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Quantum vortices of type II superconductors in curved space-time and Cattaneo's projection method

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In this paper we show that the covariant dynamical equations of quantum vortices in type II superconductors may be brought into the same form as the classical equations written in 3-vector forms and can then be interpreted by using the familiar language of physics. This result is attained by introducing the physically measurable quantities by means of the corresponding standard relative quantities as defined by Cattaneo and by using his operator of transverse derivation, partial or covariant, with respect to the x^4 lines. The standard equations thus obtained have a tensorial character and are covariant under the sole changes of coordinates leaving invariant the x^4 lines, i.e. changes of coordinates internal to the physical system of reference S associated with the coordinates (x^i). Expressions of the electric and magnetic fields induced inside a type II superconductor at rest in a curved space-time are obtained. The generation of these fields is influenced not only by the presence of a gravitational field but also by the presence of vortices. Comparison is made with the results predicted by the method of anholonomic frames.

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I. INTRODUCTION

In a previous paper [1] we have shown that the generalization of the covariant London equation

$$\partial_{[i}U_{j]} + \frac{Q}{mc}F_{ij} = 0 \qquad (i, j = 1, 2, 3, 4), \tag{1.1}$$

to type II superconductors may be written in the following form:

$$\partial_{[i}U_{j]} + \frac{Q}{mc}F_{ij} = \frac{Q}{mc}sS_{ij}, \tag{1.2}$$

where F_{ij} $(i, j = 1, 2, 3, 4)$ and U_j are, respectively, the electromagnetic field tensor and the 4-velocity $(U_jU^j = -c^2)$ of the superelectron of proper mass m and charge Q $(Q < 0)$. The antisymmetric tensor S_{ij} embodies the properties of the system of vortices and is defined by

$$S_{ij} = \tfrac{1}{2}s\eta_{ijkl}D^{kl}, \tag{1.3a}$$
$$D^{kl} = -u^k(L)w^l(L) + u^l(L)w^k(L), \tag{1.3b}$$

where $u^k(L)$ and $w^l(L)$ are, respectively, the 4-velocity of the vortex and the unit spacelike vector defining the direction of the vortex. The scalar s $(s^2 = \frac{1}{2}S^{ij}S_{ij})$ is proportional to the proper density of vortices. The permutation tensor η is in contravariant and covariant forms given by

$$\eta_{ijkl} = \sqrt{-g}\varepsilon_{ijkl}, \qquad \eta^{ijkl} = -\frac{1}{\sqrt{-g}}\varepsilon_{ijkl}, \tag{1.4}$$

and by virtue of Eqs. (1.2) and (1.3) may also be exhibited in the following form:

$$\partial_{[i}U_{j]} + \frac{Q}{mc}F_{ij} = -\frac{Qs}{mc}\eta_{ijkl}u^k(L)w^l(L). \tag{1.5}$$

The physical interpretation of Eqs. (1.1) and (1.2) is not immediate; it necessitates the knowledge of the relations connecting the absolute tensorial quantities appearing in these equations to the physically measurable quantities. For instance, one must be able to relate the components of the absolute tensor F_{ij} to the physically measurable electric and magnetic field 3-vector. A difficulty inherent to the theory of general relativity is the lack of a unique mathematical representation of the physically measurable quantities in terms of the corresponding absolute quantities. In a recent note [2] the physical interpretation of Eq. (1.1) in general relativity and a discussion of certain electrodynamic effects derived from it were developed by applying Cattaneo's projection method [3–7]. This method has a purely tensorial character; absolute tensorial equations are given a form similar to that of classical physics with added terms representing the influence of the gravitational field and are interpreted by using the familiar language of physics. This result is attained by the systematic use of a differential operation that generalizes the usual absolute differentiation and by the introduction of the physically measurable quantities by means of suitably defined "standart quantities," relative to the chosen system of reference S, which transform according to the classical tensorial law on changes of coordinates internal to the system of reference S.

The purpose of this paper is to extend Cattaneo's approach to the discussion of certain aspects of the electrodynamics of type II superconductors in curved space-time.

We shall see that the presence of vortices has an influence on the generation of the electromagnetic field inside the superconductor. Whereas the magnetic field is conditioned by the presence of vortices, the generation of an electric field, because of the phenomenon of unipolar induction, is related to the motion of magnetic vortices. The estimations of electric and magnetic fields in the superconducting proton and quark cores of neutron stars will be considered. The main source of electric field is the inhomogeneous gravitational filed, whereas the magnetic field is generated by the set of superconducting quantum vortices. The magnetic field generation takes place as a result of the collapse of the supernova remnant into the neutron star and via the ‘‘entrainment’’ of superconducting protons by superfluid neutrons [8]. In both cases they have different direction of the dipole axis and the magnetic field is concentrated in the proton or quark vortices. Section II gives the prerequisites of Cattaneo's method. The natural projection of a tensor is defined by means of the space and time projectors $\gamma_{ij} = g_{ij} + \gamma_i\gamma_j$ and $-\gamma_i\gamma_j$ (γ^i unit tangent vector to the x^4 line). The operation of covariant transverse derivation is introduced and the definitions of kinematical and electromagnetic standard relative quantities are given. In Sec. III, the standard London equations of type II superconductors relative to the system of reference S associated with the physically admissible coordinates (x^i), with the x^4 lines along the world lines of the normal part, are obtained. The expressions of the electric and magnetic fields induced, inside a type II superconductor at rest, by the presence of a gravitational field and of vortices are derived from the standard London equations. Using Eqs. (3.15) and (3.16) the order of magnitude of the electric and magnetic fields are estimated. In Sec. IV comparision is made with the results predicted by the method of anholonomic frame. Section V contains concluding remarks.

