Relativistic harmonic oscillator revisited

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The familiar Fock space commonly used to describe the relativistic harmonic oscillator, for example, as part of string theory, is insufficient to describe all the states of the relativistic oscillator. We find that there are three different vacua leading to three disconnected Fock sectors, all constructed with the same creation-annihilation operators. These have different spacetime geometric properties as well as different algebraic symmetry properties or different quantum numbers. Two of these Fock spaces include negative norm ghosts (as in string theory), while the third one is completely free of ghosts. We discuss a gauge symmetry in a worldline theory approach that supplies appropriate constraints to remove all the ghosts from all Fock sectors of the single oscillator. The resulting ghost-free quantum spectrum in $d+1$ dimensions is then classified in unitary representations of the Lorentz group $SO(d,1)$. Moreover, all states of the single oscillator put together make up a single infinite dimensional unitary representation of a hidden global symmetry $SU(d,1)$, whose Casimir eigenvalues are computed. Possible applications of these new results in string theory and other areas of physics and mathematics are briefly mentioned.

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I. INTRODUCTION

The relativistic harmonic oscillator in d space and one-time dimensions that will be discussed in this paper is the straightforward generalization of the nonrelativistic case obtained by replacing position and momentum by their relativistic counterparts x^μ, p^μ as $SO(d,1)$ vectors.

There is a long history of studies of the relativistic harmonic oscillator. Some of these were motivated by possible physical applications of the relativistic oscillator as an imperfect model [1][1] to approximate bound states of quarks in a relativistic setting. This involved solving the relativistic oscillator eigenvalue equation[2] in the space of the relative coordinate $x^\mu = x_1^\mu - x_2^\mu$,

$$\tfrac{1}{2}(-\partial^\mu \partial_\mu + x^\mu x_\mu)\psi_\lambda(x) = \lambda \psi_\lambda(x), \qquad (1.1)$$

and associating the eigenvalue λ with the mass of the bound state.

Some solutions of this equation appeared in earlier papers [2,3] and in follow-up applications [4], but the Lorentz symmetry properties of these solutions remain obscure to this day [5]. Lorentz covariant solutions based on a vacuum state $\psi_{\rm vac}(x) \sim \exp(-x^\mu x_\mu/2)$, that is a Lorentz invariant Gaussian, have a number of problems, including issues of infinite norm and negative norm states, that were suppressed with *ad hoc* arguments for the sake of going forward with the physical application [1]. More careful analyses, which paid attention to Lorentz properties by using infinite dimensional unitary representations of $SO(3,1)$ [6] relevant for this problem [7,8], suggest that there are solutions of this equation in different spacelike and timelike patches that should be matched across the light cone $x^\mu x_\mu = 0$. Several examples of this covariant approach using generalized relativistically invariant potentials $V(x^\mu x_\mu)$ that may be different in different patches were also studied [9]. Proposals to confine the solutions to only *part* of the spacelike region were also discussed [10,11].

It is fair to say that there remains open questions regarding the symmetry properties of the solutions of this differential equation. Understanding the symmetry properties of the solutions will be the focus of the present paper.

The same equation arises as a building block in string theory. The phase space $X^\mu(\tau,\sigma)$, $P^\mu(\tau,\sigma)$ of an open relativistic string can be expressed in terms of its normal modes

$$X^\mu = x_0^\mu(\tau) + \sqrt{2}\sum_{n=1}^{\infty} x_n^\mu(\tau)\cos(n\sigma),$$
$$P^\mu = \frac{1}{\pi}p_0^\mu(\tau) + \frac{\sqrt{2}}{\pi}\sum_{n=1}^{\infty} p_n^\mu(\tau)\cos(n\sigma). \qquad (1.2)$$

Except for the center of mass mode (x_0^μ, p_0^μ) that behaves like a free particle, the normal modes (x_n^μ, p_n^μ) are relativistic harmonic oscillator modes with frequency $\omega_n = n$. The quantum wave function of a string in position space depends on all of these modes,

[1]Feynman called this approach an imperfect model. Indeed, as is now known, the physically correct description of systems such as quark-antiquark bound states is formulated in the context of quantum chromodynamics. Approximations to chromodynamics for slow moving heavy quarks is handled in terms of a non-relativistic potential $V(\vec{r}) = \alpha|\vec{r}| - \beta/|\vec{r}|$, rather than the relativistic oscillator, while for fast moving light quarks this approach is not an accurate model.

[2]We absorb all dimensionful parameters as well as the frequency of the oscillator by rescaling the x^μ, p^μ.

$$\psi(X^\mu) = \psi(x_0^\mu, x_1^\mu, x_2^\mu, \cdots). \tag{1.3}$$

This is the string field that appears in string field theory [12,13]. It obeys a differential equation $(L_0 - 1)\psi(X^\mu) = 0$ where L_0 is the zeroth Virasoro operator which is basically a sum of operators $Q_n = \frac{1}{2}(p_n^2 + n^2 x_n^2)$ of the type that appears in Eq. (1.1),[3]

$$L_0 = -\partial_0^\mu \partial_{0\mu} + \sum_{n=1}^{\infty} \frac{1}{2}(-\partial_n^\mu \partial_{n\mu} + n^2 x_n^\mu x_{n\mu}) - a. \tag{1.4}$$

If this had been the only equation for the string field $\psi(X^\mu)$, then the solution would have been a direct product of solutions of Eq. (1.1) with a restriction on the sum of the eigenvalues,

$$\psi(X^\mu) \sim e^{ik\cdot x_0} \prod_{n=1}^{\infty} \psi_{\lambda_n}(x_n), \qquad \sum_{n=1}^{\infty} \lambda_n = (1 - k^2). \tag{1.5}$$

Here the center of mass momentum k^μ gives the mass squared of the relativistic string state $M^2 \equiv -k^2 = k_0^2 - \vec{k}^2$. However, $\psi(X^\mu)$ must also obey the Virasoro constraints $L_n \psi(X^\mu) = 0$. Therefore solutions for the free string field $\psi(X^\mu)$ are linear combinations of (1.5) with different λ_n's that satisfy the same mass level, taken with coefficients such that the Virasoro constraints are also obeyed. Such solutions were obtained in the covariant quantization approach, which also provided a proof of the absence of negative norm ghosts in string theory [14–16].

As will be explained in Sec. III, upon a closer examination it becomes evident that the relativistic Fock space treatment of string theory [17] inadvertently specializes to only the *spacelike* sector of every normal mode without any warning, namely,

$$x_n^\mu x_{n\mu} \geq 0 \quad \text{and} \quad p_n^\mu p_{n\mu} \geq 0 \quad \text{for every string mode } n \geq 1. \tag{1.6}$$

This can give only non-negative eigenvalues $\lambda_n \geq 0$, and hence Eq. (1.5) is solved for k^2 by mostly timelike center of mass momenta $k^\mu k_\mu < 0$, or positive M^2. The exception is the tachyon state that is forced to have spacelike momentum k^μ when all $\lambda_n = 0$, and hence $M^2 = -k^2 = -1$ gives a tachyon

$$\psi(X^\mu) \sim \langle X|0, k\rangle \sim e^{ik\cdot x_0} \exp\left(-\frac{1}{2}\sum_{n=1}^{\infty} n x_n^\mu x_{n\mu}\right) \tag{1.7}$$

when all string modes x_n^μ are in the spacelike region. For excited levels this expression is multiplied by polynomials in the various x_n^μ.

In view of the fact that the single oscillator equation (1.1) has solutions in different spacetime regions as indicated above, a natural question arises of whether there might be more general solutions to string theory beyond the spacelike region of Eq. (1.6). This is not an easy question to answer, both because there are the Virasoro constraints to deal with, and because there is still obscurity in the previously known solutions of the relativistic oscillator equations (1.1).

This brings us to the main topic of the current paper. We will investigate the single relativistic oscillator without prejudice as to its possible physical applications. Our main interest is to clarify the symmetry and unitarity or lack thereof of its various solutions in various parts of spacetime. At the end we will point out possible applications of our findings.

Our key observations will follow from hidden symmetries not discussed before. First we point out that the symmetries of Eq. (1.1) go beyond the Lorentz symmetry $SO(d,1)$. There is a hidden symmetry $SU(d,1)$ that includes $SO(d,1)$, and therefore all solutions, unitary or nonunitary, must fall into irreducible representations of $SU(d,1)$. Apparently this was never explored in previous investigations of Eq. (1.1).

After clarifying the symmetry aspects we will build three different Fock spaces by using the same relativistic harmonic oscillator creation-annihilation operators. This includes a spacelike, timelike, and mixed spacetime sectors that are distinct from each other. While the spacelike or timelike sectors have negative norm states, the mixed case is completely free of negative norm ghosts and is covariant under $SO(d,1)$ and $SU(d,1)$ in infinite dimensional unitary representations. There may be more solutions in more intricate spacetime sectors than those described in this paper, but we will not attempt to investigate them here [see comments following Eq. (A16) and footnote 17].

For the single harmonic oscillator we will also discuss a worldline gauge symmetry that removes ghosts and thereby introduces a constraint. The covariant quantization of this constrained model is in agreement with the general discussion. On the other hand, a gauge fixed quantization does not capture all the sectors but is in agreement with the sectors describable in that gauge. This simple example illustrates how a gauge fixed theory can fail to capture all the gauge invariant sectors of a gauge invariant theory.[4]

The new phenomena uncovered here both in the covariant quantization as well as the gauge fixed quantization of the relativistic oscillator may provide tools and rekindled interest to revisit string theory.

[3]The constant $a = \frac{1}{2}(d+1)\sum_n n$ subtracts the vacuum energy of all the oscillators. After this renormalization the Virasoro constraint is determined as $L_0 = 1$.

[4]Another example is that the usual treatment of the light-cone gauge in string theory fails to capture the folded string sectors of string theory [18–20].

II. RELATIVISTIC HARMONIC OSCILLATOR AND SU(d, 1)

For the sake of clarity, parts of our presentation, including this section, will include some material that may be quite familiar to many readers, but this will be compensated by simple observations that are not that familiar.

The operator $Q = \frac{1}{2}(p \cdot p + x \cdot x)$ which is being diagonalized, $Q\psi_\lambda = \lambda\psi_\lambda$, can be written, as usual, in terms of Lorentz covariant oscillators,

$$a_\mu = \frac{1}{\sqrt{2}}(x_\mu + ip_\mu), \qquad \bar{a}_\mu = \frac{1}{\sqrt{2}}(x_\mu - ip_\mu). \quad (2.1)$$

The covariant quantization rules

$$[x_\mu, p_\nu] = i\eta_{\mu\nu}, \quad (2.2)$$

with the SO(d, 1) Minkowski metric $\eta_{\mu\nu}$, lead to the relativistic quantum oscillator commutation rules

$$[a_\mu, \bar{a}_\nu] = \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, \cdots, 1). \quad (2.3)$$

In a unitary Hilbert space the operators x_μ, p_μ are Hermitian; in that case $\bar{a}_\mu$ is the Hermitian conjugate of a_μ, i.e. $\bar{a}_\mu = (a_\mu)^\dagger$. A unitary Hilbert space without ghosts (negative norm states) is possible if and only if x_μ, p_μ are Hermitian or, equivalently, if $\bar{a}_\mu = (a_\mu)^\dagger$.

In what follows we will seek unitary Hilbert spaces, but along the way we also come across nonunitary Fock spaces in which $\bar{a}_\mu \neq (a_\mu)^\dagger$. Therefore we prefer the more general notation $\bar{a}_\mu$ in order not to confuse it with the Hermitian conjugate of a_μ when such vector spaces arise.

In terms of a_μ, $\bar{a}_\mu$, the operator Q takes the form

$$Q = \frac{1}{2}(p \cdot p + x \cdot x) = \bar{a} \cdot a + \frac{d+1}{2} = a \cdot \bar{a} - \frac{d+1}{2}. \quad (2.4)$$

This operator Q has a larger symmetry than the evident Lorentz symmetry of the dot products $\bar{a} \cdot a = \eta^{\mu\nu}\bar{a}_\mu a_\nu$. The hidden symmetry is U(d, 1) whose generators are

$$U(d, 1) \text{ generators: } \bar{a}_\mu a_\nu. \quad (2.5)$$

All of these $(d + 1)^2$ generators commute with Q,

$$[Q, \bar{a}_\mu a_\nu] = [\bar{a} \cdot a, \bar{a}_\mu a_\nu] = 0; \quad (2.6)$$

hence Q has U(d, 1) symmetry, and the spectrum of Q, whether unitary or nonunitary, must be classified as irreducible representations of U(d, 1) = SU(d, 1) × U(1) unless the symmetry is broken by boundary conditions.[5] The U(1) part is just the number operator J_0,

$$J_0 \equiv \bar{a} \cdot a = a \cdot \bar{a} - (d + 1), \quad (2.7)$$

which is essentially the operator Q up to a shift. Therefore the nontrivial part is SU(d, 1) with $(d + 1)^2 - 1$ generators that correspond to the traceless tensor

$$J_{\mu\nu} = \left(\bar{a}_\mu a_\nu - \frac{1}{d+1}\eta_{\mu\nu}\bar{a} \cdot a\right) = \left(a_\nu \bar{a}_\mu - \frac{1}{d+1}\eta_{\mu\nu} a \cdot \bar{a}\right) \quad (2.8)$$

which satisfies $\eta^{\mu\nu}J_{\mu\nu} = 0$. The Lorentz generators $L_{\mu\nu}$ for SO(d, 1) correspond to the antisymmetric part of the tensor $J_{\mu\nu}$,

$$L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu = -i(\bar{a}_\mu a_\nu - \bar{a}_\nu a_\mu) = -i(a_\nu \bar{a}_\mu - a_\mu \bar{a}_\nu). \quad (2.9)$$

The $L_{\mu\nu}$ are Hermitian by construction as long as x_μ, p_μ are Hermitian. So a unitary representation of the Lorentz group will be obtained if and only if $\bar{a}_\mu = (a_\mu)^\dagger$. We know that unitary representations of noncompact groups are infinite dimensional except for the singlet. Hence $\bar{a}_\mu = (a_\mu)^\dagger$ can be satisfied only on singlets or on infinite dimensional representations of the Lorentz or the $SU(d, 1)$ symmetry.[6]

In the following we will see that there are different Fock spaces disconnected from each other, all of which contribute to the full unitary spectrum of Q. These Fock spaces are built with the same oscillators $\bar{a}_\mu$, a_ν but are based on three different vacua with different SU(d, 1) or SO(d, 1) symmetry properties as well as different spacetime geometric properties. This shows that there are some surprising features of the relativistic harmonic oscillator that are fundamentally different from the nonrelativistic one.

Our aim is to identify the physically acceptable unitary sector of the theory that contains no ghosts and find ways in which the physical sectors can be singled out by an appropriate set of constraints.

III. SYMMETRIC VACUUM, NONUNITARY FOCK SPACE

We will start with the standard approach to the relativistic oscillator Fock space used by most authors, including

[5]See the last paragraph of the Appendix for an example of how the SU(d, 1) symmetry is broken to SO(d, 1) in the purely spacelike sector.

[6]To be more accurate we should distinguish between fundamental and antifundamental representations of SU(d, 1) by using different indices to label them. For example, we can use undotted indices $a_\mu = \frac{1}{\sqrt{2}}(x_\mu + ip_\mu)$ to emphasize that a_μ is in the fundamental representation and dotted indices $\bar{a}_{\dot{\mu}} = \frac{1}{\sqrt{2}}(x_\mu - ip_\mu)$ to emphasize that $\bar{a}_{\dot{\mu}}$ is in the antifundamental representation. Indices are raised or lowered with the Minkowski metric $\eta^{\mu\dot{\nu}}$ that has mixed indices, such as $\bar{a}^\mu = \eta^{\mu\dot{\nu}}\bar{a}_{\dot{\nu}}$ and $a^{\dot{\mu}} = \eta^{\dot{\mu}\nu}a_\nu$. Because we will not have much use for it, we will forgo this more accurate notation and use the same type of indices on all creation or annihilation oscillators. The reader should understand that a lower index on the operator $\bar{a}$ is really meant to be a dotted index $\bar{a}_{\dot{\mu}}$, while an upper index on $\bar{a}$ is undotted $\bar{a}^\mu$. The opposite is true for the operators a_μ, $a^{\dot{\mu}}$.

string theorists [17]. The corresponding relativistic differential equation $(-\frac{1}{2}\partial^\mu\partial_\mu + \frac{1}{2}x^\mu x_\mu)\psi_\lambda(x) = \lambda\psi(x)$ in position space, in the purely spacelike sector, is solved in the Appendix in $1+1$ dimensions. Although the Fock space approach in this section and the position space approach of the Appendix are in full agreement, a great deal of complementary insight about the issues regarding spacetime regions is gained from considering the properties of the probability amplitude $\psi_\lambda(x)$ in position space. So the reader may benefit from studying the Appendix and comparing it to the Fock space approach in this section.

