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Non-Abelian vortices with product moduli

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Vortices of a new type, carrying non-Abelian flux moduli $CP^{n-1} \times CP^{r-1}$, are found in the context of softly broken $\mathcal{N} = 2$ supersymmetric quantum chromodynamics. By tuning the bare quark masses appropriately, we identify the vacuum in which the underlying $SU(N)$ gauge group is partially broken to $SU(n) \times SU(r) \times U(1)/\mathbb{Z}_K$, where K is the least common multiple of (n, r), and with $N_f^{su(n)} = n$ and $N_f^{su(r)} = r$ flavors of light quark multiplets. At much lower energies, the gauge group is broken completely by the squark vacuum expectation values, and vortices develop which carry non-Abelian flux moduli $CP^{n-1} \times CP^{r-1}$. For $n > r$, at the length scale at which the $SU(n)$ fluctuations become strongly coupled and Abelianize, the vortex still carries weakly fluctuating $SU(r)$ flux moduli. We discuss the possibility that these vortices are related to the light non-Abelian monopoles found earlier in the fully quantum-mechanical treatment of 4D supersymmetric quantum chromodynamics.

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I. INTRODUCTION

Attempts to understand better the mechanism of confinement of the non-Abelian variety, which is probably the case for the realistic world of QCD, has eventually led to the discovery of vortices with non-Abelian continuous flux moduli [1,2], triggering a remarkable development of research activity in related problems [3–25]. A typical system considered is a $U(n)$ theory with $N_f = n$ scalar quark flavors, whose vacuum expectation value (VEV) breaks the gauge symmetry completely, leaving however the color-flavor diagonal $SU(n)_{C+F}$ symmetry unbroken (color-flavor locking). Vortices in such a system develop continuous zero modes (moduli) parametrizing

$$SU(n)/SU(n-1) \times U(1) \sim CP^{n-1},$$

where the divisor represents the symmetry respected by individual vortices. When the vortex orientation is allowed to fluctuate along z (the direction of the vortex length) and in time t, the dynamics of such fluctuations is described by a two-dimensional CP^{n-1} sigma model [2,4,5]. If the original system is the bosonic sector of a $\mathcal{N} = 2$ supersymmetric model, the sigma model has $(2, 2)$ supersymmetry, as half of the supersymmetry is broken by the vortex. In the infrared limit, the sigma model becomes strongly coupled, and the 2D system reproduces exactly [4,5] the dynamics of the corresponding 4D gauge theory in the *Coulomb phase*, encoded by Seiberg-Witten curves [26–28], realizing thus the idea of duality between a two-dimensional sigma model and a four-dimensional gauge theory discussed earlier by Dorey [29].

Beautiful as it may be, the very result of the analysis shows that the vortices considered in Refs. [2,4,5] dynamically Abelianize to Abrikosov-Nielsen-Olesen (ANO) vortices. This fact can be seen in both two and four dimensions. In the sigma model analysis, the fluctuations inside the vortex become strongly coupled and generate the mass scale Λ; there are n degenerate ground states [7] (Witten-Cecotti-Fendley-Intriligator-Vafa index [30,31]). Monopoles appear as kinks (domain walls) connecting two adjacent vortex ground states. Each monopole is confined by two vortices carrying the "adjacent" $U(1)$ fluxes, a typical situation for a monopole arising from the breaking of $SU(2) \subset U(n)$ to $U(1)$. The global $SU(N_f) = SU(n)$ *flavor* symmetry is not spontaneously broken by the vortex dynamics[1]; this, however, does not contradict the fact that the monopoles in the infrared carry only Abelian magnetic $U(1)^n$ charges.

In four dimensions, the model considered can be seen as the (bosonic part of the) low-energy effective action of $\mathcal{N} = 2$ supersymmetric $SU(N)$, with $N = n + 1$ and with $N_f = n$ flavors. The gauge group is broken by the adjoint scalar VEV

$$\langle \phi \rangle = \mathrm{diag}\Big(m_1, m_2, \ldots, m_n, -\sum_{j=1}^{n} m_j\Big), \qquad m_i \to m, \tag{1.1}$$

to $SU(n) \times U(1)/\mathbb{Z}_n \sim U(n)$. The light monopoles and the magnetic gauge quantum numbers of these, in the limit of small m_i and μ, can be read off from the singularities of the Seiberg-Witten curves [32,33]. Semiclassically (large m_i), instead, the vacua of this theory are classified according to the number of quark flavors which remain massless due to the cancellation between the bare quark mass and the adjoint scalar VEV in the superpotential

$$\tilde{Q}(\sqrt{2}\Phi + M)Q,$$

[1]Of course, this is consistent with Coleman's theorem.

TABLE I. Confining vacua of $SU(N)$ gauge theory with N_f flavors. In the superconformal $r = N_f/2$ vacuum, relatively nonlocal monopoles and dyons both appear as the low-energy effective degrees of freedom. "Almost SCFT" means that the theory is a nontrivial superconformal theory when $\mu = 0$ but confines upon $\mu \neq 0$ perturbation. In the theory with $N_f = N$ considered here, the vacua at the "baryonic root," in free magnetic phase, are absent. They appear only for $N_f > N$, with an effective gauge group $SU(N_f - N)$.

r	Deg. freed.	Eff. gauge group	Phase	Global symmetry
0	Monopoles	$U(1)^{N-1}$	Confinement	$U(n_f)$
1	Monopoles	$U(1)^{N-1}$	Confinement	$U(N_f - 1) \times U(1)$
$2, .., [\frac{N_f-1}{2}]$	NA monopoles	$SU(r) \times U(1)^{N-r}$	Confinement	$U(N_f - r) \times U(r)$
$N_f/2$	Rel. nonloc.	$\cdots$	Almost SCFT	$U(N_f/2) \times U(N_f/2)$

where M is the (bare) quark mass matrix. The model considered in Refs. [2,4,5], as can be seen from the form of the VEV of the adjoint scalar, corresponds to the $r = n = N_f$ vacuum of the above theory.

