spectral function corresponding to ρ_T in (134a) is

$$-\frac{\rho_8(s;p^2,q^2)}{p\cdot q} = \frac{e^2}{2\pi^2}\frac{1}{p\cdot q}\int_0^1 dx\int_0^{1-x} dy(1-2x) \times (1-2y)\delta(s-S(x,y;p^2,q^2)), \tag{143}$$

but unlike (135) ρ_8 obeys the *vanishing* sum rule,

$$\int_0^\infty ds\rho_8(s;p^2,q^2) = -\frac{e^2}{2\pi^2}\int_0^1 dx\int_0^{1-x} dy(1-2x)(1-2y) = 0. \tag{144}$$

Hence, although ρ_8 can be expressed in the form,

[3]This observation was made in Ref. [34] in the context of photon pair creation by a cosmological gravitational field.

$$\rho_8(s;p^2;q^2) = \frac{p^2}{s}(\rho_4-\rho_8)+\frac{q^2}{s}(\rho_6-\rho_8) - \frac{m^2}{s}\frac{e^2}{2\pi^2}\int_0^1 dx\int_0^{1-x} dy\frac{(1-2x)(1-2y)}{xy}\delta(s-S) \tag{145}$$

analogous to (136), which vanishes pointwise in the combined limit, $p^2, q^2, m^2 \to 0^+$, it has no positivity property, and no reason to develop a δ function singularity at $s=0$ in that limit. Indeed it is not difficult to see that ρ_8 and indeed the corresponding full amplitude C_8 vanishes identically in this limit, c.f. Eq. (149b) below, consistent with the vanishing of F_2, which unlike F_1 has no pole as $p^2, q^2, m^2 \to 0^+$ [35].

We may use the general amplitude $\Gamma^{\mu\nu\alpha\beta}(p,q)$ to evaluate the matrix element of $T^{\mu\nu}$ to physical photons on shell, $p^2=q^2=0$ which are also transverse. In this case all terms with p^α and q^β vanish when contracted with the transverse photon polarization states, and the matrix element simplifies considerably. The tensor $w^{\alpha\beta}(p,q)$ and hence the tensors t_2, t_4, t_6, and t_8 vanish upon contraction with transverse photons, as do the tensors t_9, t_{10}, t_{11}, and t_{12}. The remaining relevant form factors also simplify considerably when $p^2=q^2=0$, becoming

$$F_1(k^2;0,0) = \frac{2C_7+C_8}{3}\Big|_{p^2=q^2=0} \tag{146a}$$

$$F_3(k^2;0,0) = F_5(k^2;0,0) = \frac{C_7-C_8}{12}\Big|_{p^2=q^2=0} \tag{146b}$$

$$F_7(k^2,0,0) = -4F_3(k^2;0,0) = \frac{-C_7+C_8}{3}\Big|_{p^2=q^2=0} \tag{146c}$$

$$F_{13,R}(k^2,0,0) = \frac{k^2}{4}(2C_2+C_3)|_{p^2=q^2=0}, \tag{146d}$$

for any m. The scalar coefficients C_j here are evaluated on the photon mass shell, given by

$$C_j(k^2;0,0) = \frac{e^2}{4\pi^2}\int_0^1 dx\int_0^{1-x} dy\frac{c_j(x,y)}{k^2xy+m^2}, \tag{147}$$

in which case $C_2=C_{11}$, $C_3=C_{10}$, and $C_7=C_9$. Thus, for on shell photons there are only three independent form factors, and we can write the matrix element to physical photons with transverse field amplitudes $\tilde{A}_\alpha(p)$, $\tilde{A}_\beta(q)$ in the form,

$$\langle 0|T^{\mu\nu}(0)|p,q\rangle = \{F_1(k^2g^{\mu\nu}-k^\mu k^\nu)u^{\alpha\beta}(p,q) - 2F_3[k^2g^{\mu\nu}-4(p^\mu q^\nu+q^\mu p^\nu)+2(p^\mu p^\nu+q^\mu q^\nu)]u^{\alpha\beta}(p,q) + F_{13,R}t_{13}^{\mu\nu\alpha\beta}(p,q)\}\tilde{A}_\alpha(p)\tilde{A}_\beta(q), \tag{148}$$

with F_1, F_3, F_{13R} evaluated at $p^2=q^2=0$ given by (146) and (147) and Table V. The survival of only three independent tensors when both photons are on their mass shell and have physical transverse polarizations agrees with the literature [31,35]. Each of the three tensors remaining in (148) is conserved and their contractions with k_ν vanish for photons on shell. Only the first has a nonzero trace.

Taking the $m=0$ limit gives the further simplification that

$$C_7(k^2)|_{p^2=q^2=m^2=0} = \frac{e^2}{2\pi^2k^2}\int_0^1 dx\int_0^{1-x} dy(1-2x)^2 = \frac{e^2}{12\pi^2}\frac{1}{k^2} \tag{149a}$$

$$C_8(k^2)|_{p^2=q^2=m^2=0} = 0, \tag{149b}$$

while F_{13R} contains a $\ln(k^2/m^2)$ behavior in this limit, but no pole (reflecting the need to renormalize the charge at a mass scale $\mu^2>0$ different from m^2 in the massless limit). Thus from (146) both the F_1 and F_3 form factors of the scattering amplitude to physical on shell photons exhibits a massless scalar pole in the limit of vanishing electron mass, with F_1 given by (140) and

$$\lim_{m\to 0} 2F_3(k^2;0,0) = \frac{e^2}{72\pi^2}\frac{1}{k^2}. \tag{150}$$

Thus the leading order behavior as $k^2\to 0$ of the sum of terms in the amplitude (148) to physical on shell photons is

$$\lim_{k^2\to 0}\lim_{m\to 0}\langle 0|T^{\mu\nu}(0)|p,q\rangle \to -\frac{e^2}{12\pi^2k^2}(p^\mu p^\nu+q^\mu q^\nu)u^{\alpha\beta}(p,q)\tilde{A}_\alpha(p)\tilde{A}_\beta(q) + \log(k^2), \quad \text{and finite terms}, \tag{151}$$

when the limit of vanishing electron mass is taken first. This shows that the singular massless pole behavior survives in the matrix element of the stress tensor to physical transverse photons (in its tracefree terms), while the trace remains finite in the conformal limit of vanishing electron mass and all three four-momenta $(k,\ p,\ q)$ become lightlike.

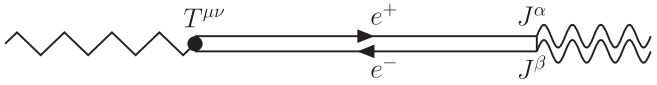

FIG. 7. The two-particle intermediate state of a collinear e^+e^- pair responsible for the δ-fn in (137).

The kinematics of the state appearing in the imaginary part and spectral function (137) in this limit is essentially $1+1$ dimensional, and can be represented as the two-particle collinear e^+e^- pair in Fig. 7. This is the only configuration possible for one particle with four-momentum k^μ converting to two particles of zero mass, with $p^2 = q^2 = 0$ as $k^2 \to 0$ as well. A detailed examination of the imaginary part of the amplitude, illustrated by the analog of Fig. 2, shows that there is a cancellation between the numerator and Feynman propagator in the denominator of the amplitude from the uncut fermion line in the triangle. Thus all particles in the real propagating intermediate state depicted in Fig. 7 are massless, on shell, and collinear. Although this special collinear kinematics is a set of vanishing measure in the two-particle phase space, the $\delta(s)$ in the spectral function (137) and finiteness of the anomaly itself shows that this pair state couples to on shell photons on the one hand, and gravitational metric perturbations on the other hand, with finite amplitude. When gravitational scattering is considered in Sec. VIII the four-momentum transfer k^μ may be timelike or spacelike, the pole terms (139) and (150) in the real part of the amplitude become relevant, and neither fermion pair nor final state photons are collinear.