II. MATHEMATICAL PRELIMINARIES: STANDARD RELATIVE QUANTITIES

For purposes of reference we briefly recall (i) some basic points concerning the operation of covariant transverse derivation, and (ii) the definitions of kinematical and electromagnetic standard relative quantities introduced by Cattaneo's [3,7,9]. A more detailed review is given in Ref. [2].

Let V_4 be the space-time manifold with

$$ds^2 = g_{ij}dx^i dx^j \qquad (i, j = 1, 2, 3, 4) \tag{2.1}$$

the fundamental quadratic form with signature +2, (x^i) a physically admissible system of coordinates i.e. the x^4 lines being timelike lines with unit tangent vectors $\vec{\gamma}$ pointing into the future and satisfying

$$\gamma^\alpha = 0 \qquad (\alpha = 1, 2, 3), \qquad \gamma^4 = \frac{1}{\sqrt{-g_{44}}}. \tag{2.2}$$

The ∞^3 ideal particles having the x^4 lines as world lines form the physical system of reference S associated with the coordinates (x^i). A physical significance will be ascribed to quantities which are covariant under transformations of coordinates leaving invariant the x^4 lines. At every event $x \in V_4$, the subspaces θ_x and Σ_x of the tangent space T_x respectively parallel and orthogonal to $\vec{\gamma}$ define the time and space associated with the event x, the tensors $-\gamma_i\gamma_j$ and $\gamma_{ij} = g_{ij} + \gamma_i\gamma_j$ acting, respectively, as time-projector and space-projector. Denoting by P_θ (P_Σ) the operations of projection on θ_x (Σ_x) the natural projections of a vector $\vec{V} \in T_x$ are given by

$$P_\theta(V_i) = -\gamma_i\gamma_k V^k, \qquad P_\Sigma(V_i) = \gamma_{ik}V^k, \qquad V_i = P_\Sigma(V_i) + P_\theta(V_i). \tag{2.3}$$

Similarly the natural projections of a tensor A_{ij} are obtained by means of the space and time projectors in the following way:

$$P_{\Sigma\Sigma}(A_{ij}) = \gamma_{ir}\gamma_{js}A^{rs}, \qquad P_{\Sigma\theta}(A_{ij}) = -\gamma_{ir}\gamma_j\gamma_s A^{rs},$$
$$P_{\theta\Sigma}(A_{ij}) = -\gamma_i\gamma_r\gamma_{js}A^{rs}, \qquad P_{\theta\theta}(A_{ij}) = \gamma_i\gamma_j\gamma_r\gamma_s A^{rs}. \tag{2.4}$$

We are particularly interested in the natural projections of an antisymmetric tensor $A_{ij} = -A_{ji}$, for such a tensor the natural decomposition may be written

$$A_{ij} = \tilde{A}_{ij} + \tilde{A}_i\gamma_j + \gamma_i\tilde{A}'_j \qquad (\tilde{A}_{ij} = -\tilde{A}_{ji}), \tag{2.5}$$

where

$$\tilde{A}_{ij} = P_{\Sigma\Sigma}(A_{ij}), \qquad \tilde{A}_i = -\gamma_{ir}\gamma_s A^{rs}, \qquad \tilde{A}'_j = -\gamma_r\gamma_{js}A^{rs}. \tag{2.6}$$

(The symbol $\sim$ denotes the spatial character of the tensors and vectors i.e. $\tilde{A}_4 = 0$, $\tilde{A}_{i4} = 0$.) The covariant transverse derivation $\tilde{\nabla}^*$ operates on a spatial vector $\tilde{v}$ or on a spatial tensor $\tilde{T}_{ij}$ and is defined by

$$\tilde{\nabla}^*_i\tilde{v}_j = \gamma^h_i\gamma^k_j\nabla_h\tilde{v}_k, \qquad \tilde{\nabla}^*_i\tilde{T}_{jk} = \gamma^h_i\gamma^k_j\gamma^r_l\nabla_h\tilde{T}_{kr}. \tag{2.7}$$