On the other hand, the light monopoles in Table I correspond to the limit $m_i \to m \to 0$, and we need to know to which quantum vacuum each semiclassical vacuum corresponds. This problem of matching the semiclassical and fully quantum-mechanical vacua one by one has been solved by using the vacuum counting and by symmetry considerations. The classical r vacua $r = 0, 1, \ldots, N_f$ found in the semiclassical regime

$$|m_i| \gg |\mu| \gg \Lambda$$

are found to correspond [33–35] to the quantum r vacua $r = 0, 1, \ldots, N_f/2$ as

$$\{r, N_f - r\}^{(\mathrm{class})} \Longleftrightarrow r, \qquad r = 0, 1, \ldots \leq N_f/2. \tag{1.2}$$

Note that the quantum r vacua, valid at

$$|m_i|, \qquad |\mu| \sim \Lambda$$

[with $SU(r)$ the non-Abelian magnetic gauge symmetry], exist only up to $r \leq N_f/2$ for dynamical reasons [36].

According to the matching [Eq. (1.2)], the model considered in Refs. [2,4,5] must correspond to the $r = 0$ quantum vacuum. The latter is characterized by the fact that all monopoles are Abelian (see Table I); furthermore, none of them carries any flavor $SU(N_f)$ quantum numbers. The condensation of the light monopoles (which occurs when the adjoint scalar mass $\mu \Phi^2$ is added in the theory) does not break $SU(N_f)$ symmetry, consistent with the finding from the vortex dynamics.[2]

On the other hand, one *knows* [32,33] that in four-dimensional $\mathcal{N} = 2$ supersymmetric QCD there appear light monopoles carrying non-Abelian $SU(r)$ charges (r vacua with $2 \leq r \leq N_f/2$ in Table I). There must be ways to understand the nature of non-Abelian monopoles, starting from the more familiar concept of semiclassical regular monopoles. In particular, one wonders whether non-Abelian vortices which do not Abelianize dynamically completely, and thus are naturally related to non-Abelian monopoles through the monopole-vortex matching argument [3,19], can be found in some appropriate semiclassical regime.

We shall show below that the new types of vortex solutions with desired properties can indeed be found. The underlying model is the same as the one discussed in Refs. [2,33]: an $\mathcal{N} = 2$ supersymmetric $SU(N)$ gauge theory with $N_f = N$ flavors. But the gauge group is broken partially down to $SU(n) \times SU(r) \times U(1)$ gauge symmetry ($N = n + r$) by the adjoint scalar VEV.

II. NON-ABELIAN VORTICES WITH MORE THAN ONE NON-ABELIAN MODULI FACTORS

The model on which we shall base our consideration is the softly broken $\mathcal{N} = 2$ supersymmetric QCD with $SU(N)$ and $N_f = N$ flavors of quark multiplets:

$$\mathcal{L} = \frac{1}{8\pi} \mathrm{Im}\tau_{\mathrm{cl}} \left[\int d^4\theta \, \mathrm{Tr}(\Phi^\dagger e^V \Phi e^{-V}) + \int d^2\theta \frac{1}{2} \mathrm{Tr}(WW) \right] + \mathcal{L}^{(\mathrm{quarks})} + \int d^2\theta \mu \, \mathrm{Tr}\Phi^2, \tag{2.1}$$

$$\mathcal{L}^{(\mathrm{quarks})} = \sum_i \left[\int d^4\theta (Q_i^\dagger e^V Q_i + \tilde{Q}_i e^{-V} \tilde{Q}_i^\dagger) + \int d^2\theta (\sqrt{2} \tilde{Q}_i \Phi Q^i + m_i \tilde{Q}_i Q^i) \right], \tag{2.2}$$

where $\tau_{\mathrm{cl}} \equiv \theta_0/\pi + 8\pi i/g_0^2$ contains the coupling constant and the theta parameter and μ is the adjoint scalar mass, breaking softly $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$. We tune the bare quark masses as

$$m_1 = \cdots = m_n = m^{(1)}; \quad m_{n+1} = m_{n+2} = \cdots = m_{n+r} = m^{(2)}, \quad N = n + r; \quad n m^{(1)} + r m^{(2)} = 0, \tag{2.3}$$

or

[2] The authors thank R. Auzzi and G. Marmorini for discussions on this point.

$$m^{(1)} = \frac{rm_0}{\sqrt{r^2+n^2}}, \qquad m^{(2)} = -\frac{nm_0}{\sqrt{r^2+n^2}}, \tag{2.4}$$

and their magnitude is taken as

$$|m_0| \gg |\mu| \gg \Lambda. \tag{2.5}$$

The adjoint scalar VEV can be taken to be

$$\langle\Phi\rangle = -\frac{1}{\sqrt{2}}\begin{pmatrix} m^{(1)}\mathbb{1}_{n\times n} & \mathbf{0} \\ \mathbf{0} & m^{(2)}\mathbb{1}_{r\times r}\end{pmatrix}. \tag{2.6}$$

Below the mass scale $v_1 \sim |m_i|$, the system thus reduces to a gauge theory with gauge group

$$G = \frac{SU(n)\times SU(r)\times U(1)}{\mathbb{Z}_K}, \qquad K = \mathrm{LCM}\{n, r\}, \tag{2.7}$$

where K is the least common multiple of n and r. The higher n color components of the first n flavors (with the bare mass $m^{(1)}$) remain massless, as well as the lower r color components of the last r flavors (with the bare mass $m^{(2)}$): They will be denoted as $q^{(1)}$ and $q^{(2)}$, respectively. They carry the charges λ_1 and $-\lambda_2$:

$$\lambda_1 \equiv \frac{r}{\sqrt{2nr(r+n)}}, \qquad \lambda_2 \equiv \frac{n}{\sqrt{2nr(r+n)}}, \tag{2.8}$$

with respect to the $U(1)$ gauge symmetry generated by

$$t^{(0)} = \begin{pmatrix} \lambda_1\mathbb{1}_{n\times n} & \mathbf{0} \\ \mathbf{0} & -\lambda_2\mathbb{1}_{r\times r}\end{pmatrix}, \qquad \mathrm{Tr}t^{(0)2} = \frac{1}{2}. \tag{2.9}$$