VII. ANOMALY EFFECTIVE ACTION AND MASSLESS SCALAR FIELDS

Having demonstrated the existence of a real massless spin-0 intermediate state in the imaginary part of the triangle amplitude, and a corresponding massless pole in the full amplitude, we turn in this section to the effective action and scalar fields which describe these massless scalar degrees of freedom. In fact, a covariant action for the trace anomaly in a general curved space has been given already in several earlier works [18,36–38]. This general effective action can be presented in the nonlocal form,

$$S_{\rm anom}[g,A] = \frac{1}{8}\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\Big(E-\frac{2}{3}\Box R\Big)_x \times \Delta_4^{-1}(x,x')\Big[2bC^2 + b'\Big(E-\frac{2}{3}\Box R\Big) + 2cF_{\mu\nu}F^{\mu\nu}\Big]_{x'} \quad (152)$$

where the b and b' parameters are the coefficients of the Weyl tensor squared, $C^2 = C_{\lambda\mu\nu\rho}C^{\lambda\mu\nu\rho} = R_{\lambda\mu\nu\rho}R^{\lambda\mu\nu\rho} - 2R_{\mu\nu}R^{\mu\nu} + \frac{R^2}{3}$ and the Euler density $E = {}^*R_{\lambda\mu\nu\rho}{}^*R^{\lambda\mu\nu\rho} = R_{\lambda\mu\nu\rho}R^{\lambda\mu\nu\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2$, respectively, of the trace anomaly in a general background curved spacetime, and the last term in (152) takes into account the anomaly in a background gauge field with coefficient c. For the present case of Dirac fermions in a classical gravitational ($g_{\mu\nu}$) and classical electromagnetic (A_α) background, $b = 1/320\pi^2$, $b' = -11/5760\pi^2$, and $c = -e^2/24\pi^2$. The notation $\Delta_4^{-1}(x,x')$ denotes the Green's function inverse of the conformally covariant fourth order differential operator defined by

$$\Delta_4 \equiv \nabla_\mu(\nabla^\mu\nabla^\nu + 2R^{\mu\nu} - \tfrac{2}{3}Rg^{\mu\nu})\nabla_\nu = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu + \tfrac{1}{3}(\nabla^\mu R)\nabla_\mu - \tfrac{2}{3}R\Box. \quad (153)$$

By varying (152) multiple times with respect to the background metric $g_{\mu\nu}$ and/or the background gauge fields A_α one can derive formulas for the trace anomaly related parts of amplitudes involving multiple insertions of the stress tensor $\langle TTT\ldots JJ\rangle$ and $\langle TTT\ldots\rangle$ in curved or flat space. We emphasize that the effective action (152) was obtained by integrating the anomaly, and is determined up to terms which are conformally invariant. Therefore one can expect it to yield correct results for the trace related parts of amplitudes such as (70), while the tracefree parts are not given uniquely by (152).

As detailed in Ref. [18] we may render the nonlocal anomaly action (152) into a local form, by the introduction of two scalar auxiliary fields φ and ψ which satisfy fourth order differential equations,

$$\Delta_4\varphi = \tfrac{1}{2}(E - \tfrac{2}{3}\Box R), \quad (154a)$$

$$\Delta_4\psi = \frac{1}{2}C_{\lambda\mu\nu\rho}C^{\lambda\mu\nu\rho} + \frac{c}{2b}F_{\mu\nu}F^{\mu\nu}, \quad (154b)$$

where we have added the last term in (154b) to take account of the background gauge field. This local effective action corresponding to (152) in a general curved space is given by

$$S_{\rm anom} = b'S^{(E)}_{\rm anom} + bS^{(F)}_{\rm anom} + \frac{c}{2}\int d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu}\varphi, \quad (155)$$

where

$$S^{(E)}_{\rm anom} \equiv \frac{1}{2}\int d^4x\sqrt{-g}\Big\{-(\Box\varphi)^2 + 2\Big(R^{\mu\nu} - \frac{R}{3}g^{\mu\nu}\Big)\times(\nabla_\mu\varphi)(\nabla_\nu\varphi) + \Big(E - \frac{2}{3}\Box R\Big)\varphi\Big\};$$

$$S^{(F)}_{\rm anom} \equiv \int d^4x\sqrt{-g}\Big\{-(\Box\varphi)(\Box\psi) + 2\Big(R^{\mu\nu} - \frac{R}{3}g^{\mu\nu}\Big)\times(\nabla_\mu\varphi)(\nabla_\nu\psi) + \frac{1}{2}C_{\lambda\mu\nu\rho}C^{\lambda\mu\nu\rho}\varphi + \frac{1}{2}\Big(E - \frac{2}{3}\Box R\Big)\psi\Big\}. \quad (156)$$

The free variation of the local action (155) and (156) with respect to ψ and φ yields the equations of motion (154).

Each of these terms when varied with respect to the background metric gives a stress-energy tensor in terms of the auxiliary fields satisfying Eqs. (154). Since we are interested here in only the first variation of the action with respect to $g_{\mu\nu}$, we may drop all terms in (155) which are second order or higher in the metric deviations from flat space. Also, if we solve (154b) formally for ψ in flat space, we find a $\Box^{-2} \to k^{-4}$ pole in this stress tensor. The simplest way to eliminate this higher order pole is to assume that φ is also first order in metric deviations from flat space, so that the entire $b'S^{(E)}$ contribution to (155) can be neglected as well. These reductions are equivalent to replacing the general nonlocal effective action of the anomaly (152) by the much simpler form,

$$S_{\rm anom}[g, A] \to -\frac{c}{6}\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}R_x\Box^{-1}_{x,x'}[F_{\alpha\beta}F^{\alpha\beta}]_{x'}, \tag{157}$$

valid to first order in metric variations around flat space, or its local equivalent (155) by

$$S_{\rm anom}[g, A; \varphi, \psi'] = \int d^4x\sqrt{-g}\Big[-\psi'\Box\varphi - \frac{R}{3}\psi' + \frac{c}{2}F_{\alpha\beta}F^{\alpha\beta}\varphi\Big], \tag{158}$$

where

$$\psi' \equiv b\Box\psi, \tag{159a}$$

$$\Box\psi' = \frac{c}{2}F_{\alpha\beta}F^{\alpha\beta}, \tag{159b}$$

$$\Box\varphi = -\frac{R}{3}. \tag{159c}$$

Then after variation we may set $\varphi = 0$ in flat space, and the only terms which remain in the stress tensor derived from (155) are those linear in ψ', viz.

$$T^{\mu\nu}[\psi'(z)] = \frac{2}{\sqrt{-g}}\frac{\delta S_{\rm anom}}{\delta g_{\mu\nu}(z)}\Big|_{\rm flat,\varphi=0} = \frac{2}{3}(g^{\mu\nu}\Box - \partial^\mu\partial^\nu)\psi'(z), \tag{160}$$

which is independent of b and b', and contain only second order differential operators, after the definition (159a). Solving (159b) formally for ψ' and substituting in (160), we find

$$T^{\mu\nu}_{\rm anom}(z) = \frac{c}{3}(g^{\mu\nu}\Box - \partial^\mu\partial^\nu)_z\int d^4x'\Box^{-1}_{z,x'}[F_{\alpha\beta}F^{\alpha\beta}]_{x'}, \tag{161}$$

a result that may be derived directly from (157) as well.