What we want to emphasize is that the familiar Fock space approach yields only part of the quantum states of this relativistic system. After explaining this, we will discuss a much larger Fock space of quantum states in the following section.

The oscillator approach begins by assuming a normalized *Lorentz invariant* vacuum state that has a *finite positive norm* and is annihilated by the operators a_μ,

$$\langle 0|0\rangle = 1, \qquad a_\mu|0\rangle = 0, \qquad L_{\mu\nu}|0\rangle = 0. \tag{3.1}$$

The U(1) quantum number or the level number of this state is zero

$$J_0|0\rangle = \bar{a}\cdot a|0\rangle = 0. \tag{3.2}$$

A usually unstated property of this vacuum is that it also requires a spacelike region for x^μ as well as for p^μ since, as a probability amplitude in position space or momentum space, it has the form

$$\langle x|0\rangle \sim e^{-x^2/2} \quad \text{and} \quad \langle p|0\rangle \sim e^{-p^2/2},\ x^\mu,\ p^\mu \quad \text{spacelike.} \tag{3.3}$$

The minus sign in the exponent follows from satisfying $a_\mu|0\rangle = 0$ in position or momentum space, namely,

$$\begin{aligned} a_\mu|0\rangle &= \frac{1}{\sqrt{2}}(x_\mu + ip_\mu)|0\rangle = 0 \\ &\leftrightarrow \frac{1}{\sqrt{2}}\left(x_\mu + \frac{\partial}{\partial x^\mu}\right)e^{-(1/2)x\cdot x} = 0, \\ &\frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial p^\mu} + p_\mu\right)e^{-(1/2)p\cdot p} = 0. \end{aligned} \tag{3.4}$$

Spacelike regions $x\cdot x > 0$ and $p\cdot p > 0$ are necessary so that the Gaussian is integrable at infinity,

$$\begin{aligned} \langle 0|0\rangle &\sim \int d^{n+1}x e^{-x^2} < \infty \quad \text{or} \\ \langle 0|0\rangle &\sim \int d^{n+1}p e^{-p^2} < \infty, \end{aligned} \tag{3.5}$$

to give a finite norm $\langle 0|0\rangle = 1$. Actually, these integrals are infinite as they stand because, unlike the Euclidean analogs in which both radial and angular integrals are finite, in the present case the "angular" part contains boost parameters with an infinite range [see e.g. the parametrization in Eq. (A1) and Fig. 1]. For a finite norm this infinity must be divided out (see footnote 10).

It is also possible to restrict to a timelike region by starting from another *Lorentz invariant* "vacuum" state $|0'\rangle$ to construct a different Fock space. This second alternative is not usually considered. The vacuum $|0'\rangle$ is defined by being annihilated by $\bar{a}_\mu$ rather than by a_μ,

$$\begin{aligned} &\langle 0'|0'\rangle = 1, \qquad \bar{a}_\mu|0'\rangle = 0, \qquad L_{\mu\nu}|0'\rangle = 0, \\ &\bar{a}_\mu|0'\rangle = \frac{1}{\sqrt{2}}(x_\mu - ip_\mu)|0'\rangle \\ &\qquad = 0 \leftrightarrow \begin{Bmatrix} \frac{1}{\sqrt{2}}(x_\mu - \frac{\partial}{\partial x^\mu})e^{(1/2)x\cdot x} = 0, \\ \frac{i}{\sqrt{2}}(\frac{\partial}{\partial p^\mu} - p_\mu)e^{(1/2)p\cdot p} = 0. \end{Bmatrix} \end{aligned}$$

It corresponds to a normalizable vacuum with x^μ and p^μ in the timelike region, $x\cdot x < 0$ and $p\cdot p < 0$, to be able to normalize $\langle 0'|0'\rangle = 1$,

$$\langle x|0'\rangle \sim e^{x^2/2} \quad \text{and} \quad \langle p|0'\rangle \sim e^{p^2/2},\ x^\mu,\ p^\mu \ \text{timelike.} \tag{3.6}$$

The U(1) quantum number or the level number of this state is $-(d+1)$,

$$J_0|0'\rangle = \bar{a}\cdot a|0'\rangle = [a\cdot\bar{a} - (d+1)]|0'\rangle = -(d+1)|0'\rangle, \tag{3.7}$$

so it is clearly distinguishable from the spacelike vacuum.

The Fock space based on the vacuum $|0'\rangle$ is not usually considered because it contains negative norm states for spacelike oscillators, but by contrast it contains positive norms for timelike oscillators. For example, the 1-particle excitation $a_\mu|0'\rangle$ has norm

$$\begin{aligned} &\langle 0'|\bar{a}_\nu a_\mu|0'\rangle = -\eta_{\mu\nu}, \\ &\quad \text{negative for spacelike}\ \mu,\nu;\ \text{positive for timelike}\ \mu,\nu \end{aligned} \tag{3.8}$$

However, we will see that the physical states in this Fock space sector always involve pairs of spacelike and timelike oscillators, such as $a\cdot a|0'\rangle$. Such paired oscillator states have a positive norm. In this respect, the spacelike or timelike vacua stand at an equal footing. We will see that while the spacelike vacuum leads to a positive spectrum for Q, the timelike case leads to a negative spectrum. Whether the negative or positive spectra are suitable in physical applications depends on the physical interpretation of the operator $Q = \frac{1}{2}(p\cdot p + x\cdot x)$ in some physical context.

This begins to show that there are several disconnected sectors of Fock spaces in the spectrum of the relativistic harmonic oscillator. As we will see below, both of these Fock spaces lead to nonunitary vector spaces from which we will need to fish out a subset of positive norm states. Furthermore, in the next section, we will discuss a completely different Fock space that is based on a Lorentz noninvariant vacuum $|\tilde{0}\rangle$ that leads to a completely unitary infinite dimensional Hilbert space.

In the rest of this section we discuss mainly the Fock space based on the spacelike vacuum $|0\rangle$ and only give results or make comments about the very similar Fock space based on the timelike vacuum $|0'\rangle$.

In either spacelike or timelike cases, since the vacuum respects the SO$(d, 1)$ symmetry, one should expect to find that all the states in either Fock space can be classified as irreducible unitary or nonunitary representations of SO$(d, 1)$. Furthermore, the restriction to a spacelike or timelike region is consistent with an SU$(d, 1)$ symmetric vacuum since we can verify that under an infinitesimal SU$(d, 1)$ transformation we obtain

$$J_{\mu\nu}|0\rangle = 0, \qquad J_{\mu\nu}|0'\rangle = 0, \tag{3.9}$$

by using the two forms of $J_{\mu\nu}$ given in Eq. (2.8). Hence the Fock spaces built on these invariant vacua must be classified as complete irreducible unitary or nonunitary representations not just of SO$(d, 1)$ but of SU$(d, 1)$.

The total level operator can be written out in more detail as

$$J_0 = \bar{a}\cdot a = (-\bar{a}_0 a_0) + \bar{a}_i a_i. \tag{3.10}$$

Note how the number operator in the timelike direction $(-\bar{a}_0 a_0)$ works to give a positive number for the level in the spacelike Fock space *even when the excitation is in the timelike direction*: $(-\bar{a}_0 a_0)[\bar{a}_0|0\rangle] = (+1)[\bar{a}_0|0\rangle]$,

$$\begin{aligned}(-\bar{a}_0 a_0)[\bar{a}_0|0\rangle] &= -\bar{a}_0[a_0, \bar{a}_0]|0\rangle = \bar{a}_0|0\rangle(-1)^2 \\ &= (+1)\bar{a}_0|0\rangle.\end{aligned} \tag{3.11}$$

Therefore the total level operator J_0 on the covariant states $\bar{a}_\mu|0\rangle$, excited in either the time or space directions μ, has J_0 eigenvalue $+1$.

Similarly, the excited states at a general level $J_0 = n$ in the spacelike Fock space are constructed by applying n creation operators either in space or time directions,

$$\bar{a}_{\mu_1}\bar{a}_{\mu_2}\cdots\bar{a}_{\mu_n}|0\rangle = \mathrm{SU}(d,1)\ \text{tensor} \sim \overbrace{\boxed{\mu_1|\mu_2|\mu_3|\cdots|\mu_n}}^{n}. \tag{3.12}$$

This is a symmetric SU$(d, 1)$ or U$(d, 1)$ tensor corresponding to a single-row Young tableau as indicated. So, this collection of states at level $J_0 = n$ form a *finite* dimensional irreducible representation of SU$(d, 1)$.

The above SU$(d, 1)$ representation can be reduced into irreducible representations of SO$(d, 1)$. This is done by decomposing the symmetric tensor above into a sum of traceless tensors (trace is defined by contracting with the Minkowski metric $\eta^{\mu\nu}$),

$$\begin{aligned}&\{(\bar{a}_{\mu_1}\bar{a}_{\mu_2}\cdots\bar{a}_{\mu_n} - \text{trace})|0\rangle + \cdots\} \\ &\quad = \mathrm{SO}(d,1)\ \text{traceless tensors.}\end{aligned} \tag{3.13}$$

For example, at level $J_0 = 2$ we have one SO$(d, 1)$ tensor of rank 2 and one of rank zero as listed below,

$$\left(\bar{a}_{\mu_1}\bar{a}_{\mu_2} - \frac{\eta_{\mu_1\mu_2}}{d+1}\bar{a}\cdot\bar{a}\right)|0\rangle \quad \text{and} \quad \bar{a}\cdot\bar{a}|0\rangle. \tag{3.14}$$

Similarly at level n there are the following irreducible tensors of rank r,

$$r = n, (n-2), (n-4), \cdots, (0 \text{ or } 1). \tag{3.15}$$

At level $J_0 = n$, each traceless tensor of rank r listed in Eq. (3.15) is the basis for a separate *finite* dimensional irreducible representation of SO$(d, 1)$.

All finite representations of noncompact groups, except the singlet, are nonunitary. Therefore, all SU$(d, 1)$ or SO$(d, 1)$ representations that emerge in this Fock space at all levels n, except the singlets, are nonunitary. Hence at every level $J_0 = n$ there are many negative norm states that are unphysical. We have to discuss the types of constraints that can eliminate the ghosts to obtain a physical theory.

Let us now identify the *negative norm states* which appear among the SU$(d, 1)$ or SO$(d, 1)$ states in Eqs. (3.13) and (3.15). These are all the ones that contain an odd number of *timelike* oscillators. For example, the state $\bar{a}_0|0\rangle$ has a negative norm[7]:

$$\text{norm} = \langle 0|a_0\bar{a}_0|0\rangle = \langle 0|[a_0, \bar{a}_0]|0\rangle = (-1)\langle 0|0\rangle = -1. \tag{3.16}$$

The states at a fixed level n that have an even number of $\bar{a}_0$'s and any number of spacelike oscillators, such as $(\bar{a}_0)^m(\bar{a}_{i_1}\bar{a}_{i_2}\cdots\bar{a}_{i_{n-m}})|0\rangle$, have a positive norm for every even $m = 0, 2, 4, \cdots (n \text{ or } n-1)$. A constraint that eliminates all negative norm states in the spacelike region is to demand a reflection symmetry from every state under the operation $\bar{a}_0 \to -\bar{a}_0$ and similarly for $a_0 \to -a_0$. This can be achieved through the operator[8] $T = \exp(i\pi\bar{a}_0 a_0)$ which gives $Ta_0T^{-1} = -a_0$ and $T\bar{a}_0T^{-1} = -\bar{a}_0$, and the boost generator changes sign, $TL^{0i}T^{-1} = -L^{0i}$. Therefore, a ghost-free spectrum is obtained by demanding the following constraint:

[7] The negative norm also implies that $\langle 0|x_0x_0|0\rangle$ and $\langle 0|p_0p_0|0\rangle$ are negative as seen from $\langle 0|x_0x_0|0\rangle = \frac{1}{2}\langle 0|(a_0 + \bar{a}_0)\times(a_0 + \bar{a}_0)|0\rangle = \frac{1}{2}\langle 0|a_0\bar{a}_0|0\rangle = -\frac{1}{2}$. If x_0 were Hermitian then x_0x_0 would have to be a positive operator with a positive expectation value. But in this Fock space x_0, p_0 are not Hermitian; equivalently, $\bar{a}_0$ is not the Hermitian conjugate of a_0, and this is why negative norms arise.

[8] A similar operator for the timelike region is $S = \exp(i\pi\bar{a}_i a_i)$.

$$T|\phi\rangle = (+1)|\phi\rangle, \Leftrightarrow \text{ghost-free, unitary subset of states, but not SO}(d,1)\text{ covariant.} \quad (3.17)$$

However, such states by themselves break the Lorentz symmetry since they cannot make up complete irreducible representations of SO(d, 1) for any nonzero n. In the absence of this constraint, in any *finite* dimensional representation of SO(d, 1), other than the singlet, there will always be states with an odd number of timelike oscillators. For example, at level 2 the irreducible tensor in Eq. (3.14) contains the negative norm states

$$\bar{a}_0 \bar{a}_i|0\rangle. \quad (3.18)$$

Therefore, to eliminate the negative norm states all finite representations of SO(d, 1) must be discarded by some consistent set of constraints. This leaves only the SO(d, 1) singlets[9] at each even level $J_0 = 2k$,

$$(\bar{a}\cdot\bar{a})^k|0\rangle, \qquad k = 0, 1, 2, 3, \cdots \text{ positive norm} \leftrightarrow \text{no ghosts.} \quad (3.19)$$

The eigenvalue of Q on these states is $\lambda = 2k + \frac{d+1}{2}$,

$$Q[(\bar{a}\cdot\bar{a})^k|0\rangle] = [(\bar{a}\cdot\bar{a})^k|0\rangle]\left(2k + \frac{d+1}{2}\right). \quad (3.20)$$

These states have positive norms since $\bar{a}\cdot\bar{a} = -\bar{a}_0\bar{a}_0 + \bar{a}_i\bar{a}_i$ ensures that every term in $(-\bar{a}_0\bar{a}_0 + \bar{a}_i\bar{a}_i)^k$ contains only an even number of $\bar{a}_0$'s. All the SO(d, 1) generators $L^{\mu\nu}$ in Eq. (2.9) annihilate these states since $[L^{\mu\nu}, \bar{a}\cdot\bar{a}] = 0$ gives

$$L^{\mu\nu}[(\bar{a}\cdot\bar{a})^k|0\rangle] = (\bar{a}\cdot\bar{a})^k L^{\mu\nu}|0\rangle = 0, \text{ Lorentz singlets.} \quad (3.21)$$

So, if the Fock space is restricted to the Lorentz invariant subset, then there are no ghosts.

The position space probability amplitude for these states is determined as

$$\psi_k^{(+)}(x) \sim \langle x|(\bar{a}\cdot\bar{a})^k|0\rangle \sim [\tfrac{1}{2}(x-\partial)\cdot(x-\partial)]^k e^{-(1/2)x^\mu x_\mu}, \text{ spacelike } x^\mu.$$

For example, for $k = 1$ it becomes

$$\psi_1^{(+)}(x) \sim (2x^2 - (d+1))e^{-(1/2)x^2}. \quad (3.22)$$

More generally, this gives the generalized Laguerre polynomial $L_k^{(d-1)/2}(x^2)$ with the argument x^2 multiplying the Gaussian $e^{-(1/2)x^\mu x_\mu}$.

$$\psi_k^{(+)}(x) = \alpha_k e^{-(1/2)x\cdot x}\sum_{m=0}^{k}(-1)^m \binom{k+\frac{d-1}{2}}{k-m}\frac{(x\cdot x)^m}{m!} = \alpha_k e^{-(1/2)x^2} L_k^{(d-1)/2}(x^2),$$

where α_k is an overall constant. It can be checked that this $\psi_k^{(+)}(x)$ is indeed a solution of the relativistic differential equation in $d+1$ dimensions, with the specified eigenvalue for every positive integer k,

$$\frac{1}{2}[-\partial^\mu\partial_\mu + x^\mu x_\mu]\psi_k^{(+)}(x) = \left(2k + \frac{d+1}{2}\right)\psi_k^{(+)}(x), \quad k = 0, 1, 2, \cdots. \quad (3.23)$$

Furthermore, these wave functions clearly have a positive norm $\int d^{d+1}x|\psi_k^{(+)}(x)|^2$ for all k. We see that according to the symmetry criteria, and unitarity, only these states are admissible as quantum states in the spacelike Fock space.[10]

Similarly, there is another set of SU(d, 1) singlet states $(a\cdot a)^k|0'\rangle$ in the timelike Fock space given by substituting a_μ instead of $\bar{a}_\mu$ and using $|0'\rangle$ instead of $|0\rangle$,

$$J_{\mu\nu}[(a\cdot a)^k|0'\rangle] = 0, \quad (3.24)$$

$$Q[(a\cdot a)^k|0'\rangle] = -\left(2k + \frac{d+1}{2}\right)[(a\cdot a)^k|0'\rangle], \quad (3.25)$$

$$\psi_k^{(-)}(x) = \tilde{\beta}_k[\tfrac{1}{2}(x+\partial)\cdot(x+\partial)]^k e^{(1/2)x^\mu x_\mu} \sim \langle x|(a\cdot a)^k|0'\rangle, \text{ timelike } x^\mu. \quad (3.26)$$

The $\psi_k^{(-)}(x)$ are related to the $\psi_k^{(+)}(x)$ by an analytic continuation of $x^2 \to -x^2$ from the spacelike to the timelike region, so they can also be expressed in terms of the

[9] This is in the case of a single oscillator, as in the current simplified problem. If there are additional degrees of freedom then one can find constraints that lead to more interesting ghost-free solutions. For example, in string theory, with an infinite number of oscillators, the Virasoro constraints eliminate ghosts while allowing nonsinglets of SO(d, 1).