Non-Abelian gauge groups are generated by the standard SU generators

$$t^a_{su(n)} = \begin{pmatrix} (t^a)_{n\times n} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{r\times r}\end{pmatrix}; \qquad t^b_{su(r)} = \begin{pmatrix} \mathbf{0}_{n\times n} & \mathbf{0} \\ \mathbf{0} & (t^b)_{r\times r}\end{pmatrix}; \tag{2.10}$$

$a = 1, 2, \ldots, n^2-1$; $b = 1, 2, \ldots, r^2-1$, with the normalization

$$\mathrm{Tr}_n(t^a t^{a'}) = \frac{\delta_{aa'}}{2}, \qquad \mathrm{Tr}_r(t^b t^{b'}) = \frac{\delta_{bb'}}{2}.$$

Our model for studying the vortices then is[3]

$$\begin{aligned}\mathcal{L} = {} & -\frac{1}{4g_0^2}F^{02}_{\mu\nu} - \frac{1}{4g_n^2}F^{n2}_{\mu\nu} - \frac{1}{4g_r^2}F^{r2}_{\mu\nu} + \frac{1}{g_0^2}|\mathcal{D}_\mu\Phi^{(0)}|^2 \\ & + \frac{1}{g_n^2}|\mathcal{D}_\mu\Phi^{(n)}|^2 + \frac{1}{g_r^2}|\mathcal{D}_\mu\Phi^{(r)}|^2 + |\mathcal{D}_\mu q^{(1)}|^2 \\ & + |\mathcal{D}_\mu\bar{\tilde{q}}^{(1)}|^2 + |\mathcal{D}_\mu q^{(2)}|^2 + |\mathcal{D}_\mu\bar{\tilde{q}}^{(2)}|^2 - V_D - V_F,\end{aligned} \tag{2.11}$$

[3]One could very well start with a model of this sort directly. The squark VEVs can be induced by a Fayet-Iliopoulos term introduced by hand. By an $SU_R(2)$ rotation, which rotates $(q, \tilde{q}^\dagger)$ as a doublet, such a model can be seen to be equivalent to the one being considered here.

plus fermionic terms, where V_D and V_F are the D-term and F-term potentials, respectively. The D-term potential V_D has the form

$$V_D = \frac{1}{8}\sum_A\left(\mathrm{Tr}t^A\left[\frac{2}{g^2}[\Phi, \Phi^\dagger] + \sum_i(Q_iQ_i^\dagger - \tilde{Q}_i^\dagger\tilde{Q}_i)\right]\right)^2, \tag{2.12}$$

where the generator A takes the values 0 for $U(1)$, $a = 1, 2, \ldots, n^2-1$ for $SU(n)$, and $b = 1, 2, \ldots r^2-1$ for $SU(r)$. V_F is of the form

$$\begin{aligned}& g_0^2|\mu\Phi^{(0)} + \sqrt{2}\tilde{Q}t^{(0)}Q|^2 + g_n^2|\mu\Phi^{(a)} + \sqrt{2}\tilde{Q}t^{(a)}_{su(n)}Q|^2 \\ & \quad + g_r^2|\mu\Phi^{(b)} + \sqrt{2}\tilde{Q}t^{(b)}_{su(r)}Q|^2 + \tilde{Q}[M+\sqrt{2}\Phi] \\ & \quad \times[M+\sqrt{2}\Phi]^\dagger\tilde{Q}^\dagger + Q^\dagger[M+\sqrt{2}\Phi]^\dagger[M+\sqrt{2}\Phi]Q,\end{aligned} \tag{2.13}$$

where

$$M = \begin{pmatrix} m^{(1)}\mathbb{1}_{n\times n} & \mathbf{0} \\ \mathbf{0} & m^{(2)}\mathbb{1}_{r\times r}\end{pmatrix}$$

is the mass matrix and the (massless) squark fields have the form

$$\begin{aligned} Q(x) &= \begin{pmatrix} q^{(1)}(x)_{n\times n} & 0 \\ 0 & q^{(2)}(x)_{r\times r}\end{pmatrix}, \\ \tilde{Q}(x) &= \begin{pmatrix} \tilde{q}^{(1)}(x)_{n\times n} & 0 \\ 0 & \tilde{q}^{(2)}(x)_{r\times r}\end{pmatrix},\end{aligned} \tag{2.14}$$

if written in a color-flavor mixed matrix notation. The light squarks (supersymmetric partners of the left-handed quarks in supersymmetric model) are summarized in Table II.

We set V_D to zero identically, in the vacuum and in the vortex configurations, by keeping

$$\tilde{q}^{(1)} = (q^{(1)})^\dagger, \qquad q^{(2)} = -(\tilde{q}^{(2)})^\dagger; \tag{2.15}$$

the redefinition

$$q^{(1)} \to \frac{1}{\sqrt{2}}q^{(1)}, \qquad \tilde{q}^{(2)} \to \frac{1}{\sqrt{2}}\tilde{q}^{(2)} \tag{2.16}$$

brings the kinetic terms for these fields back to the original form.

TABLE II. The light particles and their charges with respect to the $U(1)\times SU(n)\times SU(r)$ gauge groups in our model.