By varying (161) again with respect to the background gauge potentials, making use of (82a) and Fourier transforming, we obtain

$$\Gamma^{\mu\nu\alpha\beta}_{\rm anom}(p, q) = \int d^4x\int d^4y e^{ip\cdot x+iq\cdot y}\frac{\delta^2 T^{\mu\nu}_{\rm anom}(0)}{\delta A_\alpha(x)A_\beta(y)} = \frac{e^2}{18\pi^2}\frac{1}{k^2}(g^{\mu\nu}k^2 - k^\mu k^\nu)u^{\alpha\beta}(p, q), \tag{162}$$

which coincides with the first term of (86), with (140), and gives the full trace for massless fermions,

$$g_{\mu\nu}T^{\mu\nu}_{\rm anom} = cF_{\alpha\beta}F^{\alpha\beta} = -\frac{e^2}{24\pi^2}F_{\alpha\beta}F^{\alpha\beta}, \tag{163}$$

in agreement with (65). We observe that as in the chiral case, the strict $1/k^2$ pole in the anomalous amplitude F_1 obtained from the $D_{\psi'\varphi} = i\langle\mathcal{T}\psi'\varphi\rangle$ propagator auxiliary field applies only in the limit of (140), or equivalently for $k^2 \gg (|p^2|, |q^2|, m^2)$. The spectral representations and sum rule of the previous section show that when this condition is not satisfied, the two-particle intermediate state in the anomalous amplitude becomes a broad resonance instead of an isolated pole, as in Fig. 6, and the residue of the pole at $k^2 = 0$ vanishes when any of p^2, q^2, or m^2 are nonzero. The tree amplitude of the effective action (158) which reproduces the pole in the trace part of the $\langle TJJ\rangle$ triangle amplitude is illustrated in Fig. 8.

Most of the remarks about the auxiliary field description of the axial anomaly at the end of Sec. III apply also to the trace anomaly case. As in the case of the axial anomaly, the effective action (156) or (158) explicitly exhibiting these two scalar fields is a rewriting of a part of the nonlocal form of the effective action (152) or (157) for massless QED in curved spacetime, with the reduction to (157) correct to leading order in the metric deviation from flat space, $h_{\mu\nu} = \delta g_{\mu\nu}$. The massless degrees of freedom φ and ψ' are a necessary consequence of the trace anomaly, required by imposition of all the other symmetries. In this case these are scalar rather than pseudoscalar degrees of freedom. As in the chiral case, two independent fields are required, and the propagator appearing in the intermediate state of the triangle amplitude is a certain off diagonal $D_{\psi'\varphi} = i\langle\mathcal{T}\psi'\varphi\rangle$ term. Unlike the chiral case the general effective action (156) or (158) requires the fourth order differential

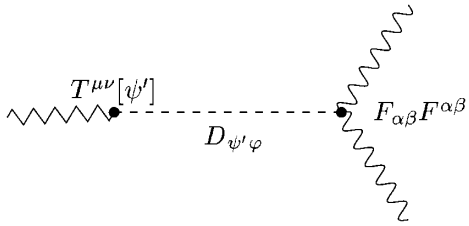

FIG. 8. Tree diagram of the effective action (158), which reproduces the trace of the triangle anomaly. The dashed line denotes the propagator $D_{\psi'\varphi} = \Box^{-1}$ of the scalar intermediate state, while the jagged line denotes the gravitational metric field variation $h_{\mu\nu} = \delta g_{\mu\nu}$.

operator Δ_4 of (153), implying that higher order amplitudes such as $\langle TTJJ\rangle$ should have double poles.

An important physical difference with the axial case is that the introduction of a chiral current J_5^μ and axial vector source $\mathcal{B}_\mu$ corresponding to it appear rather artificial, and difficult to realize in nature, whereas the trace of the stress tensor obtained by a conformal variation of the effective action is simply a particular metric variation already present in the QED Lagrangian in curved space, required by general coordinate invariance and the equivalence principle, without any additional couplings or extraneous fields. Since the stress-energy tensor couples to the universal force of gravity, we should expect that physical processes can excite the scalar φ and ψ' scalar degrees of freedom required by the trace anomaly with a gravitational coupling strength. If $m = 0$ these produce effects of arbitrarily long range. An example of this coupling to a gravitational scattering amplitude is given in the next section.

Finally we remark that strictly speaking, the anomaly action (152) or (158) and stress tensor derived from it contain no information about the nonanomalous or tracefree amplitudes F_i, $i = 3, \ldots, 13$, although in certain circumstances the addition of homogeneous solutions to the wave equations (154) can give tracefree terms in the stress tensor which have physical consequences [18]. Our detailed computation of the full amplitude (70) shows that there is also a massless pole appearing in the traceless part of the physical amplitude to two photons, (148) and (150). This traceless pole term corresponds to a term in the effective action of the form,

$$\frac{c}{6}\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}h_{\mu\nu}(x)\Box^{-1}_{x,x'} \times\left[3(\partial^\mu\partial^\nu F_{\alpha\beta})F^{\alpha\beta}+\frac{1}{4}(g^{\mu\nu}\Box-4\partial^\mu\partial^\nu)F_{\alpha\beta}F^{\alpha\beta}\right]_{x'}. \tag{164}$$

The tensor structure of this term precludes writing it as a scalar particle exchange. The pole in this amplitude with nontrivial tensor structure is clearly connected with the possible nonzero values of $\partial^\mu\partial^\nu F_{\alpha\beta}$ and $\partial^\mu\partial^\nu(F_{\alpha\beta}F^{\alpha\beta})$ in the background electromagnetic field, which breaks Lorentz invariance. Thus it appears that in this case of a nonvanishing background field which is nongravitational in origin, the Ward identities obeyed by the full amplitude (70) implies the existence of additional massless intermediate states which do not transform as spacetime scalars, and therefore cannot in general be described by the anomaly induced effective action (156) or (155). Instead these massless modes are associated with longitudinal components of the metric perturbation in a Lorentz noninvariant background, analogous to longitudinal plasmon excitations in a finite temperature electromagnetic plasma.

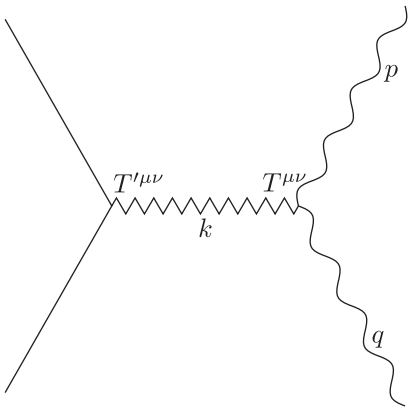

FIG. 9. Tree level gravitational scattering amplitude.

VIII. SCALAR ANOMALY POLE CONTRIBUTION TO GRAVITATIONAL SCATTERING

In order to verify the existence of the massless scalar pole in a physical process, we consider the simple tree diagram of gravitational exchange between an arbitrary conserved stress-energy source $T'^{\mu\nu}$ and photons illustrated in Fig. 9.

This process is described by the scattering amplitude [39],

$$\mathcal{M} = 8\pi G\int d^4x'\int d^4x\left[T'^{\mu\nu}(x')\left(\frac{1}{\Box}\right)_{x',x}T_{\mu\nu}(x) -\frac{1}{2}T'^{\mu}{}_{\mu}(x')\left(\frac{1}{\Box}\right)_{x',x}T^{\nu}{}_{\nu}(x)\right]. \tag{165}$$