[10] Recall the infinite integrals mentioned following Eq. (3.5). These resurface again in the norm above. For example, in the simplified case in Eq. (A13) the delta function normalization $\delta(m' - m)$ blows up for $m' = m$. This will be a common infinite factor for all Lorentz invariant wave functions. The infinity can be avoided by redefining the norm by simply not integrating over the extra boost parameters, since those parameters do not appear in the Lorentz invariant wave functions. If such a redefinition is not adapted, the infinities may be an argument to discard all of the Lorentz invariant states $\psi_k^{\pm}(x)$. By comparison, note that the unitary states based on the Lorentz noninvariant vacuum $|\tilde{0}\rangle$ discussed in Sec. IV have no infinities in their norms.

Laguerre polynomials,

$$\psi_k^{(-)}(x) = \gamma_k e^{(1/2)x\cdot x} \sum_{m=0}^{k} \binom{k+\frac{d-1}{2}}{k-m} \frac{(x\cdot x)^m}{m!} = \gamma_k e^{(1/2)x^2} L_k^{(d-1)/2}(-x^2).$$

However, it must be emphasized that, as computed[11] in Eq. (3.25), the $\psi_k^{(-)}(x)$ have the opposite signs for the eigenvalues of Q as compared to the $\psi_k^{(+)}(x)$.

All of these $\psi_k^{(\pm)}(x)$ are SO(d, 1) invariants, but what are their SU(d, 1) properties? The SU(d, 1) symmetry of Q and of the vacuum exhibited in Eqs. (2.8) and (3.9) requires that the spectrum be classified as complete SU(d, 1) multiplets. Which SU(n, 1) multiplets do these states correspond to? If we apply an infinitesimal SU(d, 1) transformation on the SO(d, 1) singlets, we find

$$J_{\mu\nu}[(\bar{a}\cdot\bar{a})^k|0\rangle] = 2k\left(\bar{a}_\mu\bar{a}_\nu - \frac{\eta_{\mu\nu}}{d+1}(\bar{a}\cdot\bar{a})\right)(\bar{a}\cdot\bar{a})^{k-1}|0\rangle. \tag{3.27}$$

We see on the right-hand side that, except for the case of $k=0$, we generate inadmissible negative norm states. This also shows that the states $(\bar{a}\cdot\bar{a})^k|0\rangle$ with $k\neq 0$ are not in a singlet of SU(d, 1) so that they must be part of nonunitary representations of SU(d, 1). Hence even though the states $(\bar{a}\cdot\bar{a})^k|0\rangle$ are unitary with respect to SO(d, 1), they are not consistent with an SU(d, 1) symmetry-consistent unitary spectrum, except for $k=0$.

What happened to the SU(d, 1) symmetry? It got broken by the boundary conditions of restricting the Fock space inadvertently to a purely spacelike region (see last paragraph of the Appendix for more insight). If one wishes to be consistent with SU(d, 1) covariance, and also restrict to the spacelike region, then only the vacuum state can be kept in the spectrum.

In a broken SU(d, 1) scenario all Lorentz singlet states $(\bar{a}\cdot\bar{a})^k|0\rangle$ are admissible. Similarly, in a broken SO(d, 1) scenario all states of the form (3.17) with an even number of a_0's can be included in the ghost-free Hilbert space. But, in an exact SU(d, 1) scenario only the vacuum state $|0\rangle$ can be included. A similar statement applies to the purely timelike sector where only the second vacuum state $|0'\rangle$ can be included.

We see that, in an SU(d, 1) symmetry-consistent spacelike or timelike Fock space, all states other than the vacuum states $|0\rangle$, $|0'\rangle$ must be thrown away by some consistent set of constraints since otherwise the theory cannot be both consistent with its SU(d, 1) symmetry and free of ghosts. One possibility is to choose the constraint to be $J_{\mu\nu}=0$ but this is too restrictive because, as we will see, it throws away the big sector of unitary states that we will discuss in the next section. Less restrictive is a constraint of the form

$$\left[\frac{1}{2}(p^2+x^2)-\lambda_0\right]=0, \quad \text{no ghosts only for } \lambda_0 = \pm\frac{d+1}{2}. \tag{3.28}$$

When $\lambda_0=\frac{d+1}{2}$ the constraint can be satisfied only by $|0\rangle$, and when $\lambda_0=-\frac{d+1}{2}$ it can be satisfied only by $|0'\rangle$. For other values of λ_0 that appeared in the spectrum above, such as $\lambda=\pm(n+\frac{d+1}{2})$, the constraint allows also negative norm states in nonunitary representations of SU(d, 1) with a Young tableau with n boxes as in Eq. (3.12), so only $n=0$ is admissible. We see that the only possible constraint of this form can only involve $\lambda_0=\pm\frac{d+1}{2}$, leading to only one of the possible states: either $|0\rangle$ or $|0'\rangle$.

A constraint of the type (3.28) with general λ_0 emerges as a natural outcome in a worldline theory as a consequence of a gauge symmetry on the worldline, as we will see in detail in Sec. VI. That kind of local symmetry is reasonable because it can be used to eliminate the ghosts that come from timelike directions, thus guaranteeing a unitary theory.

If λ_0 is in the range $-\frac{d+1}{2}<\lambda<\frac{d+1}{2}$, no state in the spacelike or timelike sectors can satisfy the constraint (3.28). So, with such a constraint all the states in the purely spacelike or purely timelike sectors, including $|0\rangle$ and $|0'\rangle$, would be excluded.

But in the next section we will find that this type of constraint is satisfied by many more states beyond those that appeared in the spacelike or timelike Fock spaces discussed in this section. There is a large sector of positive norm quantum states that cannot be built by starting from the conventional Lorentz invariant vacuum states $|0\rangle$, $|0'\rangle$, and those additional states are compatible with the SU(d, 1) symmetry, not as singlets, but as infinite dimensional unitary representations whose Casimir eigenvalues are determined by λ_0.

IV. UNITARY FOCK SPACE, NONSYMMETRIC VACUUM

We will now take a different approach to solving the eigenvalue problem $Q\psi_\lambda=\lambda\psi_\lambda$. Rather than starting with a Lorentz invariant vacuum state as is usually done, we will consider solving the differential equation

$$\tfrac{1}{2}(-\partial^\mu\partial_\mu + x^\mu x_\mu)\psi_\lambda(x) = \lambda\psi_\lambda(x), \tag{4.1}$$

without paying attention at first to its Lorentz covariance properties [2–4]. We will then clarify the symmetry properties of the solutions by appealing to the hidden symmetry SU(d, 1).

We can obtain solutions by separating the equation in the x^0, $\vec{x}$ variables,

[11] This follows from the form of $Q=a\cdot\bar{a}-\frac{d+1}{2}$ given in Eq. (2.4), and from the fact that $[a\cdot\bar{a},(a\cdot a)]=-2(a\cdot a)$.

$$\tfrac{1}{2}[(-\vec{\partial}^2+\vec{x}^2)-(-\partial_0^2+x_0^2)]\psi_\lambda(\vec{x},x_0)=\lambda\psi(\vec{x},x_0), \tag{4.2}$$

with a wave function of the form

$$\psi_\lambda(\vec{x},x_0)=A_{\lambda_a}(\vec{x})B_{\lambda_b}(x_0), \qquad \lambda=(\lambda_a-\lambda_b), \tag{4.3}$$

such that

$$\begin{aligned}\tfrac{1}{2}(-\vec{\partial}^2+\vec{x}^2)A_{\lambda_a}(\vec{x})&=\lambda_a A_{\lambda_a}(\vec{x}),\\ \tfrac{1}{2}(-\partial_0^2+x_0^2)B_{\lambda_b}(x_0)&=\lambda_b B_{\lambda_b}(x_0).\end{aligned} \tag{4.4}$$

In a unitary Hilbert space in which x^μ, p^μ are all Hermitian operators, both λ_a and λ_b must be positive since the operators $\frac{1}{2}(\vec{p}^2+\vec{x}^2)$ as well as $\frac{1}{2}(p_0^2+x_0^2)$ are positive. In fact, from the study of the Euclidean harmonic oscillator in d dimensions and one dimension, respectively, we already know all possible eigenvalues and eigenstates[12] for $(\lambda_a, A_{\lambda_a}(\vec{x}))$ and for $(\lambda_b, B_{\lambda_b}(x_0))$, where

$$\begin{aligned}\lambda_a&=n_a+\frac{d}{2}, \quad \text{with } n_a=0,1,2,3,\cdots,\\ \lambda_b&=n_b+\frac{1}{2}, \quad \text{with } n_b=0,1,2,3,\cdots.\end{aligned} \tag{4.5}$$

Furthermore, we know that the wave functions take the form shown in footnote 12, so we can write

$$\begin{aligned}A_{\lambda_a}(\vec{x})&=e^{-(1/2)\vec{x}^2}\\ &\quad\times(\text{polynomial of degree } n_a \text{ in the variables } x_i),\\ B_{\lambda_a}(x_0)&=e^{-(1/2)x_0^2}\\ &\quad\times(\text{polynomial of degree } n_b \text{ in the variable } x_0).\end{aligned} \tag{4.6}$$

In this basis there is infinite degeneracy for the same eigenvalue of $Q\to\lambda$, since eigenstates with different n_a, n_b can lead to the same eigenvalue $\lambda=\lambda_a-\lambda_b=n+\frac{d-1}{2}$. Thus with both m, n even integers or with both m, n odd integers, we can write

$$\begin{aligned}&n_a=\frac{m+n}{2}, \qquad n_b=\frac{m-n}{2}, \qquad \text{at fixed} \quad n,\\ &\qquad \text{all} \quad m\geq|n| \text{gives infinite degeneracy.}\end{aligned} \tag{4.7}$$

All solutions with the same eigenstate λ can be constructed from (infinite) linear combinations of the ones above, but they all must have the form

$$\begin{aligned}\psi_\lambda(x^\mu)&=e^{-(1/2)(\vec{x}^2+x_0^2)}\\ &\quad\times(\text{polynomials in the variables } x_\mu),\\ \lambda&=n+\frac{d-1}{2}, \quad \text{with } n=0,\pm1,\pm2,\pm3,\cdots.\end{aligned} \tag{4.8}$$

It is evident that these solutions have a positive norm since the integrals converge in all spacetime directions and they are positive,

$$\langle\psi_\lambda|\psi_\lambda\rangle=\int d^{d+1}x|\psi_\lambda(x^\mu)|^2=1. \tag{4.9}$$

We definitely have, at hand, a *unitary basis*, but what are the Lorentz symmetry properties of these solutions?

The striking contrast to the solutions in the previous section is that the exponent $(\vec{x}^2+x_0^2)$ is not Lorentz invariant, and hence these solutions and the solutions of the previous section are mutually exclusive. They each span different Hilbert spaces, and the spacetime geometric properties are very different. The Lorentz symmetry properties of the solutions (4.8) are not yet evident.

On the other hand, the operator Q is invariant under SU$(d,1)$ and its Lorentz subgroup SO$(d,1)$, so we must be able to organize the solutions at each value λ in terms of the representations of SU$(d,1)$ and any of its subgroups. These representations are automatically unitary since we have already ensured that x^μ, p^μ, and therefore the Lorentz generators $L_{\mu\nu}=x_\mu p_\nu-x_\nu p_\mu$, are Hermitian in this basis. Hence, we must expect infinite dimensional unitary representations of SO$(d,1)$ and of SU$(d,1)$ at each λ. In fact, this is in agreement with the infinite degeneracy at each λ noted above. We still need to determine what precisely these unitary representations are, and how to label states with quantum numbers within the representation.

We now address this issue. We will explain below that at each λ there is a *single irreducible unitary representation of* SU$(d,1)$ whose Casimir eigenvalues are completely determined by λ and d. We will give the detailed content of this representation in the group theoretical basis when SU$(d,1)$ is decomposed into SU$(d)\times$U(1). In this way we will be able to determine the SU(d), and the angular momentum SO$(d)\subset$SU(d), quantum numbers of each quantum state.

The starting point is a new vacuum state $|\tilde{0}\rangle$ which is different than the Lorentz invariant vacuum states $|0\rangle$, $|0'\rangle$ of the previous section. The new vacuum state is defined as the state for which the excitation numbers n_a, n_b are both

[12]The wave function of an arbitrary excited state of the d-dimensional Euclidean harmonic oscillator at eigenvalue $\lambda=n+d/2$, and SO(d) orbital angular momentum quantum number l, has the form

$$A^{nl}_{i_1i_2\cdots i_l}(\vec{x})=e^{-\vec{x}^2/2}|\vec{x}|^l L_n^{l-1+d/2}(\vec{x}^2)T_{i_1i_2\cdots i_l}(\hat{x}).$$

Here $T_{i_1i_2\cdots i_l}(\hat{x})$ is the symmetric *traceless* tensor of rank l constructed from the unit vector $\hat{x}_i\equiv x_i/|\vec{x}|$ (this is equivalent to the spherical harmonics in $d=3$ space dimensions). $L_\alpha^\beta(z)$ is the generalized Laguerre polynomial with argument $z=\vec{x}^2$, and indices $\alpha=n$ and $\beta=l-1+d/2$. The quantum numbers take the following values: The excitation level n is any positive integer $n=0,1,2,3,\cdots$, while at fixed n the allowed values of l are $l=n,(n-2),(n-4),\cdots,(1 \text{ or } 0)$.

zero. Hence, it is defined by the following equations:

$$\bar{a}_0|\tilde{0}\rangle = a_i|\tilde{0}\rangle = 0, \quad \text{so } \bar{a}_0 \text{ rather than } a_0 \text{ acts as an annihilator.} \tag{4.10}$$

The position space representation of this state justifies this definition since the oscillators a_μ, $\bar{a}_\mu$ defined in Eq. (2.1) have the following form in position space, and therefore they act on the state $|\tilde{0}\rangle$ as creators/annihilators as indicated,

$$\langle x|\tilde{0}\rangle \sim \exp\left(-\frac{x_0^2 + \vec{x}^2}{2}\right), \tag{4.11}$$

$$\text{annihilators: } \bar{a}_0 = \frac{1}{\sqrt{2}}\left(x_0 + \frac{\partial}{\partial x_0}\right), \qquad a_i = \frac{1}{\sqrt{2}}\left(x_i + \frac{\partial}{\partial x^i}\right), \tag{4.12}$$

$$\text{creators: } a_0 = \frac{1}{\sqrt{2}}\left(x_0 - \frac{\partial}{\partial x_0}\right), \qquad \bar{a}_i = \frac{1}{\sqrt{2}}\left(x_i - \frac{\partial}{\partial x^i}\right). \tag{4.13}$$

The extra sign in front of $\frac{\partial}{\partial x_0}$ in a_0, $\bar{a}_0$ is due to lowering the timelike index with the Minkowski metric $p_0 = -i\frac{\partial}{\partial x^0} = +i\frac{\partial}{\partial x_0}$. Then it is convenient to define the excitation number operators as[13]

$$\hat{N}_a = \bar{a}_i a_i, \qquad \hat{N}_b = a_0\bar{a}_0, \tag{4.14}$$

where the orders of $a_0\bar{a}_0$ are reversed compared to traditional notation. The eigenvalues (n_a, n_b) of these operators vanish on $|\tilde{0}\rangle$,

$$\hat{N}_a|\tilde{0}\rangle = 0, \qquad \hat{N}_b|\tilde{0}\rangle = 0. \tag{4.15}$$

It should be noted that the Lorentz covariant commutation rule in the timelike direction $[a_0, \bar{a}_0] = -1$ indicates that an excited state of the form $(a_0)^{n_b}|\tilde{0}\rangle$ is correctly identified as an eigenstate of $\hat{N}_b = a_0\bar{a}_0$ with eigenvalue n_b,

$$\begin{aligned}\hat{N}_b\{(a_0)^{n_b}|\tilde{0}\rangle\} &= [a_0\bar{a}_0, (a_0)^{n_b}]|\tilde{0}\rangle = a_0[\bar{a}_0, (a_0)^{n_b}]|\tilde{0}\rangle \\ &= n_b\{(a_0)^{n_b}|\tilde{0}\rangle\}.\end{aligned} \tag{4.16}$$

The general state of the form (4.5) with n_a, n_b excitations has the Fock space representation

$$|n_a, n_b\rangle = (\bar{a}_{i_1}\bar{a}_{i_2}\cdots\bar{a}_{i_{n_a}})(a_0)^{n_b}|\tilde{0}\rangle, \tag{4.17}$$

where each index i_k labels a vector of SO(d) as well as the fundamental representation of SU(d).