Fields	$U(1)$	$SU(n)$	$SU(r)$
$q^{(1)}$	λ_1	$\underline{n}$	$\underline{1}$
$\tilde{q}^{(1)}$	$-\lambda_1$	$\underline{n}^*$	$\underline{1}$
$q^{(2)}$	$-\lambda_2$	$\underline{1}$	$\underline{r}$
$\tilde{q}^{(2)}$	λ_2	$\underline{1}$	$\underline{r}^*$

The VEVs of the adjoint scalars are given by

$$\langle\Phi^{(0)}\rangle = -m_0, \qquad \langle\Phi^{(a)}\rangle = \langle\Phi^{(b)}\rangle = 0, \tag{2.17}$$

while the squark VEVs are given [from the vanishing of each term of Eq. (2.13)] by

$$\langle Q\rangle = \begin{pmatrix} v^{(1)}\mathbb{1}_{n\times n} & 0 \\ 0 & -v^{(2)*}\mathbb{1}_{r\times r}\end{pmatrix}, \qquad \langle\tilde{Q}\rangle = \begin{pmatrix} v^{(1)*}\mathbb{1}_{n\times n} & 0 \\ 0 & v^{(2)}\mathbb{1}_{r\times r}\end{pmatrix}, \tag{2.18}$$

with

$$|v^{(1)}|^2 + |v^{(2)}|^2 = \sqrt{\frac{n+r}{nr}}\mu m_0. \tag{2.19}$$

There is a continuous vacuum degeneracy; we assume that

$$v^{(1)} \neq 0, \qquad v^{(2)} \neq 0$$

in the following. The presence of the flat direction implies the existence of the so-called semilocal vortex moduli; but we shall not be concerned with these here.

"Non-Abelian" vortices exist in this theory as the vacuum breaks the gauge group G [Eq. (2.7)] completely, leaving at the same time a color-flavor diagonal symmetry

$$[SU(n)\times SU(r)\times U(1)]_{C+F} \tag{2.20}$$

unbroken. The full global symmetry, including the overall global $U(1)$, is given by

$$U(n)\times U(r). \tag{2.21}$$

The minimal vortex in this system corresponds to the smallest nontrivial loop in the G group space [Eq. (2.7)]. It is the path in the $U(1)$ space

$$\begin{pmatrix} e^{i\alpha r}\mathbb{1}_{n\times n} & 0 \\ 0 & e^{i\alpha n}\mathbb{1}_{r\times r}\end{pmatrix}, \qquad \alpha\colon 0\to\frac{2\pi}{nr}, \tag{2.22}$$

that is,

$$\mathbb{1}_{N\times N}\to\mathbb{Y}, \qquad \mathbb{Y} = \begin{pmatrix} e^{2\pi i/n}\mathbb{1}_{n\times n} & 0 \\ 0 & e^{2\pi i/r}\mathbb{1}_{r\times r}\end{pmatrix}, \tag{2.23}$$

followed by a path in the $SU(n)\times SU(r)$ manifold

$$\mathbb{1}_{n\times n}\to\mathbb{Z}_n = e^{-(2\pi i)/n}\mathbb{1}_{n\times n}, \qquad \mathbb{1}_{r\times r}\to\mathbb{Z}_r = e^{-(2\pi i)/r}\mathbb{1}_{r\times r} \tag{2.24}$$

back to the unit element. For instance, one may choose ($\beta\colon 0\to 2\pi$; $\gamma\colon 0\to 2\pi$)

$$\begin{pmatrix} e^{i\beta(n-1)/n} & 0 \\ 0 & e^{-i\beta/n}\mathbb{1}_{(n-1)\times(n-1)}\end{pmatrix}; \qquad \begin{pmatrix} e^{i\gamma(r-1)/r} & 0 \\ 0 & e^{-i\gamma/r}\mathbb{1}_{(r-1)\times(r-1)}\end{pmatrix}.$$

As

$$\mathbb{Y}^K = \mathbb{1}_{N\times N}, \qquad K = \mathrm{LCM}\{n, r\}, \tag{2.25}$$

it follows that the tension (and the winding) with respect to the $U(1)$ is $\frac{1}{K}$ of that in the standard ANO vortex.

The squark fields trace such a path asymptotically, i.e., far from the vortex core, as one goes around the vortex; at finite radius the vortex has, for instance, the form

$$q^{(1)} = \begin{pmatrix} e^{i\phi}f_1(\rho) & 0 \\ 0 & f_2(\rho)\mathbb{1}_{(n-1)\times(n-1)}\end{pmatrix}, \qquad \tilde{q}^{(2)} = \begin{pmatrix} e^{i\phi}g_1(\rho) & 0 \\ 0 & g_2(\rho)\mathbb{1}_{(r-1)\times(r-1)}\end{pmatrix}, \tag{2.26}$$

where ρ and ϕ stand for the polar coordinates in the plane perpendicular to the vortex axis and $f_{1,2}$ and $g_{1,2}$ are profile functions. The adjoint scalar fields Φ are taken to be equal to their VEVs [Eq. (2.17)]. They are accompanied by the appropriate gauge fields so that the tension is finite. The BPS equations for the squark and gauge fields and the properties of their solutions are discussed in the appendix. The behavior of numerically integrated vortex profile functions $f_{1,2}$ and $g_{1,2}$ is illustrated in Fig. 1.

We note here only that the necessary boundary conditions on the squark profile functions have the form

$$f_1(\infty) = f_2(\infty) = v^{(1)}, \qquad g_1(\infty) = g_2(\infty) = v^{(2)},$$

while at the vortex core

$$f_1(0) = 0, \qquad g_1(0) = 0, \qquad f_2(0)\neq 0, \qquad g_2(0)\neq 0. \tag{2.27}$$

The most important fact about these minimum vortices is that one of the $q^{(1)}$ *and* one of the $\tilde{q}^{(2)}$ fields must necessarily wind at infinity, simultaneously. As the indi-