The relative factor of $-\frac{1}{2}$ between the two terms is dictated by the requirement that there be no scalar or ghost state exchanged between the two sources, and is exactly the prediction of general relativity, linearized about flat space. That only a spin-2 propagating degree of freedom is exchanged between the two sources in Fig. 9 can be verified by introducing the following $3+1$ decomposition for each of the conserved stress tensors,

$$T^{00} = T_{00}, \tag{166a}$$

$$T^{0i} = -V^{\perp i} - \partial^i\frac{1}{\nabla^2}\dot{T}_{00}, \tag{166b}$$

$$T^{ij} = T^{\perp ij} + \partial^i\frac{1}{\nabla^2}\dot{V}^{\perp j} + \partial^j\frac{1}{\nabla^2}\dot{V}^{\perp i} + \frac{1}{2}\left(g^{ij}-\partial^i\frac{1}{\nabla^2}\partial^j\right) \times(T^{\mu}{}_{\mu}+T_{00}) - \frac{1}{2}\left(g^{ij}-3\partial^i\frac{1}{\nabla^2}\partial^j\right)\frac{1}{\nabla^2}\ddot{T}_{00}, \tag{166c}$$

where $\partial_i V^{\perp i} = 0$, $\partial_i T^{\perp ij} = T^{\perp i}{}_i = 0$, and ∇^{-2} denotes the static Green's function of the Laplacian operator, $\nabla^2 = \partial^i\partial_i$ in flat space. This parametrization assumes only the conservation of the stress-tensor source(s), i.e. $\partial_\mu T^{\mu\nu} = 0$, so that there remain six independent components of $T^{\mu\nu}$ which must be specified, and we have chosen these six to

be T_{00}, $V^{\perp i}$, $T^{\perp ij}$, and the total trace $T^{\mu}{}_{\mu}$, which is a spacetime scalar. Substituting the decomposition (166) into (165) gives

$$\mathcal{M} = 8\pi G \int d^4x' \int d^4x \Big[T'^{\perp}_{ij} \Big(\frac{1}{\Box}\Big)_{x',x} T^{\perp}_{ij} - 2V'^{\perp}_{i} \Big(\frac{1}{\nabla^2}\Big)_{x',x} V^{\perp}_{i} + \frac{3}{2} T'_{00} \Big(\frac{1}{(\nabla^2)^2}\Big)_{x',x} \Box T_{00} + \frac{1}{2} T'_{00} \Big(\frac{1}{\nabla^2}\Big)_{x',x} T^{\mu}{}_{\mu} + \frac{1}{2} T'^{\mu}{}_{\mu} \Big(\frac{1}{\nabla^2}\Big)_{x',x} T_{00} \Big], \tag{167}$$

which becomes

$$\mathcal{M} \to -8\pi G \Big[T'^{\perp}_{ij} \frac{1}{k^2} T^{\perp}_{ij} - 2V'^{\perp}_{i} \frac{1}{\vec{k}^2} V^{\perp}_{i} + \frac{3}{2} T'_{00} \frac{k^2}{(\vec{k}^2)^2} T_{00} + \frac{1}{2} T'_{00} \frac{1}{\vec{k}^2} T^{\mu}{}_{\mu} + \frac{1}{2} T'^{\mu}{}_{\mu} \frac{1}{\vec{k}^2} T_{00} \Big], \tag{168}$$

in momentum space. These expressions show that only the spatially transverse and tracefree components of the stress tensor, $T^{\perp}_{ij}$ exchange a physical propagating helicity ± 2 graviton in the intermediate state, characterized by a Feynman (or for classical interactions, a retarded) massless propagator $-\Box^{-1} \to k^{-2}$ pole in the first term of (167) or (168). All the other terms in either expression contain only an instantaneous Coulomb-like interaction $-\nabla^{-2} \to \vec{k}^{-2}$ or $\nabla^{-4} \to \vec{k}^{-4}$ between the sources, in which no propagating physical particle appears in the intermediate state of the cut diagram. This is the gravitational analog of the decomposition,

$$J^0 = \rho, \tag{169a}$$

$$J^i = J^{\perp i} - \partial^i \frac{1}{\nabla^2} \dot{\rho}, \tag{169b}$$

of the conserved electromagnetic current and corresponding tree level scattering amplitude,

$$\int d^4x' \int d^4x J'^{\mu}(x') \Big(\frac{1}{\Box}\Big)_{x',x} J_{\mu}(x) \to -J'^{\mu} \frac{1}{k^2} J_{\mu} = -J'^{\perp}_{i} \frac{1}{k^2} J^{\perp}_{i} + \rho' \frac{1}{\vec{k}^2} \rho, \tag{170}$$

which shows that only a helicity ± 1 photon is exchanged between the transverse components of the current, the last term in (170) being the instantaneous Coulomb interaction between the charge densities.

We now replace one of the stress-tensor sources by the matrix element (148) of the one-loop anomalous amplitude, considering first the trace term with the anomaly pole in F_1. This corresponds to the diagram in Fig. 10. We find for this term,

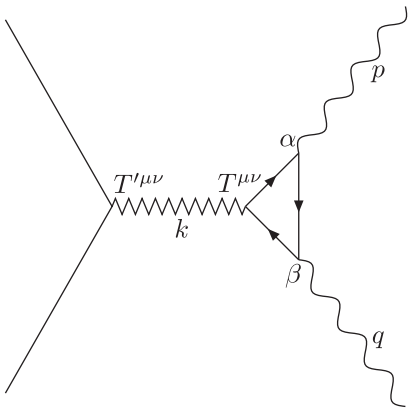

FIG. 10. Gravitational scattering of photons from the source $T'^{\mu\nu}$ via the triangle amplitude.

$$\langle 0|T_{00}|p,q\rangle_1 = -\vec{k}^2 F_1(k^2) u^{\alpha\beta}(p,q) \tilde{A}_{\alpha}(p) \tilde{A}_{\beta}(q) \tag{171a}$$

$$\langle 0|T^{\mu}_{\mu}|p,q\rangle_1 = 3k^2 F_1(k^2) u^{\alpha\beta}(p,q) \tilde{A}_{\alpha}(p) \tilde{A}_{\beta}(q) \tag{171b}$$

$$\langle 0|V^{\perp}_{i}|p,q\rangle_1 = \langle 0|T^{\perp}_{ij}|p,q\rangle_1 = 0. \tag{171c}$$

Hence the scattering amplitude (168) becomes simply,

$$\mathcal{M}_1 = 4\pi G T'^{\mu}{}_{\mu} F_1(k^2) u^{\alpha\beta}(p,q) \tilde{A}_{\alpha}(p) \tilde{A}_{\beta}(q) = \frac{4\pi G}{3} T'^{\mu}{}_{\mu} \frac{1}{k^2} \langle 0|T^{\nu}{}_{\nu}|p,q\rangle_1 \tag{172}$$

where (139) has been used. Thus for massless fermions the pole in the anomaly amplitude becomes a scalar pole in the gravitational scattering amplitude, appearing in the intermediate state as a massless scalar exchange between the traces of the energy-momentum tensors on each side. The standard gravitational interaction with the source has produced an effective interaction between the scalar auxiliary field ψ' and the trace $T'^{\mu}{}_{\mu}$ with a well-defined gravitational coupling. Thus we may equally well represent the scattering as Fig. 10 involving the fermion triangle, or as

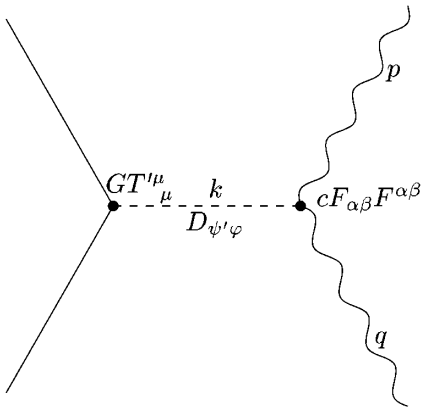

FIG. 11. Gravitational scattering of photons from the trace of a source $T'^{\mu}{}_{\mu}$ via massless scalar exchange in the effective theory of (173).

the tree level diagram Fig. 11 of the effective theory, with a scalar particle exchange.

This diagram is generated by the effective action in flat space modified from (158) to

$$S_{\rm eff}[g, A; \varphi, \psi'] = \int d^4x\sqrt{-g}\Big[-\psi'\Box\varphi + \frac{8\pi G}{3}T'^{\mu}{}_{\mu}\psi' + \frac{c}{2}F_{\alpha\beta}F^{\alpha\beta}\varphi\Big], \quad (173)$$

to include the coupling to the trace of the energy-momentum tensor of any matter $T'^{\mu}{}_{\mu}$ source. Correspondingly Eq. (159c) for φ becomes

$$\Box\varphi = \frac{8\pi G}{3}T'^{\mu}{}_{\mu}, \quad (174)$$

instead (159c). The equation of motion for ψ' remains (159b) and is unaffected. We note that if the source $T'^{\mu\nu}$ generates the curvature R by Einstein's equations, then $R = -8\pi G T'^{\mu}{}_{\mu}$, so that (173) and (174) are equivalent to (158) and (159c) at leading order in G.