[13]It may be helpful to define a new notation for the timelike oscillators, $\bar{a}_0 \equiv b$ and $a_0 \equiv \bar{b}$, so that the operators that have the bar on top, namely, $\bar{b}$, $\bar{a}_i$, are creation operators. Indeed the b, $\bar{b}$ satisfy the usual commutation rules with the $+1$ on the right-hand side: $[b, \bar{b}] = [\bar{a}_0, a_0] = +1$ similar to $[a_i, \bar{a}_j] = \delta_{ij}$. Then $N_b = \bar{b}b = a_0\bar{a}_0$ is the familiar excitation number.

In term of these, the total level operator $J_0 = \bar{a}_i a_i - \bar{a}_0 a_0$ which we identified in Eq. (2.7) becomes $J_0 = \bar{a}_i a_i - a_0\bar{a}_0 - 1$, or

$$J_0 = \hat{N}_a - \hat{N}_b - 1. \tag{4.18}$$

Therefore the total level of the vacuum state $|\tilde{0}\rangle$ is

$$J_0|\tilde{0}\rangle = (\hat{N}_a - \hat{N}_b - 1)|\tilde{0}\rangle = (-1)|\tilde{0}\rangle. \tag{4.19}$$

We contrast this (-1) eigenvalue with the J_0 eigenvalues of the vacua $|0\rangle$, $|0'\rangle$ which were 0 and $(-d-1)$ respectively, as shown in Eqs. (3.2) and (3.7). We also see that the $Q \to \lambda$ eigenvalue of the vacuum is $\lambda = \frac{d-1}{2}$,

$$Q|\tilde{0}\rangle = \left(J_0 + \frac{d+1}{2}\right)|\tilde{0}\rangle = \frac{d-1}{2}|\tilde{0}\rangle. \tag{4.20}$$

Similarly, for the general state $|n_a, n_b\rangle$ we have

$$\begin{aligned}J_0|n_a, n_b\rangle &= (n_a - n_b - 1)|n_a, n_b\rangle, \\ Q|n_a, n_b\rangle &= \left(n_a - n_b + \frac{d-1}{2}\right)|n_a, n_b\rangle\end{aligned} \tag{4.21}$$

in agreement with Eq. (4.8).

It must now be emphasized that the vacuum state $|\tilde{0}\rangle$ is neither Lorentz nor SU$(d, 1)$ invariant since the Lorentz boost operators $L_{0i} = i(\bar{a}_0 a_i - \bar{a}_i a_0)$ or the SU$(d, 1)$ generators $J_{0i} = \bar{a}_0 a_i$ contain two creation operators. So the vacuum $|\tilde{0}\rangle$ cannot be invariant under the subset of SO$(d, 1)$ or SU$(d, 1)$ infinitesimal transformations generated by the operators that contain double creation,

$$L_{0i}|\tilde{0}\rangle \neq 0, \qquad J_{0i}|\tilde{0}\rangle \neq 0. \tag{4.22}$$

However, this structure of double creators or double annihilators is tailor-made for the oscillator approach to representation theory for noncompact groups or supergroups developed in [21–24]. Using those techniques we will classify the states as parts of infinite dimensional unitary representations as explained below.

First we note that the oscillators a_μ that are in the fundamental representation of SU$(d, 1)$ contain both creation and annihilation operators (see footnote 13 for $a_0 \equiv \bar{b}$),

$$a_\mu = \begin{pmatrix} a_0 \\ a_i \end{pmatrix} = \begin{pmatrix} \bar{b} \\ a_i \end{pmatrix}. \tag{4.23}$$

Therefore, a general SU$(d, 1)$ transformation mixes creation with annihilation operators. Similarly, the antifundamental representation given by $\bar{a}_\mu = (\bar{a}_0 \quad \bar{a}_j) = (b \quad \bar{a}_j)$ has the same property, and so does the adjoint representation of SU$(d, 1)$ which classifies the generators as the traceless product of the fundamental and antifundamental representations,

$$J_{\mu\nu} = \bar{a}_\mu a_\nu - \frac{\eta_{\mu\nu}}{d+1}\bar{a}\cdot a$$
$$= \begin{pmatrix} \bar{a}_0 a_0 + \frac{\bar{a}\cdot a}{d+1} & \bar{a}_0 a_j \\ \bar{a}_i a_0 & \bar{a}_i a_j - \frac{\delta_{ij}}{d+1}\bar{a}\cdot a \end{pmatrix} \equiv \begin{pmatrix} J_{00} & J_{0j} \\ J_{i0} & J_{ij} \end{pmatrix}. \tag{4.24}$$

All of these $J_{\mu\nu}$ are symmetries of the operator Q as we noted earlier. The double annihilation part of $J_{\mu\nu}$ is the upper right corner, $J_{0j} = \bar{a}_0 a_j = b a_j$, and the double creation part is the lower left corner, $J_{i0} = \bar{a}_i \bar{b}$, of this matrix. Note that the $d \times d$ matrix J_{ij} has a traceless part q_{ij}, while its trace is related to the remaining generator J_{00} as follows:

$$J_{ij} = q_{ij} + \delta_{ij}\frac{J_{00}}{d}, \qquad J_{00} = \frac{q_0}{d+1}. \tag{4.25}$$

The generators of the subgroup $\mathrm{SU}(d) \times \mathrm{U}_q(1) \times \mathrm{U}_{J_0}(1) \subset \mathrm{SU}(d,1) \times \mathrm{U}_{J_0}(1)$ are then

$$q_{ij} = \bar{a}_i a_j - \frac{\delta_{ij}}{d}\hat{N}_a, \qquad \hat{q}_0 = \hat{N}_a + d(\hat{N}_b + 1), \qquad J_0 = \hat{N}_a - \hat{N}_b - 1. \tag{4.26}$$

The general excited state in Eq. (4.17), $|n_a, n_b\rangle$, can now be identified by its $\mathrm{SU}(d) \times \mathrm{U}_q(1) \times \mathrm{U}_{J_0}(1)$ quantum numbers, by using a Young tableau as follows:

$$|n_a, n_b\rangle = (\bar{a}_{i_1}\bar{a}_{i_2}\cdots\bar{a}_{i_{n_a}})(a_0)^{n_b}|\tilde{0}\rangle, \tag{4.27}$$

$$= \Big[\ \overbrace{\boxed{i_1|i_2|i_3|\cdots|i_{n_a}}}^{n_a},\ q_0,\ n_0\Big], \tag{4.28}$$

$$q_0 = n_a + d(n_b + 1), \qquad n_0 = n_a - n_b - 1. \tag{4.29}$$

Note that the eigenvalue q_0 is a positive integer such that $q_0 - n_a = d(n_b + 1)$ is positive, and furthermore, it is a nonzero multiple of d. The Young tableau corresponds to a completely symmetric $\mathrm{SU}(d)$ tensor of rank n_a which fully describes the $\mathrm{SU}(d)$ content of the state $|n_a, n_b\rangle$. This tensor together with the labels $\hat{q}_0 \to q_0$ and $J_0 \to n_0$, or equivalently $Q \to \lambda = n_0 + \frac{d+1}{2} = n_a - n_b + \frac{d-1}{2}$, are a complete set of quantum numbers for any representation of $\mathrm{SU}(d,1) \times \mathrm{U}_{J_0}(1)$ that appears in this theory.

The orbital angular momentum l of any state corresponds to its $\mathrm{SO}(d)$ representation. The rank l of a traceless symmetric tensor determines the angular momentum. The completely symmetric tensor of $\mathrm{SU}(d)$ in Eq. (4.27) is decomposed into traceless symmetric tensors of rank l as follows:

$$\mathrm{SO}(d) \text{ tensors: } l = n_a, (n_a - 2), (n_a - 4), \cdots, (1 \text{ or } 0), \tag{4.30}$$

where each state with angular momentum l at levels n_b and $n_a = l + 2r$ is given by

$$(\bar{a}_{i_1}\bar{a}_{i_2}\cdots\bar{a}_{i_l} - \text{trace})(\bar{a}_j\bar{a}_j)^r(a_0)^{n_b}|\tilde{0}\rangle. \tag{4.31}$$

Hence the states $|n_a, n_b\rangle$ contain a direct sum of states of the type (4.31) with the angular momenta l specified in Eq. (4.30).

Now we are ready to identify all the states in the same infinite dimensional representation of $\mathrm{SU}(d,1) \times \mathrm{U}J_0(1)$. For a fixed J_0, or equivalently a fixed $Q = J_0 + \frac{d+1}{2} \to n + \frac{d-1}{2}$, we must include all the states $|n_a, n_b\rangle$ that satisfy $n_a - n_b = n$. These may be presented as a direct sum of states, meaning any linear combination of those states,

$$\lambda = \frac{d-1}{2} + n: \begin{cases} \sum_{k=0}^{\infty} \oplus(\bar{a}_{i_1}\bar{a}_{i_2}\cdots\bar{a}_{i_{k+n}})(a_0)^k|\tilde{0}\rangle & \text{if } n \geq 0 \\ \sum_{k=0}^{\infty} \oplus(\bar{a}_{i_1}\bar{a}_{i_2}\cdots\bar{a}_{i_k})(a_0)^{k+n}|\tilde{0}\rangle & \text{if } n \leq 0. \end{cases} \tag{4.32}$$

More explicitly, we give the example of $n = 0$ by writing it out,

$$\lambda = \frac{d-1}{2}: |\tilde{0}\rangle \oplus [\bar{a}_i a_0|\tilde{0}\rangle] \oplus [\bar{a}_i\bar{a}_j(a_0)^2|\tilde{0}\rangle] \oplus [\bar{a}_i\bar{a}_j\bar{a}_k(a_0)^3|\tilde{0}\rangle] \oplus \cdots, \tag{4.33}$$

and similarly for $n = 1, -1$,

$$\lambda = \frac{d-1}{2} + 1: \bar{a}_i|\tilde{0}\rangle \oplus [\bar{a}_i\bar{a}_j a_0|\tilde{0}\rangle] \oplus [\bar{a}_i\bar{a}_j\bar{a}_k(a_0)^2|\tilde{0}\rangle] \oplus \cdots, \tag{4.34}$$

$$\lambda = \frac{d-1}{2} - 1: a_0|\tilde{0}\rangle \oplus [\bar{a}_i(a_0)^2|\tilde{0}\rangle] \oplus [\bar{a}_i\bar{a}_j(a_0)^3|\tilde{0}\rangle] \oplus \cdots. \tag{4.35}$$

Evidently, each distinct value of λ completely determines the allowed $|n_a, n_b\rangle$ and the corresponding $\mathrm{SU}(d) \times \mathrm{U}(1) \times \mathrm{U}(1)$ tensors of each infinite dimensional tower. Note also that for each λ there is a single tower.

It is easy to show that each tower at fixed λ is an irreducible representation of $\mathrm{SU}(d,1)$. Under an $\mathrm{SU}(d,1)$ group transformation $g = \exp(i\omega^{\mu\nu}J_{\mu\nu})$, towers with differing eigenvalues $\lambda \neq \lambda'$ cannot mix with each other since $J_{\mu\nu}$ commutes with Q. Hence a single tower with fixed λ is irreducible under the $\mathrm{SU}(d,1)$ group transformation. Furthermore, all the states within each tower mix because the double creation operators $J_{i0} = \bar{a}_i a_0 = \bar{a}_i\bar{b}$ and the double annihilation operators $J_{0j} = \bar{a}_0 a_j = b a_j$ applied repeatedly mix all the states under the $\mathrm{SU}(d,1)$ group transformation $g = \exp(i\omega^{\mu\nu}J_{\mu\nu})$.

In fact, all states in a given tower are obtained by repeatedly applying the double creation $\mathrm{SU}(d,1)$ group generators $J_{i0} = \bar{a}_i a_0 = \bar{a}_i\bar{b}$ on the lowest state,

$$|\text{tower}\rangle_\lambda = \left\{\sum_{k=0}^{\infty} \oplus(J_{i_10}J_{i_20}\cdots J_{i_k0})\right\}|\text{lowest}\rangle_\lambda. \tag{4.36}$$

Therefore, only the lowest state in the tower is sufficient to uniquely label the SU$(d,1)$ content of the entire tower. These unique labels correspond to the SU(d) Young tableau and the $\mathrm{U}_q(1)$ charge $\hat{q}_0 = \hat{N}_a + d(\hat{N}_b + 1)$, identified in Eq. (4.26). These are the appropriate quantum numbers for the basis $\mathrm{SU}(d) \times \mathrm{U}_q(1) \subset \mathrm{SU}(d,1)$ at a fixed λ,

$$|\text{lowest}\rangle_\lambda = \left(\overbrace{\boxed{i_1}\boxed{i_2}\boxed{i_3}\boxed{\cdots}\boxed{i_{n_a}}}^{n_a},\ q_0(\lambda) \right), \qquad q_0 = n_a(d+1) - d\left(\lambda - \frac{d+1}{2}\right). \tag{4.37}$$

We can easily compute the Casimir operators for the irreducible unitary representations identified above. The quadratic Casimir operator of SU$(d,1)$ is given by

$$\begin{aligned} C_2 &= \tfrac{1}{2} J_{\mu\nu}\eta^{\nu\lambda}J_{\lambda\sigma}\eta^{\sigma\mu} \\ &= \tfrac{1}{2}(J_{ij}J_{ji} + (J_{00})^2 - J_{i0}J_{0i} - J_{0i}J_{i0}). \end{aligned} \tag{4.38}$$

After inserting the oscillator form of the $J_{\mu\nu}$ given in Eq. (4.24), and rearranging the oscillators, we find that C_2 is rewritten as a function of only the $\mathrm{U}_{J_0}(1)$ generator,

$$C_2(\mathrm{SU}(d,1)) = \frac{dJ_0}{2}\left(\frac{J_0}{d+1} + 1\right). \tag{4.39}$$

Hence C_2 is diagonal on any state $|n_a, n_b\rangle$,

$$C_2|n_a, n_b\rangle = \frac{d(n_a - n_b - 1)(n_a - n_b + d)}{2(d+1)}|n_a, n_b\rangle, \tag{4.40}$$

and it has the same eigenvalue for all the states in the same tower as follows:

$$C_2|\text{tower}\rangle_\lambda = \frac{1}{2}d\left(\lambda - \frac{d-1}{2}\right)\left[\frac{1}{d+1}\left(\lambda - \frac{d-1}{2}\right) + 1\right] \times |\text{tower}\rangle_\lambda. \tag{4.41}$$

Similarly, all SU$(d,1)$ Casimir operators $C_n \sim \mathrm{Tr}(J)^n$ are found to be only a function of J_0, so all Casimir eigenvalues are functions of only λ.

This result on the Casimirs C_n confirms that the full SU$(d,1)$ properties of each tower are completely determined by the eigenvalue of the operator $Q \to \lambda$. Indeed, as seen explicitly in Eqs. (4.32), (4.33), and (4.34), all the states in each tower, and their $\mathrm{SU}(d) \times \mathrm{U}_q(1) \times \mathrm{U}_{J_0}(1)$ quantum numbers, are predetermined by the fixed value of λ.