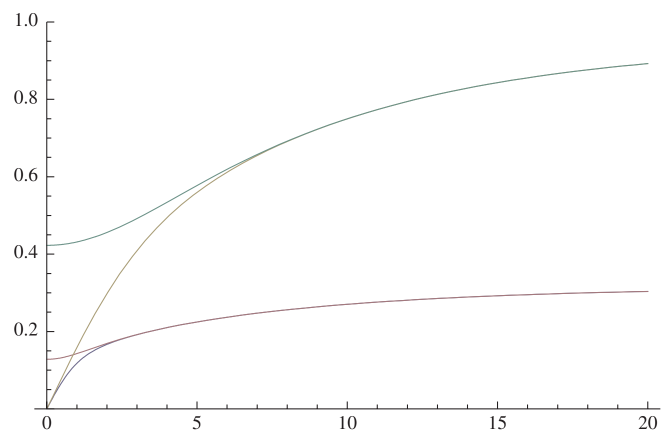

FIG. 1 (color online). Numerical result for the profile functions $f_{1,2}$ and $g_{1,2}$ as functions of the radius ρ, for $SU(3)\times SU(2)\times U(1)$ theory. The coupling constants and the ratio of the VEVs are taken to be $g_0 = 0.1$, $g_3 = 10$, $g_2 = 1$, and $|v_2|/|v_1| = 3$, respectively.

vidual vortex breaks the (global) symmetry of the vacuum as

$$[SU(n) \times SU(r) \times U(1)]_{C+F} \to SU(n-1) \times SU(r-1) \times U(1)^3, \tag{2.28}$$

the vortex acquires Nambu-Goldstone modes parametrizing

$$CP^{n-1} \times CP^{r-1}: \tag{2.29}$$

They transform under the exact color-flavor symmetry $SU(n) \times SU(r)$ as the bifundamental representation $(\underline{n}, \underline{r})$. Allowing the vortex orientation to fluctuate along the vortex length and in time, we get a $CP^{n-1} \times CP^{r-1}$ two-dimensional sigma model as an effective Lagrangian describing them. The details have been worked out in Refs. [4,5] and need not be repeated here.

The most important characteristics of these vortices is the following. Let us assume without losing generality that $n > r$, excluding the special case of $r = n$ for the moment. As has been shown in Refs. [4,5], the coupling constant of the CP^{n-1} sigma models grows precisely as the coupling constant of the $4D$ $SU(n)$ gauge theory. At the point the CP^{n-1} vortex moduli fluctuations become strong and the dynamical scale Λ gets generated, with vortex kinks (Abelian monopoles) acquiring mass of the order of Λ, the vortex still carries the unbroken $SU(r)$ fluctuation modes (CP^{r-1}), as the $SU(r)$ interactions are still weak. See Fig. 2.

Our vortex, at that mass (or length) scale, will carry one of the $U(1)$ flux arising from the dynamical breaking of $SU(n) \times U(1) \to U(1)^n$, as well as an $SU(r)$ flux. Of course, the vortex of the model [Eq. (2.11)] will eventually generate another (much lower) mass scale at which also the CP^{r-1} fluctuations become strong and Abelianize.

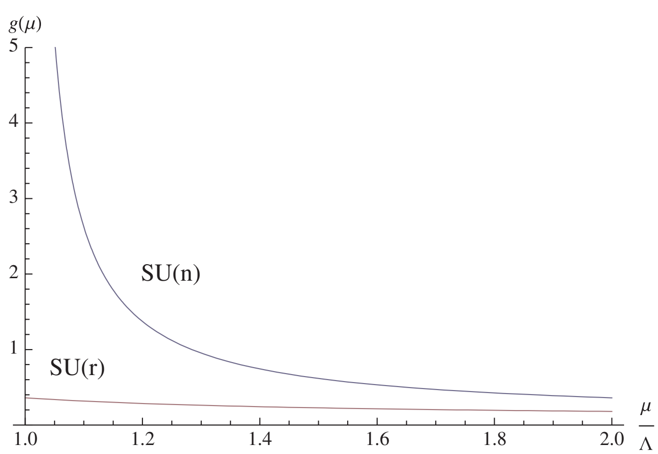

FIG. 2 (color online). A schematic representation of the running of the $SU(n)$ and $SU(r)$ coupling constants. For $n > r$ the $SU(n)$ interactions become strong first, at some mass scale Λ. At that scale, there are still weakly fluctuating $SU(r)$ moduli.

Not all of the properties of the theory will survive, however, if Eq. (2.11) is a low-energy effective action, arising from the symmetry breaking

$$SU(N) \to G = \frac{SU(n) \times SU(r) \times U(1)}{\mathbb{Z}_K},$$

at some high mass scale $\sim |m_0|$.[4] As the $SU(N)$ theory does not support vortices $[\Pi_1(SU(N)) = \mathbf{1}]$, any vortex solution appearing in the low-energy effective theory is only approximately stable: They eventually terminate at massive, regular monopoles of $\Pi_2(SU(N)/G)$. As

$$\Pi_2(SU(N)/G) \sim \Pi_1(G)$$

there is actually a one-to-one map between the vortex solution of the low-energy G theory and the regular monopoles of the high-energy theory. (See Table III and below.)

It is in this context of monopole-vortex correspondence that we propose that our vortex, carrying the CP^{r-1} modulations, is related to the monopoles the carrying $SU(r)$ charges, appearing in the r vacua of the $4D$, $SU(N)$ theory. As an indirect support for our conjecture, note that the global symmetry of our system is $U(r) \times U(N-r)$, just as in the quantum r vacua of the $4D$, $SU(N)$ theory. Another, very suggestive hint that our idea is indeed correct is that, both in the $4D$, $SU(N)$ theory and in the low-energy vortex theory, the relevant case occurs only for $r < N_f/2$.

The special case $r = 1$ corresponds to the $U(N)$ model [2,4,5,15], mentioned in the introduction, and in this case the vortices dynamically Abelianize. This is not in contradiction with the claim made above, after Eq. (1.2), that the $U(n)$ models considered in those papers corresponded to the quantum $r = 0$ vacuum of the $SU(n+1)$ model, with $N_f = n$. The point is that here we start with the underlying theory with $SU(N)$, $N_f = N$, where $N = n + r$; the classical-quantum vacuum matching condition [Eq. (1.2)] implies that the $U(n)$ models studied earlier, if embedded in our general scheme, correspond to the $r = 1$, rather than $r = 0$, vacua. The symmetry breaking pattern [Eq. (2.21)] also perfectly matches the full quantum result in Table I, as it does for generic r.