We conclude that in the conformal limit of massless electrons, the pole in the trace sector of the anomaly amplitude contributes to gravitational scattering amplitudes as would a scalar field coupled to the trace of the energy-momentum tensor of classical sources. The gravitationally coupled intermediate scalar can be understood as arising from collinear e^+e^- correlated pairs in a total spin 0^+ state. Although the result appears similar in some respects to a Jordan-Brans-Dicke scalar [40], the coupling induced by the anomaly involves two scalar fields each coupling to a *different* source, with an off diagonal propagator, $D_{\psi'\varphi}$. Hence the phenomenology of this scalar coupling will be quite different, and the observational limits on a Jordan-Brans-Dicke scalar do not apply [41]. In particular there is no direct coupling of classical energy-momentum sources $T'^{\mu}{}_{\mu}$ to $T'^{\nu}{}_{\nu}$ via scalar exchange as there would be in a classical scalar-tensor theory.

Another important difference is that as we have seen, the anomaly pole is a necessary consequence of quantum fluctuations and low energy symmetries, whereas in classical scalar-tensor theories a postulated scalar field is simply added to Einstein's general relativity. As a consequence there are one or more free parameters introduced in such an approach, whereas the effective action (173) is completely specified without any arbitrariness or free parameters, once the underlying quantum theory's matter content and couplings are given. It will be interesting to study the consequences for astrophysics and cosmology of this effective action derived from quantum first principles and fundamental low energy symmetries.

IX. SUMMARY

We have presented a complete calculation of the $\langle TJJ\rangle$ triangle amplitude in QED, for all values of the kinematic invariants and electron mass. As a consequence of the trace anomaly, this amplitude exhibits a massless pole in the conformal limit, which contributes to long range gravitational interactions, and is associated with the exchange of a massless 0^+ degree of freedom. This scalar exchange is described by a low energy local effective action (173) with two massless dynamical scalar fields φ and ψ'.

For the benefit of the reader we provide here a summary of the main results to be found in each section of the paper.

We reviewed in Sec. II the derivation of the axial anomaly in QED, showing how the finite parts of the triangle amplitude, together with the symmetry principles of Lorentz invariance, gauge invariance, and Bose exchange symmetry are sufficient to yield the complete amplitude, (7) for any mass and any value of the kinematic invariants, and determine the axial anomaly, without any explicit need of regularization of ultraviolet divergent integrals. We showed that the anomaly is closely connected to a finite sum rule of the spectral density (33) obtained by cutting the amplitude as in Fig. 2. For physical, transverse photons on shell, this spectral density vanishes pointwise for all $s > 0$, becoming proportional to $\delta(s)$ in the conformal limit of massless fermions, (35). Corresponding to the $\delta(s)$ in the spectral weight is a massless pseudoscalar pole singularity in the full amplitude and matrix element to physical photons (43). This illustrates the infrared aspect of the anomaly, and the appearance of massless states in a theory with anomalies in $3 + 1$ dimensions.

We showed next in Sec. III that the anomaly pole in the chiral case implies the existence of a nonlocal effective action, (50) which can be brought into a local form by the introduction of two pseudoscalar auxiliary fields (52) and (53). These fields and the anomaly pole can be understood as arising from a certain correlated two-particle collinear e^+e^- state in the massless limit. The local effective action of the auxiliary fields has kinetic terms, and their canonical commutation relations reproduce the anomalous current commutation relations of the underlying fermionic theory (58). Thus the auxiliary fields appear to be *bona fide* massless pseudoscalar degrees of freedom, required by the chiral anomaly.

In Sec. IV we presented a full computation of the $\langle TJJ\rangle$ triangle amplitude in QED, where the chiral current is replaced by the fermionic energy-momentum-stress tensor $T^{\mu\nu}$. Following the same method as in the chiral case, we showed how the finite parts of the triangle amplitude, together with the same symmetry principles of Lorentz invariance, gauge invariance, and Bose exchange symmetry, and the additional Ward identity following from general coordinate invariance are sufficient to yield the complete $\langle TJJ\rangle$ amplitude, (86), given by Eqs. (98)–(100) with (95) and (92) and Tables IV and V, for any value of mass or the kinematic invariants, without any need of regularization of ultraviolet divergent integrals.

In Sec. V we computed the trace and found the finite anomaly (124), equivalent to (65). The coefficient of the

second possible tensor in the trace defined in (79) is nonanomalous, but both are needed to determine the scaling violation β function at finite momentum and finite electron mass. This infrared β function is given by (128) in terms of the photon polarization which vanishes when $p^2 \ll m^2$, consistent with decoupling, and approaches the more commonly considered ultraviolet β function only in the opposite limit $p^2 \gg m^2$.

In Sec. VI we gave the spectral representation (132) for the imaginary part of the triangle amplitude, cut as in Fig. 2, for $k^2 = -s < 0$ timelike. We showed that the imaginary part of the amplitude is nonanomalous, with the anomaly in the real part arising from a cancellation between factors of $k^2 + s$ in both the numerator and denominator of (132). For the particular linear combination of spectral functions appearing in the anomalous trace, $\rho_T(s)$ defined by (134a), we derived on the one hand the finite sum rule (135), and on the other hand the representation (136), which shows that ρ_T must develop a $\delta(s)$ singularity when the fermion mass, and photon virtualities p^2 and q^2 vanish. We also exhibited this $\delta(s)$ explicitly in this limit, (137). Corresponding to the $\delta(s)$ in the spectral function ρ_T, the corresponding full amplitude (139) has a pole at $k^2 = 0$, indicating the presence of a massless scalar propagating state in the matrix element of the stress tensor to physical photons (148). As in the chiral case the massless anomaly pole can be understood as arising from a correlated two-particle collinear e^+e^- state, which because of the kinematics is essentially $1 + 1$ dimensional, c.f. Fig. 7.

In Sec. VII we showed that the trace part of the $\langle TJJ\rangle$ triangle amplitude is identical with that predicted by the covariant effective action (152), obtained in earlier work by integrating the anomaly. In particular, the variation of the simplified effective action (158) in terms of the two scalar auxiliary fields ψ' and φ yields the amplitude (162) which coincides with the first term of (70) which gives its full trace in the massless limit. In the effective action the massless scalar two-particle state of the triangle amplitude is replaced by scalar fields, whose propagator gives rise to the anomaly pole at $k^2 = 0$.

Finally, in Sec. VIII we considered the tree level gravitational scattering amplitude (165), Fig. 9, with one vertex replaced by the triangle amplitude $\langle TJJ\rangle$, and showed that the massless scalar pole in the latter survives in the physical scattering amplitude in the conformal limit of massless electrons. In the effective theory (173) it is described as a propagating massless scalar interaction, Fig. 11 between the trace parts of the energy-momentum sources.

Abstracting from the axial and trace anomaly QED examples presented here in detail, we conclude that the trace anomaly and anomaly pole of conformal fields imply the existence of new long range scalar interactions with a gravitational coupling strength to ordinary matter.

ACKNOWLEDGMENTS

We are very much indebted to Albert Roura for extensive discussions during the course of this work, which helped clarify a number of key points. We also thank L. S. Brown for several enlightening conversations and P. O. Mazur for communicating to us his discussions with A. Casher concerning the correlated two-particle state appearing in the triangle amplitude.