Now that we have determined that each $|\text{tower}\rangle_\lambda$ corresponds to a *single unitary representation of* SU$(d,1)$, what can we say about which unitary representations of the Lorentz group SO$(d,1)$ classify the quantum states? In particular, which eigenvalues of the SO$(d,1)$ Casimir $C_2 = \frac{1}{2}L_{\mu\nu}L^{\mu\nu}$ appear? This is predetermined by the group theoretical branching rules $\mathrm{SU}(d,1) \to \mathrm{SO}(d,1)$ as applied to each representation. From this it is evident that each $|\text{tower}\rangle_\lambda$ of the type (4.32) can be written as an infinite direct sum of unitary representations of SO$(d,1)$.

$$|\text{tower}\rangle_\lambda = \sum \oplus |\mathrm{SO}(d,1)\ \text{irreps}\rangle_\lambda. \tag{4.42}$$

It is not easy to see directly in the oscillator formalism precisely which eigenvalues of $C_2(\mathrm{SO}(d,1)) = \frac{1}{2}L_{\mu\nu}L^{\mu\nu}$ appear in this sum. This is because the natural Fock basis $|n_a, n_b\rangle$ we used above is labeled by the eigenvalues of the operators $\hat{N}_a, \hat{N}_b$ which are not simultaneous observables with this Casimir,

$$[\tfrac{1}{2}L_{\mu\nu}L^{\mu\nu}, \hat{N}_a] \neq 0, \qquad [\tfrac{1}{2}L_{\mu\nu}L^{\mu\nu}, \hat{N}_b] \neq 0, \tag{4.43}$$

although $\hat{N}_a - \hat{N}_b$ is. So, we do not expect that the operator $\frac{1}{2}L_{\mu\nu}L^{\mu\nu}$ would be diagonal in the basis $|n_a, n_b\rangle$. Indeed, if we construct the SO$(d,1)$ Casimir operator

$$C_2(\mathrm{SO}(d,1)) = \tfrac{1}{2}L_{\mu\nu}L^{\mu\nu} = -\tfrac{1}{2}(J_{\mu\nu} - J_{\nu\mu})(J^{\mu\nu} - J^{\nu\mu}), \tag{4.44}$$

$$= -(J_{\mu\nu}J^{\mu\nu}) + J_{\mu\nu}J^{\nu\mu} = -(J_{\mu\nu}J^{\mu\nu}) + 2C_2(\mathrm{SU}(d,1)), \tag{4.45}$$

we see that the last part $2C_2(\mathrm{SU}(d,1))$ is diagonal on each state of the $|\text{tower}\rangle_\lambda$, but the first part $J_{\mu\nu}J^{\mu\nu}$ contains double creation and double annihilation pieces, and hence it cannot be diagonal in the basis $|n_a, n_b\rangle$. However, it is guaranteed that this basis can be rearranged in the form (4.42), as a superposition of unitary representations of the Lorentz group SO$(d,1)$ with diagonal $\frac{1}{2}L_{\mu\nu}L^{\mu\nu}$, simply because at fixed n we have an irreducible representation of SU$(d,1)$. When each SO$(d,1)$ representation in (4.42) is branched down to the SO(d) subgroup of SO$(d,1)$, then the SO(d) quantum numbers must agree with those given in Eq. (4.30), namely, $l = n_a, (n_a - 2), \cdots, (0 \text{ or } 1)$. So, we can deduce that those SO$(d,1)$ representations that contain this set of angular momenta must enter in expressing $|n_a, n_b\rangle$ in terms of an SO$(d,1)$ basis.

V. UNITARITY CONSTRAINTS ON THE FULL THEORY

We have examined above three distinct Fock spaces based on the three vacua $|0\rangle$, $|0'\rangle$, $|\tilde{0}\rangle$. All the states in these Fock spaces are eigenstates of the same operator Q. After including the unitarity condition we found all the physically acceptable positive norm states.

In the quantum theory the existence of different sectors is the analog of different boundary conditions on the solutions of a given differential equation. We saw that the *unitary* sectors based on $|0\rangle$, $|0'\rangle$ are all Lorentz invariant and they are distinguished from each other by being in the spacelike or timelike regions of spacetime. On the other hand, none of the unitary states $|n_a, n_b\rangle$ or $|\text{tower}\rangle_\lambda$ based on the vacuum $|\tilde{0}\rangle$ are Lorentz singlets, since C_2 is nonvanishing on any of them. So, the different sectors may be distinguished on the basis of their Lorentz, SU$(d, 1)$, and geometric properties.

In the absence of boundary conditions that naturally emerge for a specific physical system, all sectors are *a priori* included. How can we ensure that negative norm ghosts will not appear? We saw that although the sector $|\tilde{0}\rangle$ is free of ghosts, the sectors $|0\rangle$, $|0'\rangle$ contained them. It is only by imposing unitarity by "hand," or equivalently by requiring Lorentz singlets (which may be viewed as a boundary condition), that we could distinguish the positive norm singlets in the sectors $|0\rangle$, $|0'\rangle$. However, requiring Lorentz invariants only as boundary conditions on the solutions of the entire theory also eliminates the $|\tilde{0}\rangle$ sector completely.

A more comprehensive set of constraints is of the form[14]

$$\tfrac{1}{2}(p^2 + x^2) - \lambda_0 = 0. \tag{5.1}$$

This allows states from all sectors $|0\rangle$, $|0'\rangle$, $|\tilde{0}\rangle$ as long as λ_0 is an eigenvalue of $Q = \frac{1}{2}(p^2 + x^2)$. The possible eigenvalues in each sector were

$$|0\rangle\colon\ \lambda = \frac{d+1}{2} + (\text{positive integer}), \tag{5.2}$$

$$|0'\rangle\colon\ \lambda = -\frac{d+1}{2} - (\text{positive integer}), \tag{5.3}$$

$$|\tilde{0}\rangle\colon\ \lambda = \frac{d-1}{2} + (\text{positive or negative integer}). \tag{5.4}$$

We argued in Eq. (3.28) that the only way to avoid ghosts in the spacelike or timelike sectors was to choose $\lambda_0 = \pm\frac{d+1}{2}$. Such values of λ_0 include only the vacua $|0\rangle$, $|0'\rangle$, respectively, in the spacelike and timelike sectors, and also the infinite number of states in the $|\text{tower}\rangle_{\lambda_0}$ in the $|\tilde{0}\rangle$ sector. Moreover, if we choose λ_0 in the range $\lambda_0 = 0, \pm 1, \pm 2, \cdots, \pm\frac{d-1}{2}$, we include only the corresponding towers $|\text{tower}\rangle_{\lambda_0}$ in the $|\tilde{0}\rangle$ sector, but no states at all from the spacelike or timelike sectors based on $|0\rangle$, $|0'\rangle$.

Hence, if the theory is restricted to the following range only,[15]

$$-\frac{d+1}{2} \le \frac{1}{2}(p^2 + x^2) \le \frac{d+1}{2}, \quad \text{unitary range}, \tag{5.5}$$

then it is guaranteed to be a unitary theory without any negative norm ghosts. If $\frac{1}{2}(p^2 + x^2)$ is taken outside of this range, then there will always be ghosts coming from the sectors $|0\rangle$, $|0'\rangle$. For definiteness, we list all the quantum states that satisfy this range:

$$\lambda = \frac{d+1}{2}\colon\ |0\rangle \oplus \bar{a}_i \sum_{m=0}^{\infty} \oplus (\bar{a}_{i_1}\bar{a}_{i_2}\cdots\bar{a}_{i_m})(a_0)^m|\tilde{0}\rangle, \tag{5.6}$$

$$\lambda = \frac{d-1}{2}\colon\ \sum_{m=0}^{\infty} \oplus (\bar{a}_{i_1}\bar{a}_{i_2}\cdots\bar{a}_{i_m})(a_0)^m|\tilde{0}\rangle, \tag{5.7}$$

$$\lambda = \frac{d-3}{2}\colon\ \sum_{m=0}^{\infty} \oplus (\bar{a}_{i_1}\bar{a}_{i_2}\cdots\bar{a}_{i_m})(a_0)^{m+1}|\tilde{0}\rangle, \tag{5.8}$$

$$\vdots$$

$$\lambda = -\frac{d-3}{2}\colon\ \sum_{m=0}^{\infty} \oplus (\bar{a}_{i_1}\bar{a}_{i_2}\cdots\bar{a}_{i_m})(a_0)^{m+d-2}|\tilde{0}\rangle, \tag{5.9}$$

$$\lambda = -\frac{d-1}{2}\colon\ \sum_{m=0}^{\infty} \oplus (\bar{a}_{i_1}\bar{a}_{i_2}\cdots\bar{a}_{i_m})(a_0)^{m+d-1}|\tilde{0}\rangle, \tag{5.10}$$

$$\lambda = -\frac{d+1}{2}\colon\ |0'\rangle \oplus \sum_{m=0}^{\infty} \oplus (\bar{a}_{i_1}\bar{a}_{i_2}\cdots\bar{a}_{i_m})(a_0)^{m+d}|\tilde{0}\rangle. \tag{5.11}$$

Note that the cases of $\lambda = \pm\frac{d+1}{2}$ include the Lorentz singlets $|0\rangle$, $|0'\rangle$, but these singlets do not appear for the other listed values of λ. Furthermore, note that only for $\lambda = +\frac{d+1}{2}$ is there an additional $\bar{a}_i$ outside of the sum in Eq. (5.6). This makes $|0\rangle$ evidently orthogonal to the tower at $\lambda = +\frac{d+1}{2}$. The lowest state in each case has SO(d) angular momentum zero, $l = 0$. Only the case of $\lambda = -\frac{d+1}{2}$ has *two* zero angular momentum states, one of which is an SU$(d, 1)$ singlet while the other is not.

VI. WORLDLINE THEORY WITH GAUGE SYMMETRY

A theory with constraints is obtained by constructing a gauge invariant action. Each constraint is the generator of a gauge symmetry. The gauge symmetry can be used to eliminate degrees of freedom and, in particular, it can remove ghosts and render the theory to be unitary.

A constraint of the type

$$\phi(x, p) = \tfrac{1}{2}(p^2 + x^2) - \lambda_0 = 0 \tag{6.1}$$

is obtained in the following worldline theory,

$$S(\lambda_0) = \int d\tau\left(\dot{x}^\mu p_\mu - e(\tau)\left[\frac{1}{2}(p^2 + x^2) - \lambda_0\right]\right) \tag{6.2}$$

[14] In a theory with more degrees of freedom more general constraints can also be considered; see footnote 9.

[15] We have not discussed at all the possibility of solutions in the spacelike and timelike sectors that are matched across the light cone $x^2 = 0$ as outlined following Eq. (A16). It is possible that those are already accounted for in the $|\tilde{0}\rangle$ sector, but we are not certain if there are additional ones. If those have λ's within the range in Eq. (5.5), they will be part of the constrained theory.

where $e(\tau)$ is the gauge field that plays the role of a Lagrange multiplier locally on the worldline at each instant τ. The gauge transformations with a local parameter $\Lambda(\tau)$ are

$$\delta_\Lambda x^\mu(\tau) = \Lambda(\tau)p^\mu(\tau), \qquad \delta_\Lambda p^\mu(\tau) = -\Lambda(\tau)x^\mu(\tau),$$
$$\delta_\Lambda e(\tau) = \frac{d}{d\tau}\Lambda(\tau). \tag{6.3}$$

The Lagrangian transforms to a total derivative

$$\delta_\Lambda S(\lambda_0) = \int d\tau \frac{d}{d\tau}\left(\frac{1}{2}(p^2 - x^2)\Lambda(\tau) - \lambda_0\Lambda(\tau)\right) \to 0, \tag{6.4}$$

which can be dropped in the variation of the action (note $p^2 - x^2$, not $p^2 + x^2$). Hence this action has a local gauge symmetry $\delta_\Lambda S = 0$.

One consequence of the gauge symmetry is to impose constraint (6.1) as the equation of motion for the gauge field,

$$0 = \frac{\partial S}{\partial e(\tau)} = \phi(x, p) = \frac{1}{2}(p^2 + x^2) - \lambda_0. \tag{6.5}$$

The generator of the gauge transformations is $\phi(x, p)$. Saying that $\phi(x, p)$ vanishes is equivalent to saying that the generator of gauge transformations vanishes, meaning that the sector that satisfies it must be gauge invariant.

There are various ways to quantize the theory defined by the $S(\lambda_0)$ above. The first approach is covariant quantization in which we work with the quantum rules $[x_\mu, p_\nu] = i\eta_{\mu\nu}$, in an enlarged Hilbert space that includes all the degrees freedom, including the redundant gauge degrees of freedom that are part of x^μ, p^μ. Then, among the quantum states in this enlarged space, we pick the gauge invariant physical states by demanding that they satisfy the vanishing of the gauge generator,

$$\text{gauge invariants}: [\tfrac{1}{2}(p^2 + x^2) - \lambda_0]|\text{physical}\rangle = 0. \tag{6.6}$$

If we follow this approach we see that the gauge invariant states $\langle x|\text{physical}\rangle = \psi_{\lambda_0}(x)$ are only those that satisfy the differential equation of the relativistic harmonic oscillator with a fixed eigenvalue λ_0,

$$(-\tfrac{1}{2}\partial^\mu\partial_\mu + \tfrac{1}{2}x^\mu x_\mu)\psi_{\lambda_0}(x) = \lambda_0\psi(x). \tag{6.7}$$

There is no mention of boundary conditions, and therefore we must include all sectors that solve this constraint. This is the problem we analyzed in the previous sections. From that analysis we conclude that, provided λ_0 is chosen as *one* of the quantized values in the range (5.5), then the resulting theory $S(\lambda_0)$ is guaranteed to be a ghost-free unitary theory.

Outside of this range we expect that ghosts will be present. Therefore $S(\lambda_0)$, with λ_0 fixed to any one of the values $\lambda_0 = -\frac{d+1}{2}, -\frac{d-1}{2}, \cdots, \frac{d-1}{2}, \frac{d+1}{2}$, leads to a physically acceptable unitary theory.

A second approach is noncovariant quantization in which we first choose a gauge and solve the constraint once and for all. The phase space that solves $\frac{1}{2}(p^2 + x^2) = \lambda_0$ is then automatically a parametrization of the gauge invariant sector. However, one must be careful that there may be more than one sector of phase space which can solve this equation at the classical level. If we choose a gauge in which the timelike degree of freedom is eliminated, then the remaining Euclidean degrees of freedom cannot introduce any negative norm ghosts. The quantum states are then automatically unitary, but one must check that nonlinear expressions are properly quantum ordered so as to ensure that the global symmetries of the theory have not been violated. Only if the global symmetries are treated properly—in the present case $SU(d, 1)$ and its subgroup $SO(d, 1)$—can one declare that the theory has been successfully quantized in the gauge fixed version. In what follows we show how this is done in the present theory defined by the action $S(\lambda_0)$, and how the results agree with the $SU(d, 1)$ properties of the covariant quantization approach.

VII. GAUGE FIXED QUANTIZATION

We can choose a gauge that reduces the theory to the purely spacelike harmonic oscillator. Let us first consider the following canonical transformation from $(x_0(\tau), p_0(\tau))$ to $(t(\tau), H(\tau))$ at the classical level (i.e. quantum ordering ignored),

$$x_0(\tau) = \sqrt{2H(\tau) + 2c}\,\sin(t(\tau)),$$
$$p_0(\tau) = \sqrt{2H(\tau) + 2c}\,\cos(t(\tau)), \tag{7.1}$$

where c is some constant that will be fixed later. This covers the entire (x_0, p_0) plane if $H(\tau) + c \geq 0$. The new set (t, H) is canonical as can be seen by computing the corresponding term in the Lagrangian,

$$-\dot{x}_0 p_0 = -\dot{t}H + \text{total derivatives}.$$

The total derivatives can be dropped since they are irrelevant in the action. The Lagrangian in Eq. (6.2) takes the form

$$L = -\dot{t}H + \dot{x}^i p_i - e[\tfrac{1}{2}(\vec{p}^2 + \vec{x}^2) - H - c - \lambda_0], \tag{7.2}$$

which shows that the constraint $\phi(x, p)$ that vanishes in the physical sector now has taken the form

$$\phi(x, p) = \tfrac{1}{2}(\vec{p}^2 + \vec{x}^2) - H - c - \lambda_0 = 0. \tag{7.3}$$

Next we choose the gauge

$$t(\tau) = \tau, \tag{7.4}$$

and solve the constraint $\phi(x, p) = 0$ to determine the canonical conjugate of the gauge fixed t, namely, $H(\tau)$,

$$H = \tfrac{1}{2}(\vec{p}^2 + \vec{x}^2) - c - \lambda_0. \tag{7.5}$$

The gauge fixed form of the action $S(\lambda_0)$ above describes precisely the spacelike nonrelativistic harmonic oscillator after using $\dot{t} = 1$,

$$S_{\text{fixed}}(\lambda_0) = \int d\tau \left(\partial_\tau \vec{x} \cdot \vec{p} - \left[\frac{1}{2}(\vec{p}^2 + \vec{x}^2) - c - \lambda_0 \right] \right). \tag{7.6}$$

It is possible to fix the constant c in terms of λ_0, but this is not necessary at this stage because $(-c - \lambda_0)$ seems to be an irrelevant constant that may be dropped. We will wait till we compute SU(d, 1) Casimir eigenvalues at the quantum level to learn the role of c and its relationship to λ_0 when we compare the results of covariant quantization to those of the gauge fixed quantization.