There is no difficulty in generalizing our construction and finding vortices with fluctuations corresponding to more than two non-Abelian factors

$$SU(n) \times SU(r_1) \times SU(r_2) \times \cdots,$$

[4]There are many known examples of such artifacts of an approximation in physics. The Landau pole of QED and the infrared divergence in the integration over the size of the instantons in QCD are two well known examples. In our context, the presence of the "semilocal" vortex solution is believed to be the artifact of the low-energy, strict BPS approximation, if the system is embedded in a system with a larger gauge group [37].

TABLE III. The effective low-energy degrees of freedom and their quantum numbers at the confining vacuum characterized by a magnetic dual $SU(r)$ gauge group.

	$SU(r)$	$U(1)_0$	$U(1)_1$	$\ldots$	$U(1)_{n-1}$	$U(1)_B$
$n_f \times q$	$\underline{\mathbf{r}}$	1	0	$\ldots$	0	0
e_1	$\underline{\mathbf{1}}$	0	1	$\ldots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
e_{n-1}	$\underline{\mathbf{1}}$	0	0	$\ldots$	1	0

as long as we remain in the semiclassical region with $|m_i|$, $|\mu| \gg \Lambda$. However, the main aim of this paper is to identify the semiclassical origin of the non-Abelian monopoles seen in the fully quantum effective low-energy action of the theory at $m_i \to 0$, $\mu \sim \Lambda$. In such a limit, the breaking of the gauge symmetry is a dynamical question; the result of the analysis of the $4D$ theory (Table I) suggests that in that limit the surviving non-Abelian dual group $SU(r_1) \times SU(r_2) \times \cdots$ gets enhanced to a single factor $SU(r)$. In order for gauge groups with more than one non-Abelian factor to survive dynamically at low energies, a nontrivial potential in the adjoint scalar field Φ needs to be present in the underlying theory [34].

III. CONCLUSION

In this note we have constructed vortices of a new type, having non-Abelian moduli,

$$CP^{n-1} \times CP^{r-1},$$

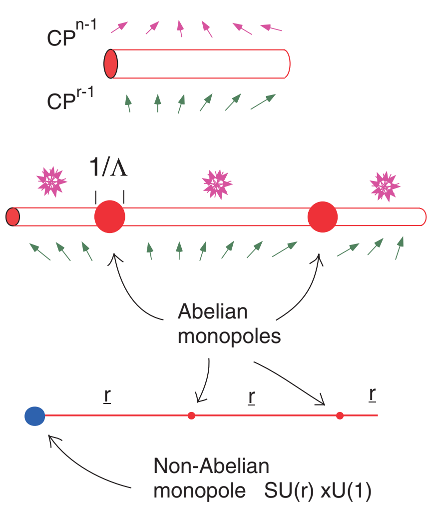

FIG. 3 (color online). Our vortex has $CP^{n-1} \times CP^{r-1}$ orientational modes which can fluctuate along the vortex length and in time (top figure). At low energies CP^{n-1} orientational modes fluctuate strongly and Abelianize, leaving the weakly fluctuating CP^{r-1} modes (middle figure). The vortex ends at a monopole which, absorbing the CP^{r-1} fluctuations, turns into a non-Abelian monopole. The latter transforms according to the fundamental representation of the dual $SU(r)$ group (bottom picture). The kink monopoles are Abelian.

resulting from the partial breaking of the $SU(n) \times SU(r) \times U(1)$ global symmetry to $SU(n-1) \times SU(r-1) \times U(1)^3$ by the vortex. For $n > r$, CP^{n-1} field fluctuations propagating along the vortex length become strongly coupled in the infrared, the $SU(n) \times U(1)$ part dynamically Abelianizes; the vortex, however, still carries weakly fluctuating $SU(r)$ modulations. See Fig. 3. In our theory where the $SU(n) \times SU(r) \times U(1)$ model emerges only as the low-energy approximation of an underlying $SU(N)$ theory, such a vortex is not stable. If these vortices end at a monopole, their CP^{r-1} orientational modes are turned into the dual $SU(r)$ color modulations of the monopole.

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APPENDIX: VORTEX CONFIGURATIONS

To be complete, we present here the vortex equations and their solutions of the model equation (2.11), in the vacuum equations (2.17) and (2.18). The action of our model, after setting Φ to its VEV [Eq. (2.17)], and after making the ansatz reduction on the squark field equations (2.15) and (2.16), takes the form

$$\begin{aligned} S = \int d^4x \Big[& \frac{1}{4g_n^2}(F^a_{\mu\nu})^2 + \frac{1}{4g_r^2}(F^b_{\mu\nu})^2 + \frac{1}{4g_0^2}(F^{(0)}_{\mu\nu})^2 \\ & + |\mathcal{D}_\mu q^{(1)}|^2 + |\mathcal{D}_\mu \tilde{q}^{(2)}|^2 + \frac{g_n^2}{2}(q^{(1)\dagger} t^a q^{(1)})^2 \\ & + \frac{g_r^2}{2}(\tilde{q}^{(2)} t^b \tilde{q}^{(2)\dagger})^2 + \frac{g_0^2}{2}(\lambda_1 q^{(1)\dagger} q^{(1)} \\ & + \lambda_2 \tilde{q}^{(2)} \tilde{q}^{(2)\dagger} - \xi)^2 \Big], \end{aligned} \tag{A1}$$