Such a derivative can be written in the same form as an usual covariant derivative in a Riemannian three-dimensional space with Christoffel symbols $\tilde{\Gamma}^{*k}_{ij}$ constructed by means of the tensor $\gamma_{\alpha\beta} = g_{\alpha\beta} + \gamma_\alpha\gamma_\beta$ instead of $g_{\alpha\beta}$ and the systematic substitution of the ordinary partial derivative ∂_α by the transverse partial derivative [3,6,7]

$$\tilde{\partial}_i = \partial_i - \frac{\gamma_\alpha}{\gamma_4}\partial_4. \tag{2.8}$$

The definition of the absolute transverse differentiation is then given by

$$\tilde{d}^*\tilde{v}_j = dx^i\tilde{\nabla}^*_i\tilde{v}_j = dx^i(\tilde{\partial}_i\tilde{v}_j - \tilde{\Gamma}^{*k}_{ij}\tilde{v}_k). \tag{2.9}$$

(The symbols $*$, $\sim$ recalling, respectively, that the metric tensor γ_{ij} must be used instead of g_{ij} and the transverse derivation $\tilde{\partial}_\alpha$ instead of the partial derivative ∂_α).

For a moving test particle of constant proper mass m, 4-velocity $u^i = \frac{dx^i}{d\tau}$, and 4-momentum $P^i = mu^i$, the following standard relative quantities are considered:

(a) Standard spatial metric tensor $\gamma_{\alpha\beta} = g_{\alpha\beta} + \gamma_\alpha\gamma_\beta$ and standard relative time

$$dT = -\frac{1}{c}\gamma_i dx^i. \tag{2.10}$$

(b) Standard relative velocity V and its space projection $\tilde{v}$, called the standard relative 3-velocity, defined, respectively, by

$$V = \frac{dx^i}{dT}e_i, \qquad \tilde{v} = \frac{dx^\rho}{dT}\tilde{e}_\rho = \gamma^i_k\frac{dx^k}{dT}e_i = \tilde{v}^i e_i. \tag{2.11}$$

The components of $\tilde{v}$ and V are thus related by

$$\tilde{v}^{\,i} = \gamma^i_k V^k \qquad \left(V^k = \frac{dx^k}{dT}\right). \tag{2.11a}$$

The components of the natural basis vectors $\tilde{e}_\alpha$ of Σ relative to the natural frame (e_i) of V_4 are given by

$$\begin{aligned}(\tilde{e}_\alpha)_i &= \gamma_{ik}(e_\alpha)^k = \gamma_{i\alpha},\\ (\tilde{e}_\alpha)^i &= g^{ih}(\tilde{e}_\alpha)_h = \delta^i_\alpha + \gamma^i\gamma_\alpha.\end{aligned} \tag{2.12}$$

The two frames being related by the formula

$$\tilde{e}_{\,\alpha} = e_\alpha + \gamma^4\gamma_\alpha e_4. \tag{2.13}$$

The spatial norm of $\tilde{v}$ and its covariant components are

$$\tilde{v}^{\,2} = \gamma_{\alpha\beta}\tilde{v}^\alpha\tilde{v}^\beta, \qquad \tilde{v}_\alpha = \gamma_{\alpha\beta}\tilde{v}^\beta. \tag{2.14}$$

(c) Standard relative mass M whose definition is similar to that of the relative mass in special relativity

$$M = m\frac{dT}{d\tau} = \frac{m}{\sqrt{1-\frac{\tilde{v}^2}{c^2}}}. \tag{2.15}$$

(d) Standard relative 3-momentum

$$\tilde{p}^{\,\alpha} = P^\alpha = M\tilde{v}^\alpha. \tag{2.16}$$

Let us note that the corresponding covariant components $\tilde{p}_\alpha = \gamma_{\alpha\beta}\tilde{p}^\beta$ differ from the components P_α of the 4-momentum since $P_\alpha = g_{\alpha i}P^i = \gamma_{\alpha\beta}P^\beta + Mc\gamma_\alpha$.

The standard relative quantities associated with the electromagnetic field tensor F_{ij} are derived from the natural decomposition (2.5) [7,9]. Writing

$$\begin{aligned}P_{\Sigma\Sigma}(F_{ij}) &= \tilde{H}_{ij} = \gamma_{ir}\gamma_{js}F^{rs},\\ P_{\Sigma\theta}(F_{ij}) &= -\gamma_{ir}\gamma_j\gamma_s F^{rs} = -\tilde{E}_i\gamma_j,\\ P_{\theta\Sigma}(F_{ij}) &= -\gamma_i\gamma_r\gamma_{js}F^{rs} = \gamma_i\tilde{E}_j,\end{aligned} \tag{2.17}$$

the familiar language of physics is introduced by calling the vector $\tilde{E}_i$ ($\tilde{E}_4 = 0$) the electric field relative to the system of reference S and the antisymmetric tensor $\tilde{H}_{ij}$ ($\tilde{H}_{4j} = 0$) the magnetic field tensor; the magnetic field vector being ascribed to the dual of $\tilde{H}_{ij}$ in Σ_x

$$\tilde{h}^{\,\alpha} = \tfrac{1}{2}\tilde{\eta}^{\alpha\rho\sigma}\tilde{H}_{\rho\sigma} \tag{2.18}$$

where

$$\tilde{\eta}_{\,\alpha\rho\sigma} = \sqrt{\gamma}\varepsilon_{\alpha\rho\sigma}, \qquad \tilde{\eta}^{\alpha\rho\sigma} = \frac{1}{\sqrt{\gamma}}\varepsilon^{\alpha\rho\sigma}, \tag{2.19}$$

$\varepsilon^{\alpha\rho\sigma} = \varepsilon_{\alpha\rho\sigma}$ being the usual permutation symbol and γ the determinant of $\gamma_{\alpha\beta}$.