The quantum states of this nonrelativistic harmonic oscillator in d Euclidean dimensions are well known. They are constructed by defining creation-annihilation operators a_i, $\bar{a}_i$ in the usual way and applying them on a vacuum $|\hat{0}\rangle$ that diagonalizes this Hamiltonian,

$$a_i|\hat{0}\rangle = 0, \qquad \langle \vec{x}|\hat{0}\rangle \sim \exp(-\tfrac{1}{2}\vec{x}^2). \tag{7.7}$$

The general quantum state is a superposition of the following states that make up a tower,

$$|\text{tower}\rangle_{\lambda_0} = \sum_{n_a=0}^{\infty} \oplus |n_a\rangle = \sum_{n_a=0}^{\infty} \oplus (\bar{a}_{i_1}\bar{a}_{i_2}\cdots\bar{a}_{i_{n_a}})|\hat{0}\rangle \tag{7.8}$$

$$\sim \sum_{n_a=0}^{\infty} \oplus \overbrace{\boxed{i_1}\boxed{i_2}\boxed{i_3}\boxed{\cdots}\boxed{i_{n_a}}}^{n_a}\,. \tag{7.9}$$

We compare this spectrum to the towers listed in Eqs. (5.7), (5.8), (5.9), and (5.10). From this comparison we see that the gauge fixed version reproduces the spectrum of the covariant quantum theory for the action $S(\lambda_0)$ at fixed values of λ_0, provided λ_0 is fixed to one of the values

$$\lambda_0 = \frac{d+1}{2}, \frac{d-1}{2}, \frac{d-3}{2}, \ldots, -\frac{d-3}{2}, -\frac{d-1}{2}, \tag{7.10}$$

but not the value $\lambda_0 = -\frac{d+1}{2}$, since in that last case there is an additional state $|0'\rangle$ in Eq. (5.11) which does not show up in Eq. (7.8).

As we will see below, the gauge fixed version (7.8) reproduces the subtlety that for $\lambda_0 = \frac{d+1}{2}$ there is a Lorentz invariant state $|0\rangle$ as listed in Eq. (5.6). That is, at $\lambda_0 = \frac{d+1}{2}$ the tower in (7.8) is actually split into two representations of SU(d, 1). But the gauge fixed version could not reproduce the other Lorentz invariant state $|0'\rangle$ at $\lambda_0 = -\frac{d+1}{2}$ in Eqs. (5.11). Similarly, the unitary sector $|n_a, n_b\rangle$ for all $n_b < n_a$ that appears in covariant quantization is entirely missed in the fixed gauge. By contrast, all the states $|n_a, n_b\rangle$ for $n_b \geq n_a$ are recovered in the gauge fixed version (7.8), even those beyond the list in (7.10).

The discrepancy between covariant quantization and gauge fixed quantization is attributable to an assumption made inadvertently when making the gauge choice. Namely, the canonical transformation (7.1) is valid only when $\sqrt{H + c}$ is real. After using Eq. (7.5), we see that the reality condition requires

$$0 \leq H + c = \tfrac{1}{2}(\vec{p}^2 + \vec{x}^2) - \lambda_0. \tag{7.11}$$

Hence, in the present gauge we have evidently limited ourselves to the quantum states that satisfy $\lambda_0 \leq \frac{1}{2}(\vec{p}^2 + \vec{x}^2)$. This explains why the gauge fixed version of the theory defined by $S_{\text{fixed}}(\lambda_0)$ can be related to the covariant theory $S(\lambda_0)$ only under this condition, and does not necessarily cover all the gauge invariant sectors of the theory defined by $S(\lambda_0)$ (for a similar example in string theory, see footnote 4). This is consistent with the fact that the gauge fixed version could not reproduce all the unitary sectors with $\lambda \geq \frac{d+1}{2}$. In the guaranteed unitary range $-\frac{d+1}{2} \leq \lambda_0 \leq \frac{d+1}{2}$, all the states except the Lorentz invariant state $|0'\rangle$ at $\lambda_0 = -\frac{d+1}{2}$ are recovered. The missing state $|0'\rangle$ should be recoverable by exploring other gauge choices, but we will not pursue this more careful gauge fixing in this paper.

VIII. SU(d, 1) AND SO(d, 1) SYMMETRY IN GAUGE FIXED THEORY

We now discuss the unitary representations of the global symmetry SU(d, 1) and SO(d, 1) in the gauge fixed version, paying attention to quantum ordering of operators. In particular, we want to show that the gauge fixed version agrees with the covariant version when we compute eigenvalues of the Casimir operator $C_2(\text{SU}(d, 1))$.

In the *gauge fixed* version, the timelike oscillator $\bar{a}_0 = \frac{1}{\sqrt{2}}(x_0 - ip_0)$ is computed in terms of the spacelike oscillators a_i, $\bar{a}_i$ after inserting the canonical transformation (7.1) and the gauge $t(\tau) = \tau$. At the classical level this takes the form

$$\bar{a}_0(\tau) = ie^{i\tau}\sqrt{H + c} = ie^{i\tau}\sqrt{\bar{a}_i(\tau)a_i(\tau) + c}. \tag{8.1}$$

At the quantum level one must address operator ordering ambiguities. Since c has not been fixed so far, we absorb all such ambiguities into c and define the quantum version of a_0 with the orders of $\bar{a}_i a_i$ as given above. We can now compute the generator of $\text{U}_{J_0}(1)$ at the quantum level in the gauge fixed version and find the constant value $J_0 = -c$,

$$J_0 = \bar{a} \cdot a = -\bar{a}_0 a_0 + \bar{a}_i a_i = -c. \tag{8.2}$$

Recall that in the covariant version $Q = J_0 + \frac{d+1}{2}$, so when Q, J_0 are fixed to $Q = \lambda_0$ and $J_0 = -c$, we determine c as

$$c = \frac{d+1}{2} - \lambda_0. \tag{8.3}$$

We see that c is positive only if $\lambda_0 \leq \frac{d+1}{2}$. This is necessary since the square root $\sqrt{\bar{a}_i a_i + c}$ was defined for all eigenvalues of the operator $\bar{a}_i a_i$ only if c is positive $c \geq 0$.

The generators of SU$(d, 1)$ can now be computed in the gauge fixed version by inserting the gauge fixed form of a_0 and $\bar{a}_0$ into the expression of $J_{\mu\nu}$ given in Eq. (2.8),

$$J_{00} = \hat{N}_a + \frac{cd}{d+1}, \qquad J_{ij} = \bar{a}_i a_j + \frac{c}{d+1}\delta_{ij}, \quad (8.4)$$

$$J_{0i} = ie^{i\tau}(\hat{N}_a + c)^{1/2} a_i, \qquad J_{i0} = -ie^{-i\tau}\bar{a}_i(\hat{N}_a + c)^{1/2}, \quad (8.5)$$

where $\hat{N}_a = \bar{a}_i a_i$ is the number operator. Note that $J_{00} = \delta^{ij} J_{ij}$ is not independent as expected from $\eta^{\mu\nu} J_{\mu\nu} = 0$. The nonlinear generators J_{0i}, J_{j0} must satisfy the following commutation rules according to the SU$(d, 1)$ algebra [the commutator is evaluated with all $\bar{a}_i(\tau)$ and $a_j(\tau)$ at equal τ],

$$[J_{0i}, J_{j0}] = \delta_{ij} J_{00} - \eta_{00} J_{ji}. \quad (8.6)$$

We can check explicitly that this commutator is indeed satisfied for any constant c. The critical point in the calculation is to use the property $a_i \hat{N}_a = (\hat{N}_a + 1) a_i$, leading to $a_i f(\hat{N}_a) = f(\hat{N}_a + 1) a_i$ for any function of $\hat{N}_a$, and similarly for the Hermitian conjugate, $\bar{a}_i f(\hat{N}_a + 1) = f(\hat{N}_a)\bar{a}_i$. Then we can compute the commutator $[J_{0i}, J_{j0}]$ as follows:

$$[J_{0i}, J_{j0}] = ((\hat{N}_a + c)^{1/2} a_i)(\bar{a}_j(\hat{N}_a + c)^{1/2}) - (\bar{a}_j(\hat{N}_a + c)^{1/2})((\hat{N}_a + c)^{1/2} a_i) \quad (8.7)$$

$$\begin{aligned} &= a_i(\hat{N}_a - 1 + c)^{1/2}(\hat{N}_a - 1 + c)^{1/2}\bar{a}_j - \bar{a}_j(\hat{N}_a + c) a_i \\ &= a_i(\hat{N}_a - 1 + c)\bar{a}_j - \bar{a}_j(\hat{N}_a + c) a_i \\ &= (\hat{N}_a + c) a_i \bar{a}_j - (\hat{N}_a - 1 + c)\bar{a}_j a_i \\ &= \delta_{ij}(\hat{N}_a + c) + \bar{a}_j a_i \\ &= \delta_{ij}\left(J_{00} + \frac{c}{d+1}\right) + \left(J_{ji} - \frac{c}{d+1}\delta_{ij}\right) = \delta_{ij} J_{00} + J_{ji}, \end{aligned} \quad (8.8)$$

in agreement with SU$(d, 1)$ as in Eq. (8.6). It is easy to check that the rest of the commutation rules for SU$(d, 1)$ are satisfied,

$$[J_{\mu\nu}, J_{\lambda\sigma}] = \eta_{\nu\lambda} J_{\mu\sigma} - \eta_{\mu\sigma} J_{\lambda\nu}. \quad (8.9)$$

Hence we have constructed correctly the SU$(d, 1)$ algebra. This implies that we have successfully quantized the theory $S(\lambda_0)$ in the gauge fixed version.

We can now learn the properties of the SU$(d, 1)$ representation by analyzing the transformation properties of the states. The Young tableaux in Eq. (7.8) already inform us about their transformation properties under the subgroup SU(d). To learn the transformation rules under the coset generators J_{i0}, J_{0i}, we apply these nonlinear forms on the states. We see that J_{i0}, J_{0i} create or annihilate excitations,

$$J_{i0}|n_a\rangle = \bar{a}_i(\hat{N}_a + c)^{1/2}|n_a\rangle \sim |n_a + 1\rangle\sqrt{n_a + c}, \quad (8.10)$$

$$J_{0i}|n_a\rangle = (\hat{N}_a + c)^{1/2} a_i|n_a\rangle \sim |n_a - 1\rangle\sqrt{n_a - 1 + c}, \quad (8.11)$$

so they mix all SU(d) Young tableaux with each other for all values of n_a. So SU$(d, 1)$ transformations connect all levels n_a to each other, thus showing that the SU$(d, 1)$ representation is infinite dimensional as long as $c > 0$.

When $c = 0$, we see that all operators $J_{\mu\nu}$ in Eqs. (8.4) and (8.5) annihilate the vacuum state,

$$[J_{\mu\nu}]_{c=0}|\hat{0}\rangle = 0. \quad (8.12)$$

Therefore, for $c = 0$ the vacuum state is SU$(d, 1)$ and Lorentz invariant, and we must identify it with the Lorentz invariant state $|0\rangle$ listed in Eq. (5.6),

$$[|\hat{0}\rangle \text{ in gauge fixed version with } c = 0] \leftrightarrow [|0\rangle \text{ in covariant version}]. \quad (8.13)$$

Furthermore, when $c = 0$, all the states starting with $n_a = 1$ form an irreducible infinite dimensional representation, so they can be written just like Eq. (5.6),

$$c = 0, \quad \text{or} \quad \lambda_0 = \frac{d+1}{2}: |\hat{0}\rangle \oplus \bar{a}_i \sum_{m=0}^{\infty} \oplus(\bar{a}_{i_1}\bar{a}_{i_2}\cdots\bar{a}_{i_m})|\hat{0}\rangle. \quad (8.14)$$

Hence at $\lambda_0 = \frac{d+1}{2}$ we have identified an SU$(d, 1)$ or SO$(d, 1)$ singlet, together with an infinite dimensional unitary representation of SU$(d, 1)$ whose lowest state has angular momentum $l = 1$. For all the other cases of $-\frac{d-1}{2} \leq \lambda_0 \leq \frac{d-1}{2}$, the lowest state has angular momentum zero $l = 0$, but it is not a Lorentz or SU$(d, 1)$ singlet. At $\lambda_0 = -\frac{d+1}{2}$, according to covariant quantization in Eq. (5.11), we should expect a Lorentz singlet together with another zero angular momentum state as part of an infinite dimensional representation, but the Lorentz invariant state $|0'\rangle$ is missed in the gauge fixed version.

It is interesting to compute the Casimir operator $C_2(\text{SU}(d, 1))$ in the gauge fixed version. To do so we insert the gauge fixed $J_{\mu\nu}$ of Eqs. (8.4) and (8.5) into Eq. (4.38) and manipulate orders of operators as in Eq. (8.7). After rearranging operators we find that C_2 is just a constant determined by c as follows,

$$C_2 = \tfrac{1}{2}(J_{ij}J_{ji} + (J_{00})^2 - J_{i0}J_{0i} - J_{0i}J_{i0}) \quad (8.15)$$

$$= \left\{\frac{1}{2}\left(\bar{a}_i a_j + \frac{c}{d+1}\delta_{ij}\right)\left(\bar{a}_j a_i + \frac{c}{d+1}\delta_{ij}\right) + \frac{1}{2}\left(\hat{N}_a + \frac{cd}{d+1}\right)^2 - \frac{1}{2}\bar{a}_i(\hat{N}_a + c)^{1/2}(\hat{N}_a + c)^{1/2}a_i - \frac{1}{2}(\hat{N}_a + c)^{1/2}a_i\bar{a}_i(\hat{N}_a + c)^{1/2}\right\} \tag{8.16}$$

$$= \frac{(-c)d}{2}\left(1 + \frac{(-c)}{d+1}\right). \tag{8.17}$$

This is the same result as the covariant approach (4.39) with J_0 fixed in the gauge fixed version to $J_0 = -c$, consistent with Eq. (8.2).

It may be interesting to discuss also the SO$(d, 1)$ content of each tower. The Hermitian Lorentz generators are

$$\mathrm{SO}(d, 1)\!: L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu = -i(J_{\mu\nu} - J_{\nu\mu}) \tag{8.18}$$

which take the following explicit forms in terms of oscillators,

$$\text{rotation}: L_{ij} = -i(\bar{a}_i a_j - \bar{a}_j a_i), \tag{8.19}$$

$$\text{boost}: L_{0i} = -i((\hat{N}_a + c)^{1/2}a_i - \bar{a}_i(\hat{N}_a + c)^{1/2}). \tag{8.20}$$

It is emphasized that these operators satisfy the SO$(d, 1)$ Lie algebra

$$[L_{\mu\nu}, L_{\lambda\sigma}] = -i(\eta_{\nu\lambda}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\lambda} - \eta_{\mu\lambda}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\lambda}), \tag{8.21}$$

and, in particular, the commutator of two boosts gives SO(d) rotations at the quantum level,

$$[L_{0i}, L_{0j}] = -iL_{ij} = i\eta_{00}L_{ij}. \tag{8.22}$$

This can be checked explicitly for our nonlinear L_{0i} by using the same methods as Eq. (8.7).

Since the $L_{\mu\nu}$ are Hermitian they act in infinite dimensional unitary representations of the Lorentz group. This implies that each tower of SU$(d, 1)$ at fixed λ_0 splits into an infinite number of irreducible SO$(d, 1)$ towers; the precise content of which SO$(d, 1)$ representations appear depends on the constant c.

In this section we exhibited new interesting *nonlinear oscillator representations* of SU$(d, 1)$ which should have generalizations to other noncompact groups. This type of oscillator representation was not previously considered in [21–24]. The new nonlinear expressions for the generators given in Eqs. (8.4) and (8.5) were obtained by starting from previous oscillator methods and then replacing some of those oscillators by nonlinear expressions in terms of the other oscillators. The same method was used to find new interesting SU(2, 3) symmetry properties based on twistors [25] that describe spinning particles in various 1T-physics systems and explain dualities among them. This nonlinear approach to constructing generators and representations of noncompact groups could be of interest in many applications in both physics and mathematics.

IX. NONRELATIVISTIC OSCILLATOR AS A RELATIVISTIC SYSTEM

While the focus in this paper was the relativistic harmonic oscillator, we were led to the nonrelativistic case as a consequence of a gauge choice. Looking at this process in reverse, this shows that the nonrelativistic oscillator provides a nonlinear realization of a relativistic system. So the nonrelativistic oscillator must have some hidden relativistic symmetry of its own. This is possibly a surprising proposition, but it is true as explained below.

In d Euclidean dimensions the nonrelativistic oscillator has evident SO(d) symmetry and also a well-known SU(d) hidden symmetry that leaves the Hamiltonian invariant. However, the discussion above suggests that we should seek an even larger hidden symmetry SU$(d, 1)$ that includes Lorentz symmetry SO$(d, 1)$.

We recall that the generator of the gauge symmetry of the relativistic action $S(\lambda_0)$ is $\phi(x, p) = Q(x, p) - \lambda_0$ as in Eq. (6.5). By using Poisson brackets $\delta_\Lambda A(x, p) = \Lambda\{A(x, p), \phi(x, p)\}$ the gauge transformation rules for all observables $A(x, p)$ are obtained. In particular, note that the gauge transformations of $\delta_\Lambda x^\mu$ and $\delta_\Lambda p^\mu$ in Eq. (6.3) follow in this way. Since the SU$(d, 1)$ generators $J_{\mu\nu}$ commute with the SU$(d, 1)$ invariant Q as shown in Eq. (2.6), it must have vanishing Poisson brackets with the gauge generator $\phi(x, p)$ when the $J_{\mu\nu}$ of Eq. (2.8) is written out in terms of phase space,

$$\{J_{\mu\nu}(x, p), \phi(x, p)\} = 0 \leftrightarrow \delta_\Lambda J_{\mu\nu} = 0. \tag{9.1}$$

Therefore, the $J_{\mu\nu}$ are gauge invariant physical observables.