$$\xi = \sqrt{2}\mu m_0. \tag{A2}$$

The tension can be written completing the squares à la Bogomolnyi [38] as

$$T = \int d^2x \Bigg(\sum_{a=1}^{n^2-1} \left[\frac{1}{2g_n} F_{ij}^{(a)} \pm \frac{g_n}{2} (q^{(1)\dagger} t^a q^{(1)}) \epsilon_{ij} \right]^2 + \sum_{b=1}^{r^2-1} \left[\frac{1}{2g_r} F_{ij}^{(b)} \pm \frac{g_r}{2} (\tilde{q}^{(2)} t^b \tilde{q}^{(2)\dagger}) \epsilon_{ij} \right]^2 + \left[\frac{1}{2g_0} F_{ij}^{(0)} \pm \frac{g_0}{2} (\lambda_1 \operatorname{Tr}_n q^{(1)\dagger} q^{(1)} + \lambda_2 \operatorname{Tr}_r \tilde{q}^{(2)\dagger} \tilde{q}^{(2)} - \xi) \epsilon_{ij} \right]^2 + \frac{1}{2} |\mathcal{D}_i q^{(1)} \pm i\epsilon_{ij} \mathcal{D}_j q^{(1)}|^2 + \frac{1}{2} |\mathcal{D}_i \tilde{q}^{(2)} \pm i\epsilon_{ij} \mathcal{D}_j \tilde{q}^{(2)}|^2 \pm \xi B^{(0)} \Bigg), \quad \text{(A3)}$$

where $B^{(0)} \equiv \frac{1}{2}\epsilon_{ij}F_{ij}^{(0)}$ is the magnetic flux density along the z direction. The first-order Bogomolnyi equations are obtained by setting to zero all square bracket terms in Eq. (A3), that is, all terms except the last, topological invariant, winding-number term. Their solutions can be elegantly expressed in terms of the moduli matrices ($z \equiv x + iy$)

$$q^{(1)} = S_n^{-1}(z, \bar{z}) e^{-\lambda_1 \psi(z,\bar{z})} H_0^{(n)}(z), \qquad \tilde{q}^{(2)} = S_r^{-1}(z, \bar{z}) e^{-\lambda_2 \psi(z,\bar{z})} H_0^{(r)}(z), \quad \text{(A4)}$$

where $H_0^{(n)}(z)$ and $H_0^{(r)}(z)$ are $n \times n$ and $r \times r$ matrices, respectively, *holomorphic* in z, while S_n (S_r) is a regular $SL(n, C)$ [$SL(r, C)$] matrix; $\psi(z, \bar{z})$ is a complex function, which can be chosen real by an appropriate choice of gauge.

$$\lambda_1 = \frac{r}{\sqrt{2nr(r+n)}}, \qquad \lambda_2 = \frac{n}{\sqrt{2nr(r+n)}} \quad \text{(A5)}$$

are the $U(1)$ charges of the $q^{(1)}$ and $\tilde{q}^{(2)}$ fields, respectively; see Eq. (2.9). S_n (S_r) corresponds to the complexified $SU(n)$ [$SU(r)$] transformations. Note that H_0's and S's are defined up to transformations of the form

$$H_0^{(n)}(z) \to V_n(z) H_0^{(n)}(z), \qquad S_n(z, \bar{z}) \to V_n(z) S_n(z, \bar{z}),$$

where $V_n(z)$ is an arbitrary regular, holomorphic $n \times n$ [*vis-à-vis*, $r \times r$ for $H_0^{(r)}(z)$, S_r] matrix of determinant one. $H_0^{(n)}(z)$ and $H_0^{(r)}(z)$, called moduli matrices, contain all of the moduli parameters [15]. $SU(n)$, $SU(r)$, and $U(1)$ gauge fields are given by ($\bar{\partial} \equiv \partial/\partial\bar{z}$)

$$A_1^{(n)} + iA_2^{(n)} = -2iS_n^{-1}(z, \bar{z}) \bar{\partial} S_n(z, \bar{z}), \qquad A_1^{(r)} + iA_2^{(r)} = -2iS_r^{-1}(z, \bar{z}) \bar{\partial} S_r(z, \bar{z}), \qquad A_1^{(0)} + iA_2^{(0)} = -2i\bar{\partial}\psi. \quad \text{(A6)}$$

These ansatz solve the matter part of the Bogomolnyi equations

$$(\mathcal{D}_1 + i\mathcal{D}_2) q^{(1)} = (\mathcal{D}_1 + i\mathcal{D}_2) \tilde{q}^{(2)} = 0 \quad \text{(A7)}$$

automatically (they reduce to $\bar{\partial} H_0 = 0$). In order to simplify the (linearized) gauge field equations, let us introduce

$$\Omega_n = S_n S_n^\dagger, \qquad \Omega_r = S_r S_r^\dagger;$$

the (Bogomolnyi) gauge field equations (sometimes called *master equations*) are[5]

$$\partial(\Omega_n^{-1}\bar{\partial}\Omega_n) = \frac{g_n^2}{4} e^{-2\lambda_1\psi} \left[\Omega_n^{-1} H_0^{(n)} H_0^{(n)\dagger} - \frac{1}{n} \operatorname{Tr}_n (\Omega_n^{-1} H_0^{(n)} H_0^{(n)\dagger}) \mathbf{1}_{n\times n} \right];$$

$$\partial(\Omega_r^{-1}\bar{\partial}\Omega_r) = \frac{g_r^2}{4} e^{-2\lambda_2\psi} \left[\Omega_r^{-1} H_0^{(r)} H_0^{(r)\dagger} - \frac{1}{r} \operatorname{Tr}_r (\Omega_r^{-1} H_0^{(r)} H_0^{(r)\dagger}) \mathbf{1}_{r\times r} \right];$$

$$\partial\bar{\partial}\psi = \frac{g_0^2}{4} \left[\lambda_1 e^{-2\lambda_1\psi} \operatorname{Tr}_n (\Omega_n^{-1} H_0^{(n)} H_0^{(n)\dagger}) + \lambda_2 e^{-2\lambda_2\psi} \operatorname{Tr}_r (\Omega_r^{-1} H_0^{(r)} H_0^{(r)\dagger}) - \xi \right].$$

Since $SU(n)$, $SU(r)$, and $U(1)$ all commute with each other, the above construction is basically just a straightforward generalization of the formulas given in the case of $U(n) \sim SU(n) \times U(1)$ theory; see, e.g., [11], except for one point. As there is just one $U(1)$ gauge group factor but two non-Abelian groups $SU(n)$ and $SU(r)$, the moduli matrices are subject to a constraint. In fact, from Eq. (A4) and the fact that S_n (S_r) belongs to $SL(n, C)$ [$SL(r, C)$], it follows that

$$e^{-2\lambda_1 n\psi} \det H_0^{(n)} H_0^{(n)\dagger} = \det(q^{(1)} q^{(1)\dagger}); \qquad e^{-2\lambda_2 r\psi} \det H_0^{(r)} H_0^{(r)\dagger} = \det(\tilde{q}^{(2)} \tilde{q}^{(2)\dagger}).$$