Remark 1.—The tensor $\tilde{\eta}_{\alpha\rho\sigma}$ can be regarded as a tensor of T_x, $\tilde{\eta}_{ijl} = \delta^\alpha_i\delta^\rho_j\delta^\sigma_l\,\tilde{\eta}_{\alpha\rho\sigma}$, is related to the tensor $\eta_{ijlm} = \sqrt{-g}\varepsilon_{ijlm}$ by the following relation:

$$\tilde{\eta}_{\,ijl} = \eta_{ijlm}\gamma^m. \tag{2.20}$$

All the above standard relative quantities, as well as the notion of transverse derivation, are invariant under changes of coordinates internal to the given physical system of reference S.

III. STANDARD RELATIVE DYNAMICAL EQUATIONS FOR TYPE II SUPERCONDUCTORS

Let us consider a universe with metric form (2.1). In this universe we have a type II superconductor with world lines of the normal part along the x^4 lines i.e. at rest with respect to the physical system of reference S associated with the coordinates (x^i). Consequently the 4-velocity $u^i(n)$ and the 4-vector $w^i(L)$, defining the direction of the vortex, satisfy respectively,

$$u^\alpha(n) = 0, \qquad g_{44}(u^4(n))^2 = -c^2, \tag{3.1a}$$

$$\gamma^i w_i(L) = \gamma^4 w_4(L) = 0 \quad \text{or} \quad w_4(L) = 0. \tag{3.1b}$$

The tensorial index i of w^i is thus purely spatial. The standard dynamical equations relative to the physical system of reference S will be derived by considering the natural projections $P_{\Sigma\Sigma}$, $P_{\Sigma\theta}$, and $P_{\theta\Sigma}$ of the absolute tensorial equation (1.2), the projection $P_{\theta\theta}$ playing no role because of the antisymmetry of the tensors. Before performing the above projections, it is convenient to introduce the natural decompositions of U_i, F_{ij}, and S_{ij}. With reference to (2.3) and (2.11) we have

$$\tilde{u}_i = P_\Sigma(U_i) = \gamma_{ir}\frac{dx^r}{dT}\frac{dT}{d\tau} = \Gamma\tilde{v}_i,$$
$$\tau_i = P_\theta(U_i) = -\gamma_i\gamma_r\frac{dx^r}{dT}\frac{dT}{d\tau} = c\Gamma\gamma_i, \tag{3.2}$$
$$\Gamma = \frac{dT}{d\tau} = \left(1 - \frac{\tilde{v}^2}{c^2}\right)^{-1/2},$$

so that, using the natural decomposition of the skew-symmetric tensors S_{ij} and F_{ij} as given, respectively, by (2.5) and (2.17), Eq. (1.2) may be exhibited in the form

$$\nabla_{[i}\tilde{u}_{j]} + \nabla_{[i}\tau_{j]} + \frac{Q}{mc}(\tilde{H}_{ij} - \tilde{E}_i\gamma_j + \tilde{E}_j\gamma_i) = \frac{Q}{mc}(\tilde{S}_{ij} + \tilde{S}_i\gamma_j + \gamma_i\tilde{S}'_j). \tag{3.3}$$

Let us now project the various terms of Eq. (3.3) on θ and Σ. The expressions of the natural projections $P_{\Sigma\Sigma}$, $P_{\Sigma\theta}$, and $P_{\theta\Sigma}$ of the covariant derivatives $\nabla_i\tilde{u}_j$ ($\nabla_i\tau_j$) are obtained from the definition of the covariant derivative of a 4-vector, $\nabla_iA_j = \partial_iA_j - \Gamma^k_{ij}A_k$, by applying the time and space projectors $-\gamma_i\gamma_j$ and γ_{ij} and may be expressed in terms of the transverse covariant and partial derivation operators (2.7) and (2.8) and of the following tensors characterizing the properties of the system of reference S [5,7].