Since both $S(\lambda_0)$ and its global symmetry generators $J_{\mu\nu}$ are gauge invariants, it must be that their gauge fixed versions $S_{\mathrm{fixed}}(\lambda_0)$, $J^{\mathrm{fixed}}_{\mu\nu}$ also maintain the same SU$(d, 1)$ global symmetry properties. That is, when written out in terms of the remaining Euclidean degrees of freedom $\vec{x}$, $\vec{p}$, we must find that $J^{\mathrm{fixed}}_{\mu\nu}$ is the generator of SU$(d, 1)$ symmetry of the nonrelativistic harmonic oscillator action,

$$S_{\mathrm{nonrel}} = \int d\tau\left(\partial_\tau\vec{x}\cdot\vec{p} - \frac{1}{2}(\vec{p}^2 + \vec{x}^2)\right). \tag{9.2}$$

The explicit form of $J^{\mathrm{fixed}}_{\mu\nu}(\vec{x}, \vec{p}, \tau)$ is obtained directly from Eqs. (8.4) and (8.5). If these $J^{\mathrm{fixed}}_{\mu\nu}$ are symmetry generators, they must be conserved when the equations of motion of the nonrelativistic oscillator are used,

$$\frac{d}{d\tau}J^{\rm fixed}_{\mu\nu}(\vec{x}(\tau),\vec{p}(\tau),\tau)=0,\quad \text{for } \dot{x}_i=p_i,\quad \dot{p}_i=-x_i \quad\text{or}\quad \dot{a}_i=-ia_i,\qquad \dot{\bar{a}}_i=+i\bar{a}_i. \tag{9.3}$$

Note that $J^{\rm fixed}_{0i}$, $J^{\rm fixed}_{i0}$ depend explicitly on τ in addition to the implicit dependence on τ that comes through $\vec{x}(\tau)$, $\vec{p}(\tau)$. Indeed, this extra dependence on τ is essential to show that the $J^{\rm fixed}_{0i}$, $J^{\rm fixed}_{i0}$ are conserved.

Since we have already shown that these $J^{\rm fixed}_{\mu\nu}(\vec{x},\vec{p},\tau)$ close to form the $SU(d,1)$ Lie algebra at the quantum level at any τ, they also satisfy the same property at the classical level under Poisson brackets. Using these generators we can define infinitesimal $SU(d,1)$ transformation laws by using Poisson brackets at any fixed τ, namely, $\delta_\omega\vec{x}=\frac{1}{2}\omega^{\mu\nu}\{\vec{x},J^{\rm fixed}_{\mu\nu}(\tau)\}$ and $\delta_\omega\vec{p}=\frac{1}{2}\omega^{\mu\nu}\{\vec{p},J^{\rm fixed}_{\mu\nu}(\tau)\}$. More explicitly, the transformation laws at any τ are

$$\delta_\omega\vec{x}(\tau)=\frac{1}{2}\omega^{\mu\nu}\frac{\partial J^{\rm fixed}_{\mu\nu}(x,p,\tau)}{\partial\vec{p}},\qquad \delta_\omega\vec{p}(\tau)=-\frac{1}{2}\omega^{\mu\nu}\frac{\partial J^{\rm fixed}_{\mu\nu}(x,p,\tau)}{\partial\vec{x}}. \tag{9.4}$$

The transformations under the $SU(d)\times U(1)$ subgroup are familiar hidden symmetry transformations of the nonrelativistic harmonic oscillator. However, the transformations generated by the classical

$$\frac{1}{\sqrt{2}}(J^{\rm fixed}_{i0}+J^{\rm fixed}_{0i})=\sqrt{\frac{1}{2}(\vec{p}^2+\vec{x}^2)+c}\,(x_i\cos\tau-p_i\sin\tau), \tag{9.5}$$

$$\frac{1}{\sqrt{2}i}(J^{\rm fixed}_{i0}-J^{\rm fixed}_{0i})=\sqrt{\frac{1}{2}(\vec{p}^2+\vec{x}^2)+c}\,(x_i\sin\tau+p_i\cos\tau) \tag{9.6}$$

are new nonlinear symmetry transformations that were not noted before. It can now be verified that the nonrelativistic harmonic oscillator action above is indeed invariant under all of the $SU(d,1)$ transformations. It can be verified that the new transformations give $\delta_\omega S_{\rm nonrel}=\int d\tau\frac{d}{d\tau}(\text{stuff})\rightarrow 0$, where the total derivative can be dropped in the transformation of the action, thus verifying the expected $SU(d,1)$ global symmetry. Again, the explicit τ dependence generated by the expressions in (9.5) and (9.6) is crucial for this result. A consequence of this symmetry via Noether's theorem is that the $J^{\rm fixed}_{i0}\pm J^{\rm fixed}_{0i}$ given in Eqs. (9.5) and (9.6) are conserved, as already claimed above in Eq. (9.3).

This hidden symmetry of the nonrelativistic harmonic oscillator was not known before. These transformations leave the *action*, not the Hamiltonian, invariant. As a consequence of the symmetry, all the states of the nonrelativistic harmonic oscillator taken together at all energy levels must fit into irreducible unitary representations of $SU(d,1)_c$ and its Lorentz subgroup $SO(d,1)$.

Note that the parameter c is used to construct the nonlinear generators $J_{0i}(c)$ and $J_{i0}(c)$ in Eq. (8.5), so the $SU(d,1)_c$ transformations are different for every c. This means different representations of $SU(d,1)$ can be realized on the same Fock space consisting of all the states in Eq. (7.8). They will transform differently as a representation basis depending on the choice of the parameter c. When $c\neq 0$, all the states form a single irreducible representation of $SU(d,1)$ with Casimir eigenvalue $C_2(SU(d,1)_c)=-\frac{cd}{2}(1-\frac{c}{d+1})$. The lowest state of this infinite tower has zero $SO(d)$ orbital angular momentum $l=0$ since it is the vacuum state $|\hat{0}\rangle$. The branching of the $SU(d,1)_c$ representation into representations of the Lorentz group $SO(d,1)$ depend on c, so we expect to describe different relativistic content by using the same nonrelativistic harmonic oscillator degrees of freedom.

The $c=0$ case is special, because then the vacuum state $|\hat{0}\rangle$ of the nonrelativistic harmonic oscillator is a singlet of $SU(d,1)_0$ and of $SO(d,1)$, so it is a Lorentz invariant as explained in Eq. (8.12). The remaining states at all energy levels given in Eq. (8.14) make up a single irreducible unitary representation of $SU(d,1)_0$ with Casimir 0. The lowest energy state of this $c=0$ infinite tower is $\bar{a}_i|\hat{0}\rangle$ which has $SO(d)$ angular momentum $l=1$. This is clearly different $SO(d,1)$ content compared to the $c\neq 0$ case for which the lowest state of the irreducible tower had angular momentum $l=0$.

This different $SU(d,1)$ or $SO(d,1)$ rearrangement of the same states for different values of c seems surprising when viewed from the perspective of the nonrelativistic oscillator. However, when compared to the corresponding $|\text{tower}\rangle_{\lambda_0}$ in Eqs. (5.6), (5.7), (5.8), (5.9), (5.10), and (5.11) in covariant quantization, the hidden information in c about the $SO(d,1)$ properties becomes evident. The comparison shows that c corresponds to the various powers of a_0 applied on the vacuum $|\tilde{0}\rangle$ to get the lowest state $(a_0)^{c-1}|\tilde{0}\rangle$ in different towers (for $c\geq 1$). The additional information gained from the Lorentz properties of a_0 in covariant quantization explains why the same nonrelativistic Fock space (7.8) relates to different relativistic $SO(d,1)$ or $SU(d,1)$ content as the value of c changes.

Note that if the starting point were the nonrelativistic oscillator, then there would be no conditions on the value of c for constructing the $SU(d,1)_c$ generators in Eq. (8.5). Of course, when c is quantized as indicated before, $c=0,1,2,\cdots,(d+1)$, the nonlinear structures J_{0i}, J_{i0} correspond to just a gauge fixed sector of the relativistic oscillator with a unitarity constraint. Other values of c on the real line seem to describe relativistic systems beyond the oscillator.

Note that c is a Lorentz invariant; therefore, in physical applications it could be related to certain relativistically invariant observables, such as the mass of a bound state.

Such relativistic properties of the *nonrelativistic* oscillator may lead to further insights.

X. MORE REVISITS?

We have shed new light on the symmetries and the quantum sectors of the relativistic harmonic oscillator. Since much of this was not noted before, it may lead to additional new observations in old or new applications of this commonly used dynamical system.

Of course, for each physical system there may be various sets of new constraints not discussed in this paper that would influence the allowed physical states as noted in footnote 9. In particular, the richer structure of the many oscillators in string theory leads to the Virasoro constraints for removing ghosts rather than those in Eq. (3.28). Whatever the ghost killing constraints may be, it would be of interest to reanalyze the relevant systems to find out whether the additional Fock spaces discussed in this paper lead to additional quantum states that may reveal new physical properties.

This paper is not focused on string theory, but rather on the single relativistic harmonic oscillator. Our initial aim was to clarify some facts about the symmetry aspects of the relativistic oscillator that appeared confusing. The clarification provided here leads us to ask, what happens in string theory? In what follows we provide some brief *preliminary remarks* on this topic.

Past work in string theory has been carried out by relying on the Fock space built exclusively from the covariant *spacelike* vacuum $|0\rangle$ of Sec. III, while being unaware of the other Fock space sectors with more general geometry discussed in Secs. III and IV. As is well known from previous studies of string theory, although not made previously explicit, the spacelike sector is completely consistent. Its results have been reproduced in many approaches, leading to the remarkable properties of string scattering amplitudes.

The question that arises now is not whether anything was wrong with that treatment of strings, but whether there might be more physical phenomena in string theory beyond the usual self-consistent spacelike sector, and hence beyond the Veneziano amplitudes. The question is natural since the conventional relativistic Fock space used in string theory inadvertently excludes a huge sector of unitary quantum states for each single mode as discussed in Sec. IV. As made clear following Eq. (A16), the relativistic oscillator actually likes to cross between spacelike and timelike regions. Such allowed motions of each single mode simply have never entered the discussion, and therefore there is much room for investigation.

In that connection, it is worth noting that from the earliest period of string theory there have been indications that the light-cone gauge fails to capture all of the gauge invariant physics in string theory (see footnote 4). A similar phenomenon of missing gauge invariant sectors was seen in the gauge fixed relativistic oscillator discussed in this paper. Therefore, gauge fixed treatments, while being quite revealing, cannot be trusted as being complete.

These observations provide new motivation to revisit the covariant quantization of string theory to see whether the concepts discussed in this paper play a role. In the standard treatment of string theory each mode is associated with the spacelike vacuum $|0\rangle$, so the standard overall string vacuum is $|0, 0, 0, \cdots\rangle$, where each 0 corresponds to a mode. Is it possible to have string configurations built on more complicated vacua, such as $|0, \tilde{0}, 0', \cdots\rangle$, etc. where the various modes could be in various spacetime regions? It is not so easy to answer this question because of the Virasoro constraints.

The sector with all the modes in the timelike Fock space based on $|0', 0', 0', \cdots\rangle$, abbreviated as $|0'\rangle$, is not difficult to decipher because the analysis is parallel to the usual treatment. The only change is that in this sector all creation-annihilation operators α_n^μ, α_{-n}^μ switch roles relative to the familiar spacelike sector. Then we find that this sector has a lot of serious problems. The eigenvalues of $Q_n = \frac{1}{2}(p_n^2 + n^2 x_n^2)$ are strictly negative and $L_0' = p_0^2 + \sum_n Q_n + a$, which is normal ordered relative to $|0'\rangle$ (see footnote 3), has only negative eigenvalues. Hence the Virasoro constraint $L_0' = 1$ gives only tachyons. The Virasoro constraints $L_{-n}|\phi'\rangle$ with $n > 0$ (not L_n) can be satisfied by using the same arguments as [14–16] but switching α_n^μ with α_{-n}^μ at every step. However, the solutions still have ghosts at every mass level because the oscillators α_n^i in d space dimensions produce negative norm states (as opposed to only one time component α_{-n}^0 in the usual arguments). Evidently, this sector is not acceptable on physical grounds and must be eliminated with some consistent set of gauge symmetries or other arguments. The supersymmetric version of string theory may avoid this sector altogether, but this needs to be investigated more explicitly.

A more interesting case is the ghost-free fully unitary sector based on the vacuum of type $|\tilde{0}, \tilde{0}, \tilde{0}, \cdots\rangle$ which we abbreviate as $|\tilde{0}\rangle$. For example, the string state $|k, \tilde{0}\rangle$ has a spacetime configuration of the form [note the relative + sign in $(x_{n0}^2 + \vec{x}_n^2)$]

$$\psi(X) \sim \langle X|k, \tilde{0}\rangle \sim e^{ik\cdot x_0} \exp\left(-\frac{1}{2}\sum_{n=1}^{\infty} n(x_{n0}^2 + \vec{x}_n^2)\right), \tag{10.1}$$

where x_n^μ can be in any spacetime region, unlike the usual string field in Eq. (1.7) where x_n^μ was strictly spacelike. This is one of the eigenstates of L_0. There are now an infinite number of eigenstates for each eigenvalue of $Q_n = \frac{1}{2}(p_n^2 + n^2 x_n^2)$, as explained in Sec. IV, leading to the same eigenvalue of L_0. All of these states are in infinite dimensional unitary representations of $SU(d, 1)$. After applying the Virasoro constraints the solutions get rearranged into representations of the overall Poincaré symmetry.[16] The good thing is that there are no ghosts at all in this Fock space. However, it is not straightforward to solve the Virasoro constraints for string states built on $|k, \tilde{0}\rangle$ because the creation-annihilation operators in the time direction α_n^0, α_{-n}^0 have their roles inverted while those in the space directions $\vec{\alpha}_n$, $\vec{\alpha}_{-n}$ remain the same. Solutions seem likely to exist but none are known at this stage. If solutions of the Virasoro constraints can be exhibited, they would be of great interest in string theory. This seems to be a challenging problem that we leave to future work.

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APPENDIX: SO(1, 1) OSCILLATOR IN POSITION SPACE

In this appendix we solve the differential equation $(-\frac{1}{2}\partial^\mu\partial_\mu + \frac{1}{2}x^\mu x_\mu)\psi_\lambda(x) = \lambda\psi_\lambda(x)$ in the purely spacelike region[17] and show that we arrive at the same conclusion as the oscillator approach using the Fock space methods of Sec. III. For simplicity, we will concentrate on one-space and one-time dimensions. Therefore, the Lorentz symmetry is SO(1, 1) while the larger hidden symmetry is SU(1, 1).