As $\lambda_1 n = \lambda_2 r$ [see Eq. (A5)], these are consistent with the asymptotic behavior

$$q^{(1)} q^{(1)\dagger} \sim |v_1|^2 \mathbb{1}_{n\times n}, \qquad \tilde{q}^{(2)} \tilde{q}^{(2)\dagger} \sim |v_2|^2 \mathbb{1}_{r\times r},$$

if a constraint

$$\frac{\det H_0^{(n)} H_0^{(n)\dagger}}{\det H_0^{(r)} H_0^{(r)\dagger}} \sim \frac{|v_1|^{2n}}{|v_2|^{2r}} \quad \text{(A8)}$$

is satisfied at large $|z|$. So for a vortex of winding number k,

$$\det H_0^{(n)} H_0^{(n)\dagger} \propto \det H_0^{(r)} H_0^{(r)\dagger} \sim |z|^{2k},$$

i.e., the same winding in q and $\tilde{q}$ fields, but with the condition Eq. (A8).

[5]For instance, the $SU(n)$ gauge field components can be written from Eq. (A6) as

$$A_1 = -i(S^{-1}\bar{\partial}S + S^\dagger \partial (S^\dagger)^{-1}); \qquad A_2 = -(S^{-1}\bar{\partial}S - S^\dagger \partial (S^\dagger)^{-1}).$$

By a straightforward algebra one finds then ($F_{12} = \partial_1 A_2 - \partial_2 A_1 + i[A_1, A_2]$)

$$(S^\dagger)^{-1} F_{12} S^\dagger = -2\partial(\Omega^{-1}\bar{\partial}\Omega), \qquad \Omega = SS^\dagger.$$

The tension for the minimum vortex ($k=1$) can be worked out easily as follows. A typical such vortex has the form Eq. (A4), where the moduli matrices can be brought to the form locally, e.g.,

$$H_0^{(n)}(z) = \begin{pmatrix} c_1 z & \mathbf{0} \\ \mathbf{0} & \mathbb{1}_{(n-1)\times(n-1)} \end{pmatrix},$$
$$H_0^{(r)}(z) = \begin{pmatrix} c_2 z & \mathbf{0} \\ \mathbf{0} & \mathbb{1}_{(r-1)\times(r-1)} \end{pmatrix},$$

with

$$\frac{c_1}{c_2} = \frac{v_1^n}{v_2^r}. \tag{A9}$$

Note that one of c_1 and c_2, for instance, c_1, can be set to unity by an appropriate choice of S_n and ψ. The other is then fixed uniquely. In order for the behavior (by setting $c_1 = 1$)

$$H_0^{(n)}(z)H_0^{(n)}(z)^\dagger = \begin{pmatrix} \rho^2 & \mathbf{0} \\ \mathbf{0} & \mathbb{1}_{(n-1)\times(n-1)} \end{pmatrix}$$

to be consistent with $q^{(1)}q^{(1)\dagger} \sim |v_1|^2 \mathbb{1}_{n\times n}$, the large ρ behavior of ψ and S_n must be such that

$$e^{-2\lambda_1\psi} S_n^{-1}(S_n^\dagger)^{-1} \sim \begin{pmatrix} 1/\rho^2 & \mathbf{0} \\ \mathbf{0} & \mathbb{1}_{(n-1)\times(n-1)} \end{pmatrix};$$

this is possible if

$$S_n \sim \begin{pmatrix} e^{(n-1)\lambda_1\psi} & \mathbf{0} \\ \mathbf{0} & e^{-\lambda_1\psi}\mathbb{1}_{(n-1)\times(n-1)} \end{pmatrix}$$

and

$$e^{-2n\lambda_1\psi} \sim 1/\rho^2, \quad \therefore\ \psi \sim \sqrt{\frac{n+r}{2nr}}\log\rho^2.$$

Of course, the same conclusion for ψ is reached by considering the asymptotic behavior of $\tilde{q}^{(2)}$ and S_r. As $F_{12}^{(0)} = -4\bar{\partial}\partial\psi$,

$$T = \xi\int d^2x F_{12}^{(0)} = \xi\int d^2x \nabla^2\psi = 4\pi\sqrt{\frac{n+r}{2nr}}\xi$$
$$= 4\pi\sqrt{\frac{n+r}{nr}}\mu m_0,$$

that is

$$T = 4\pi(|v^{(1)}|^2 + |v^{(2)}|^2).$$

An (axially symmetric) vortex of generic $SU(n)\times SU(r)$ orientations can be represented by the moduli matrix of the form

$$H_0^{(n)}(z) = \begin{pmatrix} c_1 z & \mathbf{0} \\ \mathbb{b} & \mathbb{1}_{(n-1)\times(n-1)} \end{pmatrix},$$
$$H_0^{(r)}(z) = \begin{pmatrix} c_2 z & \mathbf{0} \\ \mathbb{c} & \mathbb{1}_{(r-1)\times(r-1)} \end{pmatrix},$$

where $\mathbb{b}$ ($\mathbb{c}$) is an $(n-1)$-component [$(r-1)$-component] complex vector, representing the inhomogeneous coordinates of CP^{n-1} (of CP^{r-1}). Under the color-flavor $SU(n)$ [$SU(r)$] symmetry group, they transform as in the fundamental representation of $SU(n)$ [$SU(r)$]. This is the content of some of the claims made in the main text.

The BPS equations actually allow more general kinds of vortex solutions. The moduli space, for general winding numbers and with more general position and orientation parameters, shows a very rich and interesting spectrum. This and other questions will be discussed elsewhere.

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