(a) The space vortex tensor

$$\tilde{\Omega}_{ij} = P_{\Sigma\Sigma}(\Omega_{ij}) = \gamma_4\left(\tilde{\partial}_i\frac{\gamma_j}{\gamma_4} - \tilde{\partial}_j\frac{\gamma_i}{\gamma_4}\right),$$
$$\tilde{\Omega}_{i4} = -\tilde{\Omega}_{4i} = 0, \qquad \Omega_{ij} = \partial_i\gamma_j - \partial_j\gamma_i. \tag{3.4}$$

(b) The Born tensor $\tilde{K}_{ij} = P_{\Sigma\Sigma}(K_{ij}) = \gamma^4\partial_4\gamma_{ij}$ ($\tilde{K}_{4i} = \tilde{K}_{i4} = 0$, $K_{ij} = \nabla_i\gamma_j + \nabla_j\gamma_i$).

(c) The curvature vector of the x^4 line $C_i = \gamma^r\nabla_r\gamma_i$.

Remark 2.—The vanishing of C_i, or of $\tilde{\Omega}_{ij}$, or of $\tilde{K}_{ij}$, characterizes, respectively, the geodesic frames, the spatially irrotational frames, and the rigid frames in the sense of Born. The calculations of the natural projections $P_{\Sigma\Sigma}$, $P_{\Sigma\theta}$, and $P_{\theta\Sigma}$ of the covariant derivative of a purely spatial 4-vector $\tilde{u}_i(\tilde{u}_4 = 0)$ [a purely temporal 4-vector $\tau^i(\tau^\alpha = 0)$] are exposed in Refs. [6,7]; the corresponding expressions are given in the appendix. From these expressions, the projections $P_{\Sigma\Sigma}$, $P_{\Sigma\theta} + P_{\theta\Sigma}$ of the alternated derivatives $\nabla_{[i}\tilde{u}_{j]}$ ($\nabla_{[i}\tau_{j]}$) are found to be as follows:

$$P_{\Sigma\Sigma}(\nabla_{[i}\tilde{u}_{j]}) = \tilde{\nabla}^*_{[i}\tilde{u}_{j]} = (\widetilde{rot}\,\tilde{u})_{ij},$$
$$(P_{\Sigma\theta} + P_{\theta\Sigma})(\nabla_{[i}\tilde{u}_{j]}) = -\gamma_i\bar{\partial}_4\tilde{u}_j + \gamma_j\bar{\partial}_4\tilde{u}_i \qquad (\bar{\partial}_4 = \gamma^4\partial_4),$$
$$P_{\Sigma\Sigma}(\nabla_{[i}\tau_{j]}) = c\Gamma\tilde{\Omega}_{ij},$$
$$(P_{\Sigma\theta} + P_{\theta\Sigma})(\nabla_{[i}\tau_{j]}) = \gamma_j\left\{\tilde{\partial}_i(c\Gamma) - c\Gamma\left[\partial_4\left(\frac{\gamma_i}{\gamma_4}\right) - \tilde{\partial}_i\log\sqrt{-g_{44}}\right]\right\} - \gamma_i\left\{\tilde{\partial}_j(c\Gamma) - c\Gamma\left[\partial_4\left(\frac{\gamma_j}{\gamma_4}\right) - \tilde{\partial}_j\log\sqrt{-g_{44}}\right]\right\}. \tag{3.5}$$

Using (3.5), the projections $P_{\Sigma\Sigma}$, $P_{\Sigma\theta}$, and $P_{\theta\Sigma}$ of Eq. (3.3) are, respectively, given by

$$(\widetilde{rot}\,\tilde{u})_{ij} + c\Gamma\tilde{\Omega}_{ij} + \frac{Q}{mc}\tilde{H}_{ij} = \frac{Q}{mc}\tilde{S}_{ij}, \tag{3.6}$$

$$\gamma_j\left\{\bar{\partial}_4\tilde{u}_i + \tilde{\partial}_i(c\Gamma) - c\Gamma\left[\partial_4\left(\frac{\gamma_i}{\gamma_4}\right) - \tilde{\partial}_i\log\sqrt{-g_{44}}\right] - \frac{Q}{mc}\tilde{E}_i - \frac{Q}{mc}\tilde{S}_i\right\} = \gamma_i\left\{\bar{\partial}_4\tilde{u}_j + \tilde{\partial}_j(c\Gamma) - c\Gamma\left[\partial_4\left(\frac{\gamma_j}{\gamma_4}\right) - \tilde{\partial}_j\log\sqrt{-g_{44}}\right] - \frac{Q}{mc}\tilde{E}_j - \frac{Q}{mc}\tilde{S}'_j\right\}. \tag{3.7}$$

Remark 3.—Let us note that in Eqs. (3.6) and (3.7) the Latin indices can take only the values 1, 2, 3. For $i = 4$ ($j = 4$) these equations reduce to the identity $0 = 0$.