APPENDIX A: EXTRACTION OF FINITE PARTS OF $\langle TJJ\rangle$

For the amplitude (70), in order to extract the finite terms for which each of the indices $(\mu\nu\alpha\beta)$ is associated with an external momentum p or q, we may drop the $g^{\mu\nu}$ terms in the vertex $V^{\mu\nu}$, and consider only

$$-\frac{e^2}{4}\int\frac{d^4l}{(2\pi)^4}\frac{\mathrm{tr}\{[\gamma^\mu(2l+p-q)^\nu+\gamma^\mu(2l+p-q)^\mu](-\not{l}-\not{p}+m)\gamma^\alpha(-\not{l}+m)\gamma^\beta(-\not{l}+\not{q}+m)\}}{[(l+p)^2+m^2][(l-q)^2+m^2][l^2+m^2]}, \tag{A1}$$

where the continuation to Euclidean l has already been performed. Introducing the Feynman parametrization,

$$\frac{1}{(l+p)^2+m^2}\frac{1}{(l-q)^2+m^2}\frac{1}{l^2+m^2}=2\int_0^1 dx\int_0^{1-x}dy\frac{1}{(l'^2+D)^3}, \tag{A2}$$

with $l' = l + px - qy$ and D given by (20), we shift the integration variable in (A1) from l to l'. Dropping the terms involving either powers of l' or m in the numerator, since these cannot give rise to terms which are homogeneous of degree 4 in p and q in ${}^V\Gamma^{\mu\nu\alpha\beta}$, and evaluating the finite Euclidean integral,

$$\int\frac{d^4l'}{(2\pi)^4}\frac{1}{(l'^2+D)^3}=\frac{1}{32\pi^2}\frac{1}{D}, \tag{A3}$$

we obtain from (A1),

$$\frac{e^2}{32\pi^2}\int_0^1 dx\int_0^{1-x}dy\frac{1}{D}[p(1-2x)-q(1-2y)]^{(\mu}[p(1-x)+qy]_\lambda[px-qy]_\rho[px+q(1-y)]_\sigma\,\mathrm{tr}\{\gamma^{\nu)}\gamma^\lambda\gamma^\alpha\gamma^\rho\gamma^\beta\gamma^\sigma\}. \tag{A4}$$

Of the 15 terms in the γ-matrix trace, only the 6 terms,

$$\tfrac{1}{4}\,\mathrm{tr}\{\gamma^\nu\gamma^\lambda\gamma^\alpha\gamma^\rho\gamma^\beta\gamma^\sigma\} = -g^{\nu\lambda}(g^{\alpha\rho}g^{\beta\sigma} + g^{\alpha\sigma}g^{\beta\rho}) - g^{\nu\rho}(g^{\alpha\lambda}g^{\beta\sigma} - g^{\alpha\sigma}g^{\beta\lambda}) - g^{\nu\sigma}(g^{\alpha\lambda}g^{\beta\rho} + g^{\alpha\rho}g^{\beta\lambda}) + \ldots \quad \text{(A5)}$$

need to be retained, since the other nine contract at least two of the free indices $(\mu\nu\alpha\beta)$ and do not give rise to terms in which all four indices are carried by p or q. Thus we retain from (A4) only the terms,

$$\begin{aligned}
&\frac{e^2}{8\pi^2}\int_0^1 dx\int_0^{1-x} dy\,\frac{1}{D}\{-[p(1-2x)-q(1-2y)]^{(\mu}[p(1-x)+qy]^{\nu)}[px-qy]^\alpha[px+q(1-y)]^\beta\\
&\quad-[p(1-2x)-q(1-2y)]^{(\mu}[p(1-x)+qy]^{\nu)}[px-qy]^\beta[px+q(1-y)]^\alpha-[p(1-2x)-q(1-2y)]^{(\mu}\\
&\quad\times[p(1-x)+qy]^\alpha[px-qy]^{\nu)}[px+q(1-y)]^\beta+[p(1-2x)-q(1-2y)]^{(\mu}[p(1-x)+qy]^\beta[px-qy]^{\nu)}\\
&\quad\times[px+q(1-y)]^\alpha-[p(1-2x)-q(1-2y)]^{(\mu}[p(1-x)+qy]^\alpha[px-qy]^\beta[px+q(1-y)]^{\nu)}\\
&\quad-[p(1-2x)-q(1-2y)]^{(\mu}[p(1-x)+qy]^\beta[px-qy]^\alpha[px+q(1-y)]^{\nu)}\}.
\end{aligned} \quad \text{(A6)}$$

In this form it is straightforward to collect the terms which multiply each of the 12 tensors of degree 4 which are listed in Table V of the text. For example, the coefficient of the tensor $p^\mu p^\nu p^\alpha p^\beta$ from (A6) is

$$\frac{e^2}{8\pi^2}\int_0^1 dx\int_0^{1-x} dy\,\frac{1}{D}\{-4(1-2x)(1-x)x^2\}. \quad \text{(A7)}$$

When we add the Bose symmetric contribution to (A6) with p replaced by q and α replaced by β, the coefficient of $q^\mu q^\nu q^\alpha q^\beta$ will give an equal contribution to the coefficient C_1, after also interchanging the parameter integration variables x and y. Thus,

$$C_1(k^2;p^2,q^2) = \frac{e^2}{4\pi^2}\int_0^1 dx\int_0^{1-x} dy\,\frac{c_1(x,y)}{D}, \quad \text{(A8)}$$

with

$$c_1(x,y) = -4x^2(1-x)(1-2x), \quad \text{(A9)}$$

and we have verified (92) for the first entry of Table V of the text. The remainder of the Table V may be derived from (A6) in the same way.

For the amplitude $\Lambda^{\alpha\beta}$ the calculation is similar. Beginning with (108), we have

$$\Lambda^{\alpha\beta}(p,q) = e^2 m\int\frac{d^4l}{(2\pi)^4}\,\frac{\mathrm{tr}\{(-\not l-\not p+m)\gamma^\alpha(-\not l+m)\gamma^\beta(-\not l+\not q+m)\}}{[(l+p)^2+m^2][(l-q)^2+m^2][l^2+m^2]} + (p\leftrightarrow q,\,\alpha\leftrightarrow\beta). \quad \text{(A10)}$$

Since the trace of an odd number of γ matrices vanishes, only those terms with at least one additional factor of m in the numerator survive. Since we wish to extract only those finite terms homogeneous of degree 2 in the external momenta, namely $p^\alpha p^\beta$, $p^\alpha q^\beta$, $q^\alpha p^\beta$, or $q^\alpha q^\beta$, determining the other terms by vector current conservation (109), we focus only on those terms with exactly one additional factor of m. Using the Feynman parametrization (A2), shifting integration variables from l to $l' = l + px - qy$, and evaluating the momentum integral (A3) as before, we find from the first term of (A10),

$$\begin{aligned}
&\frac{e^2m^2}{16\pi^2}\int_0^1 dx\int_0^{1-x} dy\,\frac{1}{D}\{(px-qy)_\lambda[px+q(1-y)]_\mu\\
&\quad\times\mathrm{tr}(\gamma^\alpha\gamma^\lambda\gamma^\beta\gamma^\mu) - [p(1-x)+qy]_\lambda[px+q(1-y)]_\mu\\
&\quad\times\mathrm{tr}(\gamma^\lambda\gamma^\alpha\gamma^\beta\gamma^\mu) - [p(1-x)+qy]_\lambda(px-qy)_\mu\\
&\quad\times\mathrm{tr}(\gamma^\lambda\gamma^\alpha\gamma^\mu\gamma^\beta)\}.
\end{aligned} \quad \text{(A11)}$$

In the γ matrix traces we may further discard all terms involving $g^{\alpha\beta}$, which leaves the remaining terms,

$$\begin{aligned}
&\frac{e^2m^2}{4\pi^2}\int_0^1 dx\int_0^{1-x} dy\,\frac{1}{D}\{(px-qy)^\alpha[px+q(1-y)]^\beta\\
&\quad+(px-qy)^\beta[px+q(1-y)]^\alpha-[p(1-x)+qy]^\alpha\\
&\quad\times[px+q(1-y)]^\beta+[p(1-x)+qy]^\beta\\
&\quad\times[px+q(1-y)]^\alpha-[p(1-x)+qy]^\alpha(px-qy)^\beta\\
&\quad-[p(1-x)+qy]^\beta(px-qy)^\alpha\}.
\end{aligned} \quad \text{(A12)}$$