We will discuss the spacelike region shown in Fig. 1, knowing that the timelike region is similar as indicated in Sec. III. Accordingly, we parametrize x^μ as follows to ensure spacelike x^μ:

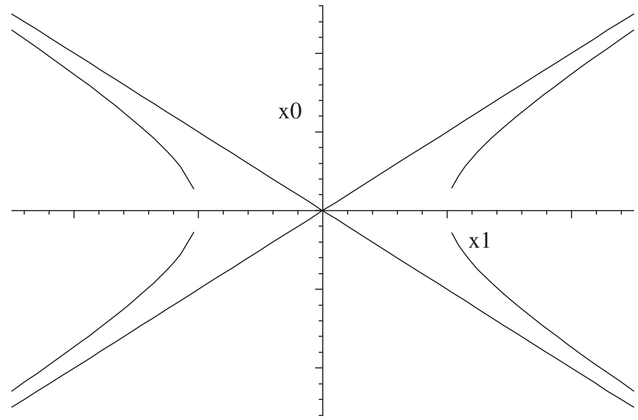

FIG. 1. Parabolas in the spacelike region of (x^0, x^1) at some fixed $x = \pm a$ and any θ.

$$x^0 = |x| \sinh\theta, \qquad x^1 = x\cosh\theta, \qquad \text{both } x, \theta \text{ range from } -\infty \text{ to } +\infty. \tag{A1}$$

This parametrization matches the parabolas in Fig. 1 for fixed positive or negative values of x, and as x is varied the entire spacelike region is covered. The differentials

$$dx^0 = \varepsilon(x)\sinh\theta dx + |x|\cosh\theta d\theta, \qquad dx^1 = \cosh\theta dx + x\sinh\theta d\theta, \tag{A2}$$

$$dx = -\varepsilon(x)\sinh\theta dx^0 + \cosh\theta dx^1, \qquad d\theta = \frac{\cosh\theta dx^0}{|x|} - \frac{\sinh\theta dx^1}{x}, \tag{A3}$$

where $\varepsilon(x) \equiv \text{sign}(x)$, are useful to compute the derivatives by using the chain rule $\frac{\partial}{\partial x^\mu} = \frac{\partial\theta}{\partial x^\mu}\partial_\theta + \frac{\partial x}{\partial x^\mu}\partial_x$ to obtain

$$\frac{\partial}{\partial x^0} = \varepsilon(x)\left[\frac{\cosh\theta}{x}\partial_\theta - \sinh\theta\partial_x\right], \qquad \frac{\partial}{\partial x^1} = -\frac{\sinh\theta}{x}\partial_\theta + \cosh\theta\partial_x. \tag{A4}$$

The SO(1, 1) boost generator becomes (note the extra sign due to raising/lowering the timelike index $p^0 = -i\partial/\partial x_0 = +i\partial/\partial x^0$)

$$L^{01} = x^0p^1 - x^1p^0 = -ix^0\frac{\partial}{\partial x^1} - ix^1\frac{\partial}{\partial x^0} = -i\varepsilon(x)\partial_\theta. \tag{A5}$$

The operator Q in x^μ space is then computed as

$$Q = \frac{1}{2}(p\cdot p + x\cdot x) = \frac{1}{2}\left[-\partial_x^2 - \frac{1}{x}\partial_x + \frac{1}{x^2}\partial_\theta^2\right] + \frac{1}{2}x^2. \tag{A6}$$

The solution of the eigenvalue equation $Q\psi_{\lambda m} = \lambda\psi_{\lambda m}$ takes the separable form

$$\psi_{\lambda m}(x, \theta) = x^{-1/2}F_{\lambda m}(x)e^{im\theta}, \tag{A7}$$

[16] The separate SU$(d, 1)$ of each single oscillator is not expected to survive in string theory because the Virasoro constraints couple all the modes, including the center of mass mode, to each other. Certainly there is at least an overall Poincaré symmetry, so that the states get rearranged into representations of Poincaré with its little group [e.g. SO(d) for massive states]. Of course, then the infinite dimensional SU$(d, 1)$ representations dissociate [they already are in the SU$(d)\times$U(1) basis in Eq. (4.32)] and are rearranged properly according to Poincaré symmetry (or a larger hidden symmetry if any such thing remains).

[17] There are more general Lorentz covariant solutions that have different forms in various spacelike and timelike regions with continuity conditions across the light cone $x^\mu x_\mu = 0$ in Fig. 1. This will become evident in the discussion following Eq. (A16). For this kind of solution the setting in Sec. IV is more convenient. In this section we will seek solutions with support only in the spacelike regions, because those are the only ones described by the standard SO$(d, 1)$ covariant Fock space approach discussed in Sec. III, to which we compare the solutions in this appendix.

where the factor of $x^{-1/2}$ is inserted for convenience. The eigenvalue m of the operator $(-i\partial_\theta)$ must be real if $L^{01}=-i\varepsilon(x)\partial_\theta$ is to be Hermitian. This condition on m *imposes unitarity*; hence only positive norms are possible (see footnote 7). The range of m is the entire continuous real line $-\infty<m<\infty$. Then $F_{\lambda m}(x)$ satisfies

$$\left\{-\partial_x^2-\frac{m^2+\frac{1}{4}}{x^2}+x^2-2\lambda\right\}F_{\lambda m}(x)=0. \tag{A8}$$

This is a one-dimensional problem with an effective potential that has an attractive (negative) component

$$V_{\rm eff}(x)=-\frac{m^2+\frac{1}{4}}{2x^2}+\frac{1}{2}x^2. \tag{A9}$$

$V_{\rm eff}(x)$ is plotted in Fig. 2. For this shape of potential we expect that there are normalizable bound states. We also need to define a normalization and *include in the spectrum only the normalizable solutions* of this equation.

We can choose the square integrable norm

$$\langle\psi_{\lambda m}|\psi_{\lambda' m'}\rangle=\int d^2x(\psi_{\lambda m}(x))^*\psi_{\lambda' m'}(x) \tag{A10}$$

$$=\int_{-\infty}^{\infty}dxF^*_{\lambda m}(x)F_{\lambda' m'}(x)\int_{-\infty}^{\infty}d\theta e^{i(m'-m)\theta} \tag{A11}$$

$$=\delta(m-m')2\pi\int_{-\infty}^{\infty}dxF^*_{\lambda m}(x)F_{k'm'}(x) \tag{A12}$$

$$=\delta(m-m')\delta_{kk'}. \tag{A13}$$

In this case we must require a finite integral in x space,

$$2\pi\int_{-\infty}^{\infty}dxF^*_{km}(x)F_{k'm}(x)=\delta_{kk'}. \tag{A14}$$

Next we solve for the allowed values of k, m. The Schrödinger equation in Eq. (A8) is related to the confluent hypergeometric equation, and the solutions are given by a linear superposition of the confluent hypergeometric functions $M(a,b,x^2)$, $U(a,b,x^2)$. The solution that is well behaved at $x^2\to\infty$ is given by

$$\psi_{\lambda m}(x)=\alpha e^{-x^2/2}x^{im}U([\tfrac{1}{2}-\tfrac{1}{2}\lambda+\tfrac{1}{2}im],[1+im],x^2), \tag{A15}$$

where α is a normalization constant. This expression is even when m is replaced by $-m$ due to the property $U(a,b,z)=z^{1-b}U(1+a-b,2-b,z)$. This is in agreement with the unitarity condition of Eq. (3.17), since the operator T in that equation also reverses the sign of the boost operator $L^{0i}\to-L^{0i}$, and hence demands that only states that are even under $m\to-m$ can appear in the unitary spectrum. Since we have already demanded unitarity of L^{0i}, it has to be true that only states even under $m\to-m$ should emerge automatically in the spectrum of Q.

The behavior at $x\to\pm\infty$ is convergent, $\psi_{\lambda m}(x)\sim|x|^{\lambda-1}e^{-(1/2)x^2}(1+O(1/x^2))$. The small $x\to0$ behavior is given by (replace m by $m\mp i\varepsilon$ with small, real ε)

$$\psi_{\lambda m}(x)\to\begin{cases}\text{if } m\neq0 & \frac{\Gamma(1\pm im)}{\Gamma(\frac{1-\lambda}{2}\pm\frac{1}{2}im)}\frac{|x|^{\pm im}}{\pm im+\varepsilon}x^{\mp2\varepsilon},\ \varepsilon\to0^+\\ \text{if } m=0 & \frac{-1}{\Gamma(\frac{1}{2}-\frac{1}{2}\lambda)}(\ln x^2+O(1)).\end{cases} \tag{A16}$$

Therefore, the norm $\int dx|x||\psi_{\lambda m}(x)|^2$ is integrable at $x=0,\pm\infty$; hence $\psi_{\lambda m}(x)$ is normalizable. This is in line with expectations on the basis of the shape of the effective potential in Fig. 2.

The probability density $|\psi_{\lambda m}(x,\theta)|^2$ does not generally vanish at $x=0$, which is everywhere at the light cone $x^\mu x_\mu=0$ in Fig. 1. The physical meaning of this result is that the oscillating particle in a spacelike region generally has a nonvanishing probability at the light cone. A similar computation in the timelike region will also show that the light cone is an allowed region of spacetime. Therefore it would make sense to match the probability amplitude in the spacetime region to the one in the timelike region at the light cone. Then we would get solutions in which the oscillating particle moves easily from the spacetime to the timelike regions and vice versa. This kind of general solution is discussed in a more convenient setting in Sec. IV.

There are, however, quantum states in which the leakage from the spacetime to the timelike regions does not occur at all. This is seen by examining Eq. (A16) and noting that for Lorentz singlets ($m=0$) the probability amplitude vanishes at the light cone when $\frac{1}{2}-\frac{1}{2}\lambda$ is a negative integer or zero. Hence only for the following quantized values of m, λ is it consistent to have a purely spacelike relativistic harmonic oscillator:

$$m=0\quad\text{and}\quad\lambda=1+2k,\quad\text{with integer } k=0,1,2,3,\cdots. \tag{A17}$$

For these values of λ the solution U reduces to a polynomial as follows:

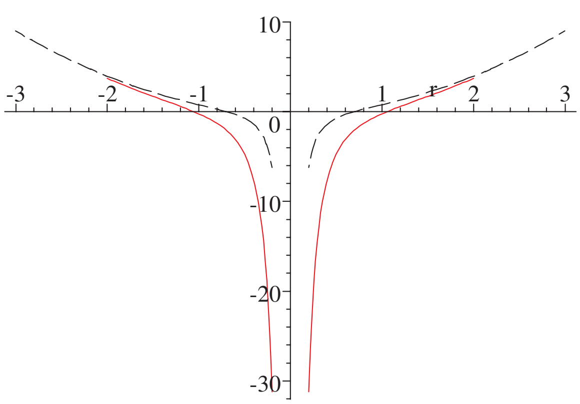

FIG. 2 (color online). The dashed line is for $m=0$, and the solid line is for $m\neq0$.

$$\psi_k = \tilde{\alpha}_k e^{-x^2/2} U(-k, 1, x^2) = \alpha_k e^{-x^2/2} L_k^0(x^2), \quad \text{(A18)}$$

where $L_k^0(x^2)$ is the Laguerre polynomial with argument x^2.

$$L_k^0(x^2) = \sum_{m=0}^{k} \frac{(-1)^m k!}{(m!)^2(k-m)!} x^{2m}, \qquad k = 0, 1, 2, 3, \cdots. \quad \text{(A19)}$$

So, the probability density $x|\psi|^2$ vanishes at the light cone.

This result in $d = 1$ is in full agreement with the oscillator approach of Sec. III for general d. The oscillator method, which was valid only for the spacelike region, also yielded only Lorentz singlets Eq. (3.20) as the only positive norm states in a unitary representation of the Lorentz group SO$(d, 1)$. Furthermore, the eigenvalues of $Q \to \lambda = 1 + 2k$ agree when specialized to $d = 1$.

What happened to the finite dimensional Lorentz representations with ghosts that showed up in the Fock space approach in Sec. III? Those had emerged in Fock space by applying oscillators $\bar{a}_\mu$ on the vacuum $|0\rangle$. What do we get if we follow the same approach in position space? To investigate this we start with the oscillators in the Cartesian basis,

$$a_0 = \frac{1}{\sqrt{2}}\left(-x^0 + \frac{\partial}{\partial x^0}\right), \qquad \bar{a}_0 = \frac{1}{\sqrt{2}}\left(-x^0 - \frac{\partial}{\partial x^0}\right), \quad \text{(A20)}$$

$$a_1 = \frac{1}{\sqrt{2}}\left(x_1 + \frac{\partial}{\partial x^1}\right), \qquad \bar{a}_1 = \frac{1}{\sqrt{2}}\left(x_1 - \frac{\partial}{\partial x^1}\right), \quad \text{(A21)}$$

and transform them to the (x, θ) basis as

$$a_0 = \frac{\varepsilon(x)}{\sqrt{2}}\left(-\sinh\theta(x + \partial_x) + \frac{\cosh\theta}{x}\partial_\theta\right), \quad \text{(A22)}$$

$$\bar{a}_0 = \frac{\varepsilon(x)}{\sqrt{2}}\left(-\sinh\theta(x - \partial_x) - \frac{\cosh\theta}{x}\partial_\theta\right), \quad \text{(A23)}$$

$$a_1 = \frac{1}{\sqrt{2}}\left(\cosh\theta(x + \partial_x) - \frac{\sinh\theta}{x}\partial_\theta\right), \quad \text{(A24)}$$

$$\bar{a}_1 = \frac{1}{\sqrt{2}}\left(\cosh\theta(x - \partial_x) + \frac{\sinh\theta}{x}\partial_\theta\right). \quad \text{(A25)}$$

Clearly, a_0, a_1 both annihilate the ground state $\psi_{vac}(x, \theta) = \langle x|0\rangle = e^{-x^2/2}$ since it is independent of θ and satisfies $(x + \partial_x)e^{-x^2/2} = 0$,

$$a_0|0\rangle \to a_0 e^{-x^2/2} = 0, \qquad a_1|0\rangle \to a_1 e^{-x^2/2} = 0. \quad \text{(A26)}$$

If we try to create states with $\bar{a}_1$, $\bar{a}_0$, we automatically obtain solutions to the differential equation, but we see that the θ dependence is not normalizable as follows:

$$\bar{a}_0|0\rangle \Rightarrow \frac{\varepsilon(x)}{\sqrt{2}}\left(-\sinh\theta(x - \partial_x) - \frac{\cosh\theta}{x}\partial_\theta\right)e^{-x^2/2} = -\sqrt{2}|x|e^{-x^2/2}\sinh\theta, \quad \text{(A27)}$$

$$\bar{a}_1|0\rangle \Rightarrow \frac{1}{\sqrt{2}}\left(\cosh\theta(x - \partial_x) + \frac{\sinh\theta}{x}\partial_\theta\right)e^{-x^2/2} = \sqrt{2}xe^{-x^2/2}\cosh\theta. \quad \text{(A28)}$$

These are solutions, but they do not have the unitary form $e^{\pm im\theta}$. (A29)

Indeed, the boost $L^{01} = -i\varepsilon(x)\partial_\theta$ is Hermitian only for the $e^{\pm im\theta}$ basis; it is not Hermitian for the $(\sinh\theta, \cosh\theta)$ or $e^{\pm\theta}$ basis. Therefore, such excited states cannot be included in the spectrum if unitarity is imposed from the beginning as was done in this section.

We emphasize that the oscillator states $\bar{a}_0|0\rangle$, $\bar{a}_1|0\rangle$ are excluded for two reasons. First, they are *not in a unitary representation of the Lorentz group* SO(1, 1) or of the hidden symmetry group SU(1, 1); second they are *not normalizable* according to the square integrable norm defined above because their norm diverges for the θ integral $\int_{-\infty}^{\infty} d\theta(\sinh\theta)^2 = \infty$, etc. It is important to emphasize that the square integrable norm above is different than the Fock space norm. On that issue note that $\bar{a}_0|0\rangle$, $\bar{a}_1|0\rangle$ are normalizable if one uses the definition of the norm in the *nonunitary* Fock space of Sec. III; however this admits negative as well as positive norms.

Following the oscillator approach in position space we obtain square integrable normalizable states only for the singlets as follows. We compute $\bar{a}\cdot\bar{a}$ and note that it is independent of θ,

$$\bar{a}\cdot\bar{a} = -\bar{a}_0\bar{a}_0 + \bar{a}_1\bar{a}_1 = \frac{1}{2}\left(x - \frac{1}{x} - \partial_x\right)(x - \partial_x).$$

Therefore, $(\bar{a}\cdot\bar{a})^k$ creates θ-independent excited states, which are Lorentz singlets. For $k = 1$ we can now compute the oscillator state in Eq. (3.20). This gives

$$(\bar{a}\cdot\bar{a})\langle x|0\rangle = \frac{1}{2}\left(x - \frac{1}{x} - \partial_x\right)(x - \partial_x)e^{-x^2/2} = 2(x^2 - 1)e^{-x^2/2}, \quad \text{(A30)}$$

which is in agreement with Eq. (A18) for $k = 1$,

$$\psi_1(x) = \alpha e^{-x^2/2}L_1^0(x^2) = \alpha e^{-x^2/2}(1 - x^2). \quad \text{(A31)}$$

More generally, we can verify that the oscillator states $(\bar{a}\cdot\bar{a})^k\langle x|0\rangle$ reproduce the Laguerre polynomials

$$\psi_k(x) \sim (\bar{a}\cdot\bar{a})^k\langle x|0\rangle \quad \text{(A32)}$$

$$= \left[\frac{1}{2}\left(x - \frac{1}{x} - \partial_x\right)(x - \partial_x)\right]^k e^{-x^2/2} \tag{A33}$$

$$\sim \alpha_k e^{-x^2/2} L_k^0(x^2). \tag{A34}$$

These are certainly normalizable in x space, and have a positive norm, so they are included in the positive norm spectrum. This is in complete agreement with the results for general d of Sec. III.

In the present approach the selection of the correct set of states emerged automatically on the basis of normalizability and unitarity of the Lorentz generator L^{01} with the chosen norm of Eqs. (A10) and (A14). Of course, this amounts to the same criterion of Sec. III.

However, in the present approach we have not seen so far why only the vacuum state $\langle x|0\rangle$ must be kept. For this, we apply the SU(1, 1) generators, such as $\bar{a}_0 a_1$ or $\bar{a}_1 a_0$ on the states $\psi_{\lambda m}(x, \theta)$, and note that this takes us out of the unitary space $e^{im\theta}$ as explained in Eqs. (A27)–(A29). This means that the restriction to only the spacelike region, *plus unitarity*, generally breaks the SU(1, 1) covariance of the problem. This is like breaking symmetries via boundary conditions. The covariance can be fully maintained only in the vacuum state. Thus, if one is to seek solutions that are consistent with SU(1, 1) covariance, then only the vacuum state can satisfy this criterion. Again, this is in agreement with the Fock space approach of Sec. III.

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