On multiplying Eq. (3.7) by γ^i we thus get

$$\left\{\bar{\partial}_4\tilde{u}_\alpha + \tilde{\partial}_\alpha(c\Gamma) - c\Gamma\left[\partial_4\left(\frac{\gamma_\alpha}{\gamma_4}\right) - \tilde{\partial}_\alpha\log\sqrt{-g_{44}}\right] - \frac{Q}{mc}\tilde{E}_\alpha\right\} = \frac{Q}{mc}\tilde{S}_\alpha. \tag{3.8}$$

Substitution of (1.3) in the definition of $\tilde{S}_i$ as given in (2.6), $\tilde{S}_i = -\gamma_{ir}\gamma_sS^{rs}$, yields the following expression for $\tilde{S}_\alpha$ in terms of the vectors $u(L)$ and $w(L)$ characterizing the vortex:

$$\tilde{S}_\alpha = -S\tilde{\eta}_{\alpha\beta\gamma}u^\beta(L)w^\gamma(L) = -S\Gamma(L)\tilde{\eta}_{\alpha\beta\gamma}\tilde{v}^\beta(L)w^\gamma(L), \tag{3.9}$$

where $\tilde{v}^\beta(L)$ is the standard relative 3-velocity of the vortex ($\tilde{v}^\beta = \gamma^\beta_rV^r = V^\beta$, $U^\beta = \frac{dx^\beta}{dT}\frac{dT}{d\tau} = V^\beta\Gamma$).

We now bring Eq. (3.6) into a form permitting an interpretation similar to that given in classical electrodynamics of superconductors. For this purpose we multiply Eq. (3.6) by $\frac{1}{2}\tilde{\eta}^{\alpha\rho\sigma}$ and introduce the dual vectors of the skew-symmetric tensors $\tilde{H}_{\rho\sigma}$, $(\widetilde{rot}\,\tilde{u})_{\rho\sigma}$, and $\tilde{S}_{\rho\sigma}$, the duality correspondence operating in Σ_x. As mentioned in Sec. II, Eq. (2.18), the dual of $\tilde{H}_{\rho\sigma}$ defines the components $\tilde{h}^\alpha$ of the magnetic field vector relative to the physical system of reference S. Following Cattaneo the dual of $(\widetilde{rot}\,\tilde{u})_{\rho\sigma}$ is written in the form

$$(\widetilde{rot}\,\tilde{u})^{\alpha} = \tfrac{1}{2}\tilde{\eta}^{\alpha\rho\sigma}(\widetilde{rot}\,\tilde{u})_{\rho\sigma} \tag{3.10}$$

and called by him the curl of the vector $\tilde{u}_j$. The dual vector of $\tilde{S}_{\rho\sigma}$ is denoted by $\tilde{*}\tilde{S}^{\alpha}$ and its expression in terms of $u(L)$ and $w(L)$ is

$$\tilde{*}\tilde{S}^{\alpha} \equiv \tfrac{1}{2}\tilde{\eta}^{\alpha\beta\gamma}\tilde{S}_{\rho\sigma} = -s(\gamma^4 u_4(L))w^{\alpha}(L). \tag{3.11}$$

With reference to the definition (2.10) of standard time, $\tilde{*}\tilde{S}^{\alpha}$ in terms of the standard relative velocity $\tilde{v}$ can be written as

$$\tilde{*}\tilde{S}^{\alpha} = S\Gamma(L)w^{\alpha}(L)\left[1 + \gamma_{\sigma}\frac{\tilde{v}^{\sigma}(L)}{c}\right] \tag{3.12}$$

so that Eq. (3.6) may be exhibited in the form

$$(\widetilde{rot}\,\tilde{u})^{\alpha} + 2\Gamma\omega^{\alpha} + \frac{Q}{mc}\tilde{h}^{\alpha} = \frac{Q}{mc}S\Gamma(L)w^{\alpha}(L) \times\left[1 + \gamma_{\sigma}\frac{\tilde{v}^{\sigma}(L)}{c}\right] \tag{3.13}$$

where

$$\omega^{\alpha} = \tfrac{1}{4}c\,\tilde{\eta}^{\alpha\rho\sigma}\tilde{\Omega}_{\rho\sigma} \tag{3.14}$$

is the local angular velocity of the system of reference S. Equations (3.8) and (3.13) may be regarded as an alternate formulation of the absolute tensor equation (1.2) written in terms of the standard electric and magnetic field vectors ($\tilde{E}_{\alpha}$ and $\tilde{h}^{\alpha}$) relative to the physical system of reference S. They have the same form as the corresponding classical dynamical equations written in 3-vector forms with added terms representing the influence of the gravitational field. The above equations may thus be used to derive certain electrodynamic and gravitational effects. As an illustrative example let us derive the expressions of the electric and magnetic fields inside a type II superconductor at rest in the presence of a gravitational field. Since there is no current inside a superconductor, setting the standard velocity $\tilde{v}$ ($\tilde{u} = \Gamma\tilde{v}$) of the superelectron equal to zero in Eqs. (3.8) and (3.13), one obtains immediately

$$\tilde{E}_{\alpha} = \frac{mc}{Q}\left[\partial_4\left(\frac{\gamma_{\alpha}}{\gamma_4}\right) - \tilde{\partial}_{\alpha}\log\sqrt{-g_{44}}\right] - S\Gamma(L)\tilde{\eta}_{\alpha\beta\gamma}\tilde{v}^{\beta}(L)w^{\gamma}(L), \tag{3.15}$$

$$\tilde{h}^{\alpha} = -2\frac{cm}{Q}\omega^{\alpha} + S\Gamma(L)w^{\alpha}(L)\left[1 + \gamma_{\alpha}\frac{\tilde{v}^{\alpha}(L)}{c}\right]. \tag{3.16}$$