Adding the Bose symmetrized term with $p\leftrightarrow q$ and $\alpha\leftrightarrow\beta$, we obtain for these finite terms,

$$\begin{aligned}
&-\frac{e^2m^2}{2\pi^2}\int_0^1 dx\int_0^{1-x} dy\,\frac{1}{D}\{2p^\alpha p^\beta x(1-2x)\\
&\quad+p^\alpha q^\beta(1-2x)(1-2y) - q^\alpha p^\beta(1-4xy)\\
&\quad+2q^\alpha q^\beta y(1-2y)\}.
\end{aligned} \quad \text{(A13)}$$

From the definitions of G_1 and G_2 in (110) and $u^{\alpha\beta}(p,q)$ and $w^{\alpha\beta}(p,q)$ in (79), it follows that the coefficient of $-q^\alpha p^\beta$ is

$$G_1(k^2;p^2,q^2) = -\frac{e^2m^2}{2\pi^2}\int_0^1 dx\int_0^{1-x} dy\,\frac{(1-4xy)}{D}, \tag{A14}$$

and the coefficient of $(p\cdot q)p^\alpha q^\beta$ is

$$G_2(k^2;p^2,q^2) = -\frac{e^2}{2\pi^2}\frac{m^2}{p\cdot q} \times\int_0^1 dx\int_0^{1-x} dy\,\frac{(1-2x)(1-2y)}{D}, \tag{A15}$$

which are Eqs. (111) of the text.

APPENDIX B: PROOF OF IDENTITIES

The coefficients of the $p^\alpha p^\beta$ and $q^\alpha q^\beta$ terms in (A13) apparently do not match those of $w^{\alpha\beta}(p,q)$ with the identification of G_2 given. This mismatch is only apparent, because of the identities,

$$\int_0^1 dx\int_0^{1-x} dy\,\frac{2x(1-2x)}{D} = -\frac{q^2}{p\cdot q}\int_0^1 dx\int_0^{1-x} dy\,\frac{(1-2x)(1-2y)}{D}, \tag{B1a}$$

$$\int_0^1 dx\int_0^{1-x} dy\,\frac{2y(1-2y)}{D} = -\frac{p^2}{p\cdot q}\int_0^1 dx\int_0^{1-x} dy\,\frac{(1-2x)(1-2y)}{D}. \tag{B1b}$$

These identities are most easily proven by considering integrals of the kind,

$$\int_0^1 dx\int_0^{1-x} dy(1-2x)\frac{\partial \ln D}{\partial y} = \int_0^1 dx\int_0^{1-x} dy(1-2x)\frac{1}{D}\frac{\partial D}{\partial y}, \tag{B2}$$

which on the one hand vanishes, because

$$\int_0^1 dx(1-2x)\ln D|_{y=0}^{y=1-x} = \int_0^1 dx(1-2x)\ln\left\{\frac{x(1-x)k^2+m^2}{x(1-x)p^2+m^2}\right\} = 0, \tag{B3}$$

due to the fact that $(1-2x)\to -(1-2x)$ is odd upon reflection about the midpoint of the integral, $x\to 1-x$, whereas $x(1-x)\to x(1-x)$ is even; while on the other hand, (B2) is equal to

$$\int_0^1 dx\int_0^{1-x} dy(1-2x)\left[\frac{(1-2y)q^2+2xp\cdot q}{D}\right]. \tag{B4}$$

Setting this expression to zero and rearranging gives (B1a). The second identity (B1b) is proven in a similar manner by exchanging x and y. When these two identities are substituted into (A13), the $p^\alpha p^\beta$, $p^\alpha q^\beta$, and $q^\alpha q^\beta$ terms become proportional to the corresponding terms in $w^{\alpha\beta}$ defined in (79), and the coefficient function (A15) is obtained in every case. We remark also that despite appearances, G_2 has no pole at $p\cdot q=0$, since the integral in (A15) multiplying it vanishes if we set $p\cdot q=0$ in the denominator D, by an argument similar to that leading to (B3).

The identities (91) are proven in a similar way. For example, from Table V and (92),

$$\begin{aligned}(p\cdot q)C_1+q^2C_4 &= -\frac{e^2}{2\pi^2}\int_0^1 dx\int_0^{1-x} dy\,\frac{x(1-x)(1-2x)}{D}[2xp\cdot q+(1-2y)q^2]\\ &= -\frac{e^2}{2\pi^2}\int_0^1 dx\,x(1-x)(1-2x)\int_0^{1-x} dy\,\frac{\partial \ln D}{\partial y}\\ &= -\frac{e^2}{2\pi^2}\int_0^1 dx\,x(1-x)(1-2x)\ln\left\{\frac{x(1-x)k^2+m^2}{x(1-x)p^2+m^2}\right\} = 0,\end{aligned} \tag{B5}$$

for the same reason (B3) vanishes. This proves (91a), with (91b) proven in exactly the same manner after interchanging p^2 and q^2, and x and y.

For the first of identities (101), we employ a similar method. In the Feynman parametrized integral representation, using (92) with Table V, we find that the linear combination,

$$p^2(C_1 - 2C_2 + C_3 - 2C_7 + 2C_8) - 2p\cdot q(C_7 - C_9) - q^2(2C_8 - 2C_9 + C_{10} - 2C_{11} + C_{12})$$

is proportional to

$$\begin{aligned}
&\int_0^1 dx \int_0^{1-x} dy \frac{(1-x-y)}{D}\{x(x+3y-1)(1-2x)p^2 - 4xy(x-y)p\cdot q - y(3x+y-1)(1-2y)q^2\} \\
&= \int_0^1 dx \int_0^{1-x} dy(1-x-y)\left\{x(x+3y-1)\frac{\partial \ln D}{\partial x} - y(3x+y-1)\frac{\partial \ln D}{\partial y}\right\} \\
&= \int_0^1 dx \int_0^{1-x} dy \ln D\left\{\frac{\partial}{\partial x}[x(1-x-y)(1-x-3y)] - \frac{\partial}{\partial y}[y(1-x-y)(1-3x-y)]\right\} = 0, \qquad \text{(B6)}
\end{aligned}$$

which vanishes identically.

To prove the second and third identities (101b) and (101c), a similar method of converting all the terms to the fourth or sixth order monomial, completely Bose symmetric in momenta, p^2q^2 or $p^2q^2p\cdot q$, respectively, must be used. This is accomplished by repeated use of the relations,

$$(1-2x)p^2 + 2yp\cdot q = \frac{\partial D}{\partial x}, \qquad \text{(B7a)}$$

$$(1-2y)q^2 + 2xp\cdot q = \frac{\partial D}{\partial y}, \qquad \text{(B7b)}$$

solved first for the p^2 or q^2 terms, and then for the $p\cdot q$ terms. Thus in (101b) we first convert $x(1-2x)p^4 \to -2xyp^2p\cdot q$ and $y(1-2y)q^4 \to -2xyq^2p\cdot q$, collect terms, and then convert $2xp^2p\cdot q \to -(1-2y)p^2q^2$ and $2yq^2p\cdot q \to -(1-2x)p^2q^2$, integrating the remaining $\frac{1}{D}\frac{\partial D}{\partial x} = \frac{\partial \ln D}{\partial x}$ and $\frac{1}{D}\frac{\partial D}{\partial y} = \frac{\partial \ln D}{\partial y}$ terms by parts. In this way we find that all terms cancel finally, and (101b) is satisfied.

For (101c), after multiplying through by p^2q^2, it contains the sixth order monomials p^6 and q^6. Hence the procedure must be carried out one additional time before all terms can be reduced to the Bose symmetric monomial $p^2q^2p\cdot q$. Thus, we first convert $x(1-2x)p^6 \to -2xyp^4p\cdot q \to y(1-2y)p^4q^2$ and $y(1-2y)q^6 \to -2xyq^4p\cdot q \to x(1-2x)p^2q^4$, combine these with the existing p^4q^2 and p^2q^4 terms in (101c), and convert both to a single Bose symmetric $p^2q^2p\cdot q$ term, whose total coefficient vanishes. Remaining from the integrations by parts of the $\frac{\partial \ln D}{\partial x}$ and $\frac{\partial \ln D}{\partial y}$ terms are two surface terms which are integrals with just one Feynman parameter, x or y. By comparing with Eq. (104), these nonzero remaining terms give exactly the $-2p^2q^2[\Pi(p^2) - \Pi(q^2)]$ contribution needed to prove (101c).