By comparison with electromagnetism $\log\sqrt{-g_{44}}$ and $\frac{\gamma_{\alpha}}{\gamma_4}$ may thus be regarded as the analogues of scalar and vector potentials [7]. In the case of a spatially irrotational frame $\tilde{\Omega}_{ij} = 0$ (the x^4 lines forming a normal congruence), so that the only surviving term in (3.16) is the contribution of the vortices. Let us also note that, for a stationary gravitational field, the transverse partial derivative $\tilde{\partial}_{\alpha}\log\sqrt{-g_{44}}$ is equal to the ordinary partial derivative $\partial_{\alpha}\log\sqrt{-g_{44}}$ [see (2.8)] hence $\tilde{E}_{\alpha}$ reads

$$\tilde{E}_{\alpha} = -\frac{mc}{Q}\partial_{\alpha}\log\sqrt{-g_{44}} - S\Gamma(L)\tilde{\eta}_{\alpha\beta\gamma}\tilde{v}^{\beta}(L)w^{\gamma}(L). \tag{3.17}$$

According to formula (3.15) the electric field is generated by inhomogeneous gravitational field and in a neutron star can be defined as $E \sim \frac{mc^2}{Q}\frac{R_g}{R^2}$, where R and R_g are, respectively, the ordinary and gravitational radii of the star. This estimation will give $E \sim 10^3$ Volt/cm. Moving vortices will also generate the additional filed, but this term will be much smaller, because $v(L)/c \sim 10^{-17}$. This term is of the order of 0.3 Volt/cm.

According to formula (3.16) the main input in the strength of the magnetic field is given by the second term, which is proportional to the density of quantum vortices. The first term of formula (3.16) is very small. If the magnetic filed is generated by collapse its mean induction B will be of the order of 10^{12} G. In the case of generation by the entrainment effect it will be of the order of 10^{14} G and will be concentrated in the central part of neutron vortices [8]. These estimations are valid for hadronic and quark cores, because both of them are type II superconductors. The difference between these two cases is the value of the superconducting gap from which the critical value of the magnetic field depends.

If the core of neutron star is not a type II but rather a type I superconductor, the estimation for electric field is the same. The magnetic filed generated in the process of collapse must be very inhomogeneous and partially strong to create normal domains through which it can penetrate. In the case of generation of the magnetic field by entrainment effect it will pass through normal cylindrical domains, which are located in the central part of the neutron vortex. The induction vector B has the same order of magnitude as in the case of the type II superconductor [10].

IV. COMPARISON WITH THE METHOD OF ANHOLONOMIC FRAMES

It is not without interest to compare the above results to those predicted by another method. In Ref. [11] the electromagnetic field generated inside a type II superconductor in a stationary gravitational field was investigated by adopting the tetrad approach. In contradistinction to the method of projections, equations describing a physical phenomenon are written in terms of scalars under arbitrary transformations of coordinates and physical significance is ascribed only to invariant components of tensors on an orthonormal tetrad $e_{\underline{k}}$ ($k = 0, 1, 2, 3$) with $e_{\underline{0}}$ timelike and future-pointing and $e_{\underline{\alpha}}$ ($\alpha = 1, 2, 3$) spacelike. Let us also note that the definition of a system of reference as a continuous distribution of hypothetical observers with 4-velocity $e_{\underline{0}}$ with each of whom is associated a frame of reference differs from that employed by Cattaneo, in particular, one does not require that this continuous distribution of observers be at rest with respect to the chosen coordinate system. The tensorial equation (1.2) with respect to the natural basis vectors, $e_i = \partial/\partial x^i$, was transformed with respect to an orthonormal tetrad $e_{\underline{k}}$ adopted to the stationary character of the Universe, i.e. $e_{\underline{0}}$ pointing along the timelike Killing vector [this choice is equivalent to identify $e_{\underline{0}}$ with the 4-velocity $u(n)$ of the normal part], with the appearance of correction terms containing the object of anholonomity [11,12] $C^{\underline{c}}_{\underline{b}\,\underline{a}} = e^j_{\underline{b}} e^i_{\underline{a}} \partial_{[j} e^{\underline{c}}_{i]}$. The nonvanishing components of this mathematical object determine the contribution of the gravitational field to the electromagnetic field generated inside the superconductor. The physical electric and magnetic field intensities as measured by an observer with 4-velocity $e_{\underline{0}} = U(n)$, introduced by means of the invariants

$$E_\alpha = F_{\underline{0}\,\underline{\alpha}} = -F^{\underline{0}\,\underline{\alpha}}, \qquad H_\gamma = \tfrac{1}{2}\varepsilon_{\gamma\alpha\beta}F_{\alpha\beta} \qquad (F_{\underline{i}\,\underline{j}} = e^l_{\underline{i}} e^m_{\underline{j}} F_{lm}) \tag{4.1}$$

are given by

$$E_\alpha = \frac{mc}{Q}\frac{1}{\zeta}\partial_{\underline{\alpha}}\zeta - \varepsilon_{\alpha\rho\sigma}u^{\underline{\rho}}(L)w^{\underline{\sigma}}(L), \tag{4.2}$$

$$\varepsilon_{\gamma\alpha\beta}H_\gamma = \frac{mc}{Q}\zeta.\nabla_{[\underline{\alpha}}\varphi_{\underline{\beta}]} + S\varepsilon_{\alpha\beta mn}u^{\underline{m}}(L)w^{\underline{n}}(L), \tag{4.3}$$

where $\partial_{\underline{\alpha}}$ is the Pfaffian derivative and $\zeta = \sqrt{g_{00}}$. $\dot{\nabla}_{\underline{\alpha}}$ is the covariant Pfaffian derivation operator related to the metric $\gamma_{\alpha\beta} = g_{\alpha\beta} - \frac{g_{0\alpha}g_{0\beta}}{g_{00}}$, the anholonomic component $\varphi_{\underline{i}} = e^l_{\underline{i}}\varphi_l$. A formal resemblance with the corresponding expressions of the standard electric and magnetic field vectors is detected when in Eqs. (3.15) and (3.16) the g's are assumed independent of the time coordinate [see Eq. (3.17)]. In both sets of equations the field intensities involve, respectively, the partial derivatives and alternated derivatives of the metric tensor.

V. CONCLUSION

Adopting Cattaneo's projection method we have presented an alternate formulation of the tensorial dynamical equations of vortices in type II superconductors. These equations have a form similar to the classical equations of motion in where the various forces acting on matter are explicitly exhibited. In Cattaneo's approach the metric tensor components enters in the definition of various standard relative quantities and in the operation of transverse derivation.

Assuming that there is no current inside superconductors, we have derived from the standard relative equations of motion the expressions of the physically measurable magnetic and electric field vectors. These fields are induced by the presence of inhomogeneous gravitational field and quantum vortices. Whereas the generation of the electric field is related to the motion of magnetic vortices (unipolar induction), the magnetic field is conditioned only by their presence and the rotation of the physical system of reference.

The results we have obtained may be used to investigate various aspects of the electrodynamics of rotating neutron stars i.e. pulsars. Since the nucleus of a neutron star consists of a proton-electron plasma, the electric field generated by the proton vortices will induce a change in the charge distribution of plasma. Problems related to this questions will be studied.

APPENDIX: NATURAL PROJECTIONS OF THE COVARIANT DERIVATIVE OF A 4-VECTOR

1. Natural projections of $\nabla_i \boldsymbol{\tau}_j$ ($\boldsymbol{\tau}_j = c\Gamma\boldsymbol{\gamma}_j \in \boldsymbol{\theta}_x$)

Projection-$\Sigma\Sigma$: $P_{\Sigma\Sigma}(\nabla_i\tau_j) = \frac{1}{2}c\Gamma(\tilde{K}_{ij} + \tilde{\Omega}_{ij})$.

Projection-$\Sigma\theta$: $P_{\Sigma\theta}(\nabla_i\tau_j) = c\gamma_j\tilde{\partial}_i\Gamma$.

Projection-$\theta\Sigma$: $P_{\theta\Sigma}(\nabla_i\tau_j) = -c\gamma_i C_j$.

Projection-$\theta\theta$: $P_{\theta\theta}(\nabla_i\tau_j) = -c\gamma^r\partial_r\Gamma.\gamma_i\gamma_j$.

2. Natural projection of $\nabla_i \tilde{\boldsymbol{u}}_j$ ($\tilde{\boldsymbol{u}}_j \in \boldsymbol{\Sigma}_x$)

Projection-$\Sigma\Sigma$: $P_{\Sigma\Sigma}(\nabla_i\tilde{u}_j) = \tilde{\nabla}^*_i\tilde{u}_j$.

Projection-$\Sigma\theta$: $P_{\Sigma\theta}(\nabla_i\tilde{u}_j) = \frac{1}{2}(\tilde{K}_{ir} + \tilde{\Omega}_{ir})\tilde{u}^r\gamma_j$.

Projection-$\theta\Sigma$: $P_{\theta\Sigma}(\nabla_i\tilde{u}_j) = \gamma_i(-\gamma^4\partial_4\tilde{u}_j + \frac{1}{2}\times[\tilde{K}_{jh} + \tilde{\Omega}_{jh}]\tilde{u}^h)$.

Projection-$\theta\theta$: $P_{\theta\theta}(\nabla_i\tilde{u}_j) = -C_r\tilde{u}^r\gamma_i\gamma_j$.

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