Lastly, the linear combination of terms in (118),

$$p^2(C_1 - 2C_2 + C_3 + 2C_4 - 4C_8) + 2p\cdot q(C_3 - 2C_5 - C_7 - C_9 + C_{10}) + q^2(2C_6 - 4C_8 + C_{10} - 2C_{11} + C_{12}),$$

after substituting for the C_j from (92) with Table V is proportional to

$$\begin{aligned}
&\int_0^1 dx \int_0^{1-x} dy \frac{1}{D}\{x[x(1-x) + y(3y-2)](1-2x)p^2 - 2xy[x(1-2x) + y(1-2y)]p\cdot q \\
&\quad + y[y(1-y) + x(3x-2)](1-2y)q^2\} \\
&= \int_0^1 dy \int_0^{1-y} dx x[x(1-x) + y(3y-2)]\frac{\partial \ln D}{\partial x} + \int_0^1 dx \int_0^{1-x} dy y[y(1-y) + x(3x-2)]\frac{\partial \ln D}{\partial y} \\
&= -\int_0^1 dy y(1-y)(1-2y)\ln[y(1-y)k^2 + m^2] - \int_0^1 dx x(1-x)(1-2x)\ln[x(1-x)k^2 + m^2] \\
&\quad + \int_0^1 dy \int_0^{1-y} dx(x-y)(3y+3x-2)\ln D - \int_0^1 dx \int_0^{1-x} dy(x-y)(3y+3x-2)\ln D = 0, \qquad \text{(B8)}
\end{aligned}$$

which also vanishes because the last two terms cancel, while the first and second terms are each separately zero by their odd parity under reflection through the midpoint of the remaining integral. This verifies (118) of the text.

All the identities and algebra were checked independently with the algebraic manipulation software package MATHEMATICA.

[1] See, e.g., P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory* (Springer, New York, 1997), and references therein.

[2] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).

[3] A. M. Polyakov, Phys. Lett. **103B**, 207 (1981); **103**, 211 (1981); Mod. Phys. Lett. A **2**, 893 (1987).

[4] V. G. Knizhnik, A. M. Polyakov, and A. B. Zamolodchikov, Mod. Phys. Lett. A **3**, 819 (1988).

[5] R. A. Bertlmann and E. Kohlprath, Phys. Lett. B **480**, 200 (2000); Ann. Phys. (N.Y.) **288**, 137 (2001).

[6] J. Schwinger, Phys. Rev. **125**, 397 (1962); **128**, 2425 (1962).

[7] L. S. Brown, Nuovo Cimento **29**, 617 (1963).

[8] R. J. Crewther, Phys. Rev. Lett. **28**, 1421 (1972); M. S. Chanowitz and J. R. Ellis, Phys. Lett. **40B**, 397 (1972); Phys. Rev. D **7**, 2490 (1973); **8**, 1917 (1973).

[9] A. D. Dolgov and V. I. Zakharov, Nucl. Phys. **B27**, 525 (1971).

[10] K. Huang, *Quarks, Leptons, and Gauge Fields* (World Scientific, Singapore, 1992).

[11] S. L. Adler, Phys. Rev. **177**, 2426 (1969); in *Lectures on Elementary Particle and Quantum Field Theory*, edited by S. Deser, M. Grisaru, and H. Pendleton (MIT Press, Cambridge, MA, 1970), Vol. 1.

[12] W. A. Bardeen, H. Fritzsch, and M. Gell-Mann, in *Scale and Conformal Symmetry in Hadron Physics*, edited by R. Gatto (Wiley, New York, 1973); T. P. Cheng and L. F. Li, *Gauge Theory of Elementary Particle Physics* (Oxford University Press, Oxford, United Kingdom, 1984); S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, England, 1995), Vol. 2.

[13] G. 't Hooft in *Recent Developments in Gauge Theories, Procedings, NATO Advanced Study Institute, Cargese*, Nato Advanced Study Institutes Series B, Physics (Plenum, New York, 1980), Vol. 59.

[14] S. R. Coleman and B. Grossman, Nucl. Phys. **B203**, 205 (1982).

[15] M. J. Duff, Nucl. Phys. **B125**, 334 (1977).

[16] L. Rosenberg, Phys. Rev. **129**, 2786 (1963).

[17] J. Hořejší, Czech. J. Phys. **42**, 241 (1992).

[18] E. Mottola and R. Vaulin, Phys. Rev. D **74**, 064004 (2006).

[19] J. S. Bell and R. Jackiw, Nuovo Cimento A **60**, 47 (1969).

[20] R. Jackiw, in *Lectures on Current Algebra and Its Applications*, edited by S. B. Treiman, R. Jackiw, and D. J. Gross (Princeton University Press, Princeton, NJ, 1972).

[21] J. Hořejší, Czechoslovak Journal of Physics, Section B **35**, 820 (1985); Phys. Rev. D **32**, 1029 (1985); J. Phys. G **12**, L7 (1986); Czech. J. Phys. **42**, 345 (1992).

[22] See, e.g., R. A. Bertlmann, *Anomalies in Quantum Field Theory* (Clarendon Press, Oxford, 1996).

[23] N. N. Achasov, Phys. Lett. B **287**, 213 (1992).

[24] Y. Frishman, A. Schwimmer, T. Banks, and S. Yankielowicz, Nucl. Phys. **B177**, 157 (1981).

[25] A. Smailagic and E. Spallucci, Phys. Rev. D **62**, 045010 (2000).

[26] J. Hořejší and O. Teryaev, Z. Phys. C **65**, 691 (1995).

[27] D. J. Gross and R. Jackiw, Nucl. Phys. **B14**, 269 (1969); see, also, S. L. Adler and D. G. Boulware, Phys. Rev. **184**, 1740 (1969); R. Jackiw and K. Johnson, Phys. Rev. **182**, 1459 (1969), for related work.

[28] S. L. Adler and W. A. Bardeen, Phys. Rev. **182**, 1517 (1969).

[29] C. Adam, R. A. Bertlmann, and P. Hofer, Riv. Nuovo Cimento Soc. Ital. Fis. **16N8**, 1 (1993).

[30] S. L. Adler, J. C. Collins, and A. Duncan, Phys. Rev. D **15**, 1712 (1977).

[31] I. T. Drummond and S. J. Hathrell, Phys. Rev. D **21**, 958 (1980).

[32] See, e.g., A. Manohar, arXiv:hep-ph/9606222.

[33] J. Hořejší and M. Schnabl, Z. Phys. C **76**, 561 (1997).

[34] A. D. Dolgov, Zh. Eksp. Teor. Fiz. **81**, 417 (1981) [Sov. Phys. JETP **54**, 223 (1981)].

[35] Y. Iwasaki, Phys. Rev. D **15**, 1172 (1977).

[36] R. J. Riegert, Phys. Lett. B **134**, 56 (1984).

[37] P. O. Mazur and E. Mottola, Phys. Rev. D **64**, 104022 (2001).

[38] I. Antoniadis, P. O. Mazur, and E. Mottola, New J. Phys. **9**, 11 (2007).

[39] R. P. Feynman, F. B. Morinigo, and W. G. Wagner, *Feynman Lectures on Gravitation* (University of Bangalore Press, New Delhi, 1995).

[40] P. Jordan, Z. Phys. **157**, 112 (1959); C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).

[41] See, e.g., C. Will, *Theory and Experiment in Gravitational Physics* (Cambridge University Press, Cambridge, England, 1993); Living Rev. Relativity **4**, 4 (